Simulating mechanical systems with frictional contact using a nonsmooth generalized-alpha method

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In this paper, we introduce a nonsmooth generalized-alpha method for the simulation of mechanical systems with frictional contact. In many engineering applications, such systems are composed of rigid and flexible bodies, which are interconnected by joints and can come into contact with each other or their surroundings. Prominent examples are automotive, wind turbine, and robotic systems. It is known form structural mechanics applications, that generalized-alpha schemes perform well for flexible multibody systems without contacts. This motivated the development of nonsmooth generalized-alpha methods for the simulation of mechanical systems with frictional contacts [2,3,5]. Typically, the Gear-Gupta-Leimkuhler approach is used to stabilize the unilateral constraint, such that numerical penetration of the contact bodies can be avoided - a big issue of the most popular time-stepping schemes such as Moreau's scheme. The nonsmooth generalized-alpha method presented in this paper is derived in [2] and in contrast to [3,5] accounts for set-valued Coulomb-type friction on both velocity and acceleration level. Finally, we validate the method using a guided flexible hopper as a benchmark mechanical systems.

Consider an n-dimensional mechanical system with frictional contact and let the coordinates $\mathbf{q}(t) \in \mathbb{R}^n$ describe the system's configuration as a function of time t. Moreover, we introduce the velocities $\mathbf{u}$ and accelerations $\mathbf{a}$ respectively satisfying $\mathbf{u} = \dot{\mathbf{q}}$ and $\mathbf{a} = \dot{\mathbf{u}}$ for almost all t. Using the generalized-α method, these relations are discretized as

$$
\begin{aligned}
&\mathbf{u}_{i+1} = \mathbf{u}_i + \Delta t\big((1-\gamma)\bar{\mathbf{a}}_i + \gamma\bar{\mathbf{a}}_{i+1}\big) + \mathbf{U}_{i+1} \\
&\mathbf{q}_{i+1} = \mathbf{q}_i + \Delta t\,\mathbf{u}_i + \frac{\Delta t^2}{2}\big((1-2\beta)\bar{\mathbf{a}}_i + 2\beta\bar{\mathbf{a}}_{i+1}\big) + \mathbf{Q}_{i+1} \\
&\alpha_m\bar{\mathbf{a}}_i + (1-\alpha_m)\bar{\mathbf{a}}_{i+1} = \alpha_f\mathbf{a}_i + (1-\alpha_f)\mathbf{a}_{i+1}\,,
\end{aligned}
\tag{1}
$$

where Δt is the time step of the scheme and β, γ, α_f and α_m are numerical parameters. Moreover, the auxiliary accelerations $\bar{\mathbf{a}}_i$ and $\bar{\mathbf{a}}_{i+1}$, as well as the discrete quantities $\mathbf{U}_{i+1}$ and $\mathbf{Q}_{i+1}$, which account for the effects of velocity jumps, have been introduced. We assume that the mechanical system is subjected to n_g ideal bilateral constraints formulated at position level as $\mathbf{g}(t,\mathbf{q}) = 0 \in \mathbb{R}^{n_g}$. The Gear–Gupta–Leimkuhler approach is used to prevent numerical constraint drift, i.e., the constraints are numerically imposed at all kinematic levels as

$$
\mathbf{g}(t_{i+1},\mathbf{q}_{i+1}) = 0, \quad \dot{\mathbf{g}}(t_{i+1},\mathbf{q}_{i+1},\mathbf{u}_{i+1}) = 0 \quad \text{and} \quad \ddot{\mathbf{g}}(t_{i+1},\mathbf{q}_{i+1},\mathbf{u}_{i+1},\mathbf{a}_{i+1}) = 0\,. \tag{2}
$$

To model the contacts occurring in the mechanical system, we assume that they can be described by n_N ideal unilateral constraints at position level $\mathbf{g}_N(t,\mathbf{q}) \geq 0$, where the inequality holds component-wise and $\mathbf{g}_N(t,\mathbf{q}) \in \mathbb{R}^{n_N}$ are the gap functions describing the distance between the tangent planes of the pairs of contact points on either contacting bodies, see [7]. Similar to the bilateral constraints, these unilateral constraints are imposed on all kinematic levels by the normal cone inclusions

$$
g^k_{N,i+1} \in \mathcal{N}_{\mathbb{R}^-_0}(-\hat{\kappa}^k_{N,i+1})\,, \quad
\begin{matrix} k \in A_{i+1}: & \xi^k_{N,i+1} \in \mathcal{N}_{\mathbb{R}^-_0}(-P^k_{N,i+1}) \\ k \in \bar{A}_{i+1}: & P^k_{N,i+1} = 0\,, \end{matrix} \quad , \quad
\begin{matrix} k \in B_{i+1}: & \ddot{g}^k_{N,i+1} \in \mathcal{N}_{\mathbb{R}^-_0}(-\lambda^k_{N,i+1}) \\ k \in \bar{B}_{i+1}: & \lambda^k_{N,i+1} = 0 \end{matrix} \quad . \tag{3}
$$

Herein, the subscript $i+1$ indicates that the corresponding quantity is evaluated at t_{i+1}, e.g., $g^k_{N,i+1} = g^k_N(t_{i+1},\mathbf{q}_{i+1})$. Moreover, similar to (1), the update formulae

$$
\begin{aligned}
&\mathbf{P}_{N,i+1} = \mathbf{\Lambda}_{N,i+1} + \Delta t\big((1-\gamma)\bar{\boldsymbol{\lambda}}_{N,i} + \gamma\bar{\boldsymbol{\lambda}}_{N,i+1}\big) \\
&\hat{\boldsymbol{\kappa}}_{N,i+1} = \boldsymbol{\kappa}_{N,i+1} + \frac{\Delta t^2}{2}\big((1-2\beta)\bar{\boldsymbol{\lambda}}_{N,i} + 2\beta\bar{\boldsymbol{\lambda}}_{N,i+1}\big) \\
&\alpha_m\bar{\boldsymbol{\lambda}}_{N,i} + (1-\alpha_m)\bar{\boldsymbol{\lambda}}_{N,i+1} = \alpha_f\boldsymbol{\lambda}_{N,i} + (1-\alpha_f)\boldsymbol{\lambda}_{N,i+1}
\end{aligned}
\tag{4}
$$

are used for the force quantities and the kinematic quantity $\xi^k_N = \dot{g}^{k+}_N + e^k_N\dot{g}^{k-}_N$ with restitution coefficient e^k_N in (3) is used to model a generalized Newton-type impact for the k-th contact pair. Finally, the index sets $A_{i+1} = \big\{k = 1,\ldots,n_N \mid g^k_{N,i+1} \leq 0\big\}$ as well as $B_{i+1} = \big\{k \in A_{i+1} \mid \xi^k_{N,i+1} \leq 0\big\}$ together with their respective complements $\bar{A}_{i+1}$ and $\bar{B}_{i+1}$ have been introduced.

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In order to model Coulomb friction in each contact, the set of admissible (negative) friction forces $C_F(\lambda_F^k) = \{\boldsymbol{\lambda}_F^k \in \mathbb{R}^2 \mid ||\boldsymbol{\lambda}_F^k|| \leq \mu^k \lambda_N^k\}$ is defined, see [7]. With that, the discretized friction laws read as

$$\begin{aligned} k \in A_{i+1}: &\quad \boldsymbol{\xi}_{F,i+1}^k \in \mathcal{N}_{C_F(P_{N,i+1}^k)}(-\mathbf{P}_{F,i+1}^k) \\ k \in \bar{A}_{i+1}: &\quad \mathbf{P}_{F,i+1}^k = 0\,, \end{aligned} \quad \text{and} \quad \begin{aligned} k \in D_{i+1}^{\mathrm{st}}: &\quad \dot{\boldsymbol{\gamma}}_{F,i+1}^k \in \mathcal{N}_{C_F(\lambda_{N,i+1}^k)}(-\boldsymbol{\lambda}_{F,i+1}^k) \\ k \in D_{i+1}^{\mathrm{sl}}: &\quad \boldsymbol{\lambda}_{F,i+1}^k = -\mu^k \lambda_{N,i+1}^k \frac{\boldsymbol{\gamma}_{F,i+1}^k}{||\boldsymbol{\gamma}_{F,i+1}^k||} \\ k \in \bar{A}_{i+1}: &\quad \boldsymbol{\lambda}_{F,i+1}^k = 0\,, \end{aligned} \tag{5}$$

where $\boldsymbol{\gamma}_F^k$ denotes the relative velocity of the contact pair in tangent direction to the contact planes and the discrete friction forces $\boldsymbol{\lambda}_{F,i+1}^k$ and $\mathbf{P}_{F,i+1}^k$ are related analogously to (4). Moreover, the index sets of sticking and slipping contacts have respectively been introduced as $D_{i+1}^{\mathrm{st}} = \{k \in A_{i+1} \mid \boldsymbol{\xi}_{F,i+1}^k = 0\}$ and $D_{i+1}^{\mathrm{sl}} = A_{i+1} \setminus D_{i+1}^{\mathrm{st}}$, and frictional contact is accounted for by defining the kinematic quantity $\xi_F^k = \dot{\boldsymbol{\gamma}}_F^{k+} + e_F^k \dot{\boldsymbol{\gamma}}_F^{k-}$ with restitution coefficient e_F^k. The description of the mechanical system is completed by stating the discretized equations of motion, which read as

$$\begin{aligned} \mathbf{M}_{i+1}\,\mathbf{a}_{i+1} &= \mathbf{h}_{i+1} + \mathbf{W}_{g,i+1}\,\boldsymbol{\lambda}_{g,i+1} + \mathbf{W}_{N,i+1}\,\boldsymbol{\lambda}_{N,i+1} + \mathbf{W}_{F,i+1}\,\boldsymbol{\lambda}_{F,i+1} \\ \mathbf{M}_{i+1}\mathbf{U}_{i+1} &= \mathbf{W}_{g,i+1}\,\boldsymbol{\Lambda}_{g,i+1} + \mathbf{W}_{N,i+1}\,\boldsymbol{\Lambda}_{N,i+1} + \mathbf{W}_{F,i+1}\,\boldsymbol{\Lambda}_{F,i+1} \\ \mathbf{M}_{i+1}\mathbf{Q}_{i+1} &= \mathbf{W}_{g,i+1}\boldsymbol{\kappa}_{g,i+1} + \mathbf{W}_{N,i+1}\boldsymbol{\kappa}_{N,i+1} + \tfrac{\Delta t}{2}\mathbf{W}_{F,i+1}\boldsymbol{\Lambda}_{F,i+1}\,, \end{aligned} \tag{6}$$

where $\mathbf{M}$ denotes the mass matrix and $\mathbf{h}$ collects all forces which are not constraint or contact forces. Furthermore, the Jacobians $\mathbf{W}_g^{\mathrm{T}} = \frac{\partial \mathbf{g}}{\partial \mathbf{q}}$, $\mathbf{W}_N^{\mathrm{T}} = \frac{\partial \mathbf{g}_N}{\partial \mathbf{q}}$ and $\mathbf{W}_F^{\mathrm{T}} = \frac{\partial \boldsymbol{\gamma}_F}{\partial \mathbf{u}}$ have been introduced. A time step of the presented generalized-α scheme is described by the normal cone inclusion problem (1)–(6), which can be solved either by a semi-smooth Newton method or by fixed-point iterations after a reformulation using the proximal point function, see [1,2].

The suitability of the presented nonsmooth generalized-α scheme for the simulation of flexible multibody systems is demonstrated by simulating a guided hopper. It consists of a vertically guided main body, which is addressed by the coordinate y. At the hip H a rigid homogeneous rod is attached to the main body. The orientation of the rod is prescribed by the angle $\alpha(t) = \frac{\pi}{3} - \frac{\pi}{30}\big(1 - \cos(4\pi\, t)\big)$. A straight planar Euler–Bernoulli beam [4] with undeformed length L is connected to the knee K of the rod by an actuated rotational joint with prescribed actuation angle $\beta(t) = \pi - 2\alpha(t)$, which is modeled as a bilateral constraint. We follow [6] and discretize the centerline of the beam with B-Spline shape functions. The percussions,

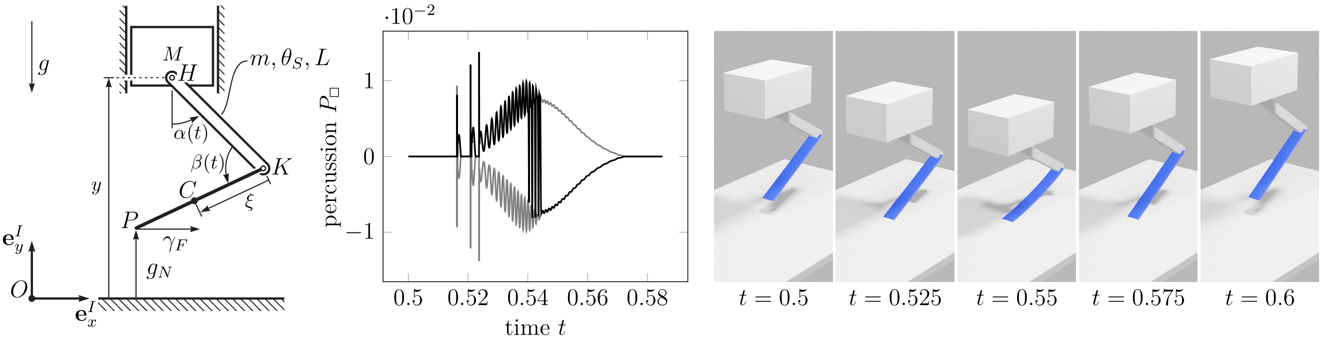

Fig. 1: Left to right: 1. Sketch of the guided hopper. 2. Simulated time evolution of the percussions (black: P_F, gray: $\pm\mu P_N$). 3. Visualization of contact phase.

plotted in Figure 1, show that the generalized-α scheme can cope with the complex contact dynamics arising in multibody systems containing flexible parts and time dependent bilateral constraints. This makes the presented scheme well suited for engineering applications.

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