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On the Foundations of Continuum Mechanics and its Application to Beam Theories

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To My Family

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Abstract

The foundations of mechanics deal with the identification of the fundamental objects and the postulation of its principles. The mechanical principles together with constitutive laws enable the description and prediction of the motion of mechanical systems and processes. This work is concerned with fundamental questions on the foundations of continuum mechanics and with the application of these concepts to beam theories.

Due to the high level of abstraction, the mathematical discipline of intrinsic differential geometry seems to be best suited for the description of continuum mechanics. Step by step, additional mathematical structure can be introduced and motivated by the underlying physics. A geometric description of continuum mechanics is on the one hand coordinate independent and on the other hand a priori metric independent. In this thesis, body and space as the central objects of continuum mechanics are introduced as smooth manifolds. Whereas balance of linear and angular momentum in integral form are not applicable on manifolds, it is possible to formulate the principle of virtual work within this generalized setting. The virtual displacement field is defined as an element of the tangent space of the infinite dimensional configuration manifold constituted by the set of all embeddings of the body into the space. Moreover, the set of forces of a continuous body in the sense of duality and its representation are discussed. With further assumptions on internal and external force contributions, the principle of virtual work of classical continuum mechanics can be postulated from an intrinsic differential geometric point of view. Especially the concept of stress is set in a new light. Insofar, the variational stress is considered as the linear map of the covariant derivative of the virtual displacement field to a volume form of the body manifold.

The theory of beams is another branch of mechanics which shows the strength of a variational formulation of a continuous body given by its virtual work principle. It is possible to consider a beam as a continuous body with a constrained position field guaranteed by a perfect constraint stress field. Defining a constrained position field and applying the restricted kinematics to the principle of virtual work of a continuous body, the constraint stresses are eliminated due to the principle of d'Alembert–Lagrange and the weak variational form of an appropriate beam theory is induced. Such an approach to beam theory relates the point of view of beams as generalized one-dimensional continua to the theory of continuous bodies. In this work all classical beam theories, where the cross sections remain rigid and plain, are presented. Additionally, augmented beam theories, where cross section deformation is allowed, are derived using the very same procedure. All theories are suitable for large displacements and large rotations.

Zusammenfassung

Die Axiomatisierung der Mechanik handelt von der Identifikation der fundamentalen Objekte und der Formulierung der Grundprinzipien. Zusammen mit konstitutiven Gesetzen ermöglichen mechanische Prinzipien die Beschreibung und die Voraussage der Bewegungen von mechanischen Systemen und Prozessen. Diese Arbeit beschäftigt sich mit grundlegenden Fragen zur Axiomatisierung der Kontinuumsmechanik und mit der Anwendung der Grundprinzipien auf die Balkentheorie.

Aufgrund der hohen Abstraktionsstufe erscheint die intrinsische Differentialgeometrie die geeignete mathematische Disziplin für die Beschreibung der Kontinuumsmechanik zu sein. Zusätzliche mathematische Strukturen können Schritt für Schritt physikalisch motiviert eingeführt werden. Eine geometrische Beschreibung der Kontinuumsmechanik ist einerseits koordinatenunabhängig und andererseits a priori metrikunabhängig. Als zentrale Objekte werden Körper und Raum in dieser Arbeit als glatte Mannigfaltigkeiten eingeführt. Während die Impuls- und Drehimpulsbilanzgleichungen auf Mannigfaltigkeiten nicht angewendet werden können, ist es unter diesen verallgemeinerten Annahmen möglich das Prinzip der virtuellen Arbeit zu formulieren. Die Menge aller Einbettungen vom Körper in den Raum bildet die unendlich-dimensionale Konfigurationsmannigfaltigkeit. Ein Element des Tangentialraumes an diese entspricht einem gesamten virtuellen Verschiebungsfeld des Körpers. Im Sinne der Dualität werden Kontinuumskräfte eingeführt und deren Representationsmöglichkeit besprochen. Mit zusätzlichen Annahmen an innere und äussere Kräfte wird des Weiteren das Prinzip der virtuellen Arbeit für das klassische Kontinuum in einer intrinsisch differentialgeometrischen Form postuliert. Insbesondere das Konzept der Spannung wird in ein neues Licht gerückt. So wird die variationelle Spannung als lineare Abbildung von der kovarianten Ableitung des virtuellen Verschiebungsfeldes auf eine Volumenform des Körpers aufgefasst.

Die Balkentheorie entspricht einem weiteren Teil der Mechanik, welcher die Stärke einer variationellen Formulierung der Kontinuumsmechanik durch das Prinzip der virtuellen Arbeit aufzeigt. Es ist möglich einen Balken als gebundenes Kontinuum zu betrachten, dessen eingeschränkte Kinematik durch ein Zwangsspannungsfeld garantiert wird. Durch die Definition eines gebundenen Verschiebungsfeldes und durch das Anwenden der eingeschränkten Kinematik auf das Prinzip der virtuellen Arbeit des Kontinuums, wird unter Verwendung des Prinzips von d'Alembert–Lagrange die schwache variationelle Form einer zugehörigen Balkentheorie induziert. Ein solcher Zugang zur Balkentheorie verbindet die Ansicht von Balken als eindimensionale generalisierte Kontinua mit der Kontinuumstheorie. In dieser Arbeit werden alle klassischen Theorien mit starren und ebenen

Schnittebenen vorgestellt. Zusätzlich werden erweiterte Balkentheorien, welche Schnittebenendeformationen zulassen, mit derselben Methodik hergeleitet. Alle Theorien lassen grosse Verschiebungen und grosse Rotationen zu.

Contents

Chapter 1

Introduction

This monograph is concerned with fundamental questions on the foundations of continuum mechanics and its application to beam theories. It does not pretend to be in any way 'complete', but merely serves as a discussion about novel approaches applied to these very classical fields of mechanics.

This first chapter starts with a short introduction and motivation for the thesis. Subsequently, Section 1.2 sheds some light on the virtual work in mechanics. After a literature survey in Section 1.3, the aim and scope of the thesis is presented in Section 1.4. An outline of this thesis is given in Section 1.5.

1.1 Motivation

One of the main goals of mechanics is the description and the prediction of the motion of mechanical devices, machines and mechanical processes. To meet this aim, abstract mechanical theories are formulated, thereby applying concepts from mathematical science. In such a determinism, a strict separation between reality and mathematical abstraction, called the model, has to be considered. The modeling process, being the procedure of the mathematical abstraction, is an interaction between the choice of the assumed mathematical structure and the description of observations in the real world within this mathematical framework. Hence, a mechanical theory can be developed on different levels of mathematical abstraction. The higher the level of mathematical abstraction, the less mathematical objects are involved and the more general a mechanical theory is. By increasing the level of abstraction in a mechanical theory, we try to extract the essential mechanical objects. An important step on that route of abstraction is the description of mechanics with as little mathematical structure as necessary, to recognize the fundamental laws of mechanics. There exists a vast amount of specific mechanical theories in which many assumptions on the kinematics of the system and on constitutive level are taken. For instance, we may distinguish between rigid body mechanics, beam theories, shell theories, theory of elasticity, theory of fluids, finite degree of freedom mechanics to name a few. A fundamental question emerges: does a mechanical theory on a high level of abstraction exist which is able to induce these specific theories? The question immediately asks for the assumptions and concepts to arrive in a rigorous way at all these specific theories. The question of the embedding of well-known specific theories in a more general mechanical theory is one of the major challenges of modern classical mechanics. Such an embedding of theories leads to a more compact formulation of the vast field of classical mechanics. It leads to a deeper understanding of mechanics and will eventually allow treating more complex mechanical systems. This is what can be understood as scientific progress.

1.2 The Virtual Work

A rather novel insight in analytical mechanics is that the virtual work of a mechanical system is invariant with respect to the change of coordinates. This is directly related to the fact that there is a (coordinate free) differential geometric definition of the virtual work. To explain the basic idea, consider the case of finite degree of freedom mechanics, where the configuration manifold fully describes the kinematic state of the mechanical system. A generalized virtual displacement is a tangent vector of the configuration manifold. A covector of the configuration manifold as an element of the cotangent space constitutes a generalized force. The virtual work is defined as the real number obtained by the evaluation of a generalized force acting on a generalized virtual displacement. This geometric definition of the virtual work is completely free of any choice of coordinates and does not require any further geometric structure such as a metric. With the geometrical point of view in mind, the determination of the configuration manifold, being the kinematic description of the mechanical system, induces the space of generalized forces of the mechanical system. In a nutshell, the choice of kinematics defines, in the sense of duality, what kind of forces we may expect.

An illustrating example is a moving particle in the Euclidean three-space, where the very same space corresponds to the configuration manifold of the particle. Consequently, the generalized forces are elements of the cotangent space of the Euclidean three-space. In the Euclidean three-space there exist two important isomorphisms. One isomorphism is a canonical isomorphism between the tangent space and the Euclidean three-space. The second isomorphism is the isomorphism between tangent and cotangent space induced by the Euclidean metric. Using both isomorphisms, a generalized force on the particle can be identified with an element from the Euclidean three-space. This corresponds to the very classical understanding of a force as a geometric object from the Euclidean three-space satisfying the parallelogram law. As a side note, it is meaningless to speak of such thing as a couple of the particle, since there is no kinematic counterpart in the description of a particle.

Being the invariant object in mechanics, the virtual work almost naturally emerges as a central element in the postulation of the fundamental laws of mechanics. The virtual work of a mechanical system is the sum of the virtual work contributions of all forces of the mechanical system. The principle of virtual work, stated as an axiom, claims that the virtual work of a mechanical system has to vanish for all virtual displacements. Hence, the principle of virtual work as a fundamental mechanical law is a coordinate free and metric independent formulation. Introducing more geometric structure as e.g. a metric, it is possible to formulate constitutive laws which relate force quantities with kinematic quantities and to arrive at more specific mechanical theories. For instance, a metric is required to define the strain of a continuous body which is necessary for the formulation of a material law. Another example is the formulation of the dynamics of a particle moving in the Euclidean three-space. The linear relation between the velocity of the particle and the linear momentum needs a metric of the space and the mass of the particle as a proportionality factor. Thus, from a differential geometric point of view, the definition of the linear momentum can be considered as an assumption on constitutive level.

In computational mechanics for infinite dimensional systems, the principle of virtual work in the form of weak variational formulations is a fully accepted concept. It is used to perform existence and uniqueness proofs on the one hand, and to develop numerical schemes on the other hand. As a variational formulation, the principle of virtual work provides the only possibility within classical mechanics to mathematically define perfect bilateral constraints. The latter is done in form of a variational equality, known as the principle of d'Alembert–Lagrange, which puts the constraint forces into the annihilator space of the admissible virtual displacements. The concept of perfect constraints is omnipresent in each branch of mechanics and is quintessential to induce more specific theories from a general mechanical theory.

Many specific mechanical theories can be considered as special cases of the theory of continuous bodies. Rigid body mechanics, for instance, is the dynamics of a continuous body whose deformation is constrained such that the position field of the body can be described by a displacement of one material point of the body and a rotation of the body only. Hence, the rigid body can be considered as a constrained continuous body. As discussed above, the principle of virtual work as a variational formulation is the only way to treat perfect bilateral constraints. Consequently, to induce a specific mechanical theory from the theory of a continuous body by imposing further constraints on the mechanical system, a variational formulation of the dynamics of a continuous body is inevitable.

In order to obtain an intrinsic theory of a continuous body in variational form, we have to use the concepts of analytical mechanics, where the forces are induced by the choice of the kinematics of the mechanical system. Before starting with a play, the actors and the scene have to be determined. Here, the body plays the role of a single actor and the scene is given by the model of the physical space. The play, i.e. how the body performs on the scene, corresponds to the admissible configurations of the body in the physical space. Using appropriate definitions of the body and the physical space, the set of all maps of the body into the physical space build an infinite dimensional manifold, called configuration manifold. This configuration manifold induces as in the finite dimensional setting the space of forces in the sense of duality. Applying the principle of virtual work together with further assumptions, which forces are involved, how these forces are represented and what their virtual work contribution is, this leads us directly to the fundamental law of a continuous body in a variational setting.

The section is closed with a list of several reasons why a general mechanical theory should be formulated variationally by the principle of virtual work.

– The space of forces of a mechanical system is induced by its kinematics. Hence, the forces cannot be defined regardless of the underlying kinematics.

- Set-valued force laws, including perfect bilateral constraints, can only be formulated variationally. This argument follows directly the slogan of P. D. Panagiotopoulos "In mechanics, there are forces and force laws".
- Many specific mechanical theories can be obtained by constraining the position field of a more general theory. To treat the perfect bilateral constraints, a variational formulation is inevitable.
- The most successful numerical methods, as e.g. finite element methods, rely on variational formulations.
- Under special assumptions on the mechanical system, variational problems and energy methods are directly obtained from the principle of virtual work.
- An intrinsic differential geometric formulation of mechanics requires the virtual work. Insofar, a more general definition of a body and the physical space is possible. Thus, the physical space is not restricted to be a Euclidean space and can be modeled, for instance, as a space-time vector bundle.

1.3 Literature Survey

In this section, a short literature survey on the foundations of continuum mechanics and on beam theories is given. To understand some developments in the foundations of continuum mechanics, it is tried to bring the literature on this field into a rough historical context. The survey on beam theories is merely intended to give some references which might be helpful in getting more detailed information.

Foundations of Continuum Mechanics

Over the past two centuries continuum mechanics has become one of the cornerstones of classical mechanics. Evidently, there exist an immense and unmanageable number of publications on the foundations of continuum mechanics. After the celebrated theorem about the existence of a stress tensor of Cauchy (1827) and the derivation of Cauchy's first law of motion in Cauchy (1828), the 19th century was to a large extent occupied with continuum mechanics for very specific material laws. The theory of elasticity, i.e. continuum mechanics for solids with infinitesimal deformations and linear elastic material laws, was the predominated paradigm for solids. This very specific theory allows to find analytic solutions for many problems. Hence, the theory of linear elasticity has been the basis to further develop the theory of strength of materials.

For a detailed historical overview of continuum mechanics in the 20th century, we refer to Maugin (2013). In the first part of this century, there has been an increase in popularity in the description of the behavior of solids undergoing finite deformations as variational problems. This trend is manifested in a series of publications such as Murnaghan (1937), Reissner (1953) or Doyle and Ericksen (1956). The drawback of a formulation of continuum mechanics as a variational problem is, that the possible material laws of the continuum are restricted to the very specific subset of hyperelastic material laws. This drawback has been eliminated in the seminal treatise of Truesdell and Toupin (1960) in which the theory of classical field theories is based on the balance of linear and angular momentum. The treatment therein has been mainly influenced by the system of axioms formulated in Noll (1958) and Noll (1959), a former student of Truesdell. The balance of linear and angular momentum, completed by the balance of energy and the conservation of mass, are generally referred to as the balance laws. Soon after the publication of the classical field theories, the theory of continuum mechanics has been enriched in Truesdell and Noll (1965), completing the former work by an extensive treatise on material laws. The influence of Truesdell and Noll on continuum mechanics has led to a wealth of textbooks on continuum mechanics which follow the very same philosophy, e.g. Malvern (1965), Gurtin (1981), Chadwick (1999), Holzapfel (2001), Liu (2002), Spencer (2004), Dvorkin and Goldschmit (2006). For a treatment in curvilinear coordinates, we refer to Ogden (1997), Ciarlet (1988) and Başar and Weichert (2000). The approach of Truesdell and Noll to continuum mechanics is revealed in the list of contents of the very technical treatise on rational continuum mechanics, Truesdell (1977): "I. Bodies, Forces, and Motions", "II. Kinematics", "III. The Stress Tensor" and "IV. Constitutive Relations". Very outstanding in this approach to continuum mechanics is the strong division between balance laws and constitutive laws. The attitude of Truesdell concerning variational principles is clarified in Truesdell and Toupin (1960), Par. 231, where he distances himself from variational principles as fundamental equations of mechanics and regard them merely as derivative and subservient to the balance laws. Even stronger words can be found in Truesdell (1964) where he claims, that Lagrange has misunderstood or neglected general principles and concepts of mechanics.

According to Truesdell and Toupin (1960), the first application of the virtual work to a continuum can be found in Piola (1833). Eighty years later, Hellinger (1914) based continuum mechanics on the principle of virtual work and emphasized the benefits of the invariance of the virtual work. Therein, the virtual work of the continuum is formulated as the duality between the 1st Piola-Kirchhoff stress and the gradient of the virtual displacements. As one of the last classic books on theoretical mechanics in the German literature, also Hamel (1967) applied the principle of virtual work to continuous bodies.

In the French literature there has been a renaissance of the concept of virtual work, or more precisely "Les puissances virtuelles", in the context of continuum mechanics induced by the publications of Germain, i.e. Germain (1972), Germain (1973a), Germain (1973b). Therein, first gradient and second gradient theories as well as continua with microstructure have been applied by the postulation of virtual work principles. One important contribution is the "Axiom of Power of Internal Forces" which corresponds to the variational formulation of the law of interaction. Another important contribution is the recognition, that also external stress contributions can be considered in a gradient theory. A rather mathematical approach to the idea of duality in mechanics has been developed in Nayroles (1971). A discussion about the stress tensor as dual object to a strain distribution has been given by Moreau (1979). The treatment of continuum mechanics using the principle of virtual work as its fundamental law of mechanics can be found in introductory textbook form by Germain (1986) and Salençon (2001).

The principle of virtual work is often used, when a coupling between different theories is demanded. Maugin (1980) has formulated a coupling between the theory of electromagnetism and mechanics formulating a virtual work principle. A coupling between non-equilibrium thermodynamics and mechanics has been formulated by Biot (1974). In the publication of Del Piero (2009), the internal virtual work is deduced from an invariance of the virtual work of the external force under change of observer. The equivalence of the principle of virtual work and the integral laws under certain regularity conditions has been shown in Antman and Osborn (1979).

Noll already recognized the importance of a differential geometric point of view on continuum mechanics and introduced the idea to regard a body as a smooth manifold, see Noll (1959) . In the well-known treatise on the foundations of elasticity, Marsden and Hughes (1983) have formulated body and space as Riemannian manifolds. The fundamental law of covariant elasticity has been considered as an invariance principle of energy, proposed in a non-differential geometric setting by Green and Rivlin (1964). An application of the covariant theory to solids, rods and plates has been treated by Simo et al. (1988). In Kanso et al. (2007) a new differential geometric interpretation of the stress tensor as a covector-valued differential two-form is given. A concise differential geometric consideration of the kinematics of the body and the space as manifolds has been presented by Aubram (2009).

A formulation of continuum mechanics in an intrinsic differential geometric setting is discussed in the seminal work of Segev (1986b). In the sense of analytical mechanics, forces of nth gradient theories are defined by duality. Using the concepts of jet-bundles and covariant derivatives, force representations for the gradient theories have been found. The virtual work principle is formulated as a mathematical compatibility condition between a force of the continuum and a stress representation. When the physical space is equipped with a connection, the variational stress of a first gradient theory is obtained as a linear map of the covariant derivative of the virtual displacement field to a volume form of the body. To date, perhaps the only existing work on intrinsic differential geometric formulation of continuum mechanics stems from Segev and his supervisor Epstein. The first steps in the development of this theory can be found in Epstein and Segev (1980) and in the dissertation of Segev (1981). Segev (1984) gives an application of the intrinsic theory to the special case, where the physical space is assumed as $\mathbb{R}^3$ and the body as a closed subset of the former. More explanations and focus on the fundamental questions of the intrinsic theory, can be found in Segev (1986a), Segev (2000), Segev and Rodnay (2000). An application of the theory to micro-structure has been presented in Segev (1994). In a recent review article Segev (2013) summarizes most of his publications. Due to the high level of abstraction in the intrinsic formulation, it is possible to contribute also in completely different fields, as the application to general relativity of Segev (2002) shows. Due to the relaxation of the continuity assumptions and due to the generalization of the concept of stress, completely new fields, such as fractal mechanics (see Epstein and Elzanowski (2007)), have been developed. Recently, an introductory textbook on the geometric understanding of continuum mechanics has been published by Epstein (2010).

Beam Theory

There exists a vast amount of treatises on the topic of beam theory. A very classical treatment of the mathematical theory of elasticity with application to beams is given by Love (1944). The beam equations are obtained by applying the balance of linear and angular momentum at an infinitesimal beam element. A textbook with plenty of applications and examples on beams is Sokolnikoff (1946). Villaggio (2005) introduces beams on the one hand as an approximation of the three-dimensional elastic theory and on the other hand as directed curves. Classical linear beam theories are discussed in Bauchau and Craig (2009). For linear theories of beams, including beams with warping fields, we refer to Hjelmstad (2005). An extensive treatise on nonlinear beam theories is given in Antman (2005), where almost any possible interpretation of beams is discussed. Outstanding is the chapter on generalized beam theories which relies on Antman (1976), in which beams are considered as constrained continuous bodies. A concise introduction to intrinsic special Cosserat beam theory is given by Ballard and Millard (2009). A discussion about two-director Cosserat beams also dealing with beam constitutive laws is part of Rubin (2000). A theory of beams deduced at an infinitesimal beam element and reformulated to a virtual work expression can be found in Wempner (1973). For a textbook including more involved cross section deformations we refer to Hodges (2006).

The plane and linear Timoshenko beam has originally been developed in Timoshenko (1921) and Timoshenko (1922). The treatment of the same kinematical assumption for large displacements but small strains has been given by Reissner (1972) for the plane and by Reissner (1981) for the spatial case. Another derivation for the spatial Timoshenko beam has been obtained by Simo (1985). Considering the Timoshenko beam as a constrained continuum, Clerici (2001) induces the weak and strong variational form of the Timoshenko beam from the virtual work principle of a three-dimensional continuous body. The same approach is proposed by Auricchio et al. (2008).

The plane Euler–Bernoulli beam has been formulated as an induced theory by Epstein and Murray (1976a). A spatial version is discussed in Hodges et al. (1980). The Kirchhoff beam, originating from Kirchhoff (1876), is treated in a more modern version by Dill (1992).

Augmented beam theories are theories, in which the cross sections are not restricted to remain plane and rigid. Classically, as proposed by Cosserat and Cosserat (1909), such theories are formulated by intrinsic director theories, where the equations of motion are obtained by an invariance principle of a stated action. The theory of one-dimensional Cosserat media are included in Naghdi (1980) and Cohen (1966) and a theory of directed curves with further constraints is developed in Naghdi and Rubin (1984). An intrinsic and an induced theory for more than two directors are discussed by Epstein (1979) and Epstein and Murray (1976b). For beam theories with warping fields we refer to Hodges (1987), Danielson and Hodges (1988), Danielson and Hodges (1987) and Simo and Vu-Quoc (1991). Beam theories with in-plane warping are applied in Papes (2012) and Bauchau and Han (2014).

1.4 Aim and Scope

As stated in Section 1.1, to obtain the essential objects of mechanics and to recognize the fundamental laws of mechanics, a high level of mathematical abstraction is aspired. An intrinsic differential geometric description, as proposed in the contributions of Segev, seems the appropriate level of abstraction for the formulation of continuum mechanics. Just as important are the formulation of further assumptions and concepts to arrive in a rigorous way at very specific theories. The scope of this thesis lies in an intrinsic differential geometric approach to first gradient continuum mechanics. For the case of a Euclidean three-space as the physical space, the discussion on beam theories serves as a playground to show how specific theories can be induced. The aims of this research monograph are:

- to introduce the reader to the differential geometric objects required for an intrinsic differential geometric description of a first gradient continuum,
- to combine the intrinsic differential geometric approach of Segev with the mechanical principles of a first gradient theory stated by Germain,
- to define beams, in an induced sense, as three-dimensional continuous bodies with constrained position fields,
- to show that the principle of virtual work of a continuous body is the adequate principle to induce arbitrary beam theories, classical as well as augmented beam theories.

The main philosophy of this thesis is that *the virtual work is THE invariant quantity in mechanics.*

1.5 Outline

This monograph is divided in two parts which can be read independently. Part I is devoted to the foundations of continuum mechanics formulated in an intrinsic differential geometric way. Part II deals with beam theories, which are considered as induced theories from a three-dimensional theory of a continuous body.

Part I begins in Chapter 2 with an intertwined introduction to differential geometric concepts together with the definition of required mechanical objects. First, the body and the physical space are introduced as differentiable manifolds. Subsequently, the configuration manifold of all embeddings of the body into the physical space and the representation of its tangent vectors, the virtual displacements, are discussed. Finally, the notion of an affine connection is treated which serves as an additional geometric structure for the physical space. Applying the concept of the virtual work, forces are defined as linear functionals on the space of virtual displacements. Chapter 3 discusses the representation of forces of a first gradient continuum in accordance with the achievements of Segev. Furthermore the principle of virtual work for the case of classical continuum mechanics is formulated and applied to the Euclidean space as a choice of the physical space.

Part II begins with Chapter 4, which first repeats some results from the previous part about the dynamics of a continuous body within the Euclidean space. Subsequently, perfect bilateral constraint stresses which may guarantee constrained position fields of a continuous body are discussed. Finally, different approaches to beam theories are presented. Using the constrained position field of a classical nonlinear beam, Chapter 5, induces the weak and the strong variational form of the classical beam from the virtual work principle of the continuous body. The equations of motion of the beam are then completed in a semi-induced sense by an intrinsic constitutive law, relating internal generalized forces of the beam with generalized strain measures. Imposing further constraints on the Timoshenko beam, Euler–Bernoulli and Kirchhoff beams are obtained. Chapter 6 presents the linearization of the classical nonlinear beam theory around a reference configuration which leads to the classical linear beam theory, valid for small displacements and small rotations. The constitutive laws are formulated as well in an intrinsic setting and relate the internal generalized forces with the linearized generalized strains. Similar to the nonlinear theory, Timoshenko, Euler–Bernoulli and Kirchhoff beams are induced by imposing further constraints. As an example of a fully induced theory, Chapter 7 derives the weak and the strong variational form of the classical linearized beam theories in the case of planar motion. Applying non-admissible virtual displacements to the principle of virtual work, the total stress field of the constrained continuous body is obtained up to certain indeterminacies. Chapter 8 is devoted to augmented beam theories which allow for cross section deformation. Applying the same procedure as for the classical nonlinear theory, the weak and the strong variational form of the nonlinear two-director Cosserat beam and the nonlinear Saint–Venant beam are induced from the principle of virtual work of a continuous body.

Separated into the two parts of the monograph, finally, concluding remarks on the thesis and an outlook on further scientific questions are given in Chapter 9. Moreover, the merit of the thesis is discussed in detail.

Part I

On the Foundations of Continuum Mechanics

"In the concept of force lies the chief difficulty in the whole of mechanics."

Hamel, letter to Truesdell, 14. Oct. 1952.

Chapter 2

Kinematics

In this chapter we discuss the admissible kinematics of a continuous body in the physical space from a differential geometric point of view, as it is proposed by Epstein and Segev (1980) and Segev (1986b). A major part of the chapter deals with the introduction of the necessary differential geometric concepts. These geometric concepts are then directly applied to the description of a first gradient continuum as a model of a deformable body.

Section 2.1 introduces the objects of continuum mechanics, the body and the physical space as manifolds. The idea to regard a body as a smooth manifold originates from Noll (1959) and is applied explicitly by Epstein and Segev (1980). In Section 2.2, tangent bundles, vector fields and global flows are defined to formulate the idea of a smooth spatial virtual displacement field. In Section 2.3, we introduce the configuration as a mapping between manifolds and discuss the infinite dimensional manifold structure of the set of all differentiable mappings. Furthermore, we introduce pullback tangent bundles which are required to represent elements of the tangent space of the configuration manifold, i.e. virtual displacement fields. In Section 2.4, we give a brief introduction to affine connections.

2.1 Body and Space

Many definitions of differential geometric concepts require notions from point set topology. We refer to textbooks like Munkres (2000) for a detailed treatise on that topic. For the sake of completeness, we briefly introduce the necessary terminology of topology.

A *topology* on a set X is a collection $\mathcal{T}$ of subsets of X having the three properties that (i) the empty set $\varnothing$ and the set X itself are elements of $\mathcal{T}$, (ii) the union of the elements of any subcollection of $\mathcal{T}$ is contained in $\mathcal{T}$, and (iii) the intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$. A *topological space* is the ordered pair $(X, \mathcal{T})$ consisting of a set X and a topology $\mathcal{T}$ on X. Elements of $\mathcal{T}$ are called *open sets*, their complements *closed sets*. An open set $U \in \mathcal{T}$ containing $P \in X$ is called an *open neighborhood* of P. A topological space $(X, \mathcal{T})$ is called a *Hausdorff space* if for each pair of distinct points of X, there exist open neighborhoods of these points, that are disjoint. A space X is said to be *compact* if any open covering of X contains a finite subcollection that also covers X. A function $x\colon X_1 \to X_2$ between two topological spaces $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ is said to be continuous if for each open subset V of X_2, its preimage under x, i.e. $x^{-1}(V)$, is an open subset of X_1. If a function $x\colon X_1 \to X_2$ is continuous and bijective with continuous inverse, then x is called a *homeomorphism.* The function x is said to be *proper* if for every compact set $K \subset X_2$, the preimage $x^{-1}(K)$ is compact.

We define the *closed n-dimensional upper half-space* $\mathbb{H}^n \subset \mathbb{R}^n$ as the set

$$\mathbb{H}^n := \{(a^1, \ldots, a^n) \in \mathbb{R}^n \,|\, a^n \geq 0\} \,.$$

For $n > 0$, we denote the *interior* and the *boundary* of $\mathbb{H}^n$ by $\operatorname{Int}\mathbb{H}^n$ and $\partial\mathbb{H}^n$, respectively, which are defined as

$$\begin{aligned}\operatorname{Int}\mathbb{H}^n &:= \{(a^1, \ldots, a^n) \in \mathbb{R}^n \,|\, a^n > 0\} \,, \\ \partial\mathbb{H}^n &:= \{(a^1, \ldots, a^n) \in \mathbb{R}^n \,|\, a^n = 0\} \,.\end{aligned}$$

For the case $n = 0$, $\mathbb{H}^0 := \mathbb{R}^0 = \{0\}$, so $\operatorname{Int}\mathbb{H}^0 = \mathbb{R}^0$ and $\partial\mathbb{H}^0 = \varnothing$.

Definition 2.1 (Topological Manifold with Boundary)**.** An *n-dimensional topological manifold with boundary* $\mathcal{M}$ is a Hausdorff space $(X, \mathcal{T})$ with a countable basis and the property, that every point P of X has an open neighborhood $U(P) \subset \mathcal{M}$, which is homeomorphic to an open set of $\mathbb{H}^n$.

The pair (U, x) consisting of an open neighborhood $U \subset \mathcal{M}$ and a homeomorphism x, which maps the open neighborhood U to an open set of $\mathbb{H}^n$, is called a *coordinate chart* on $\mathcal{M}$. We call (U, x) an *interior chart* if $x(U)$ is an open subset of $\mathbb{H}^n$ such that $x(U) \cap \partial\mathbb{H}^n = \varnothing$, and we call it a *boundary chart* if $x(U)$ is an open subset of $\mathbb{H}^n$ such that $x(U) \cap \partial\mathbb{H}^n \neq \varnothing$. A point $P \in \mathcal{M}$ is called an *interior point of* $\mathcal{M}$ if it is in the domain of some interior chart. It is a *boundary point of* $\mathcal{M}$ if it is in the domain of a boundary chart that maps P to $\partial\mathbb{H}^n$. The *boundary of* $\mathcal{M}$, denoted by $\partial\mathcal{M}$, is the set of all boundary points. The *interior of* $\mathcal{M}$ is the set of all interior points, denoted by $\operatorname{Int}\mathcal{M}$. For an interior chart (U, x), the canonical projection $\pi^i\colon \mathbb{R}^n \to \mathbb{R}$, $(a^1, \ldots, a^n) \mapsto a^i$ induces the function $x^i\colon U(P) \to V \subset \mathbb{R}$, $x^i := \pi^i \circ x$, which extracts the i-th component of the homeomorphism x and is called the *component function of* x. The n-tuple $(x^1(P), \ldots, x^n(P)) \in \mathbb{H}^n$ is called the *coordinate description of* P. For a boundary point $Q \in \partial\mathcal{M}$ the coordinate description is the n-tuple $(x^1(Q), \ldots, x^{n-1}(Q), 0) \in \mathbb{H}^n$ where the n-th component is zero.

If (U, x) and $(\tilde{U}, \tilde{x})$ are two charts such that $U \cap \tilde{U} \neq \varnothing$, the composite map $\tilde{x} \circ x^{-1}\colon x(U \cap \tilde{U}) \to \tilde{x}(U \cap \tilde{U})$ is called the *transition map from x to $\tilde{x}$.* The transition map relates two different coordinate descriptions of the same point on the manifold which is referred to as *change of coordinates.* Many of the discussed concepts are depicted in Figure 2.1 at the example of a 2-dimensional topological manifold with boundary.

If U and V are open subsets of $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively, a function $\hat{\gamma}\colon U \to V$ is said to be *C^k-continuous* or in short C^k if each of its component functions is k-times continuously differentiable. The function $\hat{\gamma}$ is called *smooth* or C^∞ if all its component functions have continuous partial derivatives of all orders. If a C^k-continuous function is also bijective and has a C^k-continuous inverse map, it is called a *C^k-diffeomorphism.* For the case, that a bijective function and its inverse map are smooth, the function is called a *diffeomorphism.* Let $\hat{\rho}$ be a map from a subset, possibly closed, $D \subset \mathbb{R}^n$ to $\mathbb{R}^n$. The

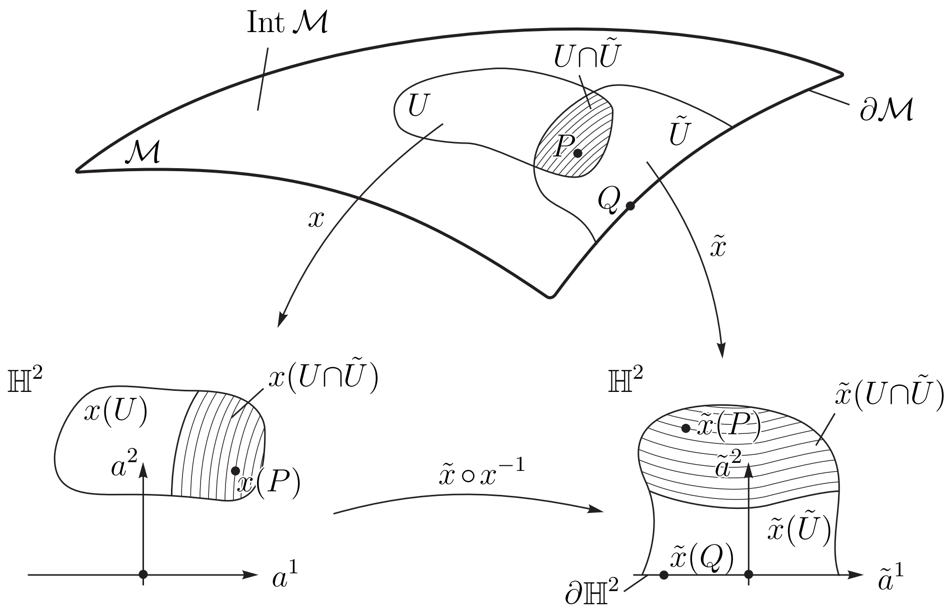

Figure 2.1: Illustration of a 2-dimensional topological manifold with boundary. The chart (U, x) and $(\tilde{U}, \tilde{x})$ are interior and boundary charts, respectively. The point P is an interior point, the point Q is a boundary point.

function $\hat{\rho}$ is called a (C^k-)diffeomorphism if at each point $x \in D$, it admits an extension to a (C^k-)diffeomorphism, defined on an open neighborhood of x in $\mathbb{R}^n$, cf. Lee (2012), App. C.

Two charts (U, x) and $(\tilde{U}, \tilde{x})$ are said to be *smoothly compatible* if either $U \cap \tilde{U} = \varnothing$ or the transition map $\tilde{x} \circ x^{-1}$ is a diffeomorphism. We define an *atlas for* $\mathcal{M}$ to be a collection of charts whose domains cover $\mathcal{M}$. An atlas $\mathcal{A}$ is called a *smooth atlas* if any two charts in $\mathcal{A}$ are smoothly compatible with each other. A smooth atlas $\mathcal{A}$ on $\mathcal{M}$ is *maximal* when any chart that is smoothly compatible with every chart in $\mathcal{A}$, is already contained in $\mathcal{A}$.

Definition 2.2 (Smooth Manifold with Boundary)**.** An *n-dimensional smooth manifold with boundary* (or in short *smooth manifold*) is an n-dimensional topological manifold with boundary with a maximal smooth atlas $\mathcal{A}$.

One possibility to define an *n-dimensional smooth manifold without boundary* is to exchange the upper half space $\mathbb{H}^n$ by $\mathbb{R}^n$ in the previous definitions about smooth manifolds with boundary. Another possibility, which we choose here, is to define an *n-dimensional smooth manifold without boundary* as a smooth manifold with boundary, whose boundary $\partial\mathcal{M}$ is the empty set $\varnothing$.

Definition 2.3 (Body)**.** A *body* is a compact m-dimensional smooth manifold with boundary. Typically, a body will be denoted by $\mathcal{B}$ and its dimension by m. A point P of the body $\mathcal{B}$ is called a *material point* of the body.

Neither $\mathbb{R}^m$ nor the upper half space $\mathbb{H}^m$ are compact sets with respect to the standard topology. Hence, these cannot be bodies by Definition 2.3. Nevertheless, non-compact bodies are often used in linear elasticity, cf. for instance Landau and Lifshitz (1986). In the following, we rely on some important mathematical results which do not allow relaxing the compactness assumption. Furthermore, it is worth noticing that the geometric definition of a body does not require metric concepts, such as length or angles. These are information of the body which are obtained by an embedding of the body into the physical space, which is defined in the following way.

Definition 2.4 (Physical Space)**.** Let $n \geq m$. The *physical space* is an n-dimensional smooth manifold $\mathcal{S}$ without boundary. A point Q of the physical space $\mathcal{S}$ is called a *space point*.

2.2 Spatial Virtual Displacement Field

When not stated differently, $\mathcal{M}$ and $\mathcal{N}$ are henceforth smooth manifolds of dimensions m and n, respectively. Let $\gamma\colon \mathcal{N} \to \mathcal{M}$ be a map, $P \in \mathcal{N}$ and (V, x) be a chart on $\mathcal{M}$ such that $\gamma(P) \in V$. Furthermore, let (U, θ) be a chart on $\mathcal{N}$ with $P \in U$ and $\gamma(U) \subset V$. Then γ has, as depicted in Figure 2.2, a *local representation around* P by the composition map $\hat{\gamma} := x \circ \gamma \circ \theta^{-1}\colon \mathbb{H}^n \to \mathbb{H}^m$. The function γ is said to be C^k*-continuous* or in short C^k if for each $P \in \mathcal{N}$ the local representation $\hat{\gamma}$ is C^k. The function γ is called *smooth* or C^∞, if the local representation for each point $P \in \mathcal{N}$ is smooth. The set of all C^k and C^∞ functions between $\mathcal{N}$ and $\mathcal{M}$ are denoted by $C^k(\mathcal{N}, \mathcal{M})$ and $C^\infty(\mathcal{N}, \mathcal{M})$, respectively. If $\gamma \in C^k(\mathcal{N}, \mathcal{M})$ is bijective with a C^k-continuous inverse map, the function is called a C^k*-diffeomorphism*. In the case of a smooth function with a smooth inverse, the function is called a *diffeomorphism*. We denote the set of all smooth real-valued functions by $C^\infty(\mathcal{M}) := C^\infty(\mathcal{M}, \mathbb{R})$.

Definition 2.5 (Germ)**.** Let U, V and $W \subset U \cap V$ be open neighborhoods of a point $P \in \mathcal{M}$. Given real-valued smooth functions $f\colon U \to \mathbb{R}$ and $g\colon V \to \mathbb{R}$, we define an equivalence relation $\sim_P$ as follows:

$$f \sim_P g \Leftrightarrow \exists\, W \text{ open neighborhood of } P\colon f \equiv g \text{ on } W\ .$$

A *germ of* f *at* P is the equivalence class

$$[f]_P := \{g\colon V \to \mathbb{R} \mid g \text{ smooth function in } P,\ (g, V) \sim_P (f, U)\}\ .$$

The set of all germs at P is denoted by $C^\infty_P(\mathcal{M})$.

Let $[f]_P$ and $[g]_P$ be germs at P and $\lambda \in \mathbb{R}$. With the operations

$$\begin{aligned} \lambda[f]_P + [g]_P &= [\lambda f + g]_P\ , \\ [f]_P[g]_P &= [fg]_P\ , \\ [f]_P(P) &= f(P)\ , \end{aligned}$$

the set of all germs $C^\infty_P(\mathcal{M})$ constitute a real vector space.

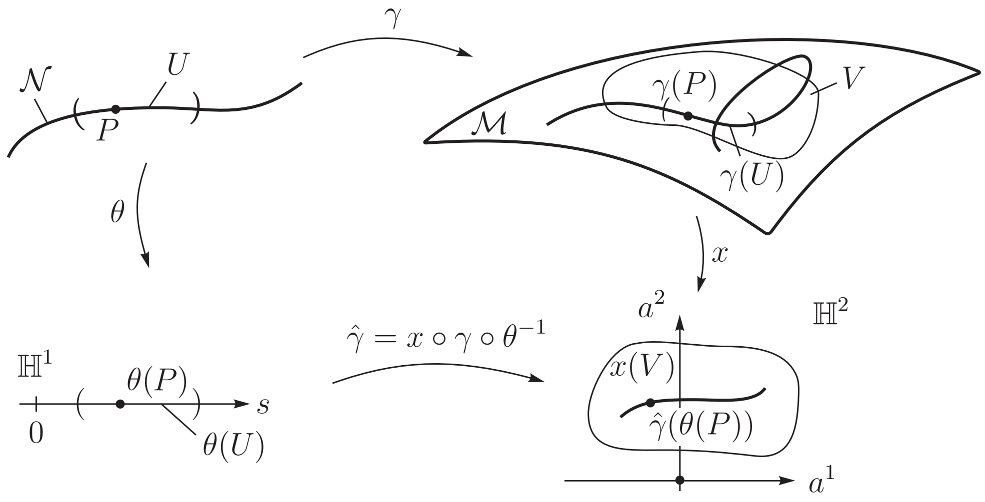

Figure 2.2: Illustration of a function between a one- and a two-dimensional manifold.

Definition 2.6 (Tangent Space)**.** A linear map $\mathbf{v}\colon C_P^\infty(\mathcal{M}) \to \mathbb{R}$ is called a *derivation on* $C_P^\infty(\mathcal{M})$, if for all $[f]_P, [g]_P \in C_P^\infty(\mathcal{M})$ the Leibniz rule

$$\mathbf{v}([fg]_P) = f(P)\mathbf{v}([g]_P) + \mathbf{v}([f]_P)g(P) \tag{2.1}$$

holds. The set $T_P\mathcal{M}$ of all derivations on $C_P^\infty(\mathcal{M})$ is called the *tangent space of* $\mathcal{M}$ *at* P.

Proposition 2.1. *Let* $\mathbf{u}, \mathbf{v} \in T_P\mathcal{M}$, $[f]_P \in C_P^\infty(\mathcal{M})$ *and* $\lambda \in \mathbb{R}$*. Defining addition and scalar multiplication as*

$$\begin{aligned} (\mathbf{u} + \mathbf{v})([f]_P) &\coloneqq \mathbf{u}([f]_P) + \mathbf{v}([f]_P)\ , \\ (\lambda\mathbf{u})([f]_P) &\coloneqq \lambda\mathbf{u}([f]_P)\ , \end{aligned} \tag{2.2}$$

the tangent space at P *is a vector space.*

Proof. Let $\mathbf{u}, \mathbf{v} \in T_P\mathcal{M}$, $[f]_P, [g]_P \in C_P^\infty(\mathcal{M})$ and $\lambda \in \mathbb{R}$. We need to show that an arbitrary linear combination $\lambda\mathbf{u} + \mathbf{v}$ is linear and satisfies the Leibniz rule (2.1). Linearity of $\lambda\mathbf{u} + \mathbf{v}$ follows directly from the definitions of addition and scalar multiplication (2.2). The Leibniz rule for the linear combination follows by linearity and straight forward computation:

$$\begin{aligned} (\lambda\mathbf{u} + \mathbf{v})([f]_P[g]_P) &= \lambda\mathbf{u}([f]_P[g]_P) + \mathbf{v}([f]_P[g]_P) \\ &\overset{(2.1)}{=} \lambda f(P)\mathbf{u}([g]_P) + \lambda\mathbf{u}([f]_P)g(P) + f(P)\mathbf{v}([g]_P) + \mathbf{v}([f]_P)g(P) \\ &= f(P)(\lambda\mathbf{u} + \mathbf{v})([g]_P) + (\lambda\mathbf{u} + \mathbf{v})([f]_P)g(P)\ . \end{aligned}$$ □

Definition 2.7 (Induced Partial Derivative)**.** Let (U, x) be a chart on $\mathcal{M}$, $P \in U$ and $f\colon U \to \mathbb{R}$ a smooth function. We define an *induced partial derivative at* P on $\mathcal{M}$ for $i \in \{1, \dots, m\}$ as

$$\partial_{x^i}|_P([f]_P) \coloneqq \partial_i(f \circ x^{-1})|_{x(P)}\ , \tag{2.3}$$

where ∂_i denotes the i-th partial derivative on $\mathbb{R}^m$.

Using the definition of the induced partial derivative together with the product rule of $\mathbb{R}^m$, it can easily be shown that the induced partial derivative at P is a linear map which satisfies the Leibniz rule (2.1) and consequently is a derivation on $C_P^\infty(\mathcal{M})$.

Theorem 2.1. *Let (U, x) be a chart on $\mathcal{M}$ and $P \in U$. The derivations $(\partial_{x^1}|_P, \ldots, \partial_{x^m}|_P)$ form a basis of the tangent space $T_P\mathcal{M}$. Consequently, applying a vector $\mathbf{v} \in T_P\mathcal{M}$ on a germ $[f]_P \in C_P^\infty(\mathcal{M})$, the vector can be represented as a linear combination*

$$\mathbf{v}([f]_P) = \mathbf{v}([x^i]_P)\partial_{x^i}|_P([f]_P) = v^i\partial_{x^i}|_P([f]_P) \ , \tag{2.4}$$

where summation over repeated indices is applied and the components v^i are defined as $\mathbf{v}([x^i]_P)$.

Proof. For the proof we refer to Michor (2008), Sec. 1.8 or to Kühnel (2013), Sec. 5.6. □

Excluded analytic functions, each germ of a smooth function has a representative which is defined on the whole $\mathcal{M}$, cf. Michor (2008). Thus, we henceforth omit the brackets designating the equivalence class, defining a germ of a smooth function at a point on a manifold.

The definition of tangent vectors of $\mathcal{M}$ at a point P as the set of all derivations on $C_P^\infty(\mathcal{M})$ is a coordinate free and consequently chart independent definition. Nevertheless, in applications, charts have to be chosen and it is of major interest how objects transform under a change of coordinates. In the following, we show how the basis and the components of a tangent vector transform. Let (U, x) and $(\tilde{U}, \tilde{x})$ be charts of $\mathcal{M}$ and let $P \in U \cap \tilde{U}$. The definition of the induced partial derivative (2.3) together with the chain rule from higher dimensional calculus implies a transformation rule for a change of coordinates. Let $f \in C^\infty(\mathcal{M})$, then by a telescopic expansion it follows

$$\begin{aligned}
\partial_{\tilde{x}^i}|_P(f) &\overset{(2.3)}{=} \partial_i(f \circ \tilde{x}^{-1})|_{\tilde{x}(P)} = \partial_i(f \circ x^{-1} \circ x \circ \tilde{x}^{-1})|_{\tilde{x}(P)} \\
&= \partial_j(f \circ x^{-1})|_{x(P)}\partial_i(x^j \circ \tilde{x}^{-1})|_{\tilde{x}(P)} = \Lambda_i^j \partial_{x^j}|_P(f) \ ,
\end{aligned}$$

where we have recognized the transformation matrix $\Lambda_i^j := \partial_i(x^j \circ \tilde{x}^{-1})|_{\tilde{x}(P)}$. By an abuse of notation, where a point in $\mathbb{H}^m$ is named by the coordinate function $\tilde{x}^i$, the transformation matrix is often introduced as $\Lambda_i^j = \frac{\partial x^j}{\partial \tilde{x}^i}$, cf. for instance Göckeler and Schücker (1989). The transformation is independent of the choice of the smooth function f and we summarize the important result as follows:

$$\partial_{\tilde{x}^i}|_P = \Lambda_i^j \partial_{x^j}|_P \ , \quad \Lambda_i^j := \partial_i(x^j \circ \tilde{x}^{-1})|_{\tilde{x}(P)} \ . \tag{2.5}$$

Let $\mathbf{v} \in T_P\mathcal{M}$. The components $\tilde{v}^i = \mathbf{v}(\tilde{x}^i)$ of the coordinate representation in the chart $(\tilde{U}, \tilde{x})$ can be transformed further using the component representation of $\mathbf{v}$ in the chart (U, x), i.e.

$$\tilde{v}^i = \mathbf{v}(\tilde{x}^i) \overset{(2.4)}{=} v^j \partial_{x^j}|_P(\tilde{x}^i) \overset{(2.3)}{=} \partial_j(\tilde{x}^i \circ x^{-1})|_{x(P)} v^j = \tilde{\Lambda}_j^i v^j \ ,$$

with the transformation matrix $\tilde{\Lambda}_j^i := \partial_j(\tilde{x}^i \circ x^{-1})|_{x(P)}$. Hence, the transformation rule for the components of a tangent vector is

$$\tilde{v}^i = \tilde{\Lambda}_j^i v^j \ , \quad \tilde{\Lambda}_j^i := \partial_j(\tilde{x}^i \circ x^{-1})|_{x(P)} \ .$$

Definition 2.8 (Cotangent Space)**.** For each $P \in \mathcal{M}$, the *cotangent space at* P, denoted by $T_P^*\mathcal{M}$, is the dual space to $T_P\mathcal{M}$. An element of the cotangent space is called a *covector.*

Let $\mathrm{d}x^i|_P \in T_P^*\mathcal{M}$ denote a dual basis to $\partial_{x^j}|_P$ which satsifies $\mathrm{d}x^i|_P(\partial_{x^j}|_P) = \delta^i_j$. According to (A.5), a covector $\boldsymbol{\omega} \in T_P^*\mathcal{M}$ can be represented as a linear combination

$$\boldsymbol{\omega} = \omega_i \mathrm{d}x^i|_P \ ,$$

with the components $\omega_i = \boldsymbol{\omega}(\partial_{x^i}|_P)$. Let (U, x) and $(\tilde{U}, \tilde{x})$ be charts of $\mathcal{M}$ and let $P \in U \cap \tilde{U}$. Using (2.5), the transformation rule of the component $\tilde{\omega}_i$ follows by linearity and duality of the base vectors

$$\tilde{\omega}_i = \boldsymbol{\omega}(\partial_{\tilde{x}^i})|_P \overset{(2.5)}{=} \omega_k \mathrm{d}x^k|_P(\Lambda^j_i \partial_{x^j}|_P) = \Lambda^j_i \omega_j \ .$$

Thus, the transformation rule for the components of a covector is

$$\tilde{\omega}_i = \Lambda^j_i \omega_j \ , \quad \Lambda^j_i = \partial_i(x^j \circ \tilde{x}^{-1})|_{\tilde{x}(P)} \ , \tag{2.6}$$

which is the same as for the base vectors of a tangent vector. Since the transformation (2.6) is performed by Λ^j_i, i.e. the 'inverse' of $\tilde{\Lambda}^j_i$, it is classically called contravariant transformation. A covector $\boldsymbol{\omega}$ has its representation as a linear combination for any chart. Hence, the transformation of the components of a covector (2.6) immediately implies the transformation rule of the dual base vectors $\mathrm{d}x^i|_P$ by

$$\boldsymbol{\omega} = \omega_j \mathrm{d}x^j|_P = \tilde{\Lambda}^i_j \tilde{\omega}_i \mathrm{d}x^j|_P = \tilde{\omega}_i \mathrm{d}\tilde{x}^i|_P \ .$$

The transformation rule for the dual base vectors is

$$\mathrm{d}\tilde{x}^i|_P = \tilde{\Lambda}^i_j \mathrm{d}x^j|_P \ , \quad \tilde{\Lambda}^i_j = \partial_j(\tilde{x}^i \circ x^{-1})|_{x(P)} \ ,$$

which is the same transformation rule as for the components of a tangent vector, i.e. a covariant transformation.

Definition 2.9 (Tangent Bundle)**.** The *tangent bundle of* $\mathcal{M}$ is the triple $(T\mathcal{M}, \pi_{\mathcal{M}}, \mathcal{M})$, where $T\mathcal{M}$ denotes the disjoint union of the tangent spaces at all points of $\mathcal{M}$

$$T\mathcal{M} := \bigcup_{P \in \mathcal{M}} \{P\} \times T_P\mathcal{M} \ .$$

The manifold $\mathcal{M}$ is *the base space* and $\pi_{\mathcal{M}}$ denotes the natural projection $\pi_{\mathcal{M}} \colon T\mathcal{M} \to \mathcal{M}$. The bundle projection maps $\mathbf{v} \in T\mathcal{M}$ to its *base point* $P \in \mathcal{M}$.

Definition 2.10 (Cotangent Bundle)**.** The *cotangent bundle of* $\mathcal{M}$ is the triple $(T^*\mathcal{M}, \pi_{\mathcal{M}}, \mathcal{M})$, where $T^*\mathcal{M}$ denotes the disjoint union of the cotangent spaces at all points of $\mathcal{M}$

$$T^*\mathcal{M} := \bigcup_{P \in \mathcal{M}} \{P\} \times T_P^*\mathcal{M} \ .$$

The manifold $\mathcal{M}$ is *the base space* and $\pi_{\mathcal{M}}$ denotes the natural projection $\pi_{\mathcal{M}} \colon T^*\mathcal{M} \to \mathcal{M}$. The bundle projection maps $\boldsymbol{\omega} \in T\mathcal{M}$ to its *base point* $P \in \mathcal{M}$.

The tangent and cotangent bundle have again the structure of a manifold, cf. Lee (2012) or Michor (2008). All upcoming operations on elements of the tangent bundle $T\mathcal{M}$ do not act on the base points. Hence, we often use the slight abuse of notation by referring to the vectorial part of $\mathbf{v} \in T\mathcal{M}$ by the same symbol, i.e. "$\mathbf{v} = (P, \mathbf{v})$". For any other bundle structure we do the same. From the context, however, it will be clear which object is meant.

Definition 2.11 (Vector Field)**.** A *vector field on* $\mathcal{M}$ is a section of the map $\pi_{\mathcal{M}} \colon T\mathcal{M} \to \mathcal{M}$. That means, it is a continuous map $\mathbf{v} \colon \mathcal{M} \to T\mathcal{M}$ with the property that

$$\pi_{\mathcal{M}} \circ \mathbf{v} = \mathrm{Id}_{\mathcal{M}} \ .$$

The set of C^k-continuous sections on $T\mathcal{M}$ is denoted by $C^k(T\mathcal{M})$. The set of smooth sections is denoted by $\Gamma(T\mathcal{M})$.

Let (U, x) be a chart on $\mathcal{M}$ and $\mathbf{v} \in \Gamma(T\mathcal{M})$, then the value of $\mathbf{v}$ can be represented at any point $P \in U$ in coordinates as

$$\mathbf{v}(P) = (x(P), v^i(P)\partial_{x^i}|_P) \ .$$

This defines m functions $v^i \colon U \to \mathbb{R}$, called the *component functions of* $\mathbf{v}$ in the given chart.

Definition 2.12 (Smooth Global Flow)**.** A *smooth global flow* on $\mathcal{M}$ is a smooth map $\varphi \colon \mathbb{R} \times \mathcal{M} \to \mathcal{M}$ satisfying the following properties for all $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$ and $P \in \mathcal{M}$:

$$\varphi(\varepsilon_1, \varphi(\varepsilon_2, P)) = \varphi(\varepsilon_1 + \varepsilon_2, P) \ , \quad \varphi(0, P) = P \ . \tag{2.7}$$

Let $f \in C^\infty(\mathcal{M})$ and $P \in \mathcal{M}$, then a smooth global flow $\varphi \colon \mathbb{R} \times \mathcal{M} \to \mathcal{M}$ induces a smooth vector field $\delta\boldsymbol{\varphi} \in \Gamma(T\mathcal{M})$ defined by

$$\delta\boldsymbol{\varphi}(P)(f) = (\varphi(0, P), \delta\boldsymbol{\varphi}(P)(f)) := \big(P, \partial_1(f \circ \varphi)|_{(0,P)}\big) \ . \tag{2.8}$$

The smooth vector field $\delta\boldsymbol{\varphi}$ is called the *infinitesimal generator of* φ. We want to emphasize, that the δ-sign does not act as an operator and remains mainly as a decoration due to historical reasons.

Let (U, x) be a chart on $\mathcal{M}$ and $P \in U$, then the infinitesimal generator is represented at P as

$$\begin{aligned} \delta\boldsymbol{\varphi}(P)(f) &= \partial_1(f \circ x^{-1} \circ x \circ \varphi)|_{(0,P)} = \partial_i(f \circ x^{-1})|_{x(P)}\partial_1(x^i \circ \varphi)|_{(0,P)} \\ &= \partial_1(x^i \circ \varphi)|_{(0,P)}\partial_{x^i}|_P(f) = \delta\varphi^i(P)\partial_{x^i}|_P(f) \ , \end{aligned} \tag{2.9}$$

where the component functions of the infinitesimal generator evaluated at P are identified as $\delta\varphi^i(P) := \partial_1(x^i \circ \varphi)|_{(0,P)}$.

Definition 2.13 (Spatial Virtual Displacement Field)**.** Let $\varphi \colon \mathbb{R} \times \mathcal{S} \to \mathcal{S}$ be a smooth global flow on the physical space $\mathcal{S}$ with an associated infinitesimal generator $\delta\boldsymbol{\varphi} \in \Gamma(T\mathcal{S})$. The infinitesimal generator of φ is called the *spatial virtual displacement field*.

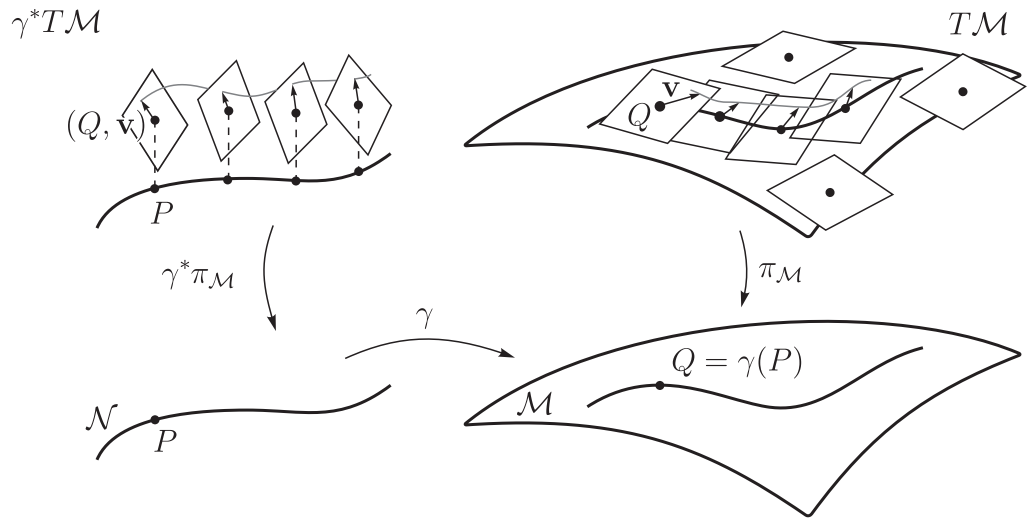

Figure 2.3: Illustration of a pullback tangent bundle $\gamma^*T\mathcal{M}$ over a one-dimensional base manifold $\mathcal{N}$. Loosely, the pullback bundle can be thought of as a base manifold $\mathcal{N}$, in which at all points P on $\mathcal{N}$ and for $Q = \gamma(P)$, the tangent space $T_Q\mathcal{M}$ is attached.

2.3 Configuration Space

The following definition of the pullback bundle is illustrated in Figure 2.3.

Definition 2.14 (Pullback Tangent Bundle)**.** Let $(T\mathcal{M}, \pi_{\mathcal{M}}, \mathcal{M})$ be the tangent bundle and $\gamma\colon \mathcal{N} \to \mathcal{M}$ be a map. The *pullback tangent bundle by* γ is the bundle $(\gamma^*T\mathcal{M}, \gamma^*\pi_{\mathcal{M}}, \mathcal{N})$, where the total space is defined as

$$\gamma^*T\mathcal{M} \coloneqq \{(P, \mathbf{v}) \in \mathcal{N} \times T\mathcal{M} : \pi_{\mathcal{M}}(\mathbf{v}) = \gamma(P)\}$$

and the projection $\gamma^*\pi_{\mathcal{M}}$ of the pullback tangent bundle is defined as

$$(\gamma^*\pi_{\mathcal{M}})\,(P, \mathbf{v}) = P \ .$$

It can be shown that the pullback tangent bundle is a fiber bundle. For the definition of a fiber bundle we refer to textbooks like Saunders (1989) or Husemöller (1994). Let $\gamma \in C^k(\mathcal{N}, \mathcal{M})$ and $\mathbf{v} \in \Gamma(T\mathcal{M})$. Then the *pullback section* $\gamma^*\mathbf{v}$ is a C^k-section of $\gamma^*T\mathcal{M}$. The evaluation of the section at P is

$$\gamma^*\mathbf{v}(P) = (P, \mathbf{v}(\gamma(P))) \ .$$

Let $P \in \mathcal{N}$ and (V, x) be a chart on $\mathcal{M}$ with $\gamma(P) \in V$. Then for each $P \in \mathcal{N}$ the evaluation of $\gamma^*\mathbf{v}$ at P can be represented as

$$\gamma^*\mathbf{v}(P) = \big(P, \big((x \circ \gamma)(P), (v^i \circ \gamma)(P)(\partial_{x^i} \circ \gamma)|_P\big)\big) \ .$$

Let $\tilde{\mathbf{v}}\colon \mathcal{N} \to T\mathcal{M}$ be a C^k-continuous function such that $\pi_{\mathcal{M}}(\tilde{\mathbf{v}}) = \gamma$, then $\tilde{\mathbf{v}}$ is called a *vector field along* γ. For an appropriate chart (U, x) on $\mathcal{M}$ and for each $P \in \mathcal{N}$ the vector field along γ is represented in coordinates as

$$\tilde{\mathbf{v}}(P) = \big((x \circ \gamma)(P), \tilde{v}^i(P)(\partial_{x^i} \circ \gamma)|_P\big) \ .$$

The pullback section $\gamma^*\mathbf{v}$ and the vector field $\tilde{\mathbf{v}}$ along γ differ only in the additional base point in the pullback section. Hence, the isomorphism between the set of pullback sections $C^k(\gamma^*T\mathcal{M})$ and the set of vector fields along γ is obvious. Since a pullback section contains more geometric structure than a vector field along γ, we prefer in the following the pullback section.

Definition 2.15 (Differential)**.** Let $k > 0$ and $\gamma\colon \mathcal{N} \to \mathcal{M}$ be a C^k-continuous map. The *differential* $D\gamma(P)$ *of* γ *at* P is a linear map

$$D\gamma(P)\colon T_P\mathcal{N} \to T_{\gamma(P)}\mathcal{M}$$

such that for $\mathbf{v} \in T_P\mathcal{N}$ and $f \in C^\infty(\mathcal{M})$

$$D\gamma(P)\mathbf{v}(f) = \mathbf{v}(f \circ \gamma)\ . \tag{2.10}$$

Let $P \in \mathcal{N}$, (V, x) be a chart on $\mathcal{M}$ with $\gamma(P) \in V$ and let (U, θ) be a chart on $\mathcal{N}$ with $P \in U$ and $\gamma(U) \subset V$. Then the coordinate representation of the differential $D\gamma(P)$ applied to a tangent vector $\mathbf{v} \in T_P\mathcal{N}$ is derived using the local representation $\hat{\gamma} \coloneqq x \circ \gamma \circ \theta^{-1}$ as follows:

$$\begin{aligned} D\gamma(P)\mathbf{v}(f) &\overset{(2.10)}{=} \mathbf{v}(f \circ \gamma) = \mathbf{v}(f \circ x^{-1} \circ \hat{\gamma} \circ \theta) \overset{(2.4)}{=} v^i \partial_{\theta^i}|_P(f \circ x^{-1} \circ \hat{\gamma} \circ \theta) \\ &\overset{(2.3)}{=} v^i \partial_i (f \circ x^{-1} \circ \hat{\gamma})|_{\theta(P)} = v^i \partial_j (f \circ x^{-1})|_{x(\gamma(P))} \partial_i \hat{\gamma}^j|_{\theta(P)} \\ &\overset{(2.3)}{=} \partial_i \hat{\gamma}^j|_{\theta(P)} v^i \partial_{x^i}|_{\gamma(P)}(f) = F_i^j(P) v^i \partial_{x^i}|_{\gamma(P)}(f)\ , \end{aligned} \tag{2.11}$$

where in the last line we have made use of the component functions $F_i^j \coloneqq \partial_i \hat{\gamma}^j \circ \theta$.

Definition 2.16 (Tangent Map)**.** Let $k > 0$ and $\gamma\colon \mathcal{N} \to \mathcal{M}$ be a C^k-continuous map inducing the pullback tangent bundle $(\gamma^*T\mathcal{M}, \gamma^*\pi_{\mathcal{M}}, \mathcal{N})$. Then the *tangent map* $T\gamma$ is defined as the bundle homomorphism over $\mathcal{N}$

$$\begin{aligned} T\gamma\colon T\mathcal{N} &\to \gamma^*T\mathcal{M} \\ (P, \mathbf{v}) &\mapsto (P, (\gamma(P), D\gamma(P)\mathbf{v}(P)))\ , \end{aligned} \tag{2.12}$$

satisfying the commutative diagram:

$$\begin{tikzcd} T\mathcal{N} \arrow[rr, "T\gamma"] \arrow[dr, "\pi_{\mathcal{N}}"'] & & \gamma^*T\mathcal{M} \arrow[dl, "\gamma^*\pi_{\mathcal{M}}"] \\ & \mathcal{N} & \end{tikzcd}$$

Definition 2.17 (Embedding)**.** A C^k-continuous and proper map $\gamma\colon \mathcal{N} \to \mathcal{M}$ is called a C^k*-embedding* if its tangent map $T\gamma$ is injective. The set of all C^k-embeddings is denoted by $\mathrm{Emb}^k(\mathcal{N}, \mathcal{M})$.

The analysis of mappings between manifolds is an important part of the theory of global analysis, cf. Palais (1968), Michor (1980), Kriegl and Michor (1997). For a short historical overview of the theory of manifolds of mappings, which started in the late fifties with Eells (1958), we refer to Marsden (1974). The beginning of global analysis was strongly influenced by the works of Eells (1966), Eliasson (1967) and Palais (1968). The special case of embeddings is treated in Binz and Fischer (1981). For the application of global analysis in physics, we refer to Marsden (1974) and Binz et al. (1988).

Theorem 2.2 (Manifold Structure of $C^k(\mathcal{N},\mathcal{M})$, Binz et al. (1988), Thm. 5.4.1)**.** *Given two smooth manifolds $\mathcal{N}$ and $\mathcal{M}$ of which $\mathcal{N}$ is compact and $\mathcal{M}$ without boundary. Then for each integer $k<\infty$ the set $C^k(\mathcal{N},\mathcal{M})$ is a smooth manifold modeled over Banach spaces, i.e. $C^k(\mathcal{N},\mathcal{M})$ is a Banach manifold.*

Proof. For a proof and a discussion about the topology of $C^k(\mathcal{N},\mathcal{M})$, we refer to Binz et al. (1988). □

Definition 2.18 (Configuration)**.** Let $\mathcal{B}$ be a body and $\mathcal{S}$ the physical space. We define the *configuration of a first gradient continuum* (or *continuous body*) to be a C^1-embedding κ of the body $\mathcal{B}$ into the physical space $\mathcal{S}$. The set of all C^1-embeddings, i.e. $\mathrm{Emb}^1(\mathcal{B},\mathcal{S})$, is called the *configuration manifold* $\mathcal{Q}$.

As recognized by Segev (1986b), the requirement that a configuration of a body into physical space is an embedding, is based upon two classical principles, cf. Truesdell and Toupin (1960), Sec. 16. These are, the *permanence of matter* and the *principle of impenetrability.* The former states that no region of positive finite volume is deformed into one of zero or infinite volume. The latter states that one portion of matter never penetrates within another. In order that the set of configurations admits the structure of a manifold, Theorem 2.2 requires a body $\mathcal{B}$ to be a compact manifold.

Definition 2.19 (Virtual Displacement Field)**.** Let $\delta\boldsymbol{\varphi}\in\Gamma(T\mathcal{S})$ be the spatial virtual displacement field and $\kappa\in\mathcal{Q}$. Then the *virtual displacement field* of a continuous body is defined as the pullback section $\delta\boldsymbol{\kappa}=\kappa^*\delta\boldsymbol{\varphi}\in C^1(\gamma^*T\mathcal{S})$.

Theorem 2.3 (Tangent Space of $C^k(\mathcal{N},\mathcal{M})$)**.** *Let $\mathcal{N}$ and $\mathcal{M}$ be manifolds of which $\mathcal{N}$ is compact and $\mathcal{M}$ without boundary. For any map $\gamma\in C^k(\mathcal{N},\mathcal{M})$, the tangent space at γ $T_\gamma C^k(\mathcal{N},\mathcal{M})$ is isomorphic to the set of pullback sections $C^k(\gamma^*T\mathcal{M})$.*

The identification of the tangent space at γ with C^k-sections of the pullback tangent bundle, is stated in Segev (1986b). For a proof it is referred to Palais (1968), Eliasson (1967) and Michor (1980). Also in Simo et al. (1988) the same identification without a proof is stated with reference to Abraham et al. (1988) and Ebin and Marsden (1969). In Abraham and Smale (1963) the isomorphism is mentioned merely as a note of Thm. 11.1 without proof. Nevertheless, a complete proof for the above stated assumptions could neither be found nor can be given in this thesis by the author. Strongly related results with proof can be found in Binz et al. (1988), Thm. 5.4.3, for the case of smooth mappings γ. Using the assumption of a Riemannian manifold $\mathcal{N}$, Eliasson (1967) “Corollaries for C^k”, serves as a reference. Inspired by Binz et al. (1988), we prove one direction which should support the reasonability of the theorem.

Idea of Proof. Let $\varphi\colon \mathbb{R}\times\mathcal{M} \to \mathcal{M}$ be a global flow on $\mathcal{M}$. Then the composition function

$$\tilde{\varphi}\colon \mathbb{R} \times \mathcal{N} \to \mathcal{M}\ , \quad (\varepsilon, P) \mapsto \tilde{\varphi}(\varepsilon, P) = \varphi(\varepsilon, \gamma(P))$$

defines a smooth curve through the $C^k(\mathcal{N}, \mathcal{M})$ manifold. The poperties of a global flow (2.7) imply that

$$\tilde{\varphi}(0, \cdot) = \gamma\ .$$

Let $f \in C^\infty(\mathcal{M})$ and $P \in \mathcal{N}$. Then the composition function $\tilde{\varphi}$ induces the section $\gamma^*\delta\tilde{\boldsymbol{\varphi}} \in C^k(\gamma^*T\mathcal{M})$ defined by

$$\gamma^*\delta\tilde{\boldsymbol{\varphi}}(P)(f) = (P, (\tilde{\varphi}(0, P), \delta\tilde{\boldsymbol{\varphi}}(P)(f))) = \big(P, \big(\gamma(P), \partial_1(f \circ \tilde{\varphi})|_{(0,P)}\big)\big)\ .$$

Let (U, x) be a chart on $\mathcal{M}$ and $\gamma(P) \in U$. Then by (2.9), the section through the pullback tangent bundle can locally be represented as

$$\gamma^*\delta\tilde{\boldsymbol{\varphi}}(P) = \big(P, \big((x \circ \gamma)(P), (\delta\varphi^i \circ \gamma)(P)(\partial_{x^i} \circ \gamma)|_P\big)\big)\ .$$

A tangent vector can alternatively be defined, cf. Aubin (2001), by an equivalence class of curves which pass with the same velocity through the same point on the manifold. The composition function $\tilde{\varphi}$ is such a curve through $C^k(\mathcal{N}, \mathcal{M})$. Since the section $\gamma^*\delta\tilde{\boldsymbol{\varphi}}$ is obtained by taking the velocity of the smooth curve $\tilde{\varphi}$ at γ, a tangent vector of $C^k(\mathcal{N}, \mathcal{M})$ induces a section through the pullback tangent bundle $\gamma^*T\mathcal{M}$. The inverse, to show that a section $C^k(\gamma^*T\mathcal{M})$ induces a smooth curve through $C^k(\mathcal{N}, \mathcal{M})$ and that the involved mappings are bijective are necessary to finish the proof of the isomorphism rigorously. □

Corollary 2.1. *The tangent space to* $\mathrm{Emb}^k(\mathcal{N}, \mathcal{M})$ *at* γ *is isomorphic to* $T_\gamma C^k(\mathcal{N}, \mathcal{M})$.

Proof. According to Binz and Fischer (1981) the set $\mathrm{Emb}^k(\mathcal{N}, \mathcal{M})$ is open in the set of $C^k(\mathcal{N}, \mathcal{M})$. □

Due to Theorem 2.3, the virtual displacement field of a continuous body $\delta\boldsymbol{\kappa} \in C^1(\kappa^*T\mathcal{S})$ can be identified with an element of the tangent space $T_\kappa\mathcal{Q}$. This follows the tradition of analytical mechanics, where the virtual displacements are tangent vectors to the finite-dimensional configuration space, cf. Arnold (1989).

2.4 Affine Connection

Definition 2.20 (Affine Connection)**.** Let $\mathbf{u}, \mathbf{v} \in \Gamma(T\mathcal{M})$ and $f \in C^\infty(\mathcal{M})$. An *(affine) connection* on $\mathcal{M}$ is a mapping ∇ which assigns to every pair $\mathbf{u}, \mathbf{v}$ another vector field $\nabla_\mathbf{u}\mathbf{v} \in \Gamma(T\mathcal{M})$ with the following properties:

$$\begin{aligned} &(a)\ \ \nabla_\mathbf{u}\mathbf{v} \text{ is bilinear in } \mathbf{u} \text{ and } \mathbf{v}\ , \\ &(b)\ \ \nabla_{f\mathbf{u}}\mathbf{v} = f\nabla_\mathbf{u}\mathbf{v}\ , \\ &(c)\ \ \nabla_\mathbf{u}(f\mathbf{v}) = f\nabla_\mathbf{u}\mathbf{v} + \mathbf{u}(f)\mathbf{v}\ . \end{aligned} \tag{2.13}$$

We call $\nabla_\mathbf{u}\mathbf{v}$ the *covariant derivative of* $\mathbf{v}$ *along* $\mathbf{u}$.

Let (U, x) be a chart on $\mathcal{M}$, then we define the m^3 functions Γ^k_{ij} by

$$\nabla_{\partial_{x^i}} (\partial_{x^j}) = \Gamma^k_{ij} \partial_{x^k} \ . \tag{2.14}$$

The Γ^k_{ij} are called the *Christoffel symbols* of the connection ∇.

Definition 2.21 (Covariant Derivative)**.** Let $\boldsymbol{\omega} \in \Gamma(T^*\mathcal{M})$ and $\mathbf{u} \in \Gamma(T\mathcal{M})$. For every vector field $\mathbf{v} \in \Gamma(T\mathcal{M})$ we consider the tensor field $\nabla\mathbf{v} \in \Gamma(T\mathcal{M} \otimes T^*\mathcal{M})$ defined by

$$\nabla\mathbf{v}(\boldsymbol{\omega}, \mathbf{u}) \coloneqq \boldsymbol{\omega}(\nabla_{\mathbf{u}}\mathbf{v}) \ . \tag{2.15}$$

The tensor field $\nabla\mathbf{v}$ is called the *covariant derivative of* $\mathbf{v}$.

Let (U, x) be a chart on $\mathcal{M}$, then $\mathbf{v} = v^i \partial_{x^i}$ and $\nabla\mathbf{v} = v^i_{;j} \partial_{x^i} \otimes \mathrm{d}x^j$. Notice the semicolon in the component of the covariant derivative. This has its origin from index notation, in which only components of the tensors are written. The semicolon distinguishes between partial derivative, i.e. application of the base vectors to the components of a vector, and covariant derivative of a vector field. According to the representation of a tensor as a linear combination (A.9) together with (2.13b) and (2.14), we obtain the component functions of the tensor field as

$$\begin{aligned} v^i_{;j} &= \nabla\mathbf{v}(\mathrm{d}x^i, \partial_{x^j}) \overset{(2.15)}{=} \mathrm{d}x^i(\nabla_{\partial_{x^j}}(v^k \partial_{x^k})) \\ &= \mathrm{d}x^i(\partial_{x^j}(v^k)\partial_{x^k} + v^k \Gamma^l_{jk} \partial_{x^l}) = \partial_{x^j}(v^i) + \Gamma^i_{jk} v^k \ . \end{aligned}$$

Definition 2.22 (Covariant Derivative of Pullback Section)**.** Let $\gamma \in C^k(\mathcal{N}, \mathcal{M})$, $\mathbf{a} \in \Gamma(T\mathcal{N})$, $\mathbf{v} \in \Gamma(T\mathcal{M})$ with the associated pullback section $\gamma^*\mathbf{v} \in C^k(\gamma^*T\mathcal{M})$ and $\boldsymbol{\omega} \in C^k(\gamma^*T^*\mathcal{M})$. Let $\mathcal{M}$ be equipped with an affine connection ∇. Then, for every pullback section $\gamma^*\mathbf{v}$, the tensor field $(\gamma^*\nabla)(\gamma^*\mathbf{v}) \in C^k(\gamma^*T\mathcal{M} \otimes T^*\mathcal{N})$ over $\mathcal{N}$ is defined as

$$(\gamma^*\nabla)(\gamma^*\mathbf{v})(\boldsymbol{\omega}, \mathbf{a}) \coloneqq \boldsymbol{\omega}(\gamma^*(\nabla_{T\gamma\mathbf{a}}\mathbf{v})) \ . \tag{2.16}$$

The tensor field $(\gamma^*\nabla)(\gamma^*\mathbf{v})$ is called *covariant derivative of* $\gamma^*\mathbf{v}$.

Let (U, θ) be a chart on $\mathcal{N}$ and let (V, x) be a chart on $\mathcal{M}$ such that $\gamma(U) \subset V$. Let $\mathbf{v} \in \Gamma(T\mathcal{M})$ be defined on the whole of V. Then the covariant derivative of the pullback section $\gamma^*\mathbf{v}$ corresponds to a tensor field $(\gamma^*\nabla)(\gamma^*\mathbf{v}) = (\gamma^*v^i)_{;j}(\partial_{x^i} \circ \gamma) \otimes \mathrm{d}\theta^j$. The computation of the component functions of the tensor field follows (A.9), i.e.

$$(\gamma^*v^i)_{;j} = (\gamma^*\nabla)(\gamma^*\mathbf{v})(\mathrm{d}x^i \circ \gamma, \partial_{\theta^j}) \overset{(2.16)}{=} (\mathrm{d}x^i \circ \gamma)(\gamma^*(\nabla_{T\gamma\partial_{\theta^j}}\mathbf{v})) \ .$$

Let $\hat{\gamma} = x \circ \gamma \circ \theta^{-1}$ be the local representation of γ around $P \in U$. Using (2.11), the vectorial part of the tangent map $T\gamma$ of a vector field $\partial_{\theta^j} \in \Gamma(T\mathcal{N})$ can locally be represented as

$$D\gamma\,\partial_{\theta^j} = (\partial_j \hat{\gamma}^i \circ \theta)\,\partial_{x^i}|_{\gamma(\cdot)} = F^i_j(\partial_{x^i} \circ \gamma) \ . \tag{2.17}$$

Let $P \in U$. Using a telescopic expansion and applying the chain rule, we show the following identity:

$$\begin{aligned} \partial_{\theta^j}|_P(v^i \circ \gamma) &\overset{(2.3)}{=} \partial_j(v^i \circ x^{-1} \circ x \circ \gamma \circ \theta^{-1})|_{\theta(P)} = \partial_k(v^i \circ x^{-1})|_{(x(\gamma(P))}(\partial_j \hat{\gamma}^k \circ \theta)(P) \\ &= \partial_{x^k}|_{\gamma(P)}(v^i)\, F^k_j(P)\ . \end{aligned} \tag{2.18}$$

Using property (2.13b) and the local representation by the Christoffel symbols (2.14) we compute:

$$\begin{aligned} \gamma^*(\nabla_{T\gamma\partial_{\theta^j}}\mathbf{v}) &\overset{(2.17)}{=} F^i_j\gamma^*((\partial_{x^i}(v^k)\partial_{x^k} + v^k\Gamma^r_{ik}\partial_{x^r})|_{\gamma(\cdot)}) \\ &\overset{(2.18)}{=} (\partial_{\theta^j}(v^k \circ \gamma) + (v^r \circ \gamma)(\Gamma^k_{ir} \circ \gamma)F^i_j)(\partial_{x^k} \circ \gamma)\ . \end{aligned}$$

Hence, the component functions of the covariant derivative of $\gamma^*\mathbf{v}$ are represented locally as

$$(\gamma^* v^i)_{;j} = \partial_{\theta^j}(v^k \circ \gamma) + (v^r \circ \gamma)(\Gamma^k_{ir} \circ \gamma)F^i_j\ . \tag{2.19}$$

Example 2.1. Let $\mathcal{N} = I$ be an interval of $\mathbb{R}$, $\gamma\colon I \to \mathcal{M}$ be a curve on $\mathcal{M}$ and $\mathbf{v} \in \Gamma(T\mathcal{M})$. An illustrative application of the covariant derivative of a pullback section is its correlation to the covariant derivative of $\mathbf{v}$ along a curve γ, denoted by $\nabla_{\dot\gamma}\mathbf{v}$. For the definition of a covariant derivative of $\mathbf{v}$ along a curve γ we refer to Abraham and Marsden (1978), Def. 2.7.3. Let $(I, \theta = \mathrm{Id_I})$ and (U, x) be charts on I and $\mathcal{M}$, respectively, then $F^i_1 = \partial_1(x^i \circ \gamma)$. Using (2.16) and (2.19), we obtain a vector field $\mathbf{v}$ along γ when taking the covariant derivative of $\gamma^*\mathbf{v}$ along ∂_θ, i.e.

$$(\gamma^*\nabla)(\gamma^*\mathbf{v})(\cdot, \partial_\theta) = \big(\partial_\theta(v^k \circ \gamma) + (v^r \circ \gamma)(\Gamma^k_{ir} \circ \gamma)\partial_1(x^i \circ \gamma)\big)\,(\partial_{x^k} \circ \gamma) = \nabla_{\dot\gamma}\mathbf{v}\ .$$

Since θ is the identity map, the induced partial derivative ∂_θ and the partial derivative ∂_1 coincide. For every $t \in I$, the covariant derivative of $\gamma^*\mathbf{v}$ along ∂_θ

$$(\gamma^*\nabla)(\gamma^*\mathbf{v})(\cdot, \partial_\theta)(t) = \big(\partial_1(v^k \circ \gamma)|_t + v^r(\gamma(t))\Gamma^k_{ir}(\gamma(t))\partial_1(x^i \circ \gamma)|_t\big)\,(\partial_{x^k} \circ \gamma)|_t = \nabla_{\dot\gamma(t)}\mathbf{v}$$

corresponds to the covariant derivative of $\mathbf{v}$ along γ.

Chapter 3

Force Representations

This chapter introduces the concept of force, states the principle of virtual work of a continuous body, discusses admissible force representations and concludes with the application to classical nonlinear continuum mechanics. In Section 3.1, forces are defined as linear functionals on the space of virtual displacements and the principle of virtual work for the continuous body is formulated. Subsequently, the force representation of Segev (1986b) by smooth tensor measures is introduced. In Section 3.2 the applied forces are restricted to a subclass of possible forces and the equations of motion of a continuous body mapped to the Euclidean vector space are derived.

3.1 Principle of Virtual Work

For this chapter, let $\mathcal{B}$ and $\mathcal{S}$ be the body and the physical space according to Definition 2.3 and 2.4, respectively, with dimensions $m = n = 3$. The configuration of a continuous body is a C^1-embedding, i.e. $\kappa \in \mathcal{Q} = \mathrm{Emb}^1(\mathcal{B}, \mathcal{S})$. The space of virtual displacements at a configuration κ is the tangent space $T_\kappa\mathcal{Q}$ to the infinite dimensional configuration manifold $\mathcal{Q}$, which is, due to Theorem 2.3, represented by the set of pullback sections $C^1(\kappa^*T\mathcal{S})$. By pointwise scalar multiplication and pointwise addition, the set of pullback sections constitute a linear infinite dimensional vector space.

Definition 3.1 (Forces)**.** Let $C^1(\kappa^*T\mathcal{S})$ be the space of virtual displacements of the continuous body. The *space of forces* is the set of real-valued linear functionals

$$C^1(\kappa^*T\mathcal{S})^* := \{\mathbf{f}\colon C^1(\kappa^*T\mathcal{S}) \to \mathbb{R} \colon \mathbf{f} \text{ linear}\} \,. \tag{3.1}$$

An element of $C^1(\kappa^*T\mathcal{S})^*$ is called a *force of a continuous body.* Let $\delta W := \mathbf{f}(\delta\boldsymbol{\kappa})$ be the real number obtained by the evaluation of a force $\mathbf{f} \in C^1(\kappa^*T\mathcal{S})^*$ acting on a virtual displacement $\delta\boldsymbol{\kappa} \in C^1(\kappa^*T\mathcal{S})$. The real number δW is called the *virtual work of the continuous body.*

Classically, people have had difficulties to define the concept of force. Thomson and Tait (1867), Par. 217, define a force as any cause which tends to alter a body's natural state of rest, or of uniform motion in a straight line. So, force is wholly expended in the action it produces. Kirchhoff (1876) already recognized that the perception to artificially split a mechanical process in action and reaction is disadvantageous. Mechanics is primarily interested in describing the mechanical process as a whole. However, Kirchhoff (1876) has refused to give a definition of a force. In more recent literature, Noll (1959) and Truesdell (1977) have dared to define forces as vector valued measures. Due to the representation theorem of Riesz-Markov, cf. Rudin (1987), in our framework, such forces can be represented by elements of the dual space of C^0-continuous sections of the pullback bundle $\kappa^*T\mathcal{S}$. Hence, Definition 3.1 and the definition of Noll and Truesdell do not coincide.

As the fundamental principle of mechanics, we postulate the principle of virtual work of a continuous body as an axiom.

Principle 3.1 (Principle of Virtual Work of a Continuous Body)**.** *Let* $\mathbf{f} \in C^1(\kappa^*T\mathcal{S})^*$ *be a force of a continuous body* $\mathcal{B}$*. Then, the principle of virtual work states, that the virtual work of a continuous body vanishes for all virtual displacements, i.e.*

$$\delta W = \mathbf{f}(\delta\boldsymbol{\kappa}) = 0 \quad \forall \delta\boldsymbol{\kappa} \in C^1(\kappa^*T\mathcal{S}) \ .$$

A force of a continuous body in the principle of virtual work, consists of all forces acting on that body. Further specifications, representations and the introduction of force laws are the next steps in the modeling process towards a proper description of the behavior of a deformable body. It is worth noticing, that this is another viewpoint on the principle of virtual work as it is given by Epstein and Segev (1980), who interpret the principle of virtual work as a mathematical compatibility between a force of the continuous body and its stress representation. To obtain the classical and established equations of motion of a continuous body, further assumptions and choices have to be done. The first assumption is to equip the physical space with further geometrical structure and redefine it as follows.

Definition 3.2 (Physical Space)**.** Let $n \geq m$. The *physical space* is an n-dimensional smooth manifold $\mathcal{S}$ without boundary with an affine connection ∇. A point Q of the physical space $\mathcal{S}$ is called a *space point*.

Remark, that the affine connection is independent of the choice of a metric, a symmetric and positive definite covariant tensor field of rank two. If there is a metric available, then it is convenient, but not necessary, to define an affine connection as the Levi-Civita connection, which is the unique metric compatible and torsion free affine connection, cf. Kühnel (2013), Thm. 5.16. Doing so, we lose degrees of freedom to model the physical space and to describe desired mechanical behavior of the space. A chart independent formulation of accelerated frames, for instance, requires the concept of vector bundles. Such a vector bundle consists of a one-dimensional Riemannian base space, modeling the time, together with a typical fiber of a three-dimensional Euclidean vector space, modeling the real space. The acceleration of the frame can then be described by an affine connection, whose definition corresponds to the choice of an inertial frame.

According to (3.1), forces of a continuous body are from the space $C^1(\kappa^*T\mathcal{S})^*$. A relation to the definition of forces as vector valued measures is obtained by a representation theorem proposed by Segev (1986b). According to Definition 2.22, the connection ∇ of the physical space implies a covariant derivative $(\kappa^*\nabla)(\delta\boldsymbol{\kappa}) \in C^0(\kappa^*T\mathcal{S} \otimes T^*\mathcal{B})$ of the virtual displacement $\delta\boldsymbol{\kappa} \in C^1(\kappa^*T\mathcal{S})$. Hence, the covariant derivative is a C^0-section through the tensor bundle $\kappa^*T\mathcal{S} \otimes T\mathcal{B}^*$ over $\mathcal{B}$. We introduce the function

$$\begin{aligned} \overline{\nabla}\colon C^1(\kappa^*T\mathcal{S}) &\to C^0(\kappa^*T\mathcal{S} \oplus (\kappa^*T\mathcal{S} \otimes T^*\mathcal{B})) \\ \delta\boldsymbol{\kappa} &\mapsto (\delta\boldsymbol{\kappa}, (\kappa^*\nabla)(\delta\boldsymbol{\kappa}))\ , \end{aligned}$$

where $\oplus$ denotes the direct sum. With reference to Segev (1986b), for the space of linear functionals on the image of $\overline{\nabla}$, the identity

$$C^0(\kappa^*T\mathcal{S} \oplus (\kappa^*T\mathcal{S} \otimes T^*\mathcal{B}))^* = C^0(\kappa^*T\mathcal{S})^* \oplus C^0(\kappa^*T\mathcal{S} \otimes T^*\mathcal{B})^* \tag{3.2}$$

holds. Thus, using the function $\overline{\nabla}$ together with the identity (3.2) and the representation theorem of Riesz-Markov, a force of a continuous body $\mathbf{f} \in C^1(\kappa^*T\mathcal{S})$ has a representation by a collection of tensor measures $(\mathbf{f}_0, \mathbf{f}_1) \in C^0(\kappa^*T\mathcal{S})^* \oplus C^0(\kappa^*T\mathcal{S} \otimes T^*\mathcal{B})^*$. Consequently, the virtual work of a continuous body can be represented as[1]

$$\delta W = \mathbf{f}(\delta\boldsymbol{\kappa}) = \int_{\mathcal{B}} \delta\boldsymbol{\kappa} \mathrm{d}\mathbf{f}_0 + \int_{\mathcal{B}} (\kappa^*\nabla)(\delta\boldsymbol{\kappa})\mathrm{d}\mathbf{f}_1\ .$$

It is important to note, that the tensor measures $(\mathbf{f}_0, \mathbf{f}_1)$ are not uniquely determined by $\mathbf{f}$. The measure $\mathbf{f}_0$ corresponds to forces defined as vector valued measures, cf. Noll (1959). The tensor measure $\mathbf{f}_1$ includes the stress tensor of classical continuum mechanics.

Let (U, x), (V, θ) and (W, λ) be appropriate charts on $\mathcal{S}$, $\mathcal{B}$ and $\partial\mathcal{B}$, respectively. Then a smooth section $\boldsymbol{\beta} \in \Gamma(\kappa^*T^*\mathcal{S} \otimes \Lambda^3T^*\mathcal{B})$ can be represented in coordinates as

$$\boldsymbol{\beta} = \beta_{i123}(\mathrm{d}x^i \circ \kappa) \otimes \mathrm{d}\theta^1 \wedge \mathrm{d}\theta^2 \wedge \mathrm{d}\theta^3 \overset{(A.21)}{=} \beta_{i123}(\mathrm{d}x^i \circ \kappa) \otimes \mathrm{d}\theta^{123}\ , \tag{3.3}$$

which describes a *body force*, i.e. a force per volume. A smooth section $\boldsymbol{\tau} \in \Gamma(\kappa^*T^*\mathcal{S} \otimes \Lambda^2T^*\partial\mathcal{B})$ is considered as a *traction force*, i.e. a force per surface, which is locally represented as

$$\boldsymbol{\tau} = \tau_{i12}(\mathrm{d}x^i \circ \kappa) \otimes \mathrm{d}\lambda^1 \wedge \mathrm{d}\lambda^2 \overset{(A.21)}{=} \tau_{i12}(\mathrm{d}x^i \circ \kappa) \otimes \mathrm{d}\lambda^{12}\ .$$

The evaluation of the tensor field $\boldsymbol{\beta}$ of body forces (3.3) at a point on the body is a tensor of rank 4, in which the last three tensor slots are alternating. Let $\delta\boldsymbol{\kappa} \in C^1(\kappa^*T\mathcal{S})$ be a virtual displacement. By a slight abuse of notation, we introduce the convention to denote the mapping from $C^1(\kappa^*T\mathcal{S})$ to $\Gamma(\Lambda^3T^*\mathcal{B})$ by

$$\boldsymbol{\beta}(\delta\boldsymbol{\kappa}) \coloneqq \boldsymbol{\beta}(\delta\boldsymbol{\kappa}, \cdot, \cdot, \cdot)\ . \tag{3.4}$$

Hence, $\boldsymbol{\beta}(\delta\boldsymbol{\kappa}) \in \Gamma(\Lambda^3T^*\mathcal{B})$ is a volume form which can be integrated over the body $\mathcal{B}$. For the traction $\boldsymbol{\tau}$ the convention holds in a similar way.

[1] We refer to Brezis (2010), Prop. 9.20, for a similar representation theorem for functions of the Sobolev space $W_0^{1,p}(\Omega)$ on a subset $\Omega \subset \mathbb{R}^n$.

Assuming that $\mathcal{B}$ is orientable, then the smooth sections $\boldsymbol{\beta}$ and $\boldsymbol{\tau}$ induce a stress measure $\mathbf{f}_0$ by

$$\int_{\mathcal{B}} \delta\boldsymbol{\kappa}\mathrm{d}\mathbf{f}_0 = \int_{\mathcal{B}} \boldsymbol{\beta}(\delta\boldsymbol{\kappa}) + \int_{\partial\mathcal{B}} \boldsymbol{\tau}(\delta\boldsymbol{\kappa}) \ .$$

Similar to the body and traction forces, we consider the *variational stress* $\boldsymbol{\pi} \in \Gamma(\kappa^* T^*\mathcal{S} \otimes T\mathcal{B} \otimes \Lambda^3 T^*\mathcal{B})$, which can locally be represented as

$$\boldsymbol{\pi} = \pi_{i123}^{j}\,(\mathrm{d}x^i \circ \kappa) \otimes \partial_{\theta^j} \otimes \mathrm{d}\theta^1 \wedge \mathrm{d}\theta^2 \wedge \mathrm{d}\theta^3 = \pi_{i123}^{j}\,(\mathrm{d}x^i \circ \kappa) \otimes \partial_{\theta^j} \otimes \mathrm{d}\theta^{123} \ . \qquad (3.5)$$

Applying the convention (3.4) to $\boldsymbol{\pi}$ and requiring $\mathcal{B}$ to be orientable, the smooth sections $\boldsymbol{\pi}$ induce a tensor measure $\mathbf{f}_1$ by

$$\int_{\mathcal{B}} (\kappa^*\nabla)(\delta\boldsymbol{\kappa})\mathrm{d}\mathbf{f}_1 = \int_{\mathcal{B}} \boldsymbol{\pi}((\kappa^*\nabla)(\delta\boldsymbol{\kappa})) = \int_{\mathcal{B}} \boldsymbol{\pi}(\delta\mathbf{F}) \ , \qquad (3.6)$$

where $\delta\mathbf{F} \coloneqq (\kappa^*\nabla)(\delta\boldsymbol{\kappa})$ has been recognized.

Due to the high continuity assumptions on the virtual displacement field, forces such as point forces, line distributed forces, traction forces within the body an many more cannot be described. A relaxation to piecewise continuous virtual displacement fields has to be discussed to allow for a broader spectrum of forces which is inevitable in mechanics.

3.2 Classical Nonlinear Continuum Mechanics

Following the assumptions of classical nonlinear continuum mechanics, cf. Truesdell and Toupin (1960), *external forces* are assumed to be given by vector valued measures $\mathbf{f}_0$ only. The corresponding virtual work contributes negatively to the total virtual work as

$$\delta W^{\mathrm{ext}} = -\int_{\mathcal{B}} \mathrm{d}\mathbf{f}_0(\delta\boldsymbol{\kappa}) \ . \qquad (3.7)$$

Within a first gradient theory, Germain (1972) allows also external forces to be given by $\mathbf{f}_1$ tensor measures.

The internal forces are modeled as “short range forces”. Thus, the internal forces are restricted to $\mathbf{f}_1$ tensor measures. Due to (3.6), this implies the virtual work contribution

$$\delta W^{\mathrm{int}} = \int_{\mathcal{B}} \boldsymbol{\pi}((\kappa^*\nabla)(\delta\boldsymbol{\kappa})) \ . \qquad (3.8)$$

Such an identification of the force representatives of internal forces, seems rather arbitrary. Assuming the physical space $\mathcal{S}$ to be Riemannian, i.e. a manifold with a metric, the identification of internal forces can be deduced from an additional fundamental principle. The additional principle is the law of interaction, which defines the internal forces of an arbitrary subsystem $\mathcal{B}' \subset \mathcal{B}$. We give here just a short outlook and an idea without proofs, how the law of interaction can be formulated on manifolds. Let $\mathbf{g} \in \Gamma(T^*\mathcal{S} \otimes T^*\mathcal{S})$ be a metric on the physical space and $\delta\boldsymbol{\varphi} \in \Gamma(T\mathcal{S})$ be the spatial virtual displacements.

Let $\mathcal{L}_{\delta\varphi}\mathbf{g}$ denote the Lie derivative of $\mathbf{g}$ with respect to $\delta\boldsymbol{\varphi}$, cf. Lee (2012). We define a Killing vector field $\delta\boldsymbol{\varphi} \in \Gamma(T\mathcal{S})$ to be a vector field satisfying

$$\mathcal{L}_{\delta\varphi}\mathbf{g} = 0 \; . \tag{3.9}$$

The space of Killing vector fields is denoted by $\Gamma_k(T\mathcal{S})$. The requirement (3.9) can be considered as a local symmetry of the physical space. In terms of a Levi-Civita connection on $\mathcal{S}$, (3.9) can be transformed and represented locally as

$$\delta\varphi^k_{;j} g_{ki} + \delta\varphi^k_{;i} g_{kj} = \delta\varphi_{i;j} + \delta\varphi_{j;i} = 0 \; ,$$

where semicolon denotes the covariant derivative. This condition for rigidifying virtual displacement fields and also the following variational form of the law of interaction has already been formulated by Murnaghan (1937). Let the spatial virtual displacement field $\delta\boldsymbol{\varphi} \in \Gamma_k(T\mathcal{S})$ be a Killing vector field. We denote the induced virtual displacement field $\delta\boldsymbol{\kappa} = \kappa^*\delta\boldsymbol{\varphi}$ by $\delta\boldsymbol{\kappa}$ rigidifying. Then the law of interaction asks the internal forces $\mathbf{f}^{\text{int}} \in C^1(\kappa^* T\mathcal{S})^*$ to satisfy

$$\delta W^{\text{int}} = \mathbf{f}^{\text{int}}(\delta\boldsymbol{\kappa}) = 0 \quad \forall \delta\boldsymbol{\kappa} \text{ rigidifying} \; .$$

Let the Euclidean three-space $\mathbb{E}^3$ be the physical space. Then, for smooth force representatives it can be shown, that the internal virtual work has to be of the form (3.8). Additionally, a symmetry condition for the first two components of the variational stress $\boldsymbol{\pi}$ is obtained[2].

Using the virtual work contribution of the external and internal forces (3.7) and (3.8), respectively, the metric independent virtual work principle of a continuous body of (3.1) is restated for the classical choice of force representatives.

Principle 3.2 (Principle of Virtual Work of Classical Continuum Mechanics)**.** *Let* $\boldsymbol{\pi} \in \Gamma(\kappa^* T^*\mathcal{S} \otimes T\mathcal{B} \otimes \Lambda^3 T^*\mathcal{B})$ *and* $\mathbf{f}_0 \in C^0(\kappa^* T\mathcal{S})^*$ *be the force representatives of* $\mathbf{f} \in C^1(\kappa^* T\mathcal{S})$*. Then, the principle of virtual work of a continuous body* $\mathcal{B}$ *states, that the virtual work*

$$\delta W = \delta W^{\text{int}} + \delta W^{\text{ext}} = \int_{\mathcal{B}} \boldsymbol{\pi}((\kappa^*\nabla)(\delta\boldsymbol{\kappa})) - \int_{\mathcal{B}} \mathrm{d}\mathbf{f}_0(\delta\boldsymbol{\kappa}) = 0 \quad \forall \delta\boldsymbol{\kappa} \in C^1(\kappa^* T\mathcal{S}) \; . \tag{3.10}$$

vanishes for all virtual displacements.

In the local representation of the variational stress (3.5), the three last slots of the tensor are constituted by a volume form. Let $\mathrm{d}V = \rho\, \mathrm{d}\theta^1 \wedge \mathrm{d}\theta^2 \wedge \mathrm{d}\theta^3 = \rho\, \mathrm{d}\theta^{123}$ be a volume element on the body $\mathcal{B}$, i.e. a nowhere vanishing volume form on $\mathcal{B}$. Then the variational stress can be transformed to

$$\boldsymbol{\pi} = \pi^j_{i123}\, (\mathrm{d}x^i \circ \kappa) \otimes \partial_{\theta^j} \otimes \mathrm{d}\theta^{123} = \rho^{-1} \pi^j_{i123} (\mathrm{d}x^i \circ \kappa) \otimes \partial_{\theta^j}\, \mathrm{d}V = \mathbf{P}\mathrm{d}V \; , \tag{3.11}$$

[2]Notice, symmetry condition does not mean that the two first components of $\boldsymbol{\pi}$ are symmetric. Such a statement is meaningless, since both components belong to different vector spaces. Hence, the symmetry condition will include metric information as well as the tangent map $T\kappa$.

where $P_i^j = \rho^{-1}\pi_{i123}^j$ and $\mathbf{P} = P_i^j(\mathrm{d}x^i \circ \kappa) \otimes \partial_{\theta^j}$. Notice that $\mathbf{P}$ as an independent object, generally denoted as the stress tensor, is not a tensor field. By splitting-off of the volume element $\mathrm{d}V$ from the variational stress $\boldsymbol{\pi}$, the tensor property is destroyed. At most, $\mathbf{P}$ can be considered as a tensor valued density. This explains the awkward transformation rules for the different stress tensors in classical nonlinear continuum mechanics, cf. Truesdell and Noll (1965). Nevertheless, the internal virtual work (3.8) as a whole remains an invariant quantity and using (3.11), it can be transformed further to

$$\delta W^{\mathrm{int}} = \int_{\mathcal{B}} \mathbf{P}(\delta\mathbf{F})\,\mathrm{d}V =: \int_{\mathcal{B}} \mathbf{P} : \delta\mathbf{F}\,\mathrm{d}V\ , \tag{3.12}$$

where we introduced the commonly used notation of the double contraction.

Example 3.1. Let the physical space $\mathcal{S}$ be the three-dimensional Euclidean space $\mathbb{E}^3$ with the Levi-Civita connection ∇ and let $(\mathbb{E}^3, x)$ be a cartesian chart, i.e. the base vectors $(\partial_{x^1}, \partial_{x^2}, \partial_{x^3})$ are orthonormal with respect to the given metric. Furthermore, assume that the body can be described by a single chart $(\mathcal{B}, \theta)$ and let $\mathrm{d}V = \mathrm{d}\theta^{123} \in \Gamma(\Lambda^3 T^*\mathcal{B})$ be the volume element. Due to the cartesian chart x, the Christoffel symbols vanish in the covariant derivative (2.19). Denote the duality pairing by a dot $(\cdot)$ and introduce $\hat{\mathbf{t}}^i := P_k^i(\mathrm{d}x^k \circ \kappa)$. Then, the internal virtual work (3.12) is transformed further to

$$\delta W^{\mathrm{int}} = \int_{\mathcal{B}} \partial_{\theta^i}(\delta\kappa^j(\partial_{x^j} \circ \kappa)) \cdot P_k^i(\mathrm{d}x^k \circ \kappa)\,\mathrm{d}\theta^{123} = \int_{\mathcal{B}} \partial_{\theta^i}(\delta\boldsymbol{\kappa}) \cdot \hat{\mathbf{t}}^i\,\mathrm{d}\theta^{123}\ .$$

Let $\overline{B} := \theta(\mathcal{B})$ be the domain of the body in the chart θ and use the composite functions $\delta\boldsymbol{\xi} := \delta\boldsymbol{\kappa} \circ \theta^{-1}$ and $\mathbf{t}^i := \hat{\mathbf{t}}^i \circ \theta^{-1}$. The local representation of the internal virtual work in the body chart follows as

$$\begin{aligned}\delta W^{\mathrm{int}} &= \int_{\overline{B}} \partial_i(\delta\boldsymbol{\kappa} \circ \theta^{-1}) \cdot (\hat{\mathbf{t}}^i \circ \theta^{-1})\,\mathrm{d}\theta^1\mathrm{d}\theta^2\mathrm{d}\theta^3 = \int_{\overline{B}} \partial_i(\delta\boldsymbol{\xi}) \cdot \mathbf{t}^i\,\mathrm{d}\theta^1\mathrm{d}\theta^2\mathrm{d}\theta^3 \\ &= \int_{\overline{B}} \delta\boldsymbol{\xi}_{,i} \cdot \mathbf{t}^i\,\mathrm{d}^3\theta\ ,\end{aligned}$$

where in the last line the partial derivative ∂_i and the volume element $\mathrm{d}\theta^1\mathrm{d}\theta^2\mathrm{d}\theta^3$ are abbreviated by $(\cdot)_{,i}$ and $\mathrm{d}^3\theta$, respectively. Note, that the integration of the volume form over the body manifold $\mathcal{B}$ is defined by the integration of the chart representation in $\mathbb{R}^3$, cf. Lee (2012).

Assume furthermore that the cartesian chart of $\mathbb{E}^3$ is an inertial chart. A *motion* $\kappa_t\colon \mathcal{B} \times \mathbb{R} \to \mathbb{E}^3$ of the body is a differentiable parametrization of configurations with respect to time $t \in \mathbb{R}$. Thus, at a given instant of time t the closed subset $\overline{\Omega}_t = \kappa_t(\overline{\mathcal{B}}) \subset \mathbb{E}^3$ is covered by the body manifold. The coordinate representation of the motion is the vector valued function

$$\boldsymbol{\xi}\colon \overline{\mathrm{B}} \times \mathbb{R} \to \mathbb{E}^3,\ (\theta^k, t) \mapsto \boldsymbol{\xi} = \kappa_t \circ \theta^{-1}(\theta^k) = \boldsymbol{\xi}(\theta^k, t)\ .$$

Let $\mathrm{d}m$ be a mass distribution on $\overline{B}$. Then, we assume the inertia force to contribute as

$$\delta W^{\mathrm{dyn}} = \int_{\overline{B}} \delta\boldsymbol{\xi} \cdot \ddot{\boldsymbol{\xi}}\,\mathrm{d}m \tag{3.13}$$

to the total virtual work of the body, where the superposed dot $(\dot{\bullet})$ denotes the derivative with respect to time t. Since the cartesian chart is restricted to be an inertial chart, the introduction of inertia forces (3.13) is chart dependent and does not fit into the geometric concepts proposed otherwise Part I. To describe the virtual work of inertia forces in an intrinsic differential geometrical setting, an extension of the physical space as discussed previously below Definition 3.2 is necessary.

Example 3.2. Let the physical space $\mathcal{S}$ be the three-dimensional Euclidean space $\mathbb{E}^3$ with the Levi-Civita connection ∇. Furthermore, let $(\mathbb{E}^3, X)$ be a cartesian chart and the body $\mathcal{B}$ be a closed subset of $\mathbb{E}^3$. Hence, the body chart and the space chart coincide. The coordinate description of the boundary of the body $\partial\mathcal{B}$ is given by the chart $(\partial\mathcal{B}, Y)$. Let the volume element be $\mathrm{d}V = \mathrm{d}X^1 \wedge \mathrm{d}X^2 \wedge \mathrm{d}X^3$ and the surface element $\mathrm{d}A = \mathrm{d}Y^1 \wedge \mathrm{d}Y^2$. Assume the external forces to be given by body forces $\boldsymbol{\beta} \in \Gamma(\kappa^* T^*\mathbb{E}^3 \otimes \Lambda^3 T^*\mathbb{E}^3)$ and traction forces $\boldsymbol{\tau} \in \Gamma(\kappa^* T^*\mathbb{E}^3 \otimes \Lambda^2 T^*\partial\mathcal{B})$. Using the volume element $\mathrm{d}V$ and the surface element $\mathrm{d}A$, the body and the traction forces can be represented in the sense of (3.11) as $\boldsymbol{\beta} = \mathbf{B}\mathrm{d}V$ and $\boldsymbol{\tau} = \mathbf{T}\mathrm{d}A$. Together with the abbreviations $\mathrm{d}x^i := \mathrm{d}X^i \circ \kappa$ and $\partial_{x^i} := \partial_{X^i} \circ \kappa$, the negative virtual work contribution of the external forces can locally be represented as

$$\begin{aligned}
-\delta W^{\mathrm{ext}} &= \int_{\mathcal{B}} \mathrm{d}\mathbf{f}_0(\delta\boldsymbol{\kappa}) = \int_{\mathcal{B}} \boldsymbol{\beta}(\delta\boldsymbol{\kappa}) + \int_{\partial\mathcal{B}} \boldsymbol{\tau}(\delta\boldsymbol{\kappa}) = \int_{\mathcal{B}} \mathbf{B}(\delta\boldsymbol{\kappa})\mathrm{d}V + \int_{\partial\mathcal{B}} \mathbf{T}(\delta\boldsymbol{\kappa})\mathrm{d}A \\
&= \int_{\mathcal{B}} B_i \mathrm{d}x^i(\delta\kappa^j \partial_{x^j})\mathrm{d}V + \int_{\partial\mathcal{B}} T_i \mathrm{d}x^i(\delta\kappa^j \partial_{x^j})\mathrm{d}A \\
&= \int_{\mathcal{B}} B_i \delta\kappa^i \mathrm{d}V + \int_{\partial\mathcal{B}} T_i \delta\kappa^i \mathrm{d}A \; .
\end{aligned}$$

Since the body is a subset of the Euclidean space, the connection on the body is given by the Levi-Civita connection of the space. Within cartesian coordinates, the Christoffel symbols vanish and the internal virtual work contribution (3.12) is locally represented as

$$\begin{aligned}
\delta W^{\mathrm{int}} &= \int_{\mathcal{B}} \boldsymbol{\pi}(\delta\mathbf{F}) = \int_{\mathcal{B}} \mathbf{P}(\delta\mathbf{F})\mathrm{d}V = \int_{\mathcal{B}} (P_i^j \mathrm{d}x^i \otimes \partial_{X^j})(\partial_{X^l}(\delta\kappa^k)\partial_{x^k} \otimes \mathrm{d}X^l)\mathrm{d}V \\
&= \int_{\mathcal{B}} P_i^j \partial_{X^j}(\delta\kappa^i)\mathrm{d}V \; .
\end{aligned}$$

For the remainder of this example the derivations are performed in the chart representation. The principle of virtual work (3.10) of the continuous body in the body chart X is

$$\delta W = \int_B P_i^j \partial_j(\delta\kappa^i)\mathrm{d}V - \int_B B_i \delta\kappa^i \mathrm{d}V - \int_{\partial B} T_i \delta\kappa^i \mathrm{d}A = 0 \; , \quad \forall \delta\kappa^i \in C^1(\overline{B}) \; , \qquad (3.14)$$

where $\overline{B} = B \cup \partial B$ denotes the domain of the body in the chart. The virtual work expression (3.14) is generally known as the weak variational form of the continuous body. Using a telescopic expansion together with the product rule and the theorem of Gauss-Ostrogradsky, cf. Başar and Weichert (2000), the virtual work is transformed further to the strong variational form

$$
\begin{aligned}
\delta W &= \int_B P_i^j \partial_j(\delta\kappa^i)\mathrm{d}V - \int_B B_i \delta\kappa^i \mathrm{d}V - \int_{\partial B} T_i \delta\kappa^i \mathrm{d}A \\
&= \int_B \{ \underbrace{P_i^j \partial_j(\delta\kappa^i) - \partial_j(P_i^j \delta\kappa^i)}_{(\star)} + \partial_j(P_i^j \delta\kappa^i)\}\mathrm{d}V - \int_B B_i \delta\kappa^i \mathrm{d}V - \int_{\partial B} T_i \delta\kappa^i \mathrm{d}A \\
&= \int_B \left\{ -\partial_j(P_i^j)\delta\kappa^i + \partial_j(P_i^j \delta\kappa^i) \right\} \mathrm{d}V - \int_B B_i \delta\kappa^i \mathrm{d}V - \int_{\partial B} T_i \delta\kappa^i \mathrm{d}A \\
&= -\int_B \left\{ (\partial_j(P_i^j) + B_i)\delta\kappa^i \right\} \mathrm{d}V - \int_{\partial B} \left\{ (T_i - P_i^j N_j)\delta\kappa^i \right\} \mathrm{d}A \,,
\end{aligned}
\tag{3.15}
$$

where N_j denote the components of the normal vector to the boundary $\partial\mathcal{B}$. Since (3.15) has to vanish for all $\delta\kappa^i \in C^1(\overline{B})$, the strong variational form together with the Fundamental Lemma of Calculus of Variations leads to

$$
\begin{aligned}
0 &= \partial_j P_i^j + B_i \quad \text{in } B \,, \\
T_i &= P_i^j N_j \quad \text{on } \partial B \,.
\end{aligned}
$$

which constitute the equations of motion of the continuous body and the boundary conditions. Notice, that the derivation (3.15) relies on cartesian coordinates and for the theorem of Gauss-Ostrogradsky, introducing the normal vector, a metric is required. Hence, the strong variational formulation in (3.15) is obtained by a coordinate- and metric-dependent derivation. To formulate a complete coordinate- and metric-independent description of classical continuum mechanics, this derivation must be performed on the manifold using concepts from differential geometry. Such a coordinate- and metric-independent formulation can be found in Segev (1986b) and in a more elaborate way in Segev (2013). Segev (2013) defines the divergence of the stress tensor in the sense of the subtraction $(\star)$ which does not simplify in a coordinate free setting. This explains the artificial intermediate step of the second line in (3.15).

In Example 3.1, we have formulated the virtual work contributions of a continuous body moving in the Euclidean three-space. Therein, we have obtained the virtual work contributions which are required to start with the second part of the thesis dealing with beam theories. Concluding remarks on Part I can be found in Chapter 9.

Part II

Beam Theories

"Tatsächlich ist aber in der kurzen Zeit eines Menschenlebens und bei dem begrenzten Gedächtnis des Menschen ein nennenswertes Wissen nur durch die grösste Oekonomie der Gedanken erreichbar."

E. Mach, Die Mechanik in ihrer Entwicklung, 1883.

Chapter 4

Preliminaries

In this chapter we discuss the fundamental mechanical principles which are necessary for the formulation of induced beam theories. In order that Part II remains more or less self-contained, Section 4.1 repeats some results of the first part about the dynamics of a continuous body within the Euclidean space. Section 4.2 introduce the concept of perfect constraint stresses which are required that the motion of a continuous body follows a constrained position field. In Section 4.3, we discuss an appropriate description of a beam-like body and introduce the classification into intrinsic, induced and semi-induced beam theories.

4.1 Fundamental Principles of a Continuous Body

We adhere to the convention that pairs of Latin indices are summed from 1 to 3 and pairs of Greek indices are summed from 1 to 2. When a function depends on the three components (a^1, a^2, a^3) or on the first two components (a^1, a^2) of a triple $\mathbf{a} \in \mathbb{R}^3$, the argument is abbreviated by (a^i) or (a^β), respectively. We consider a *three-dimensional continuous body* $\mathcal{B}$ as a three-dimensional compact differentiable manifold with boundary. In order to avoid discussions about mathematical subtleties, we assume in the following that the body can be covered by a single chart θ, see Figure 4.1. Hence, every material point $P \in \mathcal{B}$ can be described by three coordinates $(\theta^1, \theta^2, \theta^3) \in \overline{\mathrm{B}} \subset \mathbb{R}^3$, where $\overline{\mathrm{B}} := \theta(\mathcal{B})$. A *configuration* $\kappa \in \mathrm{Emb}^1(\mathcal{B}, \mathbb{E}^3)$ is a C^1-embedding of the body into the Euclidean three-space $\mathbb{E}^3$, where the Euclidean three-space represents the *physical space*. The configurations are restricted to embeddings, which are proper injective immersions. Thus, the principle of impenetrability and the permanence of matter is guaranteed by the choice of the kinematics. Since the configuration maps the material points P to the Euclidean three-space, which is a vector space, the placement of a material point $\kappa(P)$ can be represented by the position vector $\boldsymbol{\xi} \in \mathbb{E}^3$. A *motion* $\kappa_t \colon \mathcal{B} \times \mathbb{R} \to \mathbb{E}^3$ of the body is a differentiable parametrization of configurations (or *current configurations*) with respect to time $t \in \mathbb{R}$. Thus, at a given instant of time t, the closed subset $\overline{\Omega}_t = \kappa_t(\mathcal{B}) \subset \mathbb{E}^3$ is covered by the body manifold.

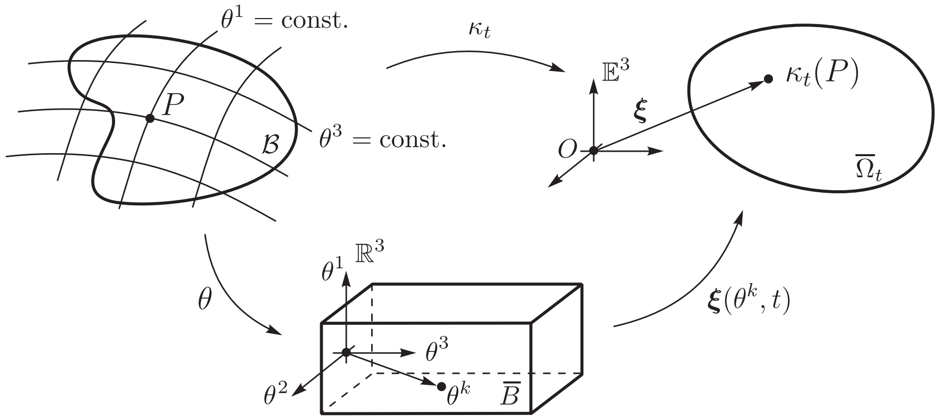

Figure 4.1: Schematic overview of the kinematics of the body manifold $\mathcal{B}$.

Using the chart θ, the coordinate representation of the motion is the vector valued function

$$\boldsymbol{\xi}\colon \overline{\mathrm{B}} \times \mathbb{R} \to \mathbb{E}^3,\ (\theta^k, t) \mapsto \boldsymbol{\xi} = \kappa_t \circ \theta^{-1}(\theta^k) = \boldsymbol{\xi}(\theta^k, t)\ ,$$

also denoted as the *position field*. Note that we are using the same symbol for the variables $\boldsymbol{\xi}$ as for the functions whose results they are. In the following we will mainly work with the coordinate representation of the motion and treat motion and coordinate representation of the motion synonymously. We consider only cartesian base vectors for $\mathbb{E}^3$ to avoid the concept of covariant derivatives. Additionally, this assumption allows to commute derivatives with respect to θ^i and derivatives with respect to a variation parameter ε.

A *variational family of the position field* is a differentiable parametrization of motions $\hat{\boldsymbol{\xi}}(\theta^k, t, \varepsilon)$ with respect to a single parameter $\varepsilon \in \mathbb{R}$. The actual motion is embedded in the family $\hat{\boldsymbol{\xi}}$ and is obtained for $\varepsilon = \varepsilon_0$, i.e. $\boldsymbol{\xi}(\theta^k, t) = \hat{\boldsymbol{\xi}}(\theta^k, t, \varepsilon_0)$. The variation of the position field $\boldsymbol{\xi}$ is defined as

$$\delta\boldsymbol{\xi}(\theta^k, t) := \frac{\partial\hat{\boldsymbol{\xi}}}{\partial\varepsilon}(\theta^k, t, \varepsilon_0)\ .$$

Let $i \in \{1, 2, 3\}$ and (i, j, k) be an even permutation of $(1, 2, 3)$, then we introduce the fields of covariant base vectors $\mathbf{g}_i(\theta^k, t)$, its corresponding variations $\delta\mathbf{g}_i(\theta^k, t)$ and its associated contravariant base vectors $\mathbf{g}^i(\theta^k, t)$ as

$$\mathbf{g}_i := \boldsymbol{\xi}_{,i}\ , \quad \delta\mathbf{g}_i = \delta\boldsymbol{\xi}_{,i}\ , \quad \mathbf{g}^i := g^{-1/2}(\mathbf{g}_j \times \mathbf{g}_k)\ , \quad g^{1/2} := \mathbf{g}_1 \cdot (\mathbf{g}_2 \times \mathbf{g}_3)\ ,$$

where partial derivatives $\partial(\cdot)/\partial\theta^k$ are abbreviated by $(\cdot)_{,k}$. The co- and contravariant base vectors fulfill the reciprocity condition $\mathbf{g}^i \cdot \mathbf{g}_j = \delta^i_j$.

The formulation of the fundamental principle of the dynamics of a continuous body demands the postulation of three contributions to the virtual work. The first contribution represents the internal virtual work δW^{int} of the continuous body which is formulated in the body chart θ as

$$\delta W^{\mathrm{int}}(\delta\boldsymbol{\xi}) := \int_{\overline{\mathrm{B}}} \boldsymbol{\sigma} : (\delta\mathbf{g}_i \otimes \mathbf{g}^i) g^{1/2} \mathrm{d}^3\theta = \int_{\overline{\mathrm{B}}} \mathbf{t}^i \cdot \delta\mathbf{g}_i \, \mathrm{d}^3\theta\ , \tag{4.1}$$

where $\mathrm{d}^3\theta = \mathrm{d}\theta^1\mathrm{d}\theta^2\mathrm{d}\theta^3$. The stress vector $\mathbf{t}^i(\theta^k, t)$ can be recognized in the Cauchy stress tensor $\boldsymbol{\sigma}(\boldsymbol{\xi}(\theta^k, t), t) = g^{-1/2}\,\mathbf{t}^i \otimes \mathbf{g}_i$. Let the indices (i, j, k) be an even permutation of $(1, 2, 3)$, then the stress vector $\mathbf{t}^i$ corresponds to the traction in the current configuration which acts at the surface element $\mathbf{g}_j \times \mathbf{g}_k\,\mathrm{d}\theta^j\mathrm{d}\theta^k = \mathbf{g}^i g^{1/2}\,\mathrm{d}\theta^j\mathrm{d}\theta^k$. A similar formulation of the internal virtual work (4.1) can also be found in Antman (2005), Sec. 16.2. The remaining two contributions to the virtual work are those of the inertia and the external forces $\mathrm{d}\mathbf{f}$ which contribute as

$$\delta W^{\mathrm{dyn}}(\delta\boldsymbol{\xi}) := \int_{\overline{\mathrm{B}}} \delta\boldsymbol{\xi} \cdot \ddot{\boldsymbol{\xi}}\,\mathrm{d}m\ , \quad \delta W^{\mathrm{ext}}(\delta\mathbf{x}) := -\int_{\overline{\mathrm{B}}} \delta\boldsymbol{\xi} \cdot \mathrm{d}\mathbf{f}\ , \tag{4.2}$$

where the superposed dot $(\dot{\bullet})$ denotes the derivative with respect to time t. We consider the mass distribution $\mathrm{d}m$ and the force distribution $\mathrm{d}\mathbf{f}$ as measures, allowing for Dirac-type contributions as well.

As the first fundamental principle of a continuous body, we postulate the principle of virtual work as an axiom.

Principle 4.1 (Principle of Virtual Work)**.** *At any instant of time t, the virtual work δW of the body $\mathcal{B}$ vanishes for all virtual displacements $\delta\boldsymbol{\xi}$, i.e.*

$$\delta W(\delta\boldsymbol{\xi}) = \delta W^{\mathrm{int}}(\delta\boldsymbol{\xi}) + \delta W^{\mathrm{dyn}}(\delta\boldsymbol{\xi}) + \delta W^{\mathrm{ext}}(\delta\boldsymbol{\xi}) = 0 \quad \forall\delta\boldsymbol{\xi}, \forall t\ . \tag{4.3}$$

Beside the virtual work principle, the law of interaction for internal forces has to be respected as a second axiom. In Glocker (2001), Sec. 2, the law of interaction is stated for force and couple distributions. A variational form of the law of interaction corresponds to the “Axiom of Power of Internal Force” formulated by Germain (1973a). In the case of particle mechanics the law of interaction coincides with “Newton’s Law of Action and Reaction”. For a variational formulation of the law of interaction, a special subset of virtual displacements is required. Virtual displacements are called *rigidifying* if they are induced by a rigid body motion of the continuous body.

Principle 4.2 (Law of Interaction)**.** *At any instant of time t, the internal virtual work of the body $\mathcal{B}$ vanishes for all rigidifying virtual displacements, i.e.*

$$\delta W^{\mathrm{int}}(\delta\boldsymbol{\xi}) = 0 \qquad \forall\delta\boldsymbol{\xi}\ \text{rigidifying}, \forall t\ . \tag{4.4}$$

Since the considered mechanical system is a continuous body and the law of interaction has to be fulfilled for all bodies, including any subbody $\mathcal{B}' \subset \mathcal{B}$, it is shown in Eugster et al. (2014), that the law of interaction for a smooth stress distribution can be formulated in the following local form:

$$\mathbf{g}_i \times \mathbf{t}^i = 0 \qquad \forall\theta^k \in \overline{\mathrm{B}}\ , \forall t\ . \tag{4.5}$$

This requirement corresponds to the symmetry condition of the Cauchy stress tensor. When the body coordinates are chosen such that they coincide with the cartesian coordinates of the Euclidean space, we can rewrite (4.5) as $\mathbf{e}_i \times \sigma^{ij}\mathbf{e}_j = 0$. Due to the orthonormality of the base vectors $\mathbf{e}_i$, we directly obtain the three symmetry conditions of the Cauchy stress, i.e. $\sigma^{12} = \sigma^{21}$, $\sigma^{13} = \sigma^{31}$ and $\sigma^{23} = \sigma^{32}$.

4.2 Constrained Position Fields

To formulate induced beam theories, one has to study constrained position fields of a continuous body. Since the constraints are treated as pointwise conditions in time, t is an inessential parameter which is omitted in the notation for the sake of clarity. For a given body chart θ and at every instant of time t, the *configuration manifold of the continuous body* is given by all possible configurations of the body which form the infinite dimensional manifold $\mathcal{K} \coloneqq \mathrm{Emb}^1(\overline{B}, \mathbb{E}^3)$. Let $\mathcal{A}$ be a finite or infinite dimensional manifold and $\mathbf{x} : \mathcal{A} \to \mathcal{K}$ be an embedding. For $\mathbf{a} \in \mathcal{A}$, the embedding $\mathbf{x}$ induces a position field $\boldsymbol{\xi} = \mathbf{x}(\mathbf{a})$ of the continuous body. The submanifold $\mathcal{C} \coloneqq \mathbf{x}(\mathcal{A}) \subset \mathcal{K}$ represents all position fields which can be described by the embedding of $\mathcal{A}$ in $\mathcal{K}$ and is called the *constraint manifold*. A configuration $\boldsymbol{\xi} \in \mathcal{C}$ is called a *constrained position field*. The tangent space at the point $\boldsymbol{\xi} \in \mathcal{C}$ to the constraint manifold is denoted by $T_{\xi}\mathcal{C}$. Elements of the tangent space $T_{\xi}\mathcal{C}$ are called *admissible virtual displacements*. Let $\mathbf{a} \in \mathcal{A}$ and $\boldsymbol{\xi} = \mathbf{x}(\mathbf{a})$, then the differential of $\mathbf{x}$

$$D\mathbf{x}(\mathbf{a})\colon T_{\mathbf{a}}\mathcal{A} \to T_{\xi}\mathcal{C} \ , \quad \delta\mathbf{a} \mapsto \delta\boldsymbol{\xi} = D\mathbf{x}(\mathbf{a})\delta\mathbf{a} \ , \tag{4.6}$$

induces admissible virtual displacements.

Assume a continuous body with a position field $\boldsymbol{\xi}$ whose dynamics is described by the principle of virtual work (4.3). To constrain the position field $\boldsymbol{\xi}$ such that it remains on the constraint manifold $\mathcal{C}$, a *constraint stress field* $\mathbf{t}_C^i(\theta^k, t)$ with a virtual work contribution

$$\delta W_C^{\mathrm{int}} \coloneqq \int_{\overline{B}} \mathbf{t}_C^i \cdot \delta\boldsymbol{\xi}_{,i} \, \mathrm{d}^3\theta$$

is introduced[1]. The stress contribution of (4.1) is renamed as $\mathbf{t}_I^i$ and is called an *impressed stress field*. Consequently, a continuous body with position field $\boldsymbol{\xi} \in \mathcal{C}$ which is enforced to follow a constrained position field, is exposed to a total stress field

$$\mathbf{t}^i(\theta^k, t) \coloneqq \mathbf{t}_I^i(\theta^k, t) + \mathbf{t}_C^i(\theta^k, t) \ , \tag{4.7}$$

which is composed by an impressed and a constraint stress field. Hence, the dynamics of a continuous body with a constrained position field is described by the principle of virtual work (4.3) with the total stress field (4.7). The constraint stresses are said to be perfect, if the constitutive law is given by the principle of d'Alembert–Lagrange, which states that the virtual work of the constraint stresses

$$\delta W_C^{\mathrm{int}}(\delta\boldsymbol{\xi}) = \int_{\overline{B}} \mathbf{t}_C^i \cdot \delta\boldsymbol{\xi}_{,i} \, \mathrm{d}^3\theta = 0 \qquad \forall \delta\boldsymbol{\xi} \in T_{\xi}\mathcal{C}, \forall t \ . \tag{4.8}$$

Let $\mathbf{a} \in \mathcal{A}$ induce the constrained position field $\boldsymbol{\xi} = \mathbf{x}(\mathbf{a})$ and assume the admissible virtual displacements $\delta\boldsymbol{\xi} = D\mathbf{x}(\mathbf{a})\delta\mathbf{a} \eqqcolon \delta\mathbf{x} \in T_{\xi}\mathcal{C}$. Then the virtual work of the perfect constraint stresses (4.8) vanishes for all $\delta\mathbf{a} \in T_{\mathbf{a}}\mathcal{A}$. To obtain the weak variational form of a constrained continuous body in a minimal description, the principle of virtual

[1] It is a choice that the constraint is guaranteed by a stress field. Alternatively the constraints can also be satisfied by constraint forces $\mathrm{d}\mathbf{z} \in C^0(\mathbb{E}^3)^*$.

work (4.3) with the total stress field (4.7) is evaluated for the constrained position field $\mathbf{x}$ together with admissible virtual displacements $\delta\mathbf{x} = D\mathbf{x}\delta\mathbf{a}$. Since the constraints are assumed to be perfect, the constraint stress contribution vanishes for the admissible virtual displacements $\delta\mathbf{x}$ by definition.

4.3 Intrinsic and Induced Beam Theories

A major challenge in beam theory is a rigorous definition of its central object, the *beam*. A beam formulation includes loads of modeling assumptions which are hard to grasp in their full diversity. A beam-like body can be considered as a model of a real body with one characteristic direction. In the case of a slender body with an isotropic material behavior, the characteristic direction coincides with the direction of the largest expansion of the body. Hence, only additional information about the body, such as geometry or material behavior, and its loading allows determining the characteristic direction of the body. Another difficulty is, that there exist several theories which call their investigated object *beam*, cf. for instance Antman (2005), Ballard and Millard (2009), Bauchau and Craig (2009), Cosserat and Cosserat (1909), Lacarbonara (2013), Love (1944), Rubin (2000), Sokolnikoff (1946), Villaggio (2005) and Wempner (1973). In this section we discuss three different classifications of beam theories which are introduced in Antman (2005). There are the *intrinsic beam theories*, the *induced* and the *semi-induced beam theories*.

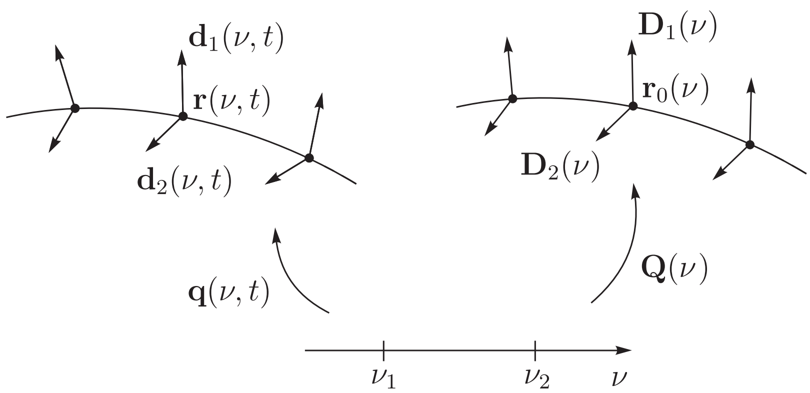

Figure 4.2: Visualization of the reference and the current configuration of the intrinsic classical beam, also denoted as Cosserat beam, $\mathbf{q}(\nu, t) = (\mathbf{r}(\nu, t), \mathbf{d}_1(\nu, t), \mathbf{d}_2(\nu, t))$ and $\mathbf{Q}(\nu) = (\mathbf{r}_0(\nu), \mathbf{D}_1(\nu), \mathbf{D}_2(\nu))$, respectively.

Basically, there exist two ways to state the dynamics of a beam-like body. The most classical way is an intrinsic beam formulation, as it is proposed by Euler (1744), Kirchhoff (1876) or Cosserat and Cosserat (1909) or more recently by Ballard and Millard (2009). Since a beam-like body has but one characteristic direction, denoted by the parameter $\nu \in [\nu_1, \nu_2] \subset \mathbb{R}$, we assume it as a generalized one-dimensional continuum. At every point ν a microstructure is attached, which is described by an N-dimensional *configuration manifold of the beam* $\mathcal{Q}$. Hence, the motion of the beam-like body is described by finitely many *generalized position functions*

$$\mathbf{q}\colon [\nu_1, \nu_2] \times \mathbb{R} \to \mathcal{Q},\ (\nu, t) \mapsto \mathbf{q}(\nu, t)\ , \tag{4.9}$$

where t parametrizes the time. The motion can schematically be visualized such as depicted in Figure 4.2. The reference configuration of the intrinsic beam is a time independent map $\mathbf{Q}\colon [\nu_1, \nu_2] \to \mathcal{Q}$, called the *reference generalized position functions.* Subsequently, the virtual work of the generalized one-dimensional continuum is stated directly. That means, we postulate the virtual work contributions of the internal, external and the inertia forces of the generalized one-dimensional continuum as integrated line densities per unit of ν and formulate a virtual work principle in the sense of (4.3) for a one-dimensional continuum. By stating the virtual work contributions, the generalized forces are defined in the sense of duality. To complete the formulation, intrinsic generalized strains have to be defined and relations between these intrinsic generalized strains and the internal generalized forces have to be stated. The benefit of an intrinsic formulation is that it is a closed and independent theory which is often free of indeterminacy. The drawback of such a formulation is, that much mechanical intuition is required for its successful application. The validation of an intrinsic formulation and the determination of constitutive parameters is done experimentally, cf. Hodges and Dowell (1975) or Dowell et al. (1977). Since an intrinsic formulation is completely decoupled from a three-dimensional theory, we cannot draw any conclusions about stress distributions of a beam in an intrinsic formulation.

In an intrinsic theory, every choice of generalized position functions $\mathbf{q}$ implies a different beam formulation. Eventually, this leads to infinitely many beam theories. In order to eliminate the intuition in the derivation of beam theories, we aspire a consistent procedure to obtain various beam theories. This possibility is given by induced beam theories. Induced beam theories are characterized by the following description of a beam:

> *A beam, in the sense of an induced theory, is a three-dimensional continuous body with one characteristic direction where the irrelevant deformations are eliminated by allowing merely constrained position fields for the bodies motion.*

We want to mention that in the above description of a beam the terms “characteristic direction” and “irrelevant” are undefined. The determination of these terms is part of the modeling process and is strongly influenced by the considered application at hand.

To formulate the constrained position fields, we use the very same generalized position functions (4.9) as introduced in the intrinsic beam formulation. At any instants of time t, let $\mathcal{A} := C^1([\nu_1, \nu_2], \mathcal{Q})$ be the set of all C^1-continuous pathes on the configuration manifold $\mathcal{Q}$. An induced beam theory states now the embedding

$$\begin{aligned} \mathbf{x}\colon \mathcal{A} &\to \mathcal{K}\,, \\ \mathbf{q}(\cdot, t) &\mapsto \boldsymbol{\xi} = \mathbf{x}(\mathbf{q}(\cdot, t))\ , \end{aligned} \tag{4.10}$$

which induces a constrained position field for the current configuration of the continuous body. In an induced theory, the body chart is chosen such that the parametrization of the characteristic direction ν equals the third body coordinate $\nu := \theta^3$. Let $\nu \in [\nu_1, \nu_2]$ and $\bar{A}(\nu) := \left\{(\theta^1, \theta^2) \,|\, (\theta^1, \theta^2, \nu) \in \overline{B}\right\}$, then we denote the collection of material points

$\boldsymbol{\xi}(\bar{A}(\nu), \nu, t)$ of the beam-like body as the *cross section* of the beam. The admissible virtual displacements

$$\delta\mathbf{x}(\cdot, t) = D\mathbf{x}(\mathbf{q}(\cdot, t))\delta\mathbf{q} \in T_{\xi}\mathcal{C} \tag{4.11}$$

are obtained by (4.6).

Formulating the embedding (4.10), the beam-like body is considered as a continuous body which is enforced by perfect constraint stresses to follow a constrained position field. Hence, the dynamics for the beam is described by (4.3) with the total stress field (4.7). According to the principle of d'Alembert–Lagrange (4.8), the virtual work contribution of the constraint stress field $\mathbf{t}^i_C$ vanishes for all admissible virtual displacement fields (4.11) and at any instant of time t. Thus, a formulation of the principle of virtual work of the constrained continuous body using the embedding $\mathbf{x}$ together with the generalized position functions $\mathbf{q}$, eliminates the constraint stresses and induces directly the weak variational form of the induced beam theory. The virtual work is obtained by an integration over the three-dimensional body which is performed by an iterated integral over the cross section areas of the body, followed by an integration along ν. Since the generalized virtual displacements only depend on (ν, t), these functions can be dragged outside the surface integral. Subsequently, we define the weighted surface integrals as resultant forces. By performing the surface integrals, the virtual work of the beam reduces to an integral of line densities only. This reduced virtual work expression of the induced beam can then be identified with the virtual work of an intrinsic theory.

The embedding (4.10) of an induced theory generates the connection between a three-dimensional theory and a corresponding generalized intrinsic theory. Within an induced beam theory, we have two possibilities to interpret the resultant forces at $\nu \in [\nu_1, \nu_2]$. Either as weighted integrals of forces and stresses of the Euclidean space which are mapped to the cotangent space $T^*_{\mathbf{q}(\nu,t)}\mathcal{Q}$ or as a generalized force of $T^*_{\mathbf{q}(\nu,t)}\mathcal{Q}$ without a relation to the Euclidean space. The virtual work in the form of integrated line densities corresponds to the weak variational form of the beam theory. If enough continuity assumptions on the line densities are required, then integration by parts is possible which leads to the strong variational form of the beam. By applying the Fundamental Lemma of Calculus of Variations the equations of motion, the boundary and transition conditions of the beam are obtained. The equations of motion are partial differential equations with one spatial variable ν only. Hence, in an intrinsic setting, it is reasonable to consider a beam as a generalized one-dimensional continuum. Nevertheless, an intrinsic formulation only makes sense, when there exists a set of constrained position fields such that the boundary value problem of an intrinsic theory is obtained by an identification with the boundary value problem of an induced theory.

An induced theory shares with a three-dimensional theory that for a complete formulation of the problem, constitutive laws for the resultant contact forces are required. In an induced theory, we have defined the resultant contact forces as weighted surface integrals which are mapped to the cotangent space of the configuration manifold of the beam. Hence, we may introduce a three-dimensional material law for $\mathbf{t}^i_I$ which depends on a three-dimensional strain measure. Such an induced theory is shown in Chapter 7. Using non-admissible virtual displacements and up to a certain indeterminacy in the constraint stress field $\mathbf{t}^i_C$, it is possible to find a correlation between generalized internal forces and the total stress field of the continuous body. In the geometrically nonlinear beam theories of Chapter 5, 6 and 8, we formulate the constitutive law in an intrinsic setting. Denoting partial derivative with respect to ν by a prime $(\cdot)'$, we define a generalized strain depending on $\mathbf{q}'$ only and state a constitutive law for the generalized internal forces directly. This form of a theory is called semi-induced theory, because we introduce the generalized forces from a three-dimensional theory, identify them with an intrinsic formulation and state the constitutive laws of the generalized internal forces in a generalized setting. Within a semi-induced theory we cut the connection of the generalized internal forces to the stress field in the Euclidean space. Hence, also in a semi-induced theory, we cannot draw any conclusions about the stresses in the beam as a continuous body. Assuming also set-valued force laws for the generalized internal forces, it is possible to impose further constraints on the beam and to develop deviated beam theories from an original formulation. This is done to show the hierarchical structure of the classical beam theories.

Chapter 5

Classical Nonlinear Beam Theories

Classical nonlinear beams from the point of view of an induced theory are continuous bodies with a constrained position field which is described by the motion of a centerline and the motion of plane rigid cross sections attached to every point at the centerline. This restricted kinematics allows to determine resultant forces at each cross section and to reduce the equations of motion of a three-dimensional continuous body to a partial differential equation with only one spatial variable. The present chapter is partly based on the publication of Eugster et al. (2014).

First, in Section 5.1, the kinematical assumptions are stated. Subsequently, in Section 5.2, the virtual work contributions of the internal forces, the inertia forces and the external forces are reformulated by the application of the restricted kinematics to the virtual work of the continuous body. Lastly, in Section 5.3 – 5.5 we present the generalized constitutive laws of the geometrically nonlinear and elastic theories of Timoshenko, Euler–Bernoulli and Kirchhoff in the form of a semi-induced beam theory.

5.1 Kinematical Assumptions

For the derivation of the classical beam theory, it is convenient to think of a slender continuous body with an isotropic material behavior as depicted in Figure 5.1. First, we assume at a given instant of time t a placement of the slender body in $\mathbb{E}^3$, at which the body covers the subset $\overline{\Omega}_t \subset \mathbb{E}^3$. We identify the characteristic direction of the slender isotropic body with an arbitrarily chosen centerline $\mathbf{r}$ which propagates along the largest expansion of the body. The property that the configuration $\boldsymbol{\xi}(\cdot, t)$ at time t is an embedding, enables us to identify every point of the continuous body in $\overline{\Omega}_t$ with a unique point in the set $\overline{B} \coloneqq \boldsymbol{\xi}(\cdot, t)^{-1}(\overline{\Omega}_t) \subset \mathbb{R}^3$. Subsequently, we choose the body chart θ such that the centerline $\mathbf{r}$ is parametrized by $\theta^3 =: \nu$ only. For a classical beam we assume the existence of a motion given by the constrained position field of the form

$$\boldsymbol{\xi}(\theta^\alpha, \nu, t) = \mathbf{x}(\mathbf{q}(\cdot, t))(\theta^\alpha, \nu) = \mathbf{r}(\nu, t) + \theta^\alpha \mathbf{d}_\alpha(\nu, t) \; , \tag{5.1}$$

where the generalized position functions $\mathbf{q}(\cdot, t)$ are recognized as $\mathbf{r}(\cdot, t)$, $\mathbf{d}_1(\cdot, t)$ and $\mathbf{d}_2(\cdot, t)$. The centerline is given by the space curve $\mathbf{r}(\cdot, t) = \boldsymbol{\xi}(0, 0, \cdot, t)$ and is bounded by its ends

$\nu = \nu_1$ and $\nu = \nu_2$ for $\nu_2 > \nu_1$. A customary choice of ν is the arc length parametrization s of the centerline $\mathbf{r}$. Since the arc length parametrization comes along with an additional constraint and may change under deformation from one instant of time to another, we do not want to restrict us to this special case. At every material point ν of the centerline $\mathbf{r}$ a

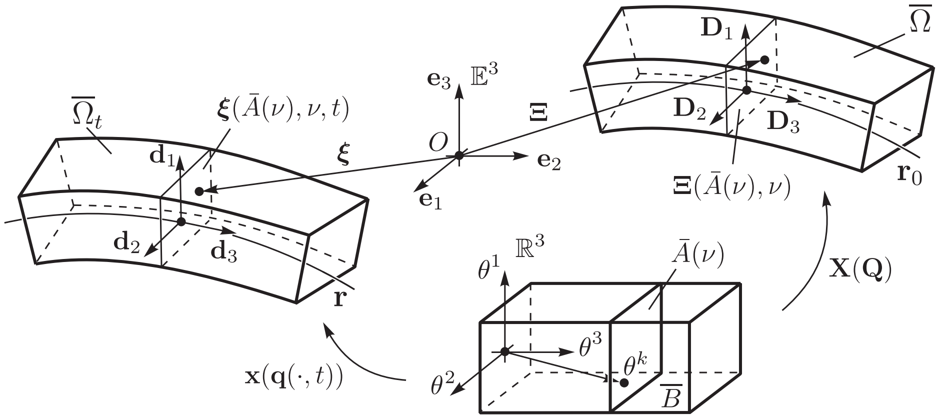

Figure 5.1: Reference and current configuration of the beam.

positively oriented orthonormal director triad $(\mathbf{d}_1(\nu,t), \mathbf{d}_2(\nu,t), \mathbf{d}_3(\nu,t))$ is attached. The two directors $\mathbf{d}_\alpha$ span the plane cross section of the beam. The current state of the cross section $\boldsymbol{\xi}(\bar{A}(\nu), \nu, t)$ is parametrized by the cartesian coordinates $(\theta^1, \theta^2) \in \bar{A}(\nu)$, where $\bar{A}(\nu) := \{(\theta^1, \theta^2) \,|\, (\theta^1, \theta^2, \nu) \in \overline{B}\}$. The restriction to cartesian coordinates is implied by the parametrization of the cross section by two orthonormal directors. For specific problems, e.g. computation of the cross section area, appropriate local reparametrizations can be performed. One could think of different descriptions of the plane which do allow for more general coordinates, but such a generalization is outside the scope of this thesis. The director triad $(\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3)$ can be related to an inertial orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ by introducing for the rotation tensor $\mathbf{R}(\nu, t) \in SO(3)$ such that

$$\mathbf{d}_k(\nu,t) = \mathbf{R}(\nu,t)\mathbf{e}_k \ , \qquad \text{with } \mathbf{R} = \mathbf{d}_k \otimes \mathbf{e}_k \ . \tag{5.2}$$

For orthonormal vector triads, we do not distinguish here between co- and contravariant vectors. In (5.1) we have identified the generalized position functions $\mathbf{q}(\cdot,t)$ with $\mathbf{r}(\cdot,t)$, $\mathbf{d}_1(\cdot,t)$, $\mathbf{d}_2(\cdot,t)$ and have constrained the directors $\mathbf{d}_1(\cdot,t)$ and $\mathbf{d}_2(\cdot,t)$ by (5.2) to remain orthonormal. Hence, the evaluation at ν of the generalized position functions $\mathbf{q}(\cdot,t)$ can be considered as a point on the 6-dimensional manifold $\mathbb{E}^3 \times SO(3)$.

Since a beam in an induced theory is treated as a continuous body with a constrained position field, one has to guarantee that the motion always requires the conditions of an embedding. As long as the density of the volume form $g^{1/2} > 0$ does not vanish for every point θ^k and the function remains injective, the permanence of matter and the principle of impenetrability are fulfilled and the motion is an embedding. As an example of how extreme such deformations can be, we assume a beam with circular cross sections of radius r where the cross sections remain orthogonal to the tangent vector of the centerline. As depicted in Figure 5.2, the beam is bent in-plane up to a bending radius R. As long

Figure 5.2: Maximal allowed deformation of a beam with cross section radius r and limit bending radius R.

as the bending radius is larger than the radius of the beam $R \geq r$, no interpenetration of the cross sections may appear. This restriction seems to be reasonable for the example at hand. Ultimately, at the configuration where the bending radius coincides with the cross section radius $r = R$, the lateral surfaces of the beam come into contact. Because of the impenetrability condition $R \geq r$, beam theories are generally limited to slender bodies (among other reasons).

In the classical beam theory, the cross section deformation is considered to be irrelevant for the deformation of the body. Consequently, the cross section is rigidified by the choice of the constrained position field (5.1). This implies that material points which are on the same cross section stay on the same cross section throughout the whole motion of the body. The choice of the body chart together with the current configuration can be denominated as a fibration of the continuous body. In the remainder of this section the kinematical expressions which are necessary for the formulation of the virtual work (4.3) of the constrained continuous body are derived.

To begin with the effective curvature, the angular velocity and the virtual rotation, which all describe the change of the directors when changing a single parameter, e.g. the parameter ν. Using (5.2), we derive

$$(\mathbf{d}_k)' = (\mathbf{R}(\nu,t)\mathbf{e}_k)' = \mathbf{R}'\mathbf{R}^{\mathrm{T}}\mathbf{d}_k =: \tilde{\mathbf{k}}\mathbf{d}_k \ , \tag{5.3}$$

in which we recognize the *effective curvature* $\tilde{\mathbf{k}} = \mathbf{R}'\mathbf{R}^{\mathrm{T}}$ and denote the partial derivative with respect to ν by a superposed prime $(\cdot)'$. The effective curvature $\tilde{\mathbf{k}}$ only coincides with the curvature of a spatial curve $\mathbf{r}(\nu,t)$ when ν corresponds to the arc length parametrization s of the spatial curve at a given instant of time t. The skew-symmetry of $\tilde{\mathbf{k}}$ can easily be shown using the $SO(3)$ properties of the rotation tensor $\mathbf{R}$:

$$\mathbf{R}\mathbf{R}^{\mathrm{T}} = \mathbb{1} \overset{(\cdot)'}{\Rightarrow} \tilde{\mathbf{k}} \overset{(5.3)}{=} \mathbf{R}'\mathbf{R}^{\mathrm{T}} = -\mathbf{R}(\mathbf{R}^{\mathrm{T}})' = -(\mathbf{R}'\mathbf{R}^{\mathrm{T}})^{\mathrm{T}} = -\tilde{\mathbf{k}}^{\mathrm{T}} \ .$$

Hence, the skew-symmetric effective curvature $\tilde{\mathbf{k}}$ has an associated axial vector $\mathbf{k}(\nu,t) \in \mathbb{E}^3$ such that

$$(\mathbf{d}_k)' = \tilde{\mathbf{k}}\mathbf{d}_k = \mathbf{k} \times \mathbf{d}_k \ , \qquad \text{with } \tilde{\mathbf{k}} = \mathbf{R}'\mathbf{R}^{\mathrm{T}} = (\mathbf{d}_i)' \otimes \mathbf{d}_i \ . \tag{5.4}$$

The tilde-operator will be used to denote the skew-symmetric tensor to an associated axial vector. The components of the effective curvature can be written using the alternating symbols ε_{ijk} as

$$k_i = \frac{1}{2}\varepsilon_{ijk}(\tilde{\mathbf{k}})_{kj} = \frac{1}{2}\varepsilon_{ijk}(\mathbf{d}_k \cdot (\mathbf{d}_j)') \ .$$

Similar to (5.4) we introduce the *angular velocity* $\tilde{\boldsymbol{\omega}}(\nu, t)$ and its associated axial vector $\boldsymbol{\omega}(\nu, t)$ as

$$\dot{\mathbf{d}}_k = \tilde{\boldsymbol{\omega}}\mathbf{d}_k = \boldsymbol{\omega} \times \mathbf{d}_k \ , \qquad \text{with } \tilde{\boldsymbol{\omega}} = \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}} = \dot{\mathbf{d}}_i \otimes \mathbf{d}_i \ . \tag{5.5}$$

Likewise, we obtain the *virtual rotation* $\delta\tilde{\boldsymbol{\phi}}(\nu, t)$ and its associated axial vector $\delta\boldsymbol{\phi}(\nu, t)$ by considering virtual variations of the directors $\mathbf{d}_k$, i.e. through derivation with respect to the variation parameter ε,

$$\delta\mathbf{d}_k = \delta\tilde{\boldsymbol{\phi}}\mathbf{d}_k = \delta\boldsymbol{\phi} \times \mathbf{d}_k \ , \quad \text{with } \delta\tilde{\boldsymbol{\phi}} = \delta\mathbf{R}\mathbf{R}^{\mathrm{T}} = \delta\mathbf{d}_i \otimes \mathbf{d}_i \ . \tag{5.6}$$

The *velocity* and *acceleration fields* are introduced by taking the total time derivative of the position field (5.1) and the kinematical relation introduced in (5.5)

$$\begin{aligned} \dot{\mathbf{x}} &= \dot{\mathbf{r}} + \boldsymbol{\omega} \times (\mathbf{x} - \mathbf{r}) = \dot{\mathbf{r}} + \boldsymbol{\omega} \times \boldsymbol{\rho} \ , \qquad \text{with } \boldsymbol{\rho} = \mathbf{x} - \mathbf{r} = \theta^{\alpha}\mathbf{d}_{\alpha} \ , \\ \ddot{\mathbf{x}} &= \ddot{\mathbf{r}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) \ . \end{aligned} \tag{5.7}$$

Using (5.1) and (5.4), the partial derivatives of the constrained position field take the form

$$\mathbf{x}_{,\alpha} = \mathbf{d}_{\alpha} \ , \qquad \mathbf{x}' = \mathbf{r}' + \mathbf{k} \times \boldsymbol{\rho} \ . \tag{5.8}$$

The variation of the constrained position field and insofar the admissible virtual displacement field is, in accordance with (5.1) and (5.6), given by

$$\delta\mathbf{x} = \delta\mathbf{r} + \delta\boldsymbol{\phi} \times \boldsymbol{\rho} \ . \tag{5.9}$$

The variation of the partial derivatives (5.8) are reformulated to

$$\delta\mathbf{x}_{,\alpha} = \delta\boldsymbol{\phi} \times \mathbf{x}_{,\alpha} \ , \quad \delta\mathbf{x}' = \delta\mathbf{r}' + \delta\mathbf{k} \times \boldsymbol{\rho} + \mathbf{k} \times (\delta\boldsymbol{\phi} \times \boldsymbol{\rho}) \ . \tag{5.10}$$

Since cartesian coordinates are chosen, the derivative with respect to ν and the variation commute, i.e. $(\delta\mathbf{d}_k)' = \delta((\mathbf{d}_k)') = \delta\mathbf{d}_k'$. By (5.4) and (5.6) we write this identity as

$$(\delta\boldsymbol{\phi} \times \mathbf{d}_k)' = \delta(\mathbf{k} \times \mathbf{d}_k) \ .$$

Applying the product rule and using again (5.4) and (5.6) yields

$$\delta\boldsymbol{\phi}' \times \mathbf{d}_k + \delta\boldsymbol{\phi} \times (\mathbf{k} \times \mathbf{d}_k) = \delta\mathbf{k} \times \mathbf{d}_k + \mathbf{k} \times (\delta\boldsymbol{\phi} \times \mathbf{d}_k) \ .$$

By subtracting the left-hand side from the right-hand side, and by applying the skew-symmetric property of the cross product and the Jacobi identity (B.1), one obtains

$$\begin{aligned} 0 &= \delta\mathbf{k} \times \mathbf{d}_k + \mathbf{k} \times (\delta\boldsymbol{\phi} \times \mathbf{d}_k) + \delta\boldsymbol{\phi} \times (\mathbf{d}_k \times \mathbf{k}) - \delta\boldsymbol{\phi}' \times \mathbf{d}_k \\ &\overset{(\mathrm{B.1})}{=} \delta\mathbf{k} \times \mathbf{d}_k - \mathbf{d}_k \times (\mathbf{k} \times \delta\boldsymbol{\phi}) - \delta\boldsymbol{\phi}' \times \mathbf{d}_k \\ &= (\delta\mathbf{k} - \delta\boldsymbol{\phi} \times \mathbf{k} - \delta\boldsymbol{\phi}') \times \mathbf{d}_k \ . \end{aligned}$$

Since the right-hand side of (5.1) has to vanish for all directors $\mathbf{d}_k \in \mathbb{E}^3$ we retrieve the important identity

$$\delta\boldsymbol{\phi}' = \delta\mathbf{k} - \delta\boldsymbol{\phi} \times \mathbf{k} \ . \tag{5.11}$$

For the formulation of constitutive laws or for the determination of mass densities it is convenient to introduce a special configuration, called *reference configuration.* Let $\mathbf{r}_0$ and $\mathbf{D}_\alpha$ be the reference generalized position functions of $\mathbf{Q}$, then the reference configuration of the beam corresponds to the constrained position field

$$\Xi(\theta^\alpha, \nu) = \mathbf{X}(\mathbf{Q})(\theta^\alpha, \nu) = \mathbf{r}_0(\nu) + \theta^\alpha \mathbf{D}_\alpha(\nu) \ . \tag{5.12}$$

We call the space curve $\mathbf{r}_0 = \Xi(0, 0, \cdot)$ the *reference curve* of the beam. At each material point of the reference curve $\mathbf{r}_0$ we have attached a positively oriented orthonormal director triad $(\mathbf{D}_1(\nu), \mathbf{D}_2(\nu), \mathbf{D}_3(\nu))$ which is related to the basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ by introducing the rotation tensor $\mathbf{R}_0(\nu) \in SO(3)$ such that

$$\mathbf{D}_k(\nu) = \mathbf{R}_0(\nu)\mathbf{e}_k \ , \qquad \text{with } \mathbf{R}_0 = \mathbf{D}_k \otimes \mathbf{e}_k \ .$$

The directors $\mathbf{D}_\alpha$ describe the reference state of the cross section $\Xi(\bar{A}(\nu), \nu)$. In the formulation of constitutive laws, the reference configuration is often defined as the stress free configuration of the body.

5.2 Virtual Work Contributions

In an induced theory, the classical nonlinear beam is a continuous body with the constrained position field (5.1). The dynamics of a continuous body with such a restricted kinematics can be described by the principle of virtual work (4.3) with the total stress field (4.7). The constraint position field (5.1) which defines the constraint manifold $\mathcal{C} \subset \mathcal{K}$ corresponds to the embedding (4.10) determining an induced theory. The admissible virtual displacements (5.9) are directly obtained by the variation of the constrained position field. Using the constrained kinematics (5.1), in the following section, the contributions of the virtual work (4.3) due to the admissible virtual displacements (5.9) are determined. Since the constraint stresses are assumed to be perfect, by the principle of d'Alembert–Lagrange (4.8), they do not contribute to the virtual work and the weak variational formulation of the classical nonlinear beam is obtained. By further continuity assumptions on the involved functions, the strong variational formulation and the corresponding boundary value problem of the classical nonlinear beam is determined.

It is important to notice, that within this formulation we lose all information about the constraint stresses which rigidify the cross sections. The fact that the constraint stresses do not appear in the equations of motion does not imply that no stresses act in the cross section.

Virtual work contributions of internal forces

Using (4.1), (5.10) and the property of the cross product of (B.2), the internal virtual work density can be written as

$$\mathbf{t}^i \cdot \delta\mathbf{x}_{,i} = \delta\boldsymbol{\phi} \cdot (\mathbf{x}_{,\alpha} \times \mathbf{t}^\alpha) + \mathbf{t}^3 \cdot \delta\mathbf{r}' + \delta\mathbf{k} \cdot (\boldsymbol{\rho} \times \mathbf{t}^3) + \mathbf{t}^3 \cdot (\mathbf{k} \times (\delta\boldsymbol{\phi} \times \boldsymbol{\rho})) \ . \tag{5.13}$$

Employing the symmetry condition (4.5), we can rewrite the first term in (5.13) as follows:

$$\delta\boldsymbol{\phi} \cdot (\mathbf{x}_{,\alpha} \times \mathbf{t}^{\alpha}) \overset{(4.5)}{=} -\delta\boldsymbol{\phi} \cdot (\mathbf{x}' \times \mathbf{t}^3) \overset{(5.8,B.2)}{=} -\mathbf{t}^3 \cdot (\delta\boldsymbol{\phi} \times \mathbf{r}' + \delta\boldsymbol{\phi} \times (\mathbf{k} \times \boldsymbol{\rho})) \ .$$

Using the above derived relation and the Jacobi identity (B.1), we can manipulate (5.13) further and obtain

$$\begin{aligned}
&\mathbf{t}^i \cdot \delta\mathbf{x}_{,i} = \\
&= -\mathbf{t}^3 \cdot (\delta\boldsymbol{\phi} \times \mathbf{r}' + \delta\boldsymbol{\phi} \times (\mathbf{k} \times \boldsymbol{\rho})) + \mathbf{t}^3 \cdot \delta\mathbf{r}' + \delta\mathbf{k} \cdot (\boldsymbol{\rho} \times \mathbf{t}^3) + \mathbf{t}^3 \cdot (\mathbf{k} \times (\delta\boldsymbol{\phi} \times \boldsymbol{\rho})) \\
&= \mathbf{t}^3 \cdot (\delta\mathbf{r}' - \delta\boldsymbol{\phi} \times \mathbf{r}') + \delta\mathbf{k} \cdot (\boldsymbol{\rho} \times \mathbf{t}^3) + \mathbf{t}^3 \cdot (\mathbf{k} \times (\delta\boldsymbol{\phi} \times \boldsymbol{\rho}) + \delta\boldsymbol{\phi} \times (\boldsymbol{\rho} \times \mathbf{k})) \\
&\overset{(B.1)}{=} \mathbf{t}^3 \cdot (\delta\mathbf{r}' - \delta\boldsymbol{\phi} \times \mathbf{r}') + \delta\mathbf{k} \cdot (\boldsymbol{\rho} \times \mathbf{t}^3) + \mathbf{t}^3 \cdot (\boldsymbol{\rho} \times (\delta\boldsymbol{\phi} \times \mathbf{k})) \\
&\overset{(B.2)}{=} \mathbf{t}^3 \cdot (\delta\mathbf{r}' - \delta\boldsymbol{\phi} \times \mathbf{r}') + (\boldsymbol{\rho} \times \mathbf{t}^3) \cdot (\delta\mathbf{k} - \delta\boldsymbol{\phi} \times \mathbf{k}) \ .
\end{aligned} \tag{5.14}$$

Since the kinematical quantities $\delta\mathbf{r}' - \delta\boldsymbol{\phi} \times \mathbf{r}'$ and $\delta\mathbf{k} - \delta\boldsymbol{\phi} \times \mathbf{k}$ depend merely on (ν, t), we split the integration over $\overline{\mathrm{B}}$ in an integration over the cross section in the body chart $\bar{\mathrm{A}}(\nu)$ and an integration along $\nu \in (\nu_1, \nu_2)$

$$\delta W^{\mathrm{int}} = \int_{\overline{\mathrm{B}}} \mathbf{t}^i \cdot \delta\mathbf{x}_{,i} \, \mathrm{d}^3\theta \overset{(5.14)}{=} \int_{\nu_1}^{\nu_2} \{\mathbf{n} \cdot (\delta\mathbf{r}' - \delta\boldsymbol{\phi} \times \mathbf{r}') + \mathbf{m} \cdot (\delta\mathbf{k} - \delta\boldsymbol{\phi} \times \mathbf{k})\} \, \mathrm{d}\nu \ . \tag{5.15}$$

Herein, the integrated kinetic quantities $\mathbf{n}$ and $\mathbf{m}$ are the *resultant contact forces* and the *resultant contact couples* of the current configuration defined by

$$\mathbf{n}(\nu, t) := \int_{\bar{\mathrm{A}}(\nu)} \mathbf{t}^3 \, \mathrm{d}^2\theta \ , \quad \mathbf{m}(\nu, t) := \int_{\bar{\mathrm{A}}(\nu)} (\boldsymbol{\rho} \times \mathbf{t}^3) \, \mathrm{d}^2\theta \ , \tag{5.16}$$

with abbreviation of the area element $\mathrm{d}^2\theta = \mathrm{d}\theta^1 \mathrm{d}\theta^2$. Due to the surface integral, the resultant contact forces and couples are independent of the cross section coordinates θ^{α}. Although not explicitly expressed in the notation, the stress distributions under the surface integral are mapped from the Euclidean cotangent space to the cotangent space of the beams configuration manifold. Nevertheless, in an induced theory, we still have the connection to the stress distribution of the Euclidean space. In order to make the connection to an intrinsic theory, it is necessary to introduce an equivalence class of forces. Force distributions in the Euclidean space which have the same resultant contact forces and contact couples are considered to be equivalent. The representatives of the equivalence class are then identified with the *internal generalized forces* of an intrinsic beam theory which postulates the right-hand side of (5.15) as its internal virtual work of the generalized one-dimensional continuum. By the definition of an equivalence class, we decouple our induced theory from the theory of a constrained three-dimensional continuous body and arrive at an intrinsic theory.

Virtual work contributions of inertia forces

For convenience, the *mass density* is introduced in the bodies reference configuration as a real valued field $\rho_0 \colon \mathbf{X}(\mathbf{Q})(\overline{B}) \subset \mathbb{E}^3 \to \mathbb{R}$ which to every point of the body in the Euclidean space assigns a local mass per volume. Together with a volume element $\mathrm{d}V = \mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3$ we obtain the mass distribution $\mathrm{d}m = \rho_0\,\mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3$. The pullback of the mass distribution to the domain $\overline{B}$ with respect to the reference configuration leads to the local description of the mass distribution as

$$\mathrm{d}m = \rho_0\, G^{1/2}\,\mathrm{d}^3\theta\;, \quad G^{1/2} = \mathbf{X}_{,1}\cdot(\mathbf{X}_{,2}\times\mathbf{X}_{,3})\;.$$

Considering the virtual work (4.3) and the virtual displacements (5.9) we can transform the virtual work contributions of the inertia terms. For the manipulation of the inertia terms we introduce some abbreviations of integral expressions which have their analogous expressions in rigid body dynamics. The *cross section mass density* per unit of ν is defined as

$$A_{\rho_0}(\nu) := \int_{\bar{\mathrm{A}}(\nu)} \rho_0\, G^{1/2}\,\mathrm{d}^2\theta\;. \tag{5.17}$$

When the centerline does not coincide with the *line of centroids* $\mathbf{r}_c(\nu,t)$, e.g. when the centerline is determined by the shear centers and the shear centers do not coincide with the centroids of the cross sections, a *coupling term* remains, which we introduce as the integrated quantity

$$\mathbf{c}(\nu,t) := A_{\rho_0}(\mathbf{r}_c - \mathbf{r}) = \int_{\bar{\mathrm{A}}(\nu)} \boldsymbol{\rho}\,\rho_0\, G^{1/2}\,\mathrm{d}^2\theta\;. \tag{5.18}$$

The *cross section inertia density* is introduced as

$$\mathbf{I}_{\rho_0}(\nu,t) := \int_{\bar{\mathrm{A}}(\nu)} \tilde{\boldsymbol{\rho}}\tilde{\boldsymbol{\rho}}^{\mathrm{T}}\rho_0\, G^{1/2}\,\mathrm{d}^2\theta\;. \tag{5.19}$$

Furthermore, it is convenient to express the time derivatives of the coupling term by the angular velocity. Using (5.5) and (5.18), the second time derivative of the coupling term is expressed by

$$\ddot{\mathbf{c}} = (\boldsymbol{\omega}\times A_{\rho_0}(\mathbf{r}_c-\mathbf{r}))^{\dot{}} = \dot{\boldsymbol{\omega}}\times A_{\rho_0}(\mathbf{r}_c-\mathbf{r}) + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times A_{\rho_0}(\mathbf{r}_c-\mathbf{r}))\;. \tag{5.20}$$

Another quantity which is going to occur, is the product of the cross section inertia density and the angular velocity

$$\mathbf{h}(\nu,t) := \mathbf{I}_{\rho_0}(\nu,t)\boldsymbol{\omega}(\nu,t)\;.$$

In the basis $\mathbf{d}_i\otimes\mathbf{d}_j$ the moment of inertia $\mathbf{I}_{\rho_0}$ is constant with respect to time t. Using a coordinate description it can easily be shown that

$$\begin{aligned}\dot{\mathbf{h}} &= ((\mathbf{I}_{\rho_0})_{ij}\omega_j\mathbf{d}_i)^{\dot{}} = (\mathbf{I}_{\rho_0})_{ij}\dot{\omega}_j\mathbf{d}_i + (\mathbf{I}_{\rho_0})_{ij}\omega_j\dot{\mathbf{d}}_i\\ &= (\mathbf{I}_{\rho_0})_{ij}\mathbf{d}_i\otimes\mathbf{d}_j(\dot{\omega}_k\mathbf{d}_k + \boldsymbol{\omega}\times\omega_k\mathbf{d}_k) + \boldsymbol{\omega}\times(\mathbf{I}_{\rho_0})_{ij}\omega_j\mathbf{d}_i\\ &= \mathbf{I}_{\rho_0}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}\times\mathbf{I}_{\rho_0}\boldsymbol{\omega}\;.\end{aligned} \tag{5.21}$$

Substitution of the admissible virtual displacements (5.9) and the accelerations (5.7) of the restricted kinematics into the virtual work expression (4.2) yields:

$$\delta W^{\mathrm{dyn}} = \int_{\overline{\mathrm{B}}} \delta\mathbf{x}\cdot\ddot{\mathbf{x}}\,\mathrm{d}m = \int_{\overline{\mathrm{B}}} \{(\delta\mathbf{r} - \tilde{\boldsymbol{\rho}}\delta\boldsymbol{\phi})\cdot(\ddot{\mathbf{r}} - \tilde{\boldsymbol{\rho}}\,\dot{\boldsymbol{\omega}} + \tilde{\boldsymbol{\omega}}\tilde{\boldsymbol{\omega}}\boldsymbol{\rho})\}\,\rho_0\, G^{1/2}\,\mathrm{d}^3\theta\;.$$

Similar to the internal virtual work contribution, the integration over $\overline{\mathrm{B}}$ is split in an integration over the cross section in the body chart $\bar{\mathrm{A}}(\nu)$ and an integration along $\nu \in (\nu_1, \nu_2)$. Together with the definitions (5.17), (5.18) and (5.19) and the property (B.5) of the cross product we obtain

$$\delta W^{\mathrm{dyn}} = \int_{\nu_1}^{\nu_2} \big\{ \delta\mathbf{r} \cdot \big(A_{\rho_0} \ddot{\mathbf{r}} + A_{\rho_0} (\tilde{\mathbf{r}}_c - \tilde{\mathbf{r}})^{\mathrm{T}} \dot{\boldsymbol{\omega}} + \tilde{\boldsymbol{\omega}}\tilde{\boldsymbol{\omega}} A_{\rho_0} (\mathbf{r}_c - \mathbf{r}) \big) + \delta\boldsymbol{\phi} \cdot \big(A_{\rho_0} (\tilde{\mathbf{r}}_c - \tilde{\mathbf{r}})\, \ddot{\mathbf{r}} + \mathbf{I}_{\rho_0}\, \dot{\boldsymbol{\omega}} + \tilde{\boldsymbol{\omega}} \mathbf{I}_{\rho_0} \boldsymbol{\omega} \big) \big\} \, \mathrm{d}\nu \; .$$

Using (5.20) and (5.21) the virtual work contribution of the inertia terms is rewritten in an even more compact form

$$\delta W^{\mathrm{dyn}} = \int_{\nu_1}^{\nu_2} \Big\{ \delta\mathbf{r} \cdot (A_{\rho_0} \ddot{\mathbf{r}} + \ddot{\mathbf{c}}) + \delta\boldsymbol{\phi} \cdot \Big(\mathbf{q} \times \ddot{\mathbf{r}} + \dot{\mathbf{h}} \Big) \Big\} \, \mathrm{d}\nu \; . \tag{5.22}$$

As for the internal virtual work expression, we have two possible points of view. Either we consider the cross section mass density, the coupling term and the cross section inertia as integrated quantities from a mass distribution of a three-dimensional continuous body or we identify them as constitutive parameters of an intrinsic theory which relate the generalized inertia forces from (5.22) with the time derivatives of the generalized position functions.

Virtual work contributions of external forces

There is a vast amount of possibilities how external forces can be impressed on the beam. Forces may occur as volume or surface forces and even point forces applied somewhere at the beam are common in engineering problems. An elegant way to be short in notation is, if we allow the force contribution $\mathrm{d}\mathbf{f}$ to contain Dirac-type contributions. Since the forces may also contribute on the boundaries, it is essential that we integrate over the closed set of the body. Using the same split of the integration as above and the admissible virtual displacements (5.9), we obtain

$$\delta W^{\mathrm{ext}} = \int_{\overline{\mathrm{B}}} \delta\mathbf{x} \cdot \mathrm{d}\mathbf{f} \overset{(5.9)}{=} \int_{[\nu_1, \nu_2]} \{ \delta\mathbf{r} \cdot \mathrm{d}\overline{\mathbf{n}} + \delta\boldsymbol{\phi} \cdot \mathrm{d}\overline{\mathbf{m}} \} \; ,$$

where the *resultant external force distribution* $\mathrm{d}\overline{\mathbf{n}}$ and the *resultant external couple distribution* $\mathrm{d}\overline{\mathbf{m}}$ are the integrated quantities

$$\mathrm{d}\overline{\mathbf{n}}(\nu, t) := \int_{\bar{\mathrm{A}}(\nu)} \mathrm{d}\mathbf{f} \; , \quad \mathrm{d}\overline{\mathbf{m}}(\nu, t) := \int_{\bar{\mathrm{A}}(\nu)} \boldsymbol{\rho} \times \mathrm{d}\mathbf{f} \; .$$

With the same equivalence class argument as for the resultant contact forces and couples, we can identify the resultant external force and couple distributions with *external generalized force distributions* of an intrinsic theory. In order to avoid cumbersome derivations, we only allow the discontinuities in the force distributions at the boundaries ν_1 and ν_2. This leads to the virtual work contribution

$$\delta W^{\text{ext}} = \int_{\nu_1}^{\nu_2} \{\delta\mathbf{r}\cdot\overline{\mathbf{n}} + \delta\boldsymbol{\phi}\cdot\overline{\mathbf{m}}\}\,\mathrm{d}\nu + \sum_{i=1}^{2}\{\delta\mathbf{r}\cdot\overline{\mathbf{n}}_i + \delta\boldsymbol{\phi}\cdot\overline{\mathbf{m}}_i\}\,|_{\nu=\nu_i}\ . \tag{5.23}$$

The resultant external forces and couples $\overline{\mathbf{n}}_i$ and $\overline{\mathbf{m}}_i$, respectively, are the resultant external forces which are impressed at ν_1 and ν_2. Whereas the unit of $\overline{\mathbf{n}}$ is [N] per unit of ν, the unit of $\overline{\mathbf{n}}_i$ is [N]. For the couples we argue in a similar way.

The boundary value problem

Taking all the transformed contributions of the virtual work for admissible virtual displacements (5.15), (5.22) and (5.23), the principle of virtual work (4.3) with the total stress (4.7), together with the principle of d'Alembert–Lagrange (4.8) leads directly to the *weak variational formulation* of the classical beam

$$\begin{aligned}\delta W = \int_{\nu_1}^{\nu_2} \Big\{&\mathbf{n}\cdot(\delta\mathbf{r}' - \delta\boldsymbol{\phi}\times\mathbf{r}') + \mathbf{m}\cdot(\delta\mathbf{k} - \delta\boldsymbol{\phi}\times\mathbf{k}) + \delta\mathbf{r}\cdot(A_{\rho_0}\ddot{\mathbf{r}} + \ddot{\mathbf{c}} - \overline{\mathbf{n}}) \\ &+\delta\boldsymbol{\phi}\cdot\Big(\mathbf{c}\times\ddot{\mathbf{r}} + \dot{\mathbf{h}} - \overline{\mathbf{m}}\Big)\Big\}\,\mathrm{d}\nu - \sum_{i=1}^{2}\{\delta\mathbf{r}\cdot\overline{\mathbf{n}}_i + \delta\boldsymbol{\phi}\cdot\overline{\mathbf{m}}_i\}\,|_{\nu=\nu_i} = 0 \quad \forall\delta\mathbf{r},\delta\boldsymbol{\phi},t\ .\end{aligned} \tag{5.24}$$

Using the identity (5.11) and integration by parts, the virtual work is expressed as

$$\begin{aligned}\delta W = &-\{\delta\mathbf{r}\cdot(\mathbf{n}+\overline{\mathbf{n}}_1) + \delta\boldsymbol{\phi}\cdot(\mathbf{m}+\overline{\mathbf{m}}_1)\}\,|_{\nu=\nu_1} \\ &+\int_{\nu_1}^{\nu_2}\Big\{\delta\mathbf{r}\cdot(A_{\rho_0}\ddot{\mathbf{r}} + \ddot{\mathbf{c}} - \overline{\mathbf{n}} - \mathbf{n}') + \delta\boldsymbol{\phi}\cdot\Big(\mathbf{c}\times\ddot{\mathbf{r}} + \dot{\mathbf{h}} - \overline{\mathbf{m}} - \mathbf{m}' - \mathbf{r}'\times\,\mathbf{n}\Big)\Big\}\,\mathrm{d}\nu \\ &+\{\delta\mathbf{r}\cdot(\mathbf{n}-\overline{\mathbf{n}}_2) + \delta\boldsymbol{\phi}\cdot(\mathbf{m}-\overline{\mathbf{m}}_2)\}\,|_{\nu=\nu_2} = 0 \quad \forall\delta\mathbf{r},\delta\boldsymbol{\phi},t\ ,\end{aligned}$$

which corresponds to the *strong variational formulation* of the classical beam. When the functions in the round brackets are continuous and when the virtual displacements $\delta\mathbf{r}$ and the virtual rotations $\delta\boldsymbol{\phi}$ are smooth enough, then by the Fundamental Lemma of Calculus of Variation, the former terms have to vanish pointwise. This leads to the complete boundary value problem with the equations of motion of the classical beam which are valid for $\nu \in (\nu_1, \nu_2)$

$$\begin{aligned}\mathbf{n}' + \overline{\mathbf{n}} &= A_{\rho_0}\ddot{\mathbf{r}} + \ddot{\mathbf{c}}\ , \\ \mathbf{m}' + \mathbf{r}'\times\,\mathbf{n} + \overline{\mathbf{m}} &= \mathbf{c}\times\ddot{\mathbf{r}} + \dot{\mathbf{h}}\ ,\end{aligned} \tag{5.25}$$

together with the boundary conditions $\mathbf{n}(\nu_1) = -\overline{\mathbf{n}}_1$, $\mathbf{m}(\nu_1) = -\overline{\mathbf{m}}_1$ and $\mathbf{n}(\nu_2) = \overline{\mathbf{n}}_2$, $\mathbf{m}(\nu_2) = \overline{\mathbf{m}}_2$. If we allow discontinuities of the force distributions at countable many points inside the beam, the domain (ν_1, ν_2) has to be divided into sets where the force distributions are continuous. The integration by parts can then only be performed on the differentiable parts. Consequently, this leads to an equation of motion (5.25) for the differentiable parts, to boundary conditions at the boundaries and to transition conditions at the points of the discontinuities.

To summarize, we have seen that the restricted kinematics of the beam allows us reducing the virtual work of the continuous body in such a way, that the equations of motion (5.25) correspond to partial differential equations with only one spatial variable. As mentioned several times, we have two different viewpoints. In an induced theory, the force contributions in (5.25) are interpreted as resultant forces, i.e. weighted surface integrals of forces and stresses of the Euclidean space mapped to the cotangent space of the beams configuration manifold. In an intrinsic theory the forces are considered as generalized forces which lose their connection to force and stress distributions of the Euclidean space.

5.3 Nonlinear Timoshenko Beam Theory

Constitutive laws for the resultant contact forces $\mathbf{n}$ and the resultant contact couples $\mathbf{m}$ are required to complete the equations of motion (5.25). In an induced theory, it is customary to choose a three-dimensional material law with an appropriate three-dimensional strain measure and integrate the corresponding stress contributions (5.16) over the cross sections. Here, however, we propose a semi-induced approach for the formulation of constitutive laws in three-dimensional beam theories. Henceforth, we interpret the resultant contact forces and couples as generalized internal forces and formulate a constitutive law between generalized strains and generalized internal forces. The generalized strains are directly determined by the generalized position functions $\mathbf{q}$. When proposing an elastic constitutive behavior, we have to show, that the variation with respect to the generalized strain measures leads to the same form of the internal virtual work (5.15) of the induced theory. This shows the compatibility between an induced and an intrinsic beam formulation. In classical beam theories, the generalized constitutive laws relate the generalized position functions of the beam, i.e. the motion of the centerline and the rotation of the cross sections, with the internal generalized forces $\mathbf{n}$ and $\mathbf{m}$. As in the three-dimensional theory, we allow the generalized internal forces to consist of an impressed and of a constraint part

$$\mathbf{n} = \mathbf{n}_I + \mathbf{n}_C \ , \qquad \mathbf{m} = \mathbf{m}_I + \mathbf{m}_C \ . \tag{5.26}$$

The subscripts $(\cdot)_I$ and $(\cdot)_C$ stand for impressed forces and constraint forces, respectively. Whereas the constitutive laws of impressed internal generalized forces are formulated by single valued force laws, the constitutive law of the constraint internal generalized forces are given by the principle of d'Alembert–Lagrange (4.8) which can be considered to be a set-valued force law.

Even though in Timoshenko (1921) and Timoshenko (1922) only the linear and plane case is treated, we call the beam theory of this section, in which no further constraints are impressed on the beam, the *nonlinear Timoshenko beam theory.* Accordingly, the constraint parts of internal generalized forces vanish, i.e.

$$\mathbf{n}_C = 0 \ , \quad \mathbf{m}_C = 0 \ . \tag{5.27}$$

There exists a multitude of other names for the same beam theory. Ballard and Millard (2009) call the beam “poutre naturelle”, Antman (2005) denotes it as “special Cosserat rod” and as “geometrically exact beam”. With reference to Reissner (1981) and Simo (1985), it is also called “Simo-Reissner beam”. In our genealogy of beam theories, we denote a beam with the same constraints by the same name. We distinguish further between a nonlinear theory, a linearized theory and a plane linearized theory.

The most basic constitutive law for a nonlinear Timoshenko beam is an elastic force law being expressed by an elastic potential $\hat{W}(\nu, t)$ for the impressed part of the generalized internal forces, such that

$$\delta W_I^{\mathrm{int}} = \delta \int_{\nu_1}^{\nu_2} \hat{W}(\nu, t)\,\mathrm{d}\nu \ .$$

We assume the elastic potential to depend on the generalized strain measures γ_i and κ_i

$$\tilde{W}(\nu, t) = W(\gamma_i(\nu, t), \kappa_i(\nu, t)) \ . \tag{5.28}$$

The generalized strain

$$\gamma_i(\nu, t) := \mathbf{d}_i \cdot \mathbf{r}' - \mathbf{D}_i \cdot \mathbf{r}_0' \ , \tag{5.29}$$

measures the difference between the deformation of the centerline in the direction $\mathbf{d}_i$ and the deformation of the reference curve in the direction $\mathbf{D}_i$. The effective reference curvature is defined as $\tilde{\mathbf{k}}_0(\nu) = \mathbf{R}_0'\mathbf{R}_0^{\mathrm{T}} = (\mathbf{D}_i)' \otimes \mathbf{D}_i$. When measuring the difference between the effective curvature and the effective reference curvature in the direction $\mathbf{d}_k, \mathbf{d}_j$ and $\mathbf{D}_k, \mathbf{D}_j$, respectively, we obtain the components $\tilde{k}_{kj} - (\tilde{k}_0)_{kj}$. Since these components are skew-symmetric, there is an associated axial vector with the components

$$\kappa_i(\nu, t) := \frac{1}{2}\varepsilon_{ijk}(\mathbf{d}_k \cdot \tilde{\mathbf{k}}\mathbf{d}_j - \mathbf{D}_k \cdot \tilde{\mathbf{k}}_0\mathbf{D}_j) = \frac{1}{2}\varepsilon_{ijk}(\mathbf{d}_k \cdot (\mathbf{d}_j)' - \mathbf{D}_k \cdot (\mathbf{D}_j)') \ . \tag{5.30}$$

In the following we demonstrate the compatibility of the intrinsic generalized strain measures with the induced theory, thereby showing that the internal virtual work expression (5.15) is obtained when varying the elastic potential (5.28), i.e. that

$$\delta W_I^{\mathrm{int}} = \int_{\nu_1}^{\nu_2} \left\{ \frac{\partial W}{\partial \gamma_i}\delta\gamma_i + \frac{\partial W}{\partial \kappa_i}\delta\kappa_i \right\} \mathrm{d}\nu$$

holds. Using (5.6) and (B.2), the variation of W with respect to γ_i takes the form

$$\frac{\partial W}{\partial \gamma_i}\delta\gamma_i \overset{(5.29)}{=} \frac{\partial W}{\partial \gamma_i}(\delta\mathbf{r}' \cdot \mathbf{d}_i + \mathbf{r}' \cdot \delta\mathbf{d}_i) = \mathbf{n}_I \cdot (\delta\mathbf{r}' - \delta\boldsymbol{\phi} \times \mathbf{r}') \ , \tag{5.31}$$

where we have recognized the resultant contact force $\mathbf{n}_I := n_{Ii}\mathbf{d}_i = \frac{\partial W}{\partial \gamma_i}\mathbf{d}_i$. By expansion with the orthonormality condition $\delta_{ij} = \mathbf{d}_i \cdot \mathbf{d}_j$ and using (5.6), the variation with respect to κ_i yields

$$\frac{\partial W}{\partial \kappa_i}\delta\kappa_i = \frac{\partial W}{\partial \kappa_i}\mathbf{d}_i \cdot \delta\kappa_j\mathbf{d}_j = \mathbf{m}_I \cdot (\delta\mathbf{k} - \delta\boldsymbol{\phi} \times \mathbf{k}) \ , \tag{5.32}$$

in which the resultant contact couple as $\mathbf{m}_I := m_{Ii}\mathbf{d}_i = \frac{\partial W}{\partial \kappa_i}\mathbf{d}_i$ has been identified. Comparison of (5.31) and (5.32) with (5.15) demonstrates the compatibility of the chosen generalized strain measures and their corresponding elastic potential.

Let E and G be the Young's and shear modulus, respectively, and let A_α be the the area of the cross sections A multiplied by a shear correction factor. Let I_1, I_2 and J be the second moments of area and polar moment, respectively. In the following we assume that the elastic potential takes the quadratic form

$$W(\gamma_i, \kappa_i) = \frac{1}{2}\,\gamma_i\,(\hat{\mathbf{D}}_1)_{ij}\,\gamma_j + \frac{1}{2}\,\kappa_i\,(\hat{\mathbf{D}}_2)_{ij}\,\kappa_j\ , \tag{5.33}$$

with

$$[\hat{\mathbf{D}}_1] = \mathrm{Diag}[GA_1, GA_2, EA]\ , \quad [\hat{\mathbf{D}}_2] = \mathrm{Diag}[EI_1, EI_2, GJ]\ ,$$

where $[\hat{\mathbf{D}}_1]$ and $[\hat{\mathbf{D}}_2]$ contain the collection of the stiffness components $(\hat{\mathbf{D}}_1)_{ij}$ and $(\hat{\mathbf{D}}_2)_{ij}$, respectively. In the elastic potential (5.33) the directors $\mathbf{d}_\alpha$ have been chosen such that they correspond to the principle axes of the cross section surfaces. Consequently, the constitutive laws for the generalized internal forces are given as

$$\mathbf{n} = \mathbf{n}_I = n_{Ii}\mathbf{d}_i = (\hat{\mathbf{D}}_1)_{ij}\gamma_j\mathbf{d}_i\ , \quad \mathbf{m} = \mathbf{m}_I = m_{Ii}\mathbf{d}_i = (\hat{\mathbf{D}}_2)_{ij}\kappa_j\mathbf{d}_i\ .$$

which coincide with the impressed part, since the constraint parts (5.27) vanish.

5.4 Nonlinear Euler–Bernoulli Beam Theory

The *nonlinear Euler–Bernoulli beam* (or *Navier–Bernoulli beam*) can be regarded as a Timoshenko beam on which additional constraints have been imposed. The cross sections, and insofar the directors $\mathbf{d}_\alpha$, have to remain orthogonal to the tangent vectors $\mathbf{r}'$ of the centerline. These constraints are formulated for every instant of time t by the two constraint functions

$$g_\alpha(\nu, t) = \mathbf{d}_\alpha \cdot \mathbf{r}' = 0\ .$$

It is convenient to let the reference configuration also to satisfy the orthonormality condition. In this case, the constraints coincide with vanishing shear deformation, i.e.

$$g_\alpha = \gamma_\alpha = \mathbf{d}_\alpha \cdot \mathbf{r}' - \mathbf{D}_\alpha \cdot \mathbf{r}_0' = 0\ . \tag{5.34}$$

The bilateral constraints are guaranteed by the constraint forces $n_{C\alpha}$. Using (5.6) and properties of the cross product, the generalized constraint forces $\mathbf{n}_C = n_{C\alpha}\mathbf{d}_\alpha$ contribute to the virtual work of the beam as

$$\delta W_C^{\mathrm{int}} = \delta g_\alpha n_{C\alpha} = (\mathbf{d}_\alpha \cdot \delta\mathbf{r}' + \delta\mathbf{d}_\alpha \cdot \mathbf{r}')n_{C\alpha} = \mathbf{n}_C \cdot (\delta\mathbf{r}' - \delta\boldsymbol{\phi} \times \mathbf{r}')\ . \tag{5.35}$$

The generalized constraint forces contribute in the same way as the generalized internal forces in (5.15). This is in accordance with the decomposition of the internal generalized forces (5.26) into an impressed and a constraint part. The force law of the generalized constraint forces, which are considered to be perfect, can only be formulated variationally by the principle of d'Alembert–Lagrange, which states that (5.35) vanishes for all virtual displacements which are admissible with respect to (5.34). Such a variational force law is described by a set-valued force law as depicted in Figure 5.3. The force law at hand

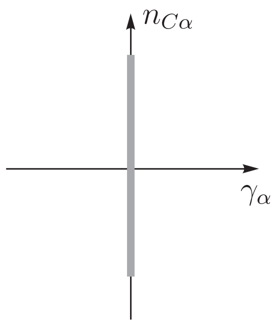

Figure 5.3: Bilateral constraint as set-valued force law.

may be cast in a normal cone inclusion $n_{C\alpha} \in \mathcal{N}_{\{0\}}(\gamma_\alpha) = \mathbb{R}$, where the *normal cone*, cf. Moreau (1966) or Rockafellar (1970), to the convex set $\{0\}$ is defined as

$$\mathcal{N}_{\{0\}}(x) = \{y \in \mathbb{R} \,|\, y(x^* - x) \leq 0\ , x \in 0,\ \forall x^* \in 0\} = \mathbb{R}\ .$$

By setting $(x^* - x) = \delta g_\alpha$ and $y = n_{C\alpha}$ in the normal cone inclusion, we readily recognize the principle of d'Alembert–Lagrange in inequality form.

For the impressed part, we assume the same quadratic form (5.33) as its elastic potential. Since the constraint forces do not allow any shear deformation γ_α, the corresponding shear stiffness components are immaterial and

$$[\hat{\mathbf{D}}_1] = \mathrm{Diag}[*, *, EA]\ , \quad [\hat{\mathbf{D}}_2] = \mathrm{Diag}[EI_1, EI_2, GJ]\ .$$

The generalized shear forces $n_{I\alpha}$ of the underlying Timoshenko beam theory have become bilateral generalized constraint forces $n_{C\alpha}$ in the Euler–Bernoulli beam theory. Hence, an elastic material law of the Euler–Bernoulli beam is given by

$$\mathbf{n} = \mathbf{n}_I + \mathbf{n}_C\ , \quad \mathbf{m} = \mathbf{m}_I\ ,$$

where the impressed parts are represented by

$$\mathbf{n}_I = n_{Ii}\mathbf{d}_i = (\hat{\mathbf{D}}_1)_{ij}\gamma_j\mathbf{d}_i\ , \quad \mathbf{m}_I = m_{Ii}\mathbf{d}_i = (\hat{\mathbf{D}}_2)_{ij}\kappa_j\mathbf{d}_i$$

and the generalized constraint forces are formulated by the normal cone inclusions

$$\mathbf{n}_C = n_{C\alpha}\mathbf{d}_\alpha\ , \qquad \text{with } n_{C\alpha} \in \mathcal{N}_{\{0\}}(\gamma_\alpha) = \mathbb{R}\ .$$

Using further concepts of convex analysis, e.g. the indicator function and the concept of the subdifferential, it is possible to also include the set-valued part in the potential (5.33), cf. Glocker (2001). This allows an alternative interpretation, that the bilateral generalized constraint forces $n_{C\alpha}$ are obtained by the limit to infinity of the shear stiffnesses GA_1 and GA_2.

5.5 Nonlinear Kirchhoff Beam Theory

The *nonlinear Kirchhoff beam* (or *nonlinear inextensible Navier–Bernoulli beam*) is an Euler–Bernoulli beam with additional inextensibility constraints. Hence, in the Kirchhoff beam theory the cross sections remain orthogonal to the tangent vectors of the centerline and the centerline is not allowed to stretch. When also the reference configuration satisfies these constraints, the set of constraints for every instant of time t is described by three bilateral constraint functions on the longitudinal and the shear strains

$$g_i(\nu, t) = \gamma_i = \mathbf{d}_i \cdot \mathbf{r}' - \mathbf{D}_i \cdot \mathbf{r}_0' = 0 \ .$$

The contribution of the generalized constraint forces $\mathbf{n}_C = n_{Ci}\mathbf{d}_i$ to the virtual work is similar to the Euler–Bernoulli beam

$$\delta W_C^{\mathrm{int}} = \delta g_i n_{Ci} = (\mathbf{d}_i \cdot \delta\mathbf{r}' + \delta\mathbf{d}_i \cdot \mathbf{r}')n_{Ci} = \mathbf{n}_C \cdot (\delta\mathbf{r}' - \delta\boldsymbol{\phi} \times \mathbf{r}') \ .$$

For the impressed part, we assume the same quadratic form (5.33) as its elastic potential. Since the generalized constraint forces do not allow any deformation γ_i, the corresponding stiffness components are immaterial and

$$[\hat{\mathbf{D}}_1] = \mathrm{Diag}[*, *, *] \ , \quad [\hat{\mathbf{D}}_2] = \mathrm{Diag}[EI_1, EI_2, GJ] \ .$$

Hence, an elastic constitutive law of the nonlinear Kirchhoff beam is given by

$$\mathbf{n} = \mathbf{n}_C \ , \quad \mathbf{m} = \mathbf{m}_I \ ,$$

where the impressed parts are represented by

$$\mathbf{m}_I = m_{Ii}\mathbf{d}_i = (\hat{\mathbf{D}}_2)_{ij}\kappa_j\mathbf{d}_i \ .$$

and the generalized constraint forces are formulated by the normal cone inclusions

$$\mathbf{n}_C = n_{Ci}\mathbf{d}_i \ , \qquad \text{with } n_{Ci} \in \mathcal{N}_{\{0\}}(\gamma_i) = \mathbb{R} \ ,$$

representing the bilateral constraints.

Chapter 6

Classical Linearized Beam Theories

In many engineering applications beams are so stiff, that only small deformations with respect to a reference configuration occur. Thus, a linear beam theory is preferred which simplifies the problem drastically. Using the nonlinear beam theory from the previous chapter, such a linear beam theory is obtained in a straight forward manner by the linearization around a reference configuration. This chapter presents the process of linearization of a nonlinear theory at the example of classical beam theories, whose results are best-known, cf. Ballard and Millard (2009).

In Section 6.1 the kinematical quantities of Chapter 5 are linearized around a reference state. Subsequently, in Section 6.2, the virtual work contributions and its corresponding differential equations are stated in its linearized form. Finally, Section 6.3 – 6.5 discuss the elastic constitutive laws of the linearized Timoshenko, Euler–Bernoulli and Kirchhoff beam theory.

6.1 Linearized Beam Kinematics

In accordance with (5.12), the reference configuration of the classical beam is given by the placement

$$\Xi(\theta^\alpha, \nu) = \mathbf{X}(\mathbf{Q})(\theta^\alpha, \nu) = \mathbf{r}_0(\nu) + \theta^\alpha \mathbf{D}_\alpha(\nu) \ .$$

For the upcoming linearization it is convenient to rewrite the motion of the beam (5.1) using its constrained position field in the form

$$\boldsymbol{\xi}(\theta^\alpha, \nu, t) = \mathbf{x}(\mathbf{q}(\cdot, t))(\theta^\alpha, \nu) = \mathbf{r}_0(\nu) + \mathbf{w}(\nu, t) + \theta^\alpha \overline{\mathbf{R}}(\nu, t)\mathbf{D}_\alpha(\nu) \ , \quad \overline{\mathbf{R}} = \mathbf{R}\mathbf{R}_0^{\mathrm{T}} \ , \tag{6.1}$$

where the generalized position functions $\mathbf{q}(\cdot, t)$ are identified with $\mathbf{w}(\cdot, t)$ and $\overline{\mathbf{R}}(\cdot, t)$. The displacement of the centerline with respect to the reference curve is represented by $\mathbf{w}$. The rotation $\overline{\mathbf{R}}$ describes the rotation of the cross section from the reference configuration to the current configuration. Within a linearized theory we assume that

$$|\mathbf{w}'| \ll 1 \ , \quad \text{and} \ \left|\overline{\mathbf{R}} - \mathbb{1}\right| \ll 1 \ .$$

We parametrize a path through the space of rotations $SO(3)$ by $\eta \in \mathbb{R}$ and demand the identity condition $\overline{\mathbf{R}}(0) = \mathbb{1}$. A Taylor expansion up to first order terms yields

$$\overline{\mathbf{R}}(\eta) = \mathbb{1} + \frac{\mathrm{d}\overline{\mathbf{R}}}{\mathrm{d}\eta}(0)\eta + \mathcal{O}\left(\eta^2\right)$$

and define the skew-symmetric matrix $\tilde{\boldsymbol{\theta}} \coloneqq \frac{\mathrm{d}\overline{\mathbf{R}}}{\mathrm{d}\eta}(0)\eta$. The skew-symmetry of $\tilde{\boldsymbol{\theta}}$ follows from the orthogonality of $\overline{\mathbf{R}}$. By taking the derivative of $\overline{\mathbf{R}}^{\mathrm{T}}\overline{\mathbf{R}} = \mathbb{1}$ with respect to η and by evaluating the functions at $\eta = 0$ we obtain the skew-symmetry property of $\tilde{\boldsymbol{\theta}}$

$$\frac{\mathrm{d}\overline{\mathbf{R}}^{\mathrm{T}}}{\mathrm{d}\eta}\overline{\mathbf{R}} = -\overline{\mathbf{R}}^{\mathrm{T}}\frac{\mathrm{d}\overline{\mathbf{R}}}{\mathrm{d}\eta} \quad \Rightarrow \quad \frac{\mathrm{d}\overline{\mathbf{R}}^{\mathrm{T}}}{\mathrm{d}\eta}(0) = -\frac{\mathrm{d}\overline{\mathbf{R}}}{\mathrm{d}\eta}(0)\ .$$

As it is also shown in Ballard and Millard (2009), from (6.1) it follows directly that the rotation up to first order terms can be approximated as

$$\overline{\mathbf{R}} \approx \mathbb{1} + \tilde{\boldsymbol{\theta}}\ , \quad |\boldsymbol{\theta}| \ll 1\ . \tag{6.2}$$

As the reference curve $\mathbf{r}_0$ is a priori known and does not depend on the motion, the variation of the centerline $\mathbf{r}$ is determined by the variation of the displacement vector only

$$\delta\mathbf{r} = \delta\mathbf{r}_0 + \delta\mathbf{w} = \delta\mathbf{w}\ .$$

According to the definition of the virtual rotations (5.6) and (6.2), the linearized virtual rotations are approximated by the variation of $\tilde{\boldsymbol{\theta}}$, i.e.

$$\delta\tilde{\boldsymbol{\phi}} = \delta\mathbf{R}\mathbf{R}^{\mathrm{T}} = \delta\overline{\mathbf{R}}\,\overline{\mathbf{R}}^{\mathrm{T}} \approx \delta\tilde{\boldsymbol{\theta}}(\mathbb{1} + \tilde{\boldsymbol{\theta}})^{\mathrm{T}} \approx \delta\tilde{\boldsymbol{\theta}}\ .$$

Assuming also small angular velocities and applying the same linearization argument as for the virtual rotations, the angular velocities are approximated in a similar way as

$$\tilde{\boldsymbol{\omega}} = \dot{\mathbf{R}}\,\mathbf{R}^{\mathrm{T}} = \dot{\overline{\mathbf{R}}}\,\overline{\mathbf{R}}^{\mathrm{T}} \approx \dot{\tilde{\boldsymbol{\theta}}}(\mathbb{1} + \tilde{\boldsymbol{\theta}})^{\mathrm{T}} \approx \dot{\tilde{\boldsymbol{\theta}}}\ .$$

The approximation of the coupling term (5.20) and the inertia term (5.21) up to first order of $\boldsymbol{\theta}$ and $\dot{\boldsymbol{\theta}}$, are determined by

$$\ddot{\mathbf{c}} \approx \ddot{\boldsymbol{\theta}} \times \mathbf{c}\ , \quad \dot{\mathbf{h}} \approx \mathbf{I}_{\rho_0}\ddot{\boldsymbol{\theta}}\ .$$

In order to linearize the generalized strain measure (5.29), we first rewrite the strain using $\mathbf{d}_i = \overline{\mathbf{R}}\mathbf{D}_i$ and apply the motion of the beam in the form (6.1). Then, by the approximation for small rotations (6.2), we obtain the linearized generalized strain measure γ_i^{lin}

$$\gamma_i \overset{(5.29,6.1)}{=} \mathbf{D}_i\cdot\left((\overline{\mathbf{R}}^{\mathrm{T}} - \mathbb{1})\mathbf{r}_0' + \overline{\mathbf{R}}^{\mathrm{T}}\mathbf{w}'\right) \overset{(6.2)}{\approx} \mathbf{D}_i\cdot(\tilde{\boldsymbol{\theta}}^{\mathrm{T}}\mathbf{r}_0' + \mathbf{w}') = \mathbf{D}_i\cdot(\mathbf{w}' - \boldsymbol{\theta}\times\mathbf{r}_0') =: \gamma_i^{\mathrm{lin}}\ . \tag{6.3}$$

For the approximation of the generalized strain measure (5.30), the effective curvature $\tilde{\bar{\mathbf{k}}}$ with respect to the rotation $\overline{\mathbf{R}}$ is required. Using the definition of the effective curvature (5.3) and applying the identity $(\mathbf{R}_0^{\mathrm{T}})'\,\mathbf{R}_0 = -\mathbf{R}_0^{\mathrm{T}}\,\mathbf{R}_0'$, we rewrite the effective curvature $\tilde{\bar{\mathbf{k}}}$ as

$$\tilde{\bar{\mathbf{k}}} = \overline{\mathbf{R}}'\overline{\mathbf{R}}^{\mathrm{T}} = (\mathbf{R}\mathbf{R}_0^{\mathrm{T}})'\mathbf{R}_0\mathbf{R}^{\mathrm{T}} = \mathbf{R}'\mathbf{R}_0^{\mathrm{T}}\mathbf{R}_0\mathbf{R}^{\mathrm{T}} - \mathbf{R}\mathbf{R}_0^{\mathrm{T}}\mathbf{R}_0{}'\mathbf{R}^{\mathrm{T}} \; . \tag{6.4}$$

In order to reformulate the generalized strain measure (5.30), we express the current directors by the reference directors $\mathbf{d}_k = \overline{\mathbf{R}}\mathbf{D}_k$ and simplify the components of the strain using the orthogonality of the rotation

$$\tilde{\kappa}_{kj} = \mathbf{D}_k \cdot (\mathbf{R}_0\mathbf{R}^{\mathrm{T}}\mathbf{R}'\mathbf{R}^{\mathrm{T}}\mathbf{R}\mathbf{R}_0^{\mathrm{T}} - \mathbf{R}_0'\mathbf{R}_0^{\mathrm{T}})\mathbf{D}_j = \mathbf{D}_k \cdot (\mathbf{R}_0\mathbf{R}^{\mathrm{T}}\mathbf{R}'\mathbf{R}_0^{\mathrm{T}} - \mathbf{R}_0'\mathbf{R}_0^{\mathrm{T}})\mathbf{D}_j \; .$$

Using a telescopic expansion by identities, we are able to express the components of the curvature with the rotations $\overline{\mathbf{R}}$ only. The linearization up to first order terms is then easily obtained by applying (6.2)

$$\begin{aligned}\tilde{\kappa}_{kj} &= \mathbf{D}_k \cdot \big(\mathbf{R}_0\mathbf{R}^{\mathrm{T}}\mathbf{R}'\mathbf{R}_0^{\mathrm{T}}(\mathbf{R}_0\mathbf{R}^{\mathrm{T}})(\mathbf{R}\mathbf{R}_0^{\mathrm{T}}) - (\mathbf{R}_0\mathbf{R}^{\mathrm{T}})(\mathbf{R}\mathbf{R}_0^{\mathrm{T}})\mathbf{R}_0'(\mathbf{R}^{\mathrm{T}}\mathbf{R})\mathbf{R}_0^{\mathrm{T}}\big)\mathbf{D}_j \\ &\overset{(6.4,6.1)}{=} \mathbf{D}_k \cdot \big(\overline{\mathbf{R}}^{\mathrm{T}}(\overline{\mathbf{R}}'\,\overline{\mathbf{R}}^{\mathrm{T}})\overline{\mathbf{R}}\big)\mathbf{D}_j \overset{(6.2)}{\approx} \mathbf{D}_k \cdot \tilde{\boldsymbol{\theta}}'\mathbf{D}_j \; .\end{aligned}$$

The components of the associated axial vector are obtained using the alternating symbols, i.e.

$$\kappa_i(\nu, t) = \frac{1}{2}\varepsilon_{ijk}\tilde{\kappa}_{kj} \approx \frac{1}{2}\varepsilon_{ijk}\mathbf{D}_k \cdot \tilde{\boldsymbol{\theta}}'\mathbf{D}_j = \mathbf{D}_i \cdot \boldsymbol{\theta}' =: \kappa_i^{\mathrm{lin}}.$$

The second generalized linearized strain κ_i^{lin} measures the change of orientation $\boldsymbol{\theta}'$ in direction of the reference directors $\mathbf{D}_i$.

6.2 The Boundary Value Problem of the Classical Linearized Beam Theory

After the linearization of the kinematic expressions, the virtual work of the linearized beam is obtained in a straight forward manner from (5.24) by replacing the nonlinear kinematic expressions by their linearized ones. As in the nonlinear case, this leads directly to the weak variational formulation of the linearized classical beam

$$\begin{aligned}\delta W = \int_{\nu_1}^{\nu_2} \Big\{ &\mathbf{n} \cdot [\delta\mathbf{w}' - \delta\boldsymbol{\theta} \times \mathbf{r}_0'] + \mathbf{m} \cdot \delta\boldsymbol{\theta}' + \delta\mathbf{w} \cdot \Big(A_{\rho_0}\ddot{\mathbf{w}} + \ddot{\boldsymbol{\theta}} \times \mathbf{c} - \overline{\mathbf{n}}\Big) \\ &+\delta\boldsymbol{\theta} \cdot \Big(\mathbf{c} \times \ddot{\mathbf{w}} + \mathbf{I}_{\rho_0}\ddot{\boldsymbol{\theta}} - \overline{\mathbf{m}}\Big)\Big\}\,\mathrm{d}\nu - \sum_{i=1}^{2}\{\delta\mathbf{w} \cdot \overline{\mathbf{n}}_i + \delta\boldsymbol{\theta} \cdot \overline{\mathbf{m}}_i\}\,|_{\nu=\nu_i} = 0 \quad \forall \delta\mathbf{w}, \delta\boldsymbol{\theta}, t \; .\end{aligned} \tag{6.5}$$

After integration by parts, the virtual work takes the form

$$\begin{aligned}\delta W = &-\{\delta\mathbf{w} \cdot (\mathbf{n} + \overline{\mathbf{n}}_1) + \delta\boldsymbol{\theta} \cdot (\mathbf{m} + \overline{\mathbf{m}}_1)\}\,|_{\nu=\nu_1} \\ &+ \int_{\nu_1}^{\nu_2} \Big\{\delta\mathbf{w} \cdot (A_\rho\ddot{\mathbf{w}} + \ddot{\boldsymbol{\theta}} \times \mathbf{c} - \overline{\mathbf{n}} - \mathbf{n}') + \delta\boldsymbol{\theta} \cdot (\mathbf{c} \times \ddot{\mathbf{w}} + \mathbf{I}_{\rho_0}\ddot{\boldsymbol{\theta}} - \overline{\mathbf{m}} - \mathbf{m}' - \mathbf{r}_0' \times \mathbf{n})\Big\}\,\mathrm{d}\nu \\ &+ \{\delta\mathbf{w} \cdot (\mathbf{n} - \overline{\mathbf{n}}_2) + \delta\boldsymbol{\theta} \cdot (\mathbf{m} - \overline{\mathbf{m}}_2)\}_{\nu=\nu_2} = 0 \quad \forall \delta\mathbf{w}, \delta\boldsymbol{\theta}, t \; ,\end{aligned}$$

which corresponds to the strong variational formulation of the linearized classical beam. Using the same smoothness arguments as in the nonlinear setting, the Fundamental Lemma of Calculus of Variation leads to the complete boundary value problem of the linearized classical beam. The boundary value problem consists of the equations of motion for the interior of the beam $\nu \in (\nu_1, \nu_2)$

$$\begin{aligned} \mathbf{n}' + \overline{\mathbf{n}} &= A_{\rho_0}\ddot{\mathbf{w}} + \ddot{\boldsymbol{\theta}} \times \mathbf{c} \;, \\ \mathbf{m}' + \mathbf{r}_0' \times \mathbf{n} + \overline{\mathbf{m}} &= \mathbf{c} \times \ddot{\mathbf{w}} + \mathbf{I}_{\rho_0}\ddot{\boldsymbol{\theta}} \;, \end{aligned} \tag{6.6}$$

and of the boundary conditions $\mathbf{n}(\nu_1) = -\overline{\mathbf{n}}_1$, $\mathbf{m}(\nu_1) = -\overline{\mathbf{m}}_1$ and $\mathbf{n}(\nu_2) = \overline{\mathbf{n}}_2$, $\mathbf{m}(\nu_2) = \overline{\mathbf{m}}_2$. The force contributions in (6.6) can be considered either as resultant force contributions with a relation to the Euclidean space or as generalized forces of a generalized one-dimensional continuum.

Let the generalized external forces $\overline{\mathbf{n}}$ and $\overline{\mathbf{m}}$ be potential forces only. Then, for static problems, it is possible to solve the equilibrium equations (6.6) for $\mathbf{n}$ and $\mathbf{m}$ with generalized force boundary conditions only. When we are interested in the displacement of the beam, a constitutive law for the generalized internal forces completes the description of the beam.

6.3 Linearized Timoshenko Beam Theory

As in the geometrically nonlinear case, we propose a semi-induced theory and look for the same form of the elastic potential

$$\hat{W}(\nu, t) = W(\gamma_i^{\mathrm{lin}}(\nu, t), \kappa_i^{\mathrm{lin}}(\nu, t))$$

as in (5.28) but exchange the nonlinear generalized strain measures by their linearizations. Also in the linearized setting, we have to show the compatibility of the chosen linearized generalized strain measures with the linearized induced theory. Defining $\mathbf{n}_I = n_{Ii}\mathbf{D}_i := \frac{\partial W}{\partial \gamma_i^{\mathrm{lin}}}\mathbf{D}_i$, the variation

$$\frac{\partial W}{\partial \gamma_i^{\mathrm{lin}}}\,\delta\gamma_i^{\mathrm{lin}} \overset{(6.3)}{=} \frac{\partial W}{\partial \gamma_i^{\mathrm{lin}}}\,\mathbf{D}_i \cdot (\delta\mathbf{w}' - \delta\boldsymbol{\theta} \times \mathbf{r}_0') = \mathbf{n}_I \cdot (\delta\mathbf{w}' - \delta\boldsymbol{\theta} \times \mathbf{r}_0') \;,$$

shows the compatibility with the virtual work contribution (6.5). Defining $\mathbf{m}_I = m_{Ii}\mathbf{D}_i := \frac{\partial W}{\partial \kappa_i^{\mathrm{lin}}}\mathbf{D}_i$, the variation

$$\frac{\partial W}{\partial \kappa_i^{\mathrm{lin}}}\,\delta\kappa_i^{\mathrm{lin}} = \frac{\partial W}{\partial \kappa_i^{\mathrm{lin}}}\,\mathbf{D}_i \cdot \delta\boldsymbol{\theta}' = \mathbf{m}_I \cdot \delta\boldsymbol{\theta}' \;,$$

shows also the compatibility with the virtual work contribution (6.5). For the elastic potential, we assume the same quadratic form (5.33) as in the nonlinear case, but replace the generalized strains by their corresponding linearizations

$$W(\gamma_i^{\mathrm{lin}}, \kappa_i^{\mathrm{lin}}) = \frac{1}{2}\,\gamma_i^{\mathrm{lin}}\,(\hat{\mathbf{D}}_1)_{ij}\,\gamma_j^{\mathrm{lin}} + \frac{1}{2}\,\kappa_i^{\mathrm{lin}}\,(\hat{\mathbf{D}}_2)_{ij}\,\kappa_j^{\mathrm{lin}} \;, \tag{6.7}$$

with the stiffness components

$$[\hat{\mathbf{D}}_1] = \mathrm{Diag}[GA_1, GA_2, EA] \ , \quad [\hat{\mathbf{D}}_2] = \mathrm{Diag}[EI_1, EI_2, GJ] \ .$$

Consequently, the constitutive laws for the generalized internal forces are

$$\mathbf{n} = \mathbf{n}_I = n_{Ii}\mathbf{D}_i = (\hat{\mathbf{D}}_1)_{ij}\gamma_j^{\mathrm{lin}}\mathbf{D}_i \ , \quad \mathbf{m} = \mathbf{m}_I = m_{Ii}\mathbf{D}_i = (\hat{\mathbf{D}}_2)_{ij}\kappa_j^{\mathrm{lin}}\mathbf{D}_i \ .$$

As in the nonlinear case, the *linearized Timoshenko beam* is described by impressed generalized internal forces only.

6.4 Linearized Euler–Bernoulli Beam Theory

The *linearized Euler–Bernoulli beam* is a linearized Timoshenko beam with the additional linearized constraints, that the cross sections remain orthogonal to the tangent vector of the centerline. The linearized version of the orthogonality constraints results in a beam, which does not allow any linearized shear deformation. Thus, for any time t, a linearized Euler–Bernoulli beam has to satisfy the following constraint functions

$$g_\alpha(\nu, t) = \gamma_\alpha^{\mathrm{lin}} = \mathbf{D}_\alpha \cdot (\mathbf{w}' - \boldsymbol{\theta} \times \mathbf{r}_0') = 0 \ .$$

The contribution of the generalized constraint forces $\mathbf{n}_C = n_{C\alpha}\mathbf{D}_\alpha$ to the virtual work is similar to the nonlinear case, i.e.

$$\delta W_C^{\mathrm{int}} = \delta g_\alpha n_{C\alpha} = \mathbf{D}_\alpha \cdot (\delta\mathbf{w}' - \delta\boldsymbol{\theta} \times \mathbf{r}_0') n_{C\alpha} = \mathbf{n}_C \cdot (\delta\mathbf{w}' - \delta\boldsymbol{\theta} \times \mathbf{r}_0') \ . \tag{6.8}$$

For the impressed part, we assume the same quadratic form (6.7) as its elastic potential. Since the generalized constraint forces do not allow any deformation γ_α, the corresponding stiffness components are immaterial and

$$[\hat{\mathbf{D}}_1] = \mathrm{Diag}[*, *, EA] \ , \quad [\hat{\mathbf{D}}_2] = \mathrm{Diag}[EI_1, EI_2, GJ] \ .$$

The elastic material law of the linearized Euler–Bernoulli beam is summarized as follows:

$$\mathbf{n} = \mathbf{n}_I + \mathbf{n}_C \ , \quad \mathbf{m} = \mathbf{m}_I \ ,$$

where the impressed parts are represented by the single-valued force law

$$\mathbf{n}_I = n_{Ii}\mathbf{D}_i = (\hat{\mathbf{D}}_1)_{ij}\gamma_j^{\mathrm{lin}}\mathbf{D}_i \ , \quad \mathbf{m}_I = m_{Ii}\mathbf{D}_i = (\hat{\mathbf{D}}_2)_{ij}\kappa_j^{\mathrm{lin}}\mathbf{D}_i \ .$$

and the generalized constraint forces are formulated by the normal cone inclusions

$$\mathbf{n}_C = n_{C\alpha}\mathbf{D}_\alpha \ , \qquad \text{with } n_{C\alpha} \in \mathcal{N}_{\{0\}}(\gamma_\alpha^{\mathrm{lin}}) = \mathbb{R} \ ,$$

representing the bilateral constraints.

6.5 Linearized Kirchhoff Beam Theory

The *linearized Kirchhoff beam* is a linearized Euler–Bernoulli beam which additionally restricts the elongation of the centerline. The constraint functions are completely described by the linearized generalized strain measure γ_i^{lin}, i.e.

$$g_i(\nu, t) = \gamma_i^{\mathrm{lin}} = \mathbf{D}_i \cdot (\mathbf{w}' - \boldsymbol{\theta} \times \mathbf{r}_0') = 0 \ .$$

The contribution of the generalized constraint forces $\mathbf{n}_C = n_{Ci}\mathbf{D}_i$ is obtained in the sense of (6.8). For the impressed part, we assume the same quadratic form (6.7) as the elastic potential. Since the generalized constraint forces do not allow any deformation γ_i^{lin}, the corresponding stiffness components are immaterial and

$$[\hat{\mathbf{D}}_1] = \mathrm{Diag}[*, *, *] \ , \quad [\hat{\mathbf{D}}_2] = \mathrm{Diag}[EI_1, EI_2, GJ] \ .$$

The elastic material law of the linearized Kirchhoff beam is summarized as follows:

$$\mathbf{n} = \mathbf{n}_C \ , \quad \mathbf{m} = \mathbf{m}_I \ ,$$

where the impressed parts of the internal generalized forces are represented by

$$\mathbf{m}_I = m_{Ii}\mathbf{D}_i = (\hat{\mathbf{D}}_2)_{ij}\kappa_j^{\mathrm{lin}}\mathbf{D}_i$$

and the force laws for the generalized constraint forces are formulated by the normal cone inclusions

$$\mathbf{n}_C = n_{Ci}\mathbf{D}_i \ , \qquad \text{with } n_{Ci} \in \mathcal{N}_{\{0\}}(\gamma_i^{\mathrm{lin}}) = \mathbb{R} \ ,$$

representing the bilateral constraints.

Chapter 7

Classical Plane Linearized Beam Theories

We speak of a classical plane linearized beam as being a classical linearized beam fulfilling the following assumptions. The cross section geometry remains the same for all cross sections, the motion is restricted to a plane, the reference configuration is straight and the material of the continuous body is described by a linear elastic material law. These assumptions on the motion of the beam and material law enable us to formulate statements which are not easily accessible for a more general configuration of a beam. One key point is that we are able to arrive at a fully induced beam theory where the integration of the stress distributions over the cross sections can be performed analytically. Hence, we recognize relations between the generalized internal forces and the three-dimensional stress field of the Euclidean space. This allows to apply concepts of the theory of strength of materials to beams which is of vital importance to solve engineering problems. In order to achieve such a connection, we restate the generalized internal forces for the plane linearized beam. The restriction to small displacements allows us to start from the internal virtual work formulated with the linearized strain. Afterwards, we proceed in a similar way as in the previous chapters. We state the constrained position field of the beam and apply it to the virtual work which leads us consequently to the boundary value problem of the beam. Using the solutions of the boundary value problem and non-admissible virtual displacements, it is possible to access in a further step the constraint stresses of the beam which guarantee the restricted kinematics of the beam.

The outline of the chapter is as follows. In Section 7.1 we repeat the principle of d'Alembert–Lagrange for linear elasticity and introduce an elastic constitutive law for the impressed stresses. In Section 7.2 – 7.4 the equations of motion and the plane stress distribution of the plane linearized Timoshenko, Euler–Bernoulli and Kirchhoff beam are determined.

7.1 Constrained Position Fields in Linear Elasticity

In the theory of linear elasticity, we linearize the configuration of the continuous body around a reference configuration $\Xi(\theta^k)$ and describe the motion $\boldsymbol{\xi}(\theta^k, t)$ of the body with the vector valued *displacement displacement field*

$$\mathbf{u}(\theta^k, t) := \boldsymbol{\xi}(\theta^k, t) - \Xi(\theta^k) \ . \tag{7.1}$$

The deformation of the body is measured by the linearized strain measure which is defined as

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}\left((\nabla\mathbf{u})^{\mathrm{T}} + \nabla\mathbf{u}\right) \ ,$$

where ∇ represents the gradient of the Euclidean space. Using the symmetry property of the Cauchy stress, the internal virtual work (4.1) can be written, for a linearized kinematics, with the variation of the displacement field $\delta\mathbf{u}$ as its virtual displacement field

$$\delta W^{\mathrm{int}} = \int_{\overline{B}} \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\delta\mathbf{u})\,\mathrm{d}V \ . \tag{7.2}$$

We assume the parametrization of the body to be given by the reference configuration of the continuous body in the Euclidean space, i.e. $\overline{B} \subset \mathbb{E}^3$. For convenience, we restrict us to cartesian coordinates (x, y, z) which parametrize the set $\overline{B}$. Hence, the volume element is given by $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ and the gradient reduces to a partial derivative with respect to (x, y, z).

As discussed in Section 4.2, a continuous body which is enforced to follow a constrained position field is exposed to the stress field

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}_I + \boldsymbol{\sigma}_C \ , \tag{7.3}$$

which is composed by an impressed part $\boldsymbol{\sigma}_I$ and a constraint part $\boldsymbol{\sigma}_C$ of perfect constraint stresses. For admissible virtual displacements and, consequently, for the strain due to virtual displacements being admissible, the virtual work due to the perfect constraint stresses

$$\delta W_C^{\mathrm{int}} = \int_{\overline{B}} \boldsymbol{\sigma}_C : \boldsymbol{\varepsilon}(\delta\mathbf{u})\mathrm{d}V = 0 \qquad \forall \delta\mathbf{u} \in T_{\xi}\mathcal{C} \ ,$$

vanishes identically.

For the impressed part of the stress, we apply a slightly modified isotropic linear elastic material law described by the Young's modulus E and the shear modulus G. We consider the beam's motion in the $\mathbf{e}_x^I$-$\mathbf{e}_z^I$ plane. Stresses in $\mathbf{e}_y^I$-direction are constituted by the constraint stresses. Since we assume the cross sections to be rigid, lateral contraction is not possible. To avoid constraint stresses due to the material law in the cross sections, we set the Poisson's ratio for the normal forces $\sigma_{Iii}, i = \{1, 2, 3\}$ to zero. The shear deformations are not influenced by the kinematical restrictions of the beam and are treated insofar in the same way as for an unconstrained continuous body. Finally, we arrive at the following linear elastic material law for the impressed part of the stress field

$$\left.\begin{array}{rcll} \sigma_{Iii} & = & E\varepsilon_{ii} \ , & \\ \sigma_{Iij} & = & 2G\varepsilon_{ij} \ , & i \neq j \end{array}\right\} i, j = \{1, 2, 3\} \ . \tag{7.4}$$

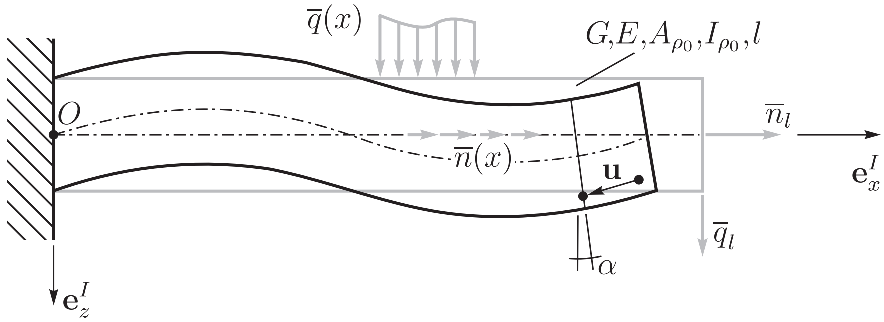

Figure 7.1: Plane linearized Timoshenko beam.

7.2 The Plane Linearized Timoshenko Beam

In the following, we investigate the plane beam theories at the example of a clamped beam, as depicted in Figure 7.1, with length l, constant cross section area A, Young's modulus E, shear modulus G, cross section mass density A_{ρ_0} and cross section inertia density I_{ρ_0}. The inertia terms arise from a homogenous mass distribution and from the definitions (5.17) and (5.19). The centerline coincides with the line of centroids which in its reference configuration corresponds to the $\mathbf{e}_x^I$-axis. The beam is loaded by applied normal forces $\overline{n}(x)$ in $\mathbf{e}_x^I$-direction and by applied shear forces $\overline{q}(x)$ in $\mathbf{e}_z^I$-direction whose force distributions in the Euclidean space are both homogenously distributed over the cross section. At $x = l$, additional forces $\overline{n}_l$ and $\overline{q}_l$, which are also homogenously distributed over the cross section, are applied.

Kinematics, virtual work and the boundary value problem

Since in a linear theory the motion of the beam is described by the displacement field (7.1), we assume the embedding (4.10) to induce a constrained displacement field. The motion of the Timoshenko beam is described by the motion of the centerline and the rotation of the cross sections. In the plane, but kinematically nonlinear case, the nonlinear displacement field $\mathbf{u}_{\mathrm{nl}}$ is constrained to

$$\mathbf{u}_{\mathrm{nl}}((x,y,z),t) = \mathbf{x}_{\mathrm{nl}}(\mathbf{q}(\cdot,t))(x,y,z) = \begin{pmatrix} u(x,t) - \sin(\alpha(x,t))z \\ 0 \\ w(x,t) + (1-\cos(\alpha(x,t))) \end{pmatrix} ,$$

where the generalized position functions $\mathbf{q}(\cdot,t)$ are identified with $u(\cdot,t)$, $w(\cdot,t)$ and $\alpha(\cdot,t)$. The *longitudinal displacement* is described by $u(\cdot,t)$, the *transverse displacement* by $w(\cdot,t)$ and the *rotation* of the cross section by $\alpha(\cdot,t)$. Due to the clamping, the displacement at $x = 0$ has to vanish, i.e. $\mathbf{u}_{\mathrm{nl}}((0,y,z),t) = 0$. A linearization around the straight reference configuration yields the constrained linearized displacement field

$$\mathbf{u}((x,y,z),t) = \begin{pmatrix} u^1(x,y,z,t) \\ u^2(x,y,z,t) \\ u^3(x,y,z,t) \end{pmatrix} = \mathbf{x}(\mathbf{q}(\cdot,t))(x,y,z) = \begin{pmatrix} u(x,t) - \alpha(x,t)z \\ 0 \\ w(x,t) \end{pmatrix} , \qquad (7.5)$$

with the clamping condition $\mathbf{u}((0,y,z),t) = 0$. A *plane linearized Timoshenko beam* is a continuous body whose motion is restricted to the displacement field (7.5). The strain of the continuous body

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \begin{pmatrix} \frac{\partial u^1}{\partial x} & \frac{1}{2}(\frac{\partial u^1}{\partial y} + \frac{\partial u^2}{\partial x}) & \frac{1}{2}(\frac{\partial u^1}{\partial z} + \frac{\partial u^3}{\partial x}) \\ & \frac{\partial u^2}{\partial y} & \frac{1}{2}(\frac{\partial u^2}{\partial z} + \frac{\partial u^3}{\partial y}) \\ \text{sym.} & & \frac{\partial u^3}{\partial z} \end{pmatrix} = \begin{pmatrix} u' - \alpha' z & 0 & \frac{1}{2}(w' - \alpha) \\ & 0 & 0 \\ \text{sym.} & & 0 \end{pmatrix}$$

is consequently constrained too such that it can be formulated in terms of u, w and α only. The admissible virtual displacements $\delta\mathbf{x}$ and the corresponding strain $\boldsymbol{\varepsilon}(\delta\mathbf{x})$ with respect to the constraint displacements (7.5) are

$$\delta\mathbf{x} = \begin{pmatrix} \delta u - \delta\alpha z \\ 0 \\ \delta w \end{pmatrix}, \quad \boldsymbol{\varepsilon}(\delta\mathbf{x}) = \begin{pmatrix} \delta u' - \delta\alpha' z & 0 & \frac{1}{2}(\delta w' - \delta\alpha) \\ & 0 & 0 \\ \text{sym.} & & 0 \end{pmatrix}. \tag{7.6}$$

Since the clamping is guaranteed, $\delta\mathbf{x}((0,y,z),t) = 0$. The material law of the impressed stress field (7.4) applied to the constrained displacement field is given by

$$\begin{aligned} \sigma_{Ixx} &= E\varepsilon_{xx} = E(u' - \alpha' z)\,, \\ \sigma_{Ixz} &= 2G\varepsilon_{xz} = G(w' - \alpha)\,. \end{aligned} \tag{7.7}$$

In order to eliminate the constraint stresses due to the kinematical restrictions, we evaluate the virtual work for the admissible virtual displacements (7.6). With the usual split of the integration, we write the internal virtual work of (7.2) for the strains of the admissible virtual displacements (7.6) as

$$\begin{aligned} \delta W^{\text{int}} &= \int_{\mathcal{B}} \{(\delta u' - \delta\alpha' z)\sigma_{Ixx} + (\delta w' - \delta\alpha')\sigma_{Ixz}\}\,\mathrm{d}V \\ &= \int_0^l \left\{\delta u' \int_A \sigma_{Ixx}\mathrm{d}A - \delta\alpha' \int_A z\sigma_{Ixx}\mathrm{d}A + (\delta w' - \delta\alpha)\int_A \sigma_{Ixz}\mathrm{d}A\right\}\mathrm{d}x \\ &= \int_0^l \{\delta u' N_I - \delta\alpha' M_I + (\delta w' - \delta\alpha)Q_I\}\,\mathrm{d}x\,. \end{aligned} \tag{7.8}$$

Herein, the *resultant contact forces* of the plane linearized Timoshenko beam have been recognized as the integrated quantities

$$N_I(x,t) := \int_A \sigma_{Ixx}\mathrm{d}A\,, \quad Q_I(x,t) := \int_A \sigma_{Ixz}\mathrm{d}A\,, \quad M_I(x,t) := \int_A z\sigma_{Ixx}\mathrm{d}A\,, \tag{7.9}$$

which we also denote as the *resultant contact normal forces*, *resultant contact shear forces* and the *resultant contact couples*. Using the linear elastic material law for the constrained displacement field (7.7), the internal virtual work is reformulated further to

$$\begin{aligned} \delta W^{\text{int}} &= \int_0^l \left\{\delta u' E\int_A \mathrm{d}A u' - \delta\alpha'\left(-E\int_A z^2\mathrm{d}A\alpha'\right) + (\delta w' - \delta\alpha)G\int_A \mathrm{d}A(w' - \alpha)\right\}\mathrm{d}x \\ &= \int_0^l \{\delta u' EA u' - \delta\alpha'(-EI\alpha') + (\delta w' - \delta\alpha)GA(w' - \alpha)\}\,\mathrm{d}x \end{aligned} \tag{7.10}$$

in which the second moment of area is abbreviated by $I := \int_A z^2 \,\mathrm{d}A$. Since the centerline corresponds to the line of centroids, the integral $\int_A z \,\mathrm{d}A = 0$ vanishes and the couple terms between u' and α' vanish in the second line of (7.10) . By comparing (7.8) and (7.10), we obtain a constitutive law for the generalized internal forces

$$N_I = EAu' \ , \quad M_I = -EI\alpha' \ , \quad Q_I = GA(w' - \alpha) \ . \tag{7.11}$$

The connection between the generalized internal forces and the impressed stress field of the continuous body

$$\begin{aligned} \sigma_{Ixx} &= E(u' - \alpha' z) = \frac{N_I}{A} + \frac{M_I}{I} z \ , \\ \sigma_{Ixz} &= G(w' - \alpha) = \frac{Q_I}{A} \ , \end{aligned} \tag{7.12}$$

is established by (7.7) and (7.11). Applying the plane kinematics to (6.5) and using the results of (7.8), we obtain the weak variational formulation of the plane linearized Timoshenko beam as

$$\begin{aligned} \delta W = \int_0^l &\{\delta u' N_I + (\delta w' - \delta\alpha) Q_I - \delta\alpha' M_I + \delta u(A\rho\ddot{u} - \overline{n}) + \delta w(A\rho\ddot{w} - \overline{q}) \\ &-\delta\alpha(-I\rho\ddot{\alpha})\} \,\mathrm{d}x - \{\delta u \overline{n}_l + \delta w \overline{q}_l\}\,|_{x=l} = 0 \ , \ \forall \delta u, \delta w, \delta\alpha, t \ , \end{aligned} \tag{7.13}$$

with $\delta u(0) = \delta w(0) = \delta\alpha(0) = 0$ in order to satisfy the clamping boundary condition. Using integration by parts, we obtain the strong variational formulation

$$\begin{aligned} \delta W = \int_0^l &\{\delta u(A\rho\ddot{u} - \overline{n} - N_I') + \delta w(A\rho\ddot{w} - \overline{q} - Q_I') - \delta\alpha(-I\rho\ddot{\alpha} - M_I' + Q_I)\} \,\mathrm{d}x \\ &- \left. (\delta u(\overline{n}_l - N_I) + \delta w(\overline{q}_l - Q_I) - \delta\alpha(-M_I)) \right|_{x=l} = 0 \ , \forall \delta u, \delta w, \delta\alpha, t \end{aligned} \tag{7.14}$$

of the plane linearized Timoshenko beam. Using the common arguments of calculus of variations, the terms in the round brackets of (7.14) have to vanish pointwise. This leads for the interior of the beam $x \in (0, l)$ to the equations of motion

$$\begin{aligned} A\rho\ddot{u} &= N_I' + \overline{n} \ , \\ A\rho\ddot{w} &= Q_I' + \overline{q} \ , \\ -I\rho\ddot{\alpha} &= M_I' - Q_I \ , \end{aligned} \tag{7.15}$$

to the kinetic boundary conditions $N_I(l) = \overline{n}_l$, $Q_I(l) = \overline{q}_l$ and $M_I(l) = 0$ and to the kinematic boundary conditions $u(0) = w(0) = \alpha(0) = 0$. In the equations of motion (7.15), we recognize that the longitudinal deformations are completely decoupled from the shear and bending deformations of the beam. With the constitutive laws of (7.11) the equations of motion of the Timoshenko beam take the form

$$\begin{aligned} \rho A\ddot{u} &= EAu'' + \overline{n} \ , \\ \rho A\ddot{w} &= GA(w'' - \alpha') + \overline{q} \ , \\ \rho I\ddot{\alpha} &= EI\alpha'' + GA(w' - \alpha) \ . \end{aligned} \tag{7.16}$$

It is possible to modify the second and the third equations of (7.16), such that the rotation angle α can be eliminated. Firstly, we take the derivative of the third equation with respect to x. Secondly, we solve the second equation for α'. Lastly, we insert α' and its further derivatives with respect to time and position into the differentiated third equation. This leads us to a forth order differential equation in the transverse displacements w

$$\begin{aligned} \rho A\ddot{u} - EAu'' &= \overline{n} \\ EIw'''' + \rho A\ddot{w} - I\rho\left(\frac{E}{G}+1\right)\ddot{w}'' + \frac{I\rho^2}{G}\ddddot{w} &= \overline{q} + \frac{\rho I}{GA}\ddot{\overline{q}} - \frac{EI}{GA}\overline{q}'' \,. \end{aligned} \tag{7.17}$$

For vanishing distributed shear forces $\overline{q}$ and a shear correction factor of 1, the second equation of (7.17) coincide with Eq. (7) of the celebrated publication of Timoshenko (1921).

Constraint stresses of the plane Timoshenko beam

In the previous subsection, (7.12) connects the generalized internal forces to the impressed stress distribution of the continuous body. Unfortunately, we do not access the total stress distribution of the beam, since the constraint stresses $\boldsymbol{\sigma}_C$ are eliminated by the principle of d'Alembert–Lagrange. Insofar, the constant shear stress σ_{Ixz} is not in contradiction with a stress free boundary at the lateral surface of the beam. The constraint stress σ_{Cxz} is guaranteeing a stress free surface. To make the constraint stresses visible, we have to evaluate the virtual work of the beam for non-admissible virtual displacements. Since the non-admissible virtual displacements do not respect the kinematical restrictions (7.5), the constraint stresses will appear in the virtual work expression. With the solution of the boundary value problem, we can determine the appearing constraint stresses up to a certain indeterminacy.

In the following, we are going to use two special functions, which are very convenient to extract the desired constraint stresses. We require the unit step function

$$h(x)\colon \mathbb{R} \to \{0,1\} \;, \quad x \mapsto \begin{cases} 0\colon x < 0 \\ 1\colon x \geq 0 \end{cases} \tag{7.18}$$

and its derivative, the Delta-Dirac distribution δ. The Delta-Dirac distribution is loosely defined for a real valued function $f(x)$ as

$$\delta(x_0)\colon \int_{\mathbb{R}} \delta(x_0) f(x)\mathrm{d}x = f(x_0) \,. \tag{7.19}$$

In accordance with the first equation of motion of (7.15), we rewrite the resultant contact normal force at $x = l$ as

$$N_I(l) = N_I(x_0) + \int_{x_0}^{l} N_I'\mathrm{d}x = N_I(x_0) + \int_{x_0}^{l} (\rho A\ddot{u} - \overline{n})\mathrm{d}x \,. \tag{7.20}$$

In order to evaluate the constraint stress distribution $\sigma_{Cxx}(x_0)$ we cut the beam, as depicted in Figure 7.2, at the position $x = x_0$ appart and virtually displace the right part of

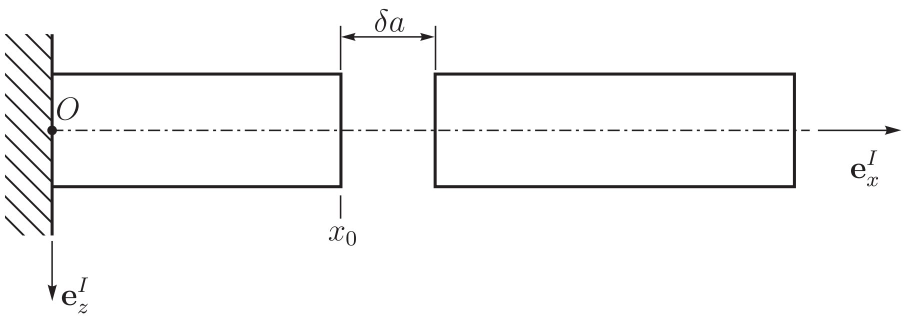

Figure 7.2: Non-admissible virtual displacements for the extraction of the constraint stresses σ_{Cxx}.

the beam by the constant value $\delta a \in \mathbb{R}$. Technically, the non-admissible virtual displacements and their corresponding strains are written as

$$\delta \mathbf{x}_{\mathrm{na}} = \begin{pmatrix} \delta a\, h(x - x_0) \\ 0 \\ 0 \end{pmatrix} , \quad \boldsymbol{\varepsilon}(\delta \mathbf{x}_{\mathrm{na}}) = \begin{pmatrix} \delta a\, \updelta(x_0) & 0 & 0 \\ & 0 & 0 \\ \text{sym.} & & 0 \end{pmatrix} , \tag{7.21}$$

where we have applied the unit step function (7.18) and the Delta-Dirac distribution (7.19). Using the internal virtual work (7.2) and the virtual work contribution of the external and inertia forces from (7.14), we obtain the virtual work for the non-admissible virtual displacements (7.21) as

$$\delta W = \int_0^l \int_A \delta a\, \updelta(x_0)(\sigma_{Ixx} + \sigma_{Cxx})\mathrm{d}A\mathrm{d}x + \int_{x_0}^l \delta a(\rho A \ddot{u} - \overline{n})\mathrm{d}x - \delta a \overline{n}_l \; .$$

Using the property of the Delta-Dirac distribution together with (7.12), we rewrite the virtual work into the form

$$\delta W = \delta a \left(\int_A \left\{ \frac{N_I}{A}(x_0) + \frac{M_I}{I}(x_0) z + \sigma_{Cxx}(x_0) \right\} \mathrm{d}A + \int_{x_0}^l \{\rho A \ddot{u} - \overline{n}\}\, \mathrm{d}x - \overline{n}_l \right) \; .$$

Since the centerline corresponds to the line of centroids, the term with the resultant contact couple M_I vanishes after integration over the cross section. The principle of virtual work states that

$$\delta W = \delta a \int_A \sigma_{Cxx}(x_0)\mathrm{d}A + \delta a \left(N_I(x_0) + \int_{x_0}^l \{\rho A \ddot{u} - \overline{n}\}\, \mathrm{d}x - \overline{n}_l \right) = 0 \quad \forall \delta a \; .$$

Using the equivalence (7.20) and the boundary condition at the end of the beam, the round bracket in the above equation vanishes. Hence, the normal constraint stresses in $\mathbf{e}_x^I$-direction integrated over the cross section have to vanish

$$\int_A \sigma_{Cxx}(x_0)\mathrm{d}A = 0 \; . \tag{7.22}$$

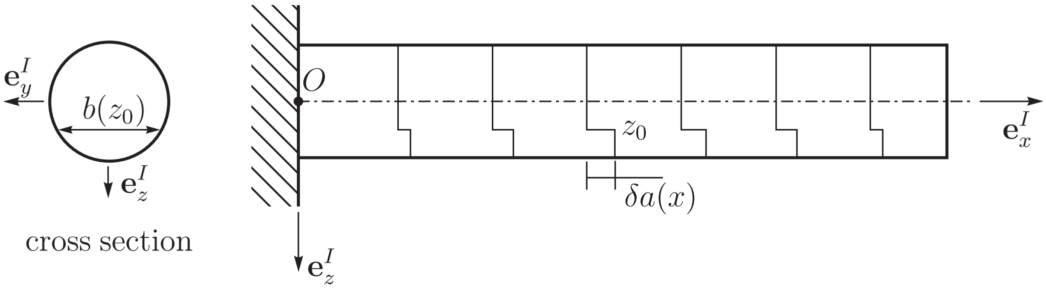

Figure 7.3: Non-admissible virtual displacements to evaluate constraint shear stresses σ_{Cxz}

Here we already recognize the first indeterminacy of the constraint stresses. It is not possible to determine the constraint stresses σ_{Cxx} uniquely. In order to obtain in the further derivations the classical results for the constraint transverse shear stresses, we choose the normal constraint stresses to vanish, i.e. $\sigma_{Cxx} = 0$.

We extract the constraint transverse shear stress in the cross section $\sigma_{xz}(z_0)$ at $z = z_0$, by shearing the beam with the non-admissible virtual displacements and its corresponding virtual strains

$$\delta \mathbf{x}_{\mathrm{na}} = \begin{pmatrix} \delta a(x) h(z - z_0) \\ 0 \\ 0 \end{pmatrix} , \quad \boldsymbol{\varepsilon}(\delta \mathbf{x}_{\mathrm{na}}) = \begin{pmatrix} \delta a' h(z - z_0) & 0 & \frac{1}{2} \delta a \, \delta(z_0) \\ & 0 & 0 \\ \text{sym.} & & 0 \end{pmatrix} , \tag{7.23}$$

where the smooth function $\delta a(x)$ vanishes at the boundary $\delta a(0) = \delta a(l) = 0$. It is convenient to introduce the following integrated quantities

$$A_{z_0} := \int_A h(z - z_0) \mathrm{d}A \, , \quad H_{z_0} := \int_A z h(z - z_0) \mathrm{d}A \, , \tag{7.24}$$

where we call H_{z_0} *first moment of area*. Using the internal virtual work (7.2) and the virtual work contribution of the external and inertia forces from (7.14), we obtain the virtual work for the non-admissible virtual displacements (7.23) as

$$\delta W = \int_0^l \int_A \left\{ \delta a' h(z - z_0) \sigma_{Ixx} + \delta a \, \delta(z_0) \sigma_{xz} + \delta a \left(\rho \ddot{u} - \frac{\overline{n}}{A} \right) h(z - z_0) \right\} \mathrm{d}A \mathrm{d}x \, .$$

In accordance with (7.12) and the abbreviations of (7.24), we integrate the virtual work over the cross section. As depicted in Figure 7.3 the size of the beam at z_0 in $\mathbf{e}_y^I$-direction is given by $b(z_0)$. Assuming that the transverse shear stresses σ_{xz} are constant in $\mathbf{e}_y$-direction, we reformulate the virtual work to

$$\delta W = \int_0^l \left\{ \delta a' \frac{M_I}{I} H_{z_0} + \delta a' \frac{N_I}{A} A_{z_0} + \delta a \, b(z_0) \sigma_{xz}(z_0) + \delta a \left(\rho \ddot{u} - \frac{\overline{n}}{A} \right) A_{z_0} \right\} \mathrm{d}x \, .$$

Using integration by parts and the connection between the resultant contact shear forces and the shear stresses of (7.12), we obtain the expression

$$\delta W = \int_0^l \delta a \left\{ \frac{-M_I'}{I} H_{z_0} + b(z_0) \left(\frac{Q_I}{A} + \sigma_{Cxz}(z_0) \right) + (A \rho \ddot{u} - \overline{n} - N_I') \frac{A_{z_0}}{A} \right\} \mathrm{d}x \, .$$

By the first equation of (7.15) the last term in the round brackets have to vanish and we state the principle of virtual work for the non-admissible virtual displacements (7.23) as follows:

$$\delta W = \int_0^l \delta a \left\{ \frac{-M_I'}{I} H_{z_0} + b(z_0) \left(\frac{Q_I}{A} + \sigma_{Cxz}(z_0) \right) \right\} \mathrm{d}x = 0 \quad \forall \delta a \ .$$

Using the common arguments of calculus of variations, the integrand has to vanish pointwise, which leads to the constraint transverse shear stress distribution

$$\sigma_{Cxz}(z_0) = \frac{M_I' H_{z_0}}{I b(z_0)} - \frac{Q_I}{A} \ , \tag{7.25}$$

where M_I' is given by the last equation of (7.15). The first term is equivalent to the transverse shear stress due to bending. We will discuss this case within the Euler–Bernoulli theory, where this result is commonly known.

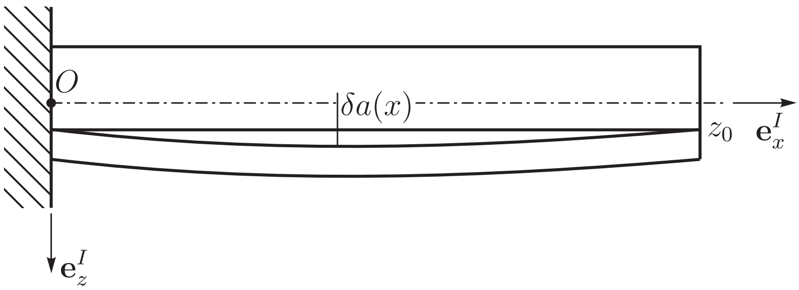

Figure 7.4: Non-admissible virtual displacements for constraint normal stresses σ_{zz}

The normal constraint stresses σ_{Czz} in the rigid cross sections is the last contribution of constraint stresses which we evaluate. These normal constraint stresses only appear in the dynamical consideration of the problem or when distributed shear forces $\overline{q}$ are imposed. In order to make statements about the constraint stress distribution we have to assume, that the shear forces $\overline{q}$ arise from a homogenous force distribution over the cross section. For the extraction of the constraint stresses, we assume the following non-admissible virtual displacements, as depicted in Figure 7.4, and its corresponding strain

$$\delta \mathbf{x}_{\mathrm{na}} = \begin{pmatrix} 0 \\ 0 \\ \delta a(x) h(z - z_0) \end{pmatrix} \ , \quad \boldsymbol{\varepsilon}(\delta \mathbf{x}_{\mathrm{na}}) = \begin{pmatrix} 0 & 0 & \frac{1}{2}\delta a' h(z - z_0) \\ & 0 & 0 \\ \mathrm{sym.} & & \delta a \delta(z_0) \end{pmatrix} \ , \tag{7.26}$$

where the smooth function $\delta a(x)$ vanishes at the boundary $\delta a(0) = \delta a(l) = 0$. Using the internal virtual work (7.2) and the virtual work contribution of the external and inertia forces from (7.14), we obtain the virtual work for the non-admissible virtual displacements (7.26) as

$$\delta W = \int_0^l \left\{ \int_A \delta a' h(z - z_0) \sigma_{xz} \mathrm{d}A + \delta a \, b(z_0) \sigma_{Czz}(z_0) + \delta a (\rho A \ddot{w} - \overline{q}) \frac{A_{z_0}}{A} \right\} \mathrm{d}x \ .$$

In accordance with (7.12), (7.25) and integration by parts, we transform the virtual work further to

$$\delta W = \int_0^l \delta a \left\{ \frac{-M_I''}{Ib(z_0)} \int_A h(z-z_0) H_{z_0} \mathrm{d}A + b(z_0)\sigma_{Czz}(z_0) + (\rho A\ddot{w} - \overline{q})\frac{A_{z_0}}{A_z} \right\} \mathrm{d}x = 0 \quad \forall \delta a \ .$$

The standard arguments lead to the constraint stress

$$\sigma_{Czz} = \frac{M_I''}{Ib(z_0)^2} \int_A h(z-z_0) H_{z_0} \mathrm{d}A - \frac{A_{z_0}}{A}(\rho A\ddot{w} - \overline{q}) \ . \tag{7.27}$$

Using (7.3), (7.12), (7.22), (7.25) and (7.27) the total stress field of the plane linearized Timoshenko beam is given by

$$\begin{aligned} \sigma_{xx} &= \frac{N_I}{A} + \frac{M_I}{I} z \ , \\ \sigma_{xz} &= \frac{M_I' H_{z_0}}{Ib} \ , \\ \sigma_{zz} &= \frac{M_I''}{Ib(z_0)^2} \int_A h(z-z_0) H_{z_0} \mathrm{d}A - \frac{A_{z_0}}{A}(\rho A\ddot{w} - \overline{q}) \ . \end{aligned} \tag{7.28}$$

Due to the constraint shear stress the boundary conditions at the lateral surface is fulfilled. When calculating the stress distribution for a clamped cantilever beam with rectangular cross section of height h under a constant shear force distribution $\overline{q}$, the static solution implies with respect to z, that $\sigma_{xx}(z)$ is a linear function, $\sigma_{xz}(z)$ is quadratic and $\sigma_{zz}(z)$ is cubic. The solutions are of the magnitude $\sigma_{xx} \sim \frac{h}{l}$, $\sigma_{xz} \sim \left(\frac{h}{l}\right)^2$ and $\sigma_{zz} \sim \left(\frac{h}{l}\right)^3$. The magnitudes of the stress distributions justify for slender bodies to neglect the normal stress contribution in $\mathbf{e}_z^I$-direction. When working with composite structures also the shear stresses become more relevant. Hence, terms up to second order are considered in engineering for the criterions of failure.

7.3 The Plane Linearized Euler–Bernoulli Beam

The Euler–Bernoulli beam assumption is that the cross sections remain orthogonal to the tangent vector of the centerline. In the plane case, the condition is fulfilled when the derivative of the lateral displacement w is related to the cross section rotation α by $w' = \tan\alpha$. For small rotations the constraint condition is written as

$$g = \alpha - w' = 0 \ . \tag{7.29}$$

In contrast to the three-dimensional Euler–Bernoulli beam theories, the plane theory allows to fulfill the additional Euler–Bernoulli constraint directly by a constrained position field of the continuous body and the constraint do not have to be guaranteed afterwards by a set-valued force law on the generalized internal forces. The constraint (7.29) leads to some specialty in the Euler–Bernoulli beam formulation. The embedding (4.10) does not only depend on the position functions $\mathbf{q}$ but also on their derivatives $\mathbf{q}'$ with respect to ν.

Kinematics, virtual work and the boundary value problem

In accordance with (7.29) and (7.5), the constrained displacement field of the Euler–Bernoulli beam is

$$\mathbf{u}((x,y,z),t) = \mathbf{x}(\mathbf{q}(\cdot,t),\mathbf{q}'(\cdot,t))(x,y,z) = \begin{pmatrix} u(x,t) - w'(x,t)z \\ 0 \\ w(x,t) \end{pmatrix} , \tag{7.30}$$

with the clamping condition $\mathbf{u}((0,y,z),t) = 0$. We recognize the generalized position functions $\mathbf{q}$ with the longitudinal and transverse displacements u and w, respectively. The strain of the continuous body

$$\boldsymbol{\varepsilon}(\mathbf{u}) = \begin{pmatrix} u' - w''z & 0 & 0 \\ & 0 & 0 \\ \text{sym.} & & 0 \end{pmatrix}$$

is due to the restricted kinematics formulated in terms of u and w only. Hence, the Euler–Bernoulli assumption leads to a beam with constrained shear deformation. The admissible virtual displacements $\delta\mathbf{x}$ and its corresponding strain $\boldsymbol{\varepsilon}(\delta\mathbf{x})$ with respect to the constrained displacements (7.30) are

$$\delta\mathbf{x} = \begin{pmatrix} \delta u - \delta w' z \\ 0 \\ \delta w \end{pmatrix} , \quad \boldsymbol{\varepsilon}(\delta\mathbf{x}) = \begin{pmatrix} \delta u' - \delta w'' z & 0 & 0 \\ & 0 & 0 \\ \text{sym.} & & 0 \end{pmatrix} . \tag{7.31}$$

Since the clamping is guaranteed, $\delta\mathbf{x}((0,y,z),t) = 0$. The internal virtual work of (7.2) for the admissible virtual strains (7.31) is written using the common split of integration as

$$\delta W^{\text{int}} = \int_0^l \left\{ \delta u' \int_A \sigma_{Ixx} \mathrm{d}A - \delta w'' \int_A z\sigma_{Ixx} \mathrm{d}A \right\} \mathrm{d}x = \int_0^l \{\delta u' N_I - \delta w'' M_I\} \,\mathrm{d}x , \tag{7.32}$$

where the internal generalized forces are defined as in (7.9). Using the same procedure as in (7.10), the constitutive laws for the generalized internal forces are obtained as

$$N_I = EAu' , \quad M_I = -EIw'' . \tag{7.33}$$

Since we treat the Euler–Bernoulli beam in the sense of an induced theory, we can formulate the connection between the generalized internal forces and the impressed stress distribution of the continuous body as

$$\begin{aligned} \sigma_{Ixx} &= E(u' - w''z) = \frac{N_I}{A} + \frac{M_I}{I} z , \\ \sigma_{Ixz} &= 0 . \end{aligned} \tag{7.34}$$

Applying the plane kinematics to (6.5) and using the results of (7.32) , we obtain the weak variational formulation of the plane linearized Euler–Bernoulli beam as

$$\begin{aligned} \delta W = \int_0^l & \{\delta u' N_I - \delta w'' M_I + \delta u(A\rho\ddot{u} - \overline{n}) + \delta w(A\rho\ddot{w} - \overline{q}) \\ & - \delta w'(-I\rho\ddot{w}')\} \,\mathrm{d}x - (\delta u \overline{n}_l + \delta w \overline{q}_l)|_{x=l} = 0 , \quad \forall \delta u, \delta w, t , \end{aligned}$$

where $\delta u(0) = \delta w(0) = \delta w'(0) = 0$ in order to satisfy the clamping boundary condition. The strong variational form is obtained by applying integration by parts, once for the $\delta u'$- and $\delta w'$-terms and twice for the $\delta w''$-terms

$$\begin{aligned}\delta W = \int_0^l &\{\delta u(A\rho\ddot{u} - \overline{n} - N_I') + \delta w(A\rho\ddot{w} - \overline{q} - M_I'' - I\rho\ddot{w}'')\}\,\mathrm{d}x \\ &- (\delta u(\overline{n}_l - N_I) + \delta w(\overline{q}_l - M_I') - \delta w'(-M_I))|_{x=l} = 0\ , \quad \forall \delta u, \delta w, t\ .\end{aligned}$$

Using the common arguments, we obtain the equations of motion of the plane linearized Euler–Bernoulli beam

$$\begin{aligned} A\rho\ddot{u} &= N_I' + \overline{n}\ , \\ A\rho\ddot{w} - I\rho\ddot{w}'' &= M_I'' + \overline{q}\ , \end{aligned} \tag{7.35}$$

together with the kinetic boundary conditions at $x = l$, $\overline{n}_l = N_D(l)$, $\overline{q}_l = M_D'(l)$ and $M_I(l) = 0$ and the kinematic boundary conditions $u(0) = w(0) = w'(0) = 0$. Using the constitutive laws (7.33), the dynamic equations of motion for the Euler–Bernoulli beam take the form

$$\begin{aligned} A\rho\ddot{u} - EAu'' &= \overline{n}\ , \\ EIw'''' + A\rho\ddot{w} - I\rho\ddot{w}'' &= \overline{q}\ . \end{aligned} \tag{7.36}$$

Constraint stresses of the plane Euler–Bernoulli beam

In the equations of motion of the Euler–Bernoulli beam (7.35) we recognize, that no traction force Q_I appears. Nevertheless, it is possible that traction forces $\overline{q}$ are equilibrated. The equilibrium is guaranteed by the constraint stresses and their corresponding internal generalized constraint forces, which in (7.35) drop out, due to the projection on admissible virtual displacements. In order to access these constraint stresses, we introduce non-admissible virtual displacements as done in the previous section. The virtual displacements (7.6), which are admissible for the Timoshenko beam, are non-admissible virtual displacements for the Euler–Bernoulli beam, i.e.

$$\delta\mathbf{x}_{\mathrm{na}} = \begin{pmatrix} \delta u - \delta\alpha z \\ 0 \\ \delta w \end{pmatrix}\ , \quad \boldsymbol{\varepsilon}(\delta\mathbf{x}_{\mathrm{na}}) = \begin{pmatrix} \delta u' - \delta\alpha' z & 0 & \frac{1}{2}(\delta w' - \delta\alpha) \\ & 0 & 0 \\ \text{sym.} & & 0 \end{pmatrix}\ . \tag{7.37}$$

Since the clamping is still guaranteed, $\delta\mathbf{x}_{\mathrm{na}}((0, y, z), t) = 0$. As for the Timoshenko beam, we choose the normal constraint stresses to vanish, i.e. $\sigma_{Cxx} = 0$. Using the vanishing impressed transverse shear stresses (7.34), we rewrite the internal virtual work of the plane linearized Euler–Bernoulli beam according to non-admissible virtual displacements (7.37) as

$$\begin{aligned}\delta W^{\mathrm{int}} &= \int_0^l \left\{\delta u' \int_A \sigma_{Ixx}\mathrm{d}A - \delta\alpha' \int_A z\sigma_{Ixx}\mathrm{d}A + (\delta w' - \delta\alpha)\int_A \sigma_{Cxz}\mathrm{d}A\right\}\mathrm{d}x \\ &= \int_0^l \{\delta u' N_I - \delta\alpha' M_I + (\delta w' - \delta\alpha)Q_C\}\,\mathrm{d}x\ ,\end{aligned}$$

in which we define the *resultant contact shear force* as

$$Q_C(x) := \int_A \sigma_{Cxz}\,\mathrm{d}A \ .$$

By substituting Q_I with Q_C and α with w' in (7.13) and (7.14), this leads to the non-minimal equations of motion

$$\begin{aligned} A\rho\ddot{u} &= N_I' + \overline{n} \ , \\ A\rho\ddot{w} &= Q_C' + \overline{q} \ , \\ -I\rho\ddot{w}' &= M_I' - Q_C \ . \end{aligned}$$

These are exactly the equations of motion which are obtained, when starting with the balance of linear and angular momentum, cf. for the static case Gross et al. (2011). From such a derivation it does not become clear, that within the Euler–Bernoulli beam theory the resultant contact shear forces $Q_C(x)$ are in fact constraint forces. The derivation of the constraint stresses is identical to the Timoshenko beam, hence it holds that

$$\sigma_{Cxx} = 0 \ , \quad \sigma_{Cxz} = \frac{M_I' H_{z_0}}{Ib} \ , \quad \sigma_{Czz} = \frac{M_I''}{Ib^2}\int_A h(z-z_0)H_{z_0}\mathrm{d}A - \frac{A_{z_0}}{A}(\rho A\ddot{w} - \overline{q}) \ .$$

The total stress distribution coincides therefore with the stress distribution of the Timoshenko beam (7.28). For the static case of the clamped beam under end load $\overline{q}_l = P$, the transverse shear stress at z_0 is obtained by

$$\sigma_{xz}(z_0) = \frac{PH_{z_0}}{Ib(z_0)} \ .$$

This is the transverse shear stress formula which is derived in all technical mechanics books and is commonly used for criterions of failure. However, it is seldom mentioned, that also constraint stresses, which cannot be determined uniquely, are considered for criteria of failure.

7.4 The Plane Linearized Kirchhoff Beam

The Kirchhoff beam has an additional inextensibility constraint

$$g = u' = 0 \ ,$$

i.e. that the derivative of longitudinal displacement has to vanish. Since the dynamics in $\mathbf{e}_x^I$-direction is eliminated, the equations of motion of the Kirchhoff beam coincide with the second line of (7.35) and (7.36). The derivation of the equations of motion works analogously to the Euler–Bernoulli beam and is therefore omitted here. The constitutive equations for the generalized internal forces is given as

$$M_I = -EIw'' \ .$$

The relationship between the three-dimensional theory and the internal generalized forces is obtained as

$$\sigma_{Ixx} = E(-w''z) = \frac{M_I}{I} z \,, \quad \sigma_{Ixz} = 0 \,, \quad \sigma_{Izz} = 0 \,.$$

With the same non-admissible virtual displacements as for the Euler–Bernoulli beam and the definition of $N_C := \int_A \sigma_{Cxx} \, \mathrm{d}A$ we obtain the non-minimal equations of motion of the Kirchhoff beam

$$\begin{aligned} A\rho\ddot{u} &= \overline{n} + N_C' \,, \\ A\rho\ddot{w} &= \overline{q} + Q_C' \,, \\ -I\rho\ddot{\alpha} &= M_I' - Q_C \,. \end{aligned}$$

By replacing N_I with N_C in (7.28), we obtain the total stress distribution of the Kirchhoff beam.

In this chapter, we have introduced the well-known equations of motion of the plane classical beam theories as induced theories using the principle of virtual work of a constrained continuous body. The virtual work together with the solution of the equations of motion and non-admissible virtual displacements have enabled us to extract the constraint stresses. It is important to notice, that the constraint stresses have only been extracted up to certain indeterminacy. Doing further assumptions on the constraint stress distribution, the total stress distribution (7.28) consequently has been uniquely determined. The main achievement of this chapter is that the very classical results of the equations of motion and the stress distributions of the plane classical beams are obtained by the principle of virtual work in a purely analytical way.

Chapter 8

Augmented Nonlinear Beam Theories

Augmented nonlinear beams are beams whose constrained position field and insofar whose cross section deformations are more involved than those in the classical theory. In classical theories the deformation of the cross sections are described by six generalized position functions, whose dual kinetic quantities are resultant contact forces and contact couples. Since the balance of linear and angular momentum hold six equations, it is also possible to derive the equations of motion of an induced theory for classical beams from the balance of linear and angular momentum, cf. Simo (1985). Assuming more complex deformation states of the cross sections using more than six generalized position functions, as for instance to describe in-plane deformation or out-of-plane warping, more complex and counterintuitive generalized resultant contact forces do appear. The postulation of the correct intrinsic equations requires much mechanical intuition. Hence, we determine the equations of motion of the nonlinear two-director Cosserat beam and the nonlinear Saint–Venant beam in a concise way in the sense of induced beam theories.

In Section 8.1, we introduce the nonlinear Cosserat beam which is intensely discussed in Naghdi (1980) and Rubin (2000). In Section 8.2, we treat a beam theory with out-of-plane warping, derived by Danielson and Hodges (1988) in its static version as an induced theory. A dynamical version of the Saint–Venant beam is obtained in Simo and Vu-Quoc (1991) as an intrinsic theory. In accordance to Saint–Venants solution of a linear elastic body under torsion, who has recognized the effect of warping fields, we call this theory Saint–Venant beam theory.

8.1 The Nonlinear Cosserat Beam

The Cosserat beam theory goes back to the celebrated work of the Cosserat brothers Cosserat and Cosserat (1909) who developed intrinsic theories for generalized one-, two- and three-dimensional continua founded on an action principle. For further historical information and an alternative derivation of the upcoming equations of motion of the nonlinear two-director Cosserat beam, we refer to the Rubin (2000)[1].

[1]Rubin (2000) induces the equations of motion of the nonlinear Cosserat beam from the balance of angular, linear and averaged linear momentum. Without recognizing, using the balance of averaged linear

Kinematical assumptions

Similar to the treatment in Chapter 5, we first assume at a given instant of time t a placement of the slender body in $\mathbb{E}^3$, at which the body covers the subset $\overline{\Omega}_t \subset \mathbb{E}^3$. We identify the characteristic direction of the slender body with an arbitrarily chosen centerline $\mathbf{r}$ which propagates along the largest expansion of the body. Subsequently, we identify every point of the continuous body in $\overline{\Omega}_t$ with a unique point of the set $\overline{B} := \boldsymbol{\xi}(\cdot,t)^{-1}(\overline{\Omega}_t) \subset \mathbb{R}^3$. Then we choose the body chart θ such that the centerline $\mathbf{r}$ is parametrized by $\theta^3 =: \nu$ only. For a Cosserat beam, we assume the existence of a motion given by the constrained position field

$$\boldsymbol{\xi}(\theta^\alpha,\nu,t) = \mathbf{x}(\mathbf{q}(\cdot,t))(\theta^\alpha,\nu) = \mathbf{r}(\nu,t) + \theta^\alpha \mathbf{d}_\alpha(\nu,t) \;, \tag{8.1}$$

where the generalized position functions $\mathbf{q}(\cdot,t)$ are recognized as $\mathbf{r}(\cdot,t)$, $\mathbf{d}_1(\cdot,t)$ and $\mathbf{d}_2(\cdot,t)$. The centerline is given by the space curve $\mathbf{r}(\cdot,t) = \boldsymbol{\xi}(0,0,\cdot,t)$ and is bounded by its ends $\nu = \nu_1$ and $\nu = \nu_2$ for $\nu_2 > \nu_1$. At every material point ν of the centerline $\mathbf{r}$ two directors $\mathbf{d}_\alpha$ are attached which span the plane cross section of the beam. In contrast to the classical beam, the directors are allowed to deform. Hence, the cross sections remain plane, but in-plane deformation may occur. The current state of the cross section $\boldsymbol{\xi}(\bar{A}(\nu),\nu,t)$ is parametrized by the coordinates $(\theta^1,\theta^2) \in \bar{A}(\nu)$, where $\bar{A}(\nu) := \big\{(\theta^1,\theta^2) \,|\, (\theta^1,\theta^2,\nu) \in \overline{B}\big\}$. In (8.1), we have identified the generalized position functions $\mathbf{q}(\cdot,t)$ with $\mathbf{r}(\cdot,t)$, $\mathbf{d}_1(\cdot,t)$ and $\mathbf{d}_2(\cdot,t)$. Hence, the generalized position functions $\mathbf{q}(\cdot,t)$ evaluated at ν can be considered as a point on the 9-dimensional manifold $\mathbb{E}^3 \times \mathbb{E}^3 \times \mathbb{E}^3$.

A body is modeled as a Cosserat beam, when the in-plane deformation is assumed to be relevant for the motion. We want to mention, that the in-plane deformation is described by the two directors only. According to the classification of Naghdi and Rubin (1984) merely normal cross section extension, tangential shear deformation and normal cross sectional shear deformation may appear.

Since the directors are unconstrained, there is no such kinematical quantity as a rotation. Rotations commonly appear in the context of rigidified and extended objects. According to the constrained position field, the *velocity* and *acceleration* of a material point are introduced by the total time derivative of the constrained position field (8.1) as

$$\dot{\mathbf{x}} = \dot{\mathbf{r}} + \theta^\alpha \dot{\mathbf{d}}_\alpha \;, \qquad \ddot{\mathbf{x}} = \ddot{\mathbf{r}} + \theta^\alpha \ddot{\mathbf{d}}_\alpha \;. \tag{8.2}$$

The partial derivatives of the constrained position field follow straight forward from (8.1)

$$\mathbf{x}_{,\alpha} = \mathbf{d}_\alpha \;, \quad \mathbf{x}' = \mathbf{r}' + \theta^\alpha (\mathbf{d}_\alpha)' \;, \tag{8.3}$$

where the partial derivative with respect to ν is still denoted by a superposed prime $(\cdot)'$. The admissible virtual displacements with respect to the constrained position field (8.1) and the corresponding partial derivatives are

$$\delta\mathbf{x} = \delta\mathbf{r} + \theta^\alpha \delta\mathbf{d}_\alpha \;, \quad \delta\mathbf{x}_{,\alpha} = \delta\mathbf{d}_\alpha \;, \quad \delta\mathbf{x}' = \delta\mathbf{r}' + \theta^\alpha (\delta\mathbf{d}_\alpha)' \;. \tag{8.4}$$

momentum, he applies the principle of virtual work for admissible virtual displacements.

For the formulation of constitutive laws or for the determination of mass densities it is convenient to introduce a special configuration, called *reference configuration.* Let $\mathbf{r}_0$ and $\mathbf{D}_\alpha$ be the reference generalized position functions of $\mathbf{Q}$, then the reference configuration of the beam corresponds to the constrained position field

$$\Xi(\theta^\alpha, \nu) = \mathbf{X}(\mathbf{Q})(\theta^\alpha, \nu) = \mathbf{r}_0(\nu) + \theta^\alpha \mathbf{D}_\alpha(\nu) \ . \tag{8.5}$$

We call the space curve $\mathbf{r}_0 = \Xi(0, 0, \cdot)$ the reference curve of the beam. The directors $\mathbf{D}_\alpha$ describe the reference state of the cross section $\Xi(\bar{A}(\nu), \nu)$ at ν.

Virtual work contribution of internal forces

In the following, we determine the contribution of the virtual work due the admissible virtual displacements (8.4). According to the principle of d'Alembert–Lagrange (4.8), the constraint stresses due to the constrained position field drop out and the boundary value problem of the nonlinear Cosserat beam is obtained.

Applying the derivatives of the admissible virtual displacements (8.4) to the internal virtual work of the continuous body (4.1), we obtain by the common split of the integration the internal virtual work of the Cosserat beam as

$$\delta W^{\text{int}} = \int_{\overline{B}} \mathbf{t}^i \cdot \delta \mathbf{x}_{,i} \, \mathrm{d}^3\theta = \int_{\nu_1}^{\nu_2} \{\delta \mathbf{d}_\alpha \cdot \mathbf{k}^\alpha + \delta \mathbf{r}' \cdot \mathbf{n} + (\delta \mathbf{d}_\alpha)' \cdot \mathbf{m}^\alpha\} \, \mathrm{d}\nu \ . \tag{8.6}$$

Herein, the integrated kinetic quantities $\mathbf{k}^\alpha$, $\mathbf{n}$ and $\mathbf{m}^\alpha$ are the *intrinsic director couples*, the *resultant contact forces* and the *resultant director contact couples* of the current configuration defined by

$$\mathbf{k}^\alpha(\nu, t) = \int_{\bar{A}(\nu)} \mathbf{t}^\alpha \mathrm{d}^2\theta \ , \quad \mathbf{m}^\alpha(\nu, t) = \int_{\bar{A}(\nu)} \theta^\alpha \mathbf{t}^3 \mathrm{d}^2\theta \ , \quad \mathbf{n}(\nu, t) = \int_{\bar{A}(\nu)} \mathbf{t}^3 \mathrm{d}^2\theta \ . \tag{8.7}$$

In order to make the connection to an intrinsic theory, it is possible to introduce an equivalence class of forces. Force distributions in the Euclidean space which have the same intrinsic director couples, the same resultant contact forces and the same director contact couples are considered to be equivalent. The representatives of this equivalence class are then identified with the *internal generalized forces* of an intrinsic Cosserat beam theory which postulates the right-hand side of (8.6) as its internal virtual work of the generalized one-dimensional continuum. By the definition of an equivalence class, we decouple our induced theory from the theory of a constrained three-dimensional continuous body and arrive at an intrinsic theory. It is worth mentioning, that since we have no rotations as kinematical quantities, also no resultant contact couples in the sense of the classical theory appears in the equations of motion of the Cosserat beam.

Virtual work contribution of inertia forces

As for the classical beam, the pullback of the mass distribution ρ_0 with respect to the reference configuration (8.5) allows us to formulate the mass distribution on the domain $\overline{B}$ as

$$\mathrm{d}m = \rho_0 \, G^{1/2} \, \mathrm{d}^3\theta \ , \quad G^{1/2} = \mathbf{X}_{,1} \cdot (\mathbf{X}_{,2} \times \mathbf{X}_{,3}) \ . \tag{8.8}$$

In accordance with the virtual work of (4.2), the accelerations (8.2) and the admissible virtual displacements (8.4), the virtual work of the inertia forces

$$\begin{aligned} \delta W^{\mathrm{dyn}} &= \int_{\overline{B}} \delta \mathbf{x} \cdot \ddot{\mathbf{x}} \, \mathrm{d}m = \int_{\nu_1}^{\nu_2} \left\{ \int_{\bar{A}(\nu)} (\delta \mathbf{r} + \theta^\alpha \delta \mathbf{d}_\alpha) \cdot (\ddot{\mathbf{r}} + \theta^\beta \ddot{\mathbf{d}}_\beta) \rho_0 \, G^{1/2} \, \mathrm{d}^2\theta \right\} \mathrm{d}\nu \\ &= \int_{\nu_1}^{\nu_2} \left\{ \delta \mathbf{r} \cdot A_{\rho_0} \ddot{\mathbf{r}} + \delta \mathbf{d}_\alpha \cdot q^\alpha_{\rho_0} \ddot{\mathbf{r}} + \delta \mathbf{r} \cdot q^\beta_{\rho_0} \ddot{\mathbf{d}}_\beta + \delta \mathbf{d}_\alpha \cdot M^{\alpha\beta}_{\rho_0} \ddot{\mathbf{d}}_\beta \right\} \mathrm{d}\nu \end{aligned} \tag{8.9}$$

is obtained, where in the last line the time independent inertia coefficients

$$\begin{aligned} A_{\rho_0}(\nu) &:= \int_{\bar{A}(\nu)} \rho_0 \, G^{1/2} \, \mathrm{d}^2\theta \ , \quad q^\alpha_{\rho_0}(\nu) := \int_{\bar{A}(\nu)} \theta^\alpha \rho_0 \, G^{1/2} \, \mathrm{d}^2\theta \ , \\ M^{\alpha\beta}_{\rho_0}(\nu) &:= \int_{\bar{A}(\nu)} \theta^\alpha \theta^\beta \rho_0 \, G^{1/2} \, \mathrm{d}^2\theta \ . \end{aligned}$$

are defined. If the centerline coincides with the line of centroids, then the inertia term $q^\alpha_{\rho_0}$ vanishes.

Virtual work contribution of external forces

As for the classical beam theory, we do allow forces $\mathrm{d}\mathbf{f}$ with Dirac-type contributions. Using the common split of integration and using the admissible virtual displacements (8.4), we obtain the external virtual work contribution

$$\delta W^{\mathrm{ext}} = \int_{\overline{B}} \delta \mathbf{x} \cdot \mathrm{d}\mathbf{f} = \int_{[\nu_1, \nu_2]} \{ \delta \mathbf{r} \cdot \mathrm{d}\overline{\mathbf{n}} + \delta \mathbf{d}_\alpha \cdot \mathrm{d}\overline{\mathbf{m}}^\alpha \} \ ,$$

where the *resultant external forces distribution* $\mathrm{d}\overline{\mathbf{n}}$ and the *resultant external director couple distribution* are the integrated quantities

$$\mathrm{d}\overline{\mathbf{n}}(\nu, t) := \int_{\bar{A}(\nu)} \mathrm{d}\mathbf{f} \ , \quad \mathrm{d}\overline{\mathbf{m}}^\alpha(\nu, t) := \int_{\bar{A}(\nu)} \theta^\alpha \mathrm{d}\mathbf{f} \ .$$

With the same equivalence class argument as for the resultant contact forces and director couples, we can identify the resultant external forces and director couples with *external generalized force distributions* of an intrinsic theory. We want to emphasize again, that naturally there appear no couples as in the classical beam theory. Nevertheless, together with constraint conditions on the directors, it is possible to assign couples from the classical theory to a Cosserat beam. As a consequence, at the point of application, the cross section is rigidified. For the sake of brevity, we only allow discontinuities in the force contributions at the boundaries ν_1 and ν_2 and obtain the virtual work contribution for external forces as

$$\delta W^{\mathrm{ext}} = \int_{\nu_1}^{\nu_2} \{ \delta \mathbf{r} \cdot \overline{\mathbf{n}} + \delta \mathbf{d}_\alpha \cdot \overline{\mathbf{m}}^\alpha \} \, \mathrm{d}\nu + \sum_{i=1}^{2} \{ \delta \mathbf{r}(\nu_i) \cdot \overline{\mathbf{n}}_i + \delta \mathbf{d}_\alpha(\nu_i) \cdot \overline{\mathbf{m}}_i \} \ . \tag{8.10}$$

The boundary value problem

Using the principle of virtual work of the continuous body (4.3) with the total stress (4.7), together with the modified virtual work contributions (8.6), (8.9), (8.10) and the principle of d'Alembert–Lagrange (4.8), we obtain to the weak variational formulation

$$\delta W = \int_{\nu_1}^{\nu_2} \Big\{ \delta\mathbf{r} \cdot (A_{\rho_0}\ddot{\mathbf{r}} + q_{\rho_0}^{\beta}\ddot{\mathbf{d}}_\beta - \overline{\mathbf{n}}) + \delta\mathbf{r}' \cdot \mathbf{n} + (\delta\mathbf{d}_\alpha)' \cdot \mathbf{m}^\alpha + \delta\mathbf{d}_\alpha \cdot (\mathbf{k}^\alpha + q_{\rho_0}^{\alpha}\ddot{\mathbf{r}} + M_{\rho_0}^{\alpha\beta}\ddot{\mathbf{d}}_\beta - \overline{\mathbf{m}}^\alpha) \Big\} \mathrm{d}\nu + \sum_{i=1}^{2} \{\delta\mathbf{r} \cdot \overline{\mathbf{n}}_i + \delta\mathbf{d}_\alpha \cdot \overline{\mathbf{m}}_i^\alpha\}|_{\nu=\nu_i} = 0 \;\forall \delta\mathbf{r}, \delta\mathbf{d}_\alpha$$

of the nonlinear Cosserat beam. Using integration by parts, the virtual work is expressed in the form

$$\begin{aligned}\delta W = &- \{\delta\mathbf{r} \cdot (\mathbf{n} + \overline{\mathbf{n}}_1) + \delta\mathbf{d}_\alpha \cdot (\mathbf{m} + \overline{\mathbf{m}}_1^\alpha)\}\,|_{\nu=\nu_1} \\ &- \int_{\nu_1}^{\nu_2} \Big\{ \delta\mathbf{r} \cdot (A_{\rho_0}\ddot{\mathbf{r}} + q_{\rho_0}^{\beta}\ddot{\mathbf{d}}_\beta - \overline{\mathbf{n}} - \mathbf{n}') + \delta\mathbf{d}_\alpha \cdot (\mathbf{k}^\alpha + q_{\rho_0}^{\alpha}\ddot{\mathbf{r}} + M_{\rho_0}^{\alpha\beta}\ddot{\mathbf{d}}_\beta - \overline{\mathbf{m}}^\alpha - (\mathbf{m}^\alpha)') \Big\} \mathrm{d}\nu \\ &+ \{\delta\mathbf{r} \cdot (\mathbf{n} - \overline{\mathbf{n}}_2) + \delta\mathbf{d}_\alpha \cdot (\mathbf{m} + \overline{\mathbf{m}}_2^\alpha)\}\,|_{\nu=\nu_2} = 0 \;\forall \delta\mathbf{r}, \delta\mathbf{d}_\alpha \;,\end{aligned}$$

which corresponds to the strong variational formulation of the Cosserat beam. If the functions in the round brackets are continuous and if the variations of the generalized position functions are smooth enough, then by the Fundamental Lemma of Calculus of Variation, the former terms have to vanish pointwise. This leads to the complete boundary value problem with the equations of motion of the Cosserat beam which are valid for $\nu \in (\nu_1, \nu_2)$

$$\begin{aligned}\mathbf{n}' + \overline{\mathbf{n}} &= A_{\rho_0}\ddot{\mathbf{r}} + q_{\rho_0}^{\beta}\ddot{\mathbf{d}}_\beta \;, \\ (\mathbf{m}^\alpha)' + \overline{\mathbf{m}}^\alpha - \mathbf{k}^\alpha &= M_{\rho_0}^{\alpha\beta}\ddot{\mathbf{d}}_\beta + q_{\rho_0}^{\alpha}\ddot{\mathbf{r}} \;,\end{aligned}$$

together with the boundary conditions $\mathbf{n}(\nu_1) = -\overline{\mathbf{n}}_1$, $\mathbf{m}^\alpha(\nu_1) = -\overline{\mathbf{m}}_1^\alpha$ and $\mathbf{n}(\nu_2) = \overline{\mathbf{n}}_2$, $\mathbf{m}^\alpha(\nu_2) = \overline{\mathbf{m}}_2^\alpha$.

Constitutive law and restrictions on internal forces

In the same spirit as for the classical beam theory, we propose a semi-induced theory for the nonlinear Cosserat beam, where we formulate an elastic generalized constitutive law relating generalized strains and generalized internal forces. Since we do not consider any Cosserat beam with further constraints, which is possible, we omit the subscript $(\cdot)_I$ for the impressed generalized internal forces.

The most basic constitutive law for a nonlinear Cosserat beam is an elastic force law in the sense of (5.28), such that

$$\hat{W}(\nu, t) = W(\gamma_i, (\delta_\alpha)_i, (\epsilon_\alpha)_i) \;,$$

where we have introduced the generalized strain measures

$$\begin{aligned}\gamma_i(\nu,t) &:= \mathbf{e}_i \cdot \mathbf{r}' - \mathbf{e}_i \cdot \mathbf{r}_0' \ ,\\ (\delta_\alpha)_i(\nu,t) &:= \mathbf{e}_i \cdot \mathbf{d}_\alpha - \mathbf{e}_i \cdot \mathbf{D}_\alpha \ ,\\ (\epsilon_\alpha)_i(\nu,t) &:= \mathbf{e}_i \cdot (\mathbf{d}_\alpha)' - \mathbf{e}_i \cdot (\mathbf{D}_\alpha)' \ ,\end{aligned}$$

with $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ being an inertial orthonormal basis of $\mathbb{E}^3$. It can readily be shown, that the chosen generalized strains are compatible with the internal virtual work (8.6) of the induced theory. The variation of the potential W is written as

$$\begin{aligned}\delta W &= \frac{\partial W}{\partial \gamma_i}\mathbf{e}_i \cdot \delta\mathbf{r}' + \frac{\partial W}{\partial (\delta_\alpha)_i}\mathbf{e}_i \cdot \delta\mathbf{d}_\alpha + \frac{\partial W}{\partial (\epsilon_\alpha)_i}\mathbf{e}_i \cdot \delta(\mathbf{d}_\alpha)' \\ &= \mathbf{n} \cdot \delta\mathbf{r}' + \mathbf{k}^\alpha \cdot \delta\mathbf{d}_\alpha + \mathbf{m}^\alpha \cdot (\delta\mathbf{d}_\alpha)' \ ,\end{aligned}$$

where it is summed over repeated latin and over repeated greek indices and the generalized internal forces

$$\mathbf{n} := \frac{\partial W}{\partial \gamma_i}\mathbf{e}_i \ , \quad \mathbf{k}^\alpha := \frac{\partial W}{\partial (\delta_\alpha)_i}\mathbf{e}_i \ , \quad \mathbf{m}^\alpha := \frac{\partial W}{\partial (\epsilon_\alpha)_i}\mathbf{e}_i$$

have been recognized. In accordance with the law of interaction (4.4), from the three-dimensional theory, we induce an additional restriction on our generalized internal forces. Since the symmetry condition (4.5) has to hold pointwise, also its integration over the cross section must be valid, i.e.

$$\int_{\bar{A}(\nu)} \mathbf{x}_{,i} \times \mathbf{t}^i \, \mathrm{d}^2\theta = 0 \quad \forall \nu$$

Using the partial derivatives of the constrained position field (8.3) together with the definition of the generalized internal forces (8.7), we identify a symmetry condition

$$\mathbf{d}_\alpha \times \mathbf{k}^\alpha + \mathbf{r}' \times \mathbf{n} + (\mathbf{d}_\alpha)' \times \mathbf{m}^\alpha = 0 \quad \forall \nu \tag{8.11}$$

for the intrinsic theory. The additional condition (8.11) on the internal generalized forces makes it elaborate to formulate a constitutive law. A very extended treatise on constitutive laws of Cosserat beams is given in the book of Rubin (2000) or in the corresponding publication Rubin (1996). Another treatise on that topic can be found in O'Reilly (1998).

8.2 The Nonlinear Saint–Venant Beam

In all previously discussed beam theories, the cross sections are assumed to remain plane during the motion of the beam. In the work of Saint–Venant, the deformation of homogenous linear elastic prismatic bodies which are loaded only near their ends are investigated. The exact solution of the torsional problem for non-circular cross sections leads to out-of-plane deformation of the cross sections. These solutions suggest that the classical beam theories are inadequate for a three-dimensional analysis of beam-like bodies. In this section, we derive the equations of motion of a beam which describes also out-of-plane warping. Since the warping field is generally related to the investigations of Saint–Venant, we call the following beam theory nonlinear Saint–Venant beam theory.

Kinematical assumptions

For the nonlinear Saint–Venant beam, we assume the following constrained position field:

$$\boldsymbol{\xi}(\theta^\alpha,\nu,t) = \mathbf{x}(\mathbf{q}(\cdot,t))(\theta^\alpha,\nu) = \mathbf{r}(\nu,t) + \theta^\alpha \mathbf{d}_\alpha(\nu,t) + \lambda(\theta^\alpha,\nu)\psi(\nu,t)\mathbf{d}_3(\nu,t)\ , \tag{8.12}$$

where the generalized position functions $\mathbf{q}(\cdot,t)$ are recognized as $\mathbf{r}(\cdot,t)$, $\mathbf{d}_1(\cdot,t)$, $\mathbf{d}_2(\cdot,t)$ and $\psi(\cdot,t)$. The centerline is given by the space curve $\mathbf{r}(\cdot,t) = \mathbf{x}(0,0,\cdot,t)$ and is bounded by its ends $\nu = \nu_1$ and $\nu = \nu_2$ for $\nu_2 > \nu_1$. At every material point ν of the centerline $\mathbf{r}$ a positively oriented orthonormal director triad $(\mathbf{d}_1(\nu,t), \mathbf{d}_2(\nu,t), \mathbf{d}_3(\nu,t))$ is attached which is related to an inertial basis in $\mathbb{E}^3$ by a rotation as introduced in (5.2). Superimposed to this rigid motion of the plane, a point on the cross section is allowed to deform additionally out-of-plane in direction of $\mathbf{d}_3$. This deformation is not arbitrary, but it is induced by an ansatz function $\lambda(\theta^\alpha,\nu)\psi(\nu,t)$ composed by the multiplication of two functions. The geometrical form of the out-of-plane displacement is given by a Saint–Venant *warping function* $\lambda(\theta^\alpha,\nu)$ which depend on the coordinates θ^α and is allowed to vary along the beam ν. The magnitude of the deformation is described by the *warp amplitude* $\psi(\nu,t)$ depending on the coordinate of the characteristic expansion ν only. The warping function is given analytically for simple cross section forms or evaluated by a precomputational step for more complex cross section forms and has to require

$$\int_{\bar{A}(\nu)} \lambda \mathrm{d}^2\theta = \int_{\bar{A}(\nu)} \lambda\theta^\alpha \mathrm{d}^2\theta = 0\ . \tag{8.13}$$

In (8.12) we have identified the generalized position functions $\mathbf{q}(\cdot,t)$ with $\mathbf{r}(\cdot,t)$, $\mathbf{d}_1(\cdot,t)$, $\mathbf{d}_2(\cdot,t)$ and $\psi(\cdot,t)$ and have constrained the directors $\mathbf{d}_1(\cdot,t)$ and $\mathbf{d}_2(\cdot,t)$ by (5.2) to remain orthonormal. Hence, the generalized position functions $\mathbf{q}(\cdot,t)$ evaluated at ν can be considered as a point on the 7-dimensional manifold $\mathbb{E}^3 \times SO(3) \times \mathbb{R}$. The large displacement of the beam is described by the motion of the centerline and the rotation of the cross sections. The out-of-plane warping field, whose magnitude is small compared to the displacements of the centerline, models an additional degree of freedom of the continuum which is especially relevant for torsional problems. Nevertheless, we still restrict in-plane warping, which would be necessary to allow an exact solution for the pure bending problem of a linear elastic continuum.

It is convenient to abbreviate the position vector from the centerline to a material point in the cross section by

$$\boldsymbol{\rho}(\theta^\alpha,\nu) = \mathbf{x} - \mathbf{r} = \theta^\alpha \mathbf{d}_\alpha + \lambda\psi\mathbf{d}_3\ . \tag{8.14}$$

The *velocity* and the *acceleration* fields

$$\begin{aligned}
\dot{\mathbf{x}} &= \dot{\mathbf{r}} + \boldsymbol{\omega} \times \boldsymbol{\rho} + \dot{\psi}\lambda\mathbf{d}_3\ ,\\
\ddot{\mathbf{x}} &= \ddot{\mathbf{r}} + \dot{\boldsymbol{\omega}} \times \boldsymbol{\rho} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}) + 2\,\boldsymbol{\omega} \times (\dot{\psi}\lambda\mathbf{d}_3) + \ddot{\psi}\lambda\mathbf{d}_3
\end{aligned} \tag{8.15}$$

are introduced by taking the total time derivative of the position field (8.12) and using the angular velocity $\boldsymbol{\omega}$ defined in (5.5). By applying the effective curvature (5.4), the partial derivatives of (8.12) are of the form

$$\mathbf{x}_{,\alpha} = \mathbf{d}_\alpha + \lambda_{,\alpha}\,\psi\mathbf{d}_3\ , \quad \mathbf{x}' = \mathbf{r}' + \mathbf{k} \times \boldsymbol{\rho} + (\lambda\psi)'\mathbf{d}_3\ . \tag{8.16}$$

With the virtual rotations (5.6), for the admissible virtual displacement field

$$\delta\mathbf{x} = \delta\mathbf{r} + \delta\boldsymbol{\phi}\times\boldsymbol{\rho} + \delta\psi\lambda\mathbf{d}_3$$

is obtained. In accordance with (5.6) and (8.14), the variation of the position vector $\boldsymbol{\rho}$ is

$$\delta\boldsymbol{\rho} = \delta\boldsymbol{\phi}\times\theta^\alpha\mathbf{d}_\alpha + \delta\psi\lambda\mathbf{d}_3 + \delta\boldsymbol{\phi}\times\lambda\psi\mathbf{d}_3 = \delta\boldsymbol{\phi}\times\boldsymbol{\rho} + \delta\psi\lambda\mathbf{d}_3\ .$$

The variations of the partial derivatives with respect to θ^α

$$\delta\mathbf{x},_\alpha = \delta\boldsymbol{\phi}\times\mathbf{x},_\alpha + \delta\psi\lambda,_\alpha\mathbf{d}_3 \tag{8.17}$$

and the variation of the partial derivative with respect to ν

$$\delta\mathbf{x}' = \delta\mathbf{r}' + \delta\mathbf{k}\times\boldsymbol{\rho} + \mathbf{k}\times\delta\boldsymbol{\rho} + \delta\psi\lambda'\mathbf{d}_3 + \delta\psi'\lambda\mathbf{d}_3 + \delta\boldsymbol{\phi}\times(\lambda\psi)'\mathbf{d}_3 \tag{8.18}$$

follow directly from (8.16).

Virtual work contribution of internal forces

The transformation of the internal force contribution follows closely to the transformation of the internal force contribution of the classical beam. Using (8.17) and (8.18) together with the property of the cross product of (B.2), the internal virtual work density (4.1) can be written as

$$\begin{aligned}\mathbf{t}^i\cdot\delta\mathbf{x},_i = {} & \delta\boldsymbol{\phi}\cdot(\mathbf{x},_\alpha\times\mathbf{t}^\alpha) + \delta\psi\lambda,_\alpha\mathbf{t}^\alpha\cdot\mathbf{d}_3 + \delta\mathbf{r}'\cdot\mathbf{t}^3 + \delta\mathbf{k}\cdot(\boldsymbol{\rho}\times\mathbf{t}^3)\\ & + \mathbf{t}^3\cdot(\mathbf{k}\times\delta\boldsymbol{\rho} + \delta\psi\lambda'\mathbf{d}_3 + \delta\psi'\lambda\mathbf{d}_3 + \delta\boldsymbol{\phi}\times(\lambda\psi)'\mathbf{d}_3)\ .\end{aligned} \tag{8.19}$$

Applying the symmetry condition (4.5), the first term of (8.19) can be written with $\mathbf{x}'$ of (8.16) and together with the property of the cross product of (B.2) the internal virtual work density takes the form

$$\begin{aligned}\mathbf{t}^i\cdot\delta\mathbf{x},_i = {} & -\mathbf{t}^3\cdot(\delta\boldsymbol{\phi}\times\mathbf{r}' + \delta\boldsymbol{\phi}\times(\mathbf{k}\times\boldsymbol{\rho}) + \delta\boldsymbol{\phi}\times(\lambda\psi)'\mathbf{d}_3) + \delta\psi\lambda,_\alpha\mathbf{t}^\alpha\cdot\mathbf{d}_3 + \delta\mathbf{r}'\cdot\mathbf{t}^3\\ & + \delta\mathbf{k}\cdot(\boldsymbol{\rho}\times\mathbf{t}^3) + \mathbf{t}^3\cdot(\mathbf{k}\times\delta\boldsymbol{\rho} + \delta\psi\lambda'\mathbf{d}_3 + \delta\psi'\lambda\mathbf{d}_3 + \delta\boldsymbol{\phi}\times(\lambda\psi)'\mathbf{d}_3)\ .\end{aligned} \tag{8.20}$$

Using (8.2) and rearranging the terms, we manipulate the expression further to

$$\begin{aligned}\mathbf{t}^i\cdot\delta\mathbf{x},_i = {} & \mathbf{t}^3\cdot(\delta\mathbf{r}' - \delta\boldsymbol{\phi}\times\mathbf{r}') + \delta\mathbf{k}\cdot(\boldsymbol{\rho}\times\mathbf{t}^3) + \mathbf{t}^3\cdot(\mathbf{k}\times(\delta\boldsymbol{\phi}\times\boldsymbol{\rho}) + \delta\boldsymbol{\phi}\times(\boldsymbol{\rho}\times\mathbf{k}))\\ & + \delta\psi(\lambda,_\alpha\mathbf{t}^\alpha\cdot\mathbf{d}_3 + \mathbf{t}^3\cdot(\mathbf{k}\times\lambda\mathbf{d}_3) + \mathbf{t}^3\cdot\lambda'\mathbf{d}_3) + \delta\psi'\mathbf{t}^3\cdot\lambda\mathbf{d}_3\ ,\end{aligned}$$

where the term $\delta\boldsymbol{\phi}\times(\lambda\psi)'\mathbf{d}_3$ cancels. The Jacobi identity (B.1) and the skew-symmetry of the cross-product finally leads to

$$\begin{aligned}\mathbf{t}^i\cdot\delta\mathbf{x},_i = {} & \mathbf{t}^3\cdot(\delta\mathbf{r}' - \delta\boldsymbol{\phi}\times\mathbf{r}') + (\boldsymbol{\rho}\times\mathbf{t}^3)\cdot(\delta\mathbf{k} - \delta\boldsymbol{\phi}\times\mathbf{k}) + \delta\psi'\mathbf{t}^3\cdot\lambda\mathbf{d}_3\\ & + \delta\psi\,\mathbf{d}_3\cdot(\lambda,_\alpha\mathbf{t}^\alpha + \lambda\mathbf{t}^3\times\mathbf{k} + \lambda'\mathbf{t}^3)\end{aligned}$$

as the internal virtual work density for admissible virtual displacements. With the usual split of the integration, the internal virtual work of the Saint–Venant beam is represented as

$$\delta W^{\text{int}} = \int_{\nu_1}^{\nu_2} \{\mathbf{n}\cdot(\delta\mathbf{r}' - \delta\boldsymbol{\phi}\times\mathbf{r}') + \mathbf{m}\cdot(\delta\mathbf{k} - \delta\boldsymbol{\phi}\times\mathbf{k}) + \delta\psi' D + \delta\psi B\}\,\mathrm{d}\nu\ . \tag{8.21}$$

Herein, the integrated kinetic quantities

$$\mathbf{n}(\nu,t) := \int_{\bar{\mathrm{A}}(\nu)} \mathbf{t}^3\,\mathrm{d}^2\theta\ ,\quad \mathbf{m}(\nu,t) := \int_{\bar{\mathrm{A}}(\nu)} (\boldsymbol{\rho}\times\mathbf{t}^3)\,\mathrm{d}^2\theta,\quad D(\nu,t) := \mathbf{d}_3\cdot\int_{\bar{\mathrm{A}}(\nu)} \lambda\mathbf{t}^3\mathrm{d}^2\theta\ ,$$
$$B(\nu,t) := \mathbf{d}_3\cdot\int_{\bar{\mathrm{A}}(\nu)} (\lambda_{,\alpha}\,\mathbf{t}^\alpha + \lambda\mathbf{t}^3\times\mathbf{k} + \lambda'\mathbf{t}^3)\mathrm{d}^2\theta\ .$$

are the *resultant contact forces*, the *resultant contact couples*, the *resultant contact bi-moments* and the *resultant contact bi-shears*, respectively, of the current configuration.[2]

Virtual work contribution of inertia forces

As for the Cosserat beam, we choose the mass distribution (8.8). For the manipulation of the inertia terms it is convenient to introduce some abbreviations of integral expressions and their properties. In order that the following transformations do not explode, we consider the centerline to be the line of centroids. According to that choice, together with the requirement (8.13) for the warping function, all terms which are linear in $\boldsymbol{\rho}$ and integrated over the cross section, vanish. The *cross section inertia density* is introduced as

$$\int_{\bar{\mathrm{A}}(\nu)} \tilde{\boldsymbol{\rho}}\tilde{\boldsymbol{\rho}}^{\mathrm{T}}\rho_0\,G^{1/2}\,\mathrm{d}^2\theta = \mathbf{I}_{\rho_0}(\nu,t) + \psi^2\Xi_{\rho_0}(\nu)\mathbf{P}_{\mathbf{d}_3}(\nu)\ ,$$

where $\mathbf{I}_{\rho_0}$ corresponds to the cross section inertia density (5.19) of the classical beam[3] and the projection $\mathbf{P}_{\mathbf{d}_3}$ and the *warping inertia density* are defined as

$$\mathbf{P}_{\mathbf{d}_3}(\nu,t) := \tilde{\mathbf{d}}_3(\nu,t)\tilde{\mathbf{d}}_3(\nu,t)^{\mathrm{T}}\ ,\quad \Xi_{\rho_0}(\nu) := \int_{\bar{A}(\nu)} \lambda^2\rho_0\,G^{1/2}\,\mathrm{d}^2\theta\ .$$

Furthermore, it is convenient to abbreviate the product of the cross section inertia density and the angular velocity by

$$\mathbf{h}(\nu,t) := (\mathbf{I}_{\rho_0}(\nu,t) + \psi^2(\nu,t)\Xi_{\rho_0}(\nu)\mathbf{P}_{\mathbf{d}_3}(\nu,t))\boldsymbol{\omega}(\nu,t)\ .$$

By considering the derivation in (5.21) and using the same arguments for the projection $\mathbf{P}_{\mathbf{d}_3}$, it can easily be shown that

$$\dot{\mathbf{h}} = (\mathbf{I}_{\rho_0} + \psi^2\Xi_{\rho_0}\mathbf{P}_{\mathbf{d}_3})\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}\times(\mathbf{I}_{\rho_0} + \psi^2\Xi_{\rho_0}\mathbf{P}_{\mathbf{d}_3})\boldsymbol{\omega} + 2\psi\dot{\psi}\Xi_{\rho_0}\mathbf{P}_{\mathbf{d}_3}\boldsymbol{\omega}\ .$$

[2] Since Danielson and Hodges (1988) introduce the constrained position field with a warping function which is constant along ν, the λ'-term in the resultant contact bi-shear vanishes in their derivation.

[3] Notice, that within the classical theory $\boldsymbol{\rho} = \theta^\alpha\mathbf{d}_\alpha$.

With the admissible virtual displacements (8.12) and the accelerations (8.15) of the restricted kinematics, the virtual work contribution of the inertia forces as

$$\delta W^{\mathrm{dyn}} = \int_{\overline{\mathrm{B}}} \delta\mathbf{x}\cdot\ddot{\mathbf{x}}\,\mathrm{d}m = \int_{\overline{\mathrm{B}}} \Big\{(\delta\mathbf{r}+\delta\boldsymbol{\phi}\times\boldsymbol{\rho}+\delta\psi\lambda\mathbf{d}_3)\cdot(\ddot{\mathbf{r}}+\dot{\boldsymbol{\omega}}\times\boldsymbol{\rho}+ \\ +\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{\rho})+2\,\boldsymbol{\omega}\times(\dot{\psi}\lambda\mathbf{d}_3)+\ddot{\psi}\lambda\mathbf{d}_3)\Big\}\,\rho_0\,G^{1/2}\,\mathrm{d}^3\theta$$

is obtained. For the sake of clarity, the inertia terms are treated separately for all the variations $\delta\mathbf{r}$, $\delta\boldsymbol{\phi}$ and $\delta\psi$. Since the terms linear in $\boldsymbol{\rho}$ integrated over the cross section vanish, the inertia forces in $\delta\mathbf{r}$-direction are of the form

$$\int_{\overline{B}} \delta\mathbf{r}\cdot\ddot{\mathbf{x}}\,\mathrm{d}m = \int_{\nu_1}^{\nu_2} \delta\mathbf{r}\cdot A_{\rho_0}\ddot{\mathbf{r}}\mathrm{d}\nu\ , \tag{8.22}$$

where the cross section mass density (5.17) has been used. The skew-symmetry of the cross product, together with (B.2) and (B.5), implies the $\delta\boldsymbol{\phi}$-terms of the inertia forces as

$$\begin{aligned}\int_{\overline{B}}(\delta\boldsymbol{\phi}\times\boldsymbol{\rho})\cdot\ddot{\mathbf{x}} &= \int_{\overline{B}}\Big\{\delta\boldsymbol{\phi}\cdot(\tilde{\boldsymbol{\rho}}\tilde{\boldsymbol{\rho}}^{\mathrm{T}}\dot{\boldsymbol{\omega}}+\tilde{\boldsymbol{\omega}}\tilde{\boldsymbol{\rho}}\tilde{\boldsymbol{\rho}}^{\mathrm{T}}\boldsymbol{\omega}+2\psi\dot{\psi}\lambda^2\mathbf{P}_{\mathbf{d}_3}\boldsymbol{\omega})\Big\}\,\rho_0G^{1/2}\mathrm{d}^3\theta \\ &= \int_{\nu_1}^{\nu_2}\delta\boldsymbol{\phi}\cdot\dot{\mathbf{h}}\,\mathrm{d}\nu\ .\end{aligned} \tag{8.23}$$

For the $\delta\psi$-terms all terms which are linear in λ and those who are orthogonal to $\mathbf{d}_3$ drop out. Using (B.3) and (B.4), the inertia contribution is of the form

$$\begin{aligned}\int_{\overline{B}}\delta\psi\lambda\mathbf{d}_3\cdot\ddot{\mathbf{x}} &= \int_{\nu_1}^{\nu_2}\Big\{\delta\psi\mathbf{d}_3\cdot\Xi_{\rho_0}(\ddot{\psi}\mathbf{d}_3+\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\psi\mathbf{d}_3))\Big\}\,\mathrm{d}\nu \\ &\overset{(B.3)}{=} \int_{\nu_1}^{\nu_2}\Big\{\delta\psi\Xi_{\rho_0}\Big(\ddot{\psi}-\psi\left((\boldsymbol{\omega}\cdot\boldsymbol{\omega})-(\boldsymbol{\omega}\cdot\mathbf{d}_3)^2\right)\Big)\Big\}\,\mathrm{d}\nu \\ &\overset{(B.4)}{=} \int_{\nu_1}^{\nu_2}\Big\{\delta\psi\Xi_{\rho_0}\Big(\ddot{\psi}-\psi(\boldsymbol{\omega}\times\mathbf{d}_3)^2\Big)\Big\}\,\mathrm{d}\nu\ .\end{aligned} \tag{8.24}$$

According to (8.22) – (8.24), the virtual work contribution

$$\delta W^{\mathrm{dyn}} = \int_{\nu_1}^{\nu_2}\Big\{\delta\mathbf{r}\cdot A_{\rho_0}\ddot{\mathbf{r}}+\delta\boldsymbol{\phi}\cdot\dot{\mathbf{h}}+\delta\psi\Xi_{\rho_0}\Big(\ddot{\psi}-\psi(\boldsymbol{\omega}\times\mathbf{d}_3)^2\Big)\Big\}\,\mathrm{d}\nu \tag{8.25}$$

can be written in compact form.

Virtual work of external forces

As for the classical beam theory, we do allow forces $\mathrm{d}\mathbf{f}$ with Dirac-type contributions. Using the usual split of the integration together with the virtual displacements (8.2), we obtain the external virtual work contribution

$$\delta W^{\mathrm{ext}} = \int_{\overline{B}}\delta\mathbf{x}\cdot\mathrm{d}\mathbf{f} = \int_{[\nu_1,\nu_2]}\Big\{\delta\mathbf{r}\cdot\mathrm{d}\overline{\mathbf{n}}+\delta\boldsymbol{\phi}\cdot\mathrm{d}\overline{\mathbf{m}}+\delta\psi\mathrm{d}\overline{D}\Big\}\ ,$$

where the generalized external force distributions

$$\mathrm{d}\overline{\mathbf{n}} := \int_{\bar{A}(\nu)} \mathrm{d}\mathbf{f}\ , \quad \mathrm{d}\overline{\mathbf{m}} := \int_{\bar{A}(\nu)} \boldsymbol{\rho} \times \mathrm{d}\mathbf{f}\ , \quad \mathrm{d}\overline{D} := \mathbf{d}_3 \cdot \int_{\bar{A}(\nu)} \lambda \mathrm{d}\mathbf{f}\ ,$$

have been recognized. For the sake of brevity, we only allow discontinuities in the force contributions at the boundaries ν_1 and ν_2 and obtain the virtual work contribution for external forces as

$$\delta W^{\mathrm{ext}} = \int_{\nu_1}^{\nu_2} \left\{\delta\mathbf{r} \cdot \overline{\mathbf{n}} + \delta\boldsymbol{\phi} \cdot \overline{\mathbf{m}} + \delta\psi\overline{D}\right\} \mathrm{d}\nu + \sum_{i=1}^{2} \left\{\delta\mathbf{r} \cdot \overline{\mathbf{n}}_i + \delta\boldsymbol{\phi} \cdot \overline{\mathbf{m}}_i + \delta\psi\overline{D}_i\right\} |_{\nu=\nu_i}\ . \quad (8.26)$$

The boundary value problem

Using the principle of virtual work of the continuous body (4.3) with the total stress (4.7), together with the modified virtual work contributions (8.21), (8.25) and (8.26), we obtain the weak variational formulation

$$\begin{aligned}\delta W = \int_{\nu_1}^{\nu_2} &\{\mathbf{n} \cdot (\delta\mathbf{r}' - \delta\boldsymbol{\phi} \times \mathbf{r}') + \mathbf{m} \cdot (\delta\mathbf{k} - \delta\boldsymbol{\phi} \times \mathbf{k}) + \delta\mathbf{r} \cdot (A_{\rho_0}\ddot{\mathbf{r}} - \overline{\mathbf{n}}) \\ &+\delta\boldsymbol{\phi} \cdot (\overline{\mathbf{m}} - \dot{\mathbf{h}}) + \delta\psi' D + \delta\psi \left(\Xi_{\rho_0}\left(\ddot{\psi} - \psi(\boldsymbol{\omega} \times \mathbf{d}_3)^2\right) + B - \overline{D}\right)\Big\}\, \mathrm{d}\nu \\ &+ \sum_{i=1}^{2} \left\{\delta\mathbf{r} \cdot \overline{\mathbf{n}}_i + \delta\mathbf{d}_\alpha \cdot \overline{\mathbf{m}}_i^\alpha + \delta\psi\overline{D}_i\right\}\Big|_{\nu_i} = 0 \quad \forall \delta\mathbf{r}, \delta\boldsymbol{\phi}, \delta\psi\end{aligned}$$

of the nonlinear Saint–Venant beam. Applying the identity (5.11) and integration by parts, the virtual work is expressed in the form

$$\begin{aligned}\delta W = &- \left\{\delta\mathbf{r} \cdot (\mathbf{n} + \overline{\mathbf{n}}_1) + \delta\boldsymbol{\phi} \cdot (\mathbf{m} + \overline{\mathbf{m}}) + \delta\psi(D + \overline{D}_1)\right\} |_{\nu=\nu_1} \\ &+ \int_{\nu_1}^{\nu_2} \Big\{\delta\mathbf{r} \cdot (A_{\rho_0}\ddot{\mathbf{r}} - \overline{\mathbf{n}} - \mathbf{n}') + \delta\boldsymbol{\phi} \cdot (\dot{\mathbf{h}} - \overline{\mathbf{m}} - \mathbf{m}' - \mathbf{r}' \times \mathbf{n}) + \\ &+\delta\psi \left(\Xi_{\rho_0}\left(\ddot{\psi} - \psi(\boldsymbol{\omega} \times \mathbf{d}_3)^2\right) + B - D' - \overline{D}\right)\Big\}\, \mathrm{d}\nu \\ &+ \left\{\delta\mathbf{r} \cdot (\mathbf{n} - \overline{\mathbf{n}}_1) + \delta\boldsymbol{\phi} \cdot (\mathbf{m} - \overline{\mathbf{m}}) + \delta\psi(D - \overline{D}_1)\right\} |_{\nu=\nu_2} = 0\ \forall \delta\mathbf{r}, \delta\boldsymbol{\phi}, \delta\psi\ ,\end{aligned}$$

which corresponds to the strong variational formulation of the Saint–Venant beam. For the common arguments of calculus of variations, this leads to the complete boundary value problem with the equations of motion of the nonlinear Saint–Venant beam which are valid for $\nu \in (\nu_1, \nu_2)$

$$\begin{aligned}\mathbf{n}' + \overline{\mathbf{n}} &= A_{\rho_0}\ddot{\mathbf{r}}\ , \\ \mathbf{m}' + \mathbf{r}' \times \mathbf{n} + \overline{\mathbf{m}} &= \dot{\mathbf{h}}\ , \\ D' - B + \overline{D} &= \Xi_{\rho_0}\left(\ddot{\psi} - \psi(\boldsymbol{\omega} \times \mathbf{d}_3)^2\right)\ ,\end{aligned}$$

together with the boundary conditions $\mathbf{n}(\nu_1) = -\overline{\mathbf{n}}_1$, $\mathbf{m}(\nu_1) = -\overline{\mathbf{m}}_1$, $D(\nu_1) = -\overline{D}_1$ and $\mathbf{n}(\nu_2) = \overline{\mathbf{n}}_2$, $\mathbf{m}(\nu_2) = \overline{\mathbf{m}}_2$, $D(\nu_2) = \overline{D}_2$.

Constitutive laws

In comparison with the classical beam formulation, two new scalar quantities appear additionally. Thus, the generalized strain measures (5.29) and (5.30) are completed by ψ and ψ'. A very straight forward elastic potential is

$$\hat{W}(\nu) = W(\gamma_i, \kappa_i, \psi', \psi) \; ,$$

The variation of the elastic potential leads to the internal virtual work

$$\begin{aligned}\delta W &= \frac{\partial W}{\partial \gamma_i} \cdot (\delta \mathbf{r}' \cdot \mathbf{d}_i + \mathbf{r}' \cdot \delta \mathbf{d}_i) + \frac{\partial W}{\partial k_i} \mathbf{d}_i \cdot \delta k_j \mathbf{d}_j + \frac{\partial W}{\partial \psi} \delta \psi + \frac{\partial W}{\partial \psi'} \delta \psi' \\ &= \mathbf{n} \cdot (\delta \mathbf{r}' - \delta \boldsymbol{\phi} \times \mathbf{r}') + \mathbf{m} \cdot (\delta \mathbf{k} - \boldsymbol{\phi} \times \mathbf{k}) + B \delta \psi + D \delta \psi' \; ,\end{aligned}$$

where we have recognized

$$\mathbf{n} := \frac{\partial W}{\partial \gamma_i} \mathbf{d}_i \; , \quad \mathbf{m} := \frac{\partial W}{\partial k_i} \mathbf{d}_i \; , \quad B := \frac{\partial W}{\partial \psi} \; , \quad D := \frac{\partial W}{\partial \psi'} \; .$$

For an explicit formulation of a constitutive law we refer to Simo and Vu-Quoc (1991).

Chapter 9

Conclusions and Outlook

The thesis has been divided into two parts. Whereas in the first part questions on the foundations of continuum mechanics are discussed, the second part applies the obtained theory to induce a number of different beam theories. To retain the purpose that the two parts may be read independently, the conclusions and the outlook of both are given in separate sections.

9.1 On the Foundations of Continuum Mechanics

In Part I of this thesis differential geometric concepts and their application to mechanical objects have been discussed. Thereby, an intrinsic differential geometric setting of a continuous body has been obtained. In the sense of analytical mechanics, the space of forces of a continuous body is defined as the set of linear functionals on the space of virtual displacements. An affine connection on the physical space which induces a covariant derivative on the space of virtual displacements allows for a non-unique representation of forces by vector and tensor valued measures. Classically, the vector and tensor valued measures represent the external and the internal forces, respectively. How internal and external forces interact is postulated in the principle of virtual work.

The scientific merits of the first part of the thesis can be summarized as follows:

– In this thesis the spatial virtual displacement field has been defined as the infinitesimal generator of a smooth global flow on the physical space. The virtual displacement field has then been defined as the pullback section of the spatial virtual displacement field with respect to the configuration of the body. Using the isomorphism between the tangent space of the configuration manifold and the set of pullback sections, the virtual displacement field has been identified with an element of the tangent space of the configuration manifold.

– A definition of the covariant derivative of a pullback section induced by an affine connection on the 'target'-manifold has been introduced. Furthermore, its local representation has been shown in this thesis.

- The forces of a continuous body and their representations have been obtained in the sense of duality as proposed by Segev. The interaction between the different classes of forces, i.e. internal and external forces, has then been introduced in this thesis by postulating a virtual work principle which is in accordance with the first gradient theory of Germain.

- A split of the variational stress into a tensor density, classically denoted as the stress tensor, and a volume element of the body has been introduced. This point of view on the classical stress, given in this thesis, clarifies the non-tensorial transformation rules between the classical stresses as e.g. the Cauchy or 1st Piola-Kirchhoff stress.

- By introducing rigidifying virtual displacements as Killing vector fields, we have introduced a concept to define internal forces in an intrinsic setting. A proof of the symmetry condition of the stress under certain regularity assumptions still needs to be given.

The author is aware that rigorous proofs for some of the statements are missing. The challenging task of an intrinsic differential geometric description of continuum mechanics is the interaction between infinite dimensional geometry, measure theory and functional analysis. The combination of all these mathematical topics forms a rather modern research field in pure mathematics. For a concise formulation of an intrinsic theory, first, the required mathematical framework has to be gathered and prepared. The following open questions and tasks are identified:

- The topology of the infinite dimensional manifold of embeddings and its mechanical interpretation have to be discussed. The possibilities to relax the continuity assumptions of the embeddings, such that the manifold structure of the set of configurations is preserved, have to be analyzed. The corresponding admissible force representatives have to be studied. For instance, for piecewise continuous virtual displacement fields it is assumed that traction forces within the body can be described.

- A complete proof of Theorem 2.3 has to be given.

- The representation theorem of $C^1(\kappa^* T\mathcal{S})^*$-forces has to be proved. A representation theorem for relaxed continuity assumptions must be discussed.

- The symmetry condition of the variational stress for rigidifying virtual displacements for the Euclidean three-space as physical space has to be derived.

- It has to be shown, that the classical stresses, as e.g. Cauchy stress, Kirchhoff stress or Biot stress, arise from the variational stress by choosing distinct configurations, volume elements and coordinate representations.

- Using Stokes' theorem on manifolds and the traction stress of Segev (2013) the strong variational form of the continuous body has to be derived.

– For hyperelastic materials the internal virtual work has to be obtained by a variation of an internal energy. Together with an intrinsic differential geometric formulation of variational calculus this leads directly to the theory of covariant elasticity.

Comparing the equilibrium equation of a linear elastic bar of infinite length with the equation of motion of a particle moving in a one-dimensional physical space, one recognizes that the differential equations coincide. Moreover, the concept of stress and the concept of linear momentum are the same from a geometric point of view. Furthermore, driven by the insights of general relativity, a vision of an intrinsic differential geometric description of classical continuum dynamics emerges. In such a description, the body is a space-time continuum which is mapped to the physical space modeled as a vector bundle. The vector bundle consists of a one-dimensional Riemannian base manifold, modeling the time, together with a typical fiber of a three-dimensional Euclidean vector space, modeling the real space. An affine connection on the vector bundle defines the inertial forces and corresponds to the choice of an inertial frame in classical mechanics. Such an invariant theory constitutes the basis of classical mechanics and the virtual work, or more appropriate the virtual action is 'The Invariant Quantity' of this theory.

9.2 Beam Theories

Starting from the principle of virtual work of a continuous body and by considering a beam as a continuous body with a constrained position field, induced beam theories have been obtained in this thesis in a systematic and concise way. A constrained position field is guaranteed by a constraint stress field, whose constitutive law can only be formulated in variational form with the principle of d'Alembert–Lagrange. Due to the principle of d'Alembert–Lagrange, the constraint stress field vanishes for all admissible virtual displacements, i.e. for variations of the constrained position field. The most convenient approach for a systematic treatment of an arbitrary induced beam theory is therefore given by the principle of virtual work of a continuous body. This emphasizes the importance of the principle of virtual work in the field of structural mechanics.

The main contributions of the second part of the thesis can be summarized as follows:

– As an example of how specific theories are induced from a general mechanical theory, beam theories have been induced in Part II from the theory of a continuous body. A variational formulation of the general theory, together with the principle of d'Alembert–Lagrange allows us to induce the specific beam theory merely by the choice of the constrained position field. This remains in the spirit, as discussed in Section 1.2, that the kinematics defines which kind of forces we may expect.

– Assuming various constrained position fields together with further constraint conditions on the level of generalized position fields, several beam theories have been induced in this thesis. The classical theories of Timoshenko, Euler–Bernoulli and Kirchhoff have been formulated in a nonlinear and linearized setting. Additionally, the derivation of augmented beam theories, such as the two-director Cosserat-beam, and the Saint–Venant beam, emphasizes the systematic procedure of an induced beam theory obtained by the principle of virtual work of the continuous body.

- The classical plane linearized beam theory is generally applied in the theory of strength of materials. By the application of non-admissible virtual displacements, the total stress distribution of the constrained continuous body is obtained. In this thesis, this analytical procedure has led to the insight that the constraint stress distribution cannot be determined uniquely and certain assumptions on the constraint stress distribution have to be taken. Hence, the total stress distribution used for the determination of failure criteria in technical mechanics is not unique.

- A valuable by-product of the formulation using the principle of virtual work is, that the weak variational form of all beam theories have been derived automatically in this thesis. A finite element discretization, as in Eugster et al. (2014), follows in a very natural way by further constraining the generalized position fields such that those can be described by finitely many degrees of freedom. Thus, also the numerical discretization fits into the concept of induced theories.

The above insight brings forth a wealth of new open questions which are to be addressed in further research. Most urgent is to derive the nonlinear theories in this thesis not only as semi-induced, but also as fully induced theories, i.e. using constitutive laws of a continuous body to determine the constitutive laws of the internal generalized forces. The interaction between the constrained position field and the constitutive law has to be discussed. For instance in the classical beam theory, the Poisson effect modeled within a three-dimensional linear elastic and isotropic material law leads only to additional constraint stresses. Especially for a numerical treatment of an induced theory, the constrained position field and the material law have to harmonize. Another topic is to derive more elaborate augmented beam theories which include in- and out-of-plane warping as proposed by Bauchau and Han (2014) and Papes (2012). Lastly, experimental work is needed to find out which beam theory is most successfully applied to a given application problem. There is no such thing as 'The Beam', every application asks for its own solution.

In the same spirit as for the beam, we are able to describe more complex structural elements as constrained three-dimensional continuous bodies:

- An *incompressible continuum* is a classical continuum with three characteristic directions with pointwise incompressibility constraints.

- A *shell* is a continuous body with two characteristic directions where the irrelevant deformations are eliminated by a constrained position field.

- A *rigid body* is a continuous body with no characteristic direction and pointwise rigidity constraints.

Hence, one should be able to induce all specific theories such as rigid body mechanics, multibody mechanics, beam theories and shell theories from the theory of a continuous body. Such a unification of mechanics will not only bring more clarity in the scientific field but will also allow to develop more complex structural elements and will lead to more efficient numerical discretizations.

Appendix A

Multilinear Algebra

This chapter presents concepts from multilinear algebra based on the basic properties of finite dimensional vector spaces and linear maps. The primary aim of the chapter is to give a concise introduction to alternating tensors which are necessary to define differential forms on manifolds. Many of the stated definitions and propositions can be found in Lee (2012), Chapters 11, 12 and 14. Some definitions and propositions are complemented by short and simple examples.

First, in Section A.1 dual and bidual vector spaces are discussed. Subsequently, in Section A.2 – A.4, tensors and alternating tensors together with operations such as the tensor and wedge product are introduced. Lastly, in Section A.5, the concepts which are necessary to introduce the wedge product are summarized in eight steps.

A.1 The Dual Space

Let V be a real vector space of finite dimension $\dim V = n$. Let $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ be a basis of V. Then every $\mathbf{v} \in V$ can be uniquely represented as a linear combination

$$\mathbf{v} = v^i \mathbf{e}_i \ , \tag{A.1}$$

where summation convention over repeated indices is applied. The coefficients $v^i \in \mathbb{R}$ are referred to as *components* of the vector $\mathbf{v}$.

Throughout the whole chapter, only finite dimensional real vector spaces, typically denoted by V, are treated. When not stated differently, summation convention is applied.

Definition A.1 (Dual Space)**.** The *dual space of V* is the set of real-valued linear functionals

$$V^* := \{\boldsymbol{\omega} \colon V \to \mathbb{R} : \boldsymbol{\omega} \text{ linear}\} \ . \tag{A.2}$$

The elements of the dual space V^* are called *linear forms* on V.

The dual space, equipped with pointwise addition and scalar multiplication is again a real vector space.

Proposition A.1. *Given any basis* $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ *for* V*, let* $\boldsymbol{\varepsilon}^1, \dots, \boldsymbol{\varepsilon}^n \in V^*$ *be the linear forms defined by*

$$\boldsymbol{\varepsilon}^i(\mathbf{e}_j) := \delta^i_j \;, \tag{A.3}$$

where δ^i_j *is the Kronecker delta symbol defined by*

$$\delta^i_j := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \;. \end{cases} \tag{A.4}$$

Then $(\boldsymbol{\varepsilon}^1, \dots, \boldsymbol{\varepsilon}^n)$ *is a basis for* V^**, called the* dual basis to $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ *and consequently* $\dim V^* = \dim V$*. Any linear form* $\boldsymbol{\omega}$ *can be uniquely represented as a linear combination*

$$\boldsymbol{\omega} = \omega_i \, \boldsymbol{\varepsilon}^i \;, \tag{A.5}$$

with components $\omega_i = \boldsymbol{\omega}(\mathbf{e}_i)$*.*

Proof. We need to show that $(\boldsymbol{\varepsilon}^1, \dots, \boldsymbol{\varepsilon}^n)$ (i) spans the dual space V^* and (ii) is linearly independent. Let $\mathbf{v} \in V$ and $\boldsymbol{\omega} \in V^*$.

(i) Due to the linearity of $\boldsymbol{\omega}$ we have

$$\boldsymbol{\omega}(\mathbf{v}) \overset{\text{(A.1)}}{=} \boldsymbol{\omega}(v^i \mathbf{e}_i) \overset{\text{(A.2)}}{=} v^i \boldsymbol{\omega}(\mathbf{e}_i) \;. \tag{A.6}$$

Writing the linear form as a linear combination (A.5) and applying it to $\mathbf{v}$, it follows by linearity of the linear form and the definition of the dual basis (A.3) that

$$\boldsymbol{\omega}(\mathbf{v}) = \omega_i \boldsymbol{\varepsilon}^i(v^j \mathbf{e}_j) \overset{\text{(A.2)}}{=} \omega_i v^j \boldsymbol{\varepsilon}^i(\mathbf{e}_j) \overset{\text{(A.3)}}{=} \omega_i v^j \delta^i_j \overset{\text{(A.4)}}{=} \omega_i v^i \;. \tag{A.7}$$

Let $\omega_i = \boldsymbol{\omega}(\mathbf{e}_i)$. Then comparison of (A.6) and (A.7) proves that $(\boldsymbol{\varepsilon}^1, \dots, \boldsymbol{\varepsilon}^n)$ spans V^*.

(ii) To show that $(\boldsymbol{\varepsilon}^1, \dots, \boldsymbol{\varepsilon}^n)$ is linearly independent, suppose some linear combination equals zero, i.e. $\boldsymbol{\omega} = \omega_i \boldsymbol{\varepsilon}^i = 0$. Applying to both sides an arbitrary vector $\mathbf{v} \in V$, it follows by the computation of (A.7) that

$$\forall \mathbf{v} \in V \colon \boldsymbol{\omega}(\mathbf{v}) = 0 \;\Rightarrow\; \forall\, v^i \in \mathbb{R} \colon v^i \omega_i = 0 \;\Rightarrow\; \omega_i = 0 \;.$$

Thus, the only linear combination of elements of $(\boldsymbol{\varepsilon}^1, \dots, \boldsymbol{\varepsilon}^n)$ that sums to zero is the trivial one. This proves the linear independency of $(\boldsymbol{\varepsilon}^1, \dots, \boldsymbol{\varepsilon}^n)$. $\square$

The application of a linear form $\boldsymbol{\omega} \in V^*$ on a vector $\mathbf{v} \in V$ is called the *duality pairing* and is expressed in components as in (A.7).

Since the dual space V^* is also a vector space, we may consider the dual space of V^*, called the *bidual space* $V^{**} := (V^*)^*$. For each vector space V, there exists a linear isomorphism

$$\phi \colon V \to V^{**} \;,\; \mathbf{v} \mapsto \phi_{\mathbf{v}} \colon \forall\, \boldsymbol{\omega} \in V^* \;,\; \phi_{\mathbf{v}}(\boldsymbol{\omega}) = \boldsymbol{\omega}(\mathbf{v}) \;.$$

Hence, we identify the vectors of the bidual space V^{**} naturally with the vectors of the vector space V. For convenience, we suppress the function ϕ in our notation and write

$$\phi_{\mathbf{v}}(\boldsymbol{\omega}) =: \mathbf{v}(\boldsymbol{\omega}) \;.$$

The duality pairing between the identified bidual basis $\mathbf{e}_i \in V^{**}$ and the dual basis $\boldsymbol{\varepsilon}^j \in V^*$ is evaluated as

$$\mathbf{e}_i(\boldsymbol{\varepsilon}^j) = \delta^j_i \;.$$

A.2 Multilinear Forms and Tensors

Definition A.2 (Multilinear Form, Tensor)**.** Suppose $V_1, \dots, V_k$ are vector spaces. A map $\mathbf{F}\colon V_1 \times \cdots \times V_k \to \mathbb{R}$ is said to be *multilinear*, if it is linear in each argument, i.e. for any $i \in \{1, \dots, k\}$, $\mathbf{v}_j \in V_j$, $\mathbf{w}_i \in V_i$, $a, b \in \mathbb{R}$

$$\mathbf{F}(\mathbf{v}_1, \dots, a\mathbf{v}_i + b\mathbf{w}_i, \dots, \mathbf{v}_k) = a\mathbf{F}(\mathbf{v}_1, \dots, \mathbf{v}_i, \dots, \mathbf{v}_k) + b\mathbf{F}(\mathbf{v}_1, \dots, \mathbf{w}_i, \dots, \mathbf{v}_k)\ .$$

We refer to multilinear $\mathbf{F}$ as *multilinear form* or *tensor of rank* k.[1] The set of such multilinear forms is denoted by $L(V_1, \dots, V_k; \mathbb{R})$.

Example A.1. Let V and U be vector spaces, then the multilinear form

$$\mathbf{F}\colon V^* \times V \times U \to \mathbb{R}, \quad (\boldsymbol{\omega}, \mathbf{v}, \mathbf{u}) \mapsto \mathbf{F}(\boldsymbol{\omega}, \mathbf{v}, \mathbf{u})$$

is a tensor of rank 3.

Example A.2. Let $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ be the basis vectors of V and $(\boldsymbol{\varepsilon}^1 \dots, \boldsymbol{\varepsilon}^n)$ the corresponding dual basis. Then the basis vectors

$$\begin{aligned} \boldsymbol{\varepsilon}^j\colon V \to \mathbb{R}\,,\ \mathbf{v} \mapsto \boldsymbol{\varepsilon}^j(\mathbf{v}) = v^j\ ,\\ \mathbf{e}_i\colon V^* \to \mathbb{R}\,,\ \boldsymbol{\omega} \mapsto \mathbf{e}_i(\boldsymbol{\omega}) = \omega_i\ , \end{aligned}$$

are tensors of rank 1, projecting the vector $\mathbf{v}$ and the covector $\boldsymbol{\omega}$ to their j-th and i-th component, respectively.

Definition A.3 (Tensor Product)**.** Let $V_1, \dots, V_k$, $W_1, \dots, W_l$ be vector spaces, and let $\mathbf{F} \in L(V_1, \dots, V_k; \mathbb{R})$ and $\mathbf{G} \in L(W_1, \dots, W_l; \mathbb{R})$. Define a function

$$\mathbf{F} \otimes \mathbf{G}\colon V_1 \times \cdots \times V_k \times W_1 \times \cdots \times W_l \to \mathbb{R}$$

by

$$(\mathbf{F} \otimes \mathbf{G})(\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{w}_1, \cdots, \mathbf{w}_k) \coloneqq \mathbf{F}(\mathbf{v}_1, \dots, \mathbf{v}_k)\mathbf{G}(\mathbf{w}_1, \cdots, \mathbf{w}_k)\ . \tag{A.8}$$

From the multilinearity of $\mathbf{F}$ and $\mathbf{G}$ it follows that $(\mathbf{F} \otimes \mathbf{G})$ is multi linear too. So, $\mathbf{F} \otimes \mathbf{G} \in L(V_1, \dots, V_k, W_1, \dots, W_l; \mathbb{R})$ and is called the *tensor product of* $\mathbf{F}$ *and* $\mathbf{G}$.

Example A.3. Let U and V be vector spaces, $\mathbf{F}\colon V \times U \to \mathbb{R}$ and $\mathbf{g}\colon V^* \to \mathbb{R}$. The tensor product of $\mathbf{F}$ and $\mathbf{g}$ is given by the multilinear form

$$\mathbf{F} \otimes \mathbf{g}\colon V \times U \times V^* \to \mathbb{R}\ ,$$

defined by its application on the vectors $(\mathbf{v}, \mathbf{u}, \boldsymbol{\omega})$ as

$$(\mathbf{v}, \mathbf{u}, \boldsymbol{\omega}) \mapsto (\mathbf{F} \otimes \mathbf{g})(\mathbf{v}, \mathbf{u}, \boldsymbol{\omega}) = \mathbf{F}(\mathbf{v}, \mathbf{u})\mathbf{g}(\boldsymbol{\omega})\ .$$

[1] There exists also an abstract definition of tensor spaces as quotient spaces of free vector spaces, cf. Lee (2012), Chap. 12. Since we are assuming the vector spaces to be finite dimensional, there exists a canonical isomorphism between the abstract tensor space and the space of multilinear forms. Accordingly, the two terms *multilinear form* and *tensor* are used synonymously.

Applying the definition of the tensor product (A.8) several times, it follows straight forward, that the tensor product is bilinear and associative. Insofar it is allowed to write the tensor products between the tensors $\mathbf{F}$, $\mathbf{G}$ and $\mathbf{H}$ without any brackets as

$$\mathbf{F} \otimes (\mathbf{G} \otimes \mathbf{H}) = (\mathbf{F} \otimes \mathbf{G}) \otimes \mathbf{H} =: \mathbf{F} \otimes \mathbf{G} \otimes \mathbf{H} .$$

With the tensor product, it is possible to find the building blocks of tensors of arbitrary rank. The following proposition is formulated and proved for $L(U, V, V^*; \mathbb{R})$, but extends on a basis of the space of multi linear forms.

Proposition A.2. *Let U, V be vector spaces of dimensions k and l with bases $(\mathbf{b}_1, \dots, \mathbf{b}_k)$ and $(\mathbf{e}_1, \dots, \mathbf{e}_l)$, respectively. Let $(\boldsymbol{\beta}^1, \dots, \boldsymbol{\beta}^k)$ and $(\boldsymbol{\varepsilon}^1, \dots, \boldsymbol{\varepsilon}^l)$ be the corresponding dual bases of U^* and V^*, respectively. Then the set*

$$\mathcal{B} = \left\{\boldsymbol{\beta}^\alpha \otimes \boldsymbol{\varepsilon}^i \otimes \mathbf{e}_j \colon 1 \le \alpha \le k, 1 \le i, j \le l\right\}$$

is a basis for $\mathbf{F} \in L(U, V, V^*; \mathbb{R})$, which therefore has dimension kl^2.

Summing repeated greek indices from 1 to k and repeated roman indices from 1 to l, any multilinear form $\mathbf{F}$ can be written as a linear combination

$$\mathbf{F} = F^j_{\alpha i}\, \boldsymbol{\beta}^\alpha \otimes \boldsymbol{\varepsilon}^i \otimes \mathbf{e}_j \ , \tag{A.9}$$

with $F^j_{\alpha i} = \mathbf{F}(\mathbf{b}_\alpha, \mathbf{e}_i, \boldsymbol{\varepsilon}^j)$.

Proof. The proof is very similar to the proof of Proposition A.1. We need to show that $\mathcal{B}$ is linearly independent and spans $L(U, V, V^*; \mathbb{R})$. Let $\mathbf{u} \in U$, $\mathbf{v} \in V$ and $\boldsymbol{\omega} \in V^*$. Due to the multilinearity of $\mathbf{F}$ it follows directly that

$$\mathbf{F}(\mathbf{u}, \mathbf{v}, \boldsymbol{\omega}) = \mathbf{F}(u^\alpha \mathbf{b}_\alpha, v^i \mathbf{e}_i, \omega_j \boldsymbol{\varepsilon}^j) = \mathbf{F}(\mathbf{b}_\alpha, \mathbf{e}_i, \boldsymbol{\varepsilon}^j) u^\alpha v^i \omega_j \ . \tag{A.10}$$

Writing the multilinear form as a linear combination (A.9) and applying it to the same vectors $\mathbf{u}, \mathbf{v}$ and $\boldsymbol{\omega}$, it follows by the definition of the tensor product that

$$\begin{aligned} \mathbf{F}(\mathbf{u}, \mathbf{v}, \boldsymbol{\omega}) &\overset{(A.9)}{=} (F^j_{\alpha i}\, \boldsymbol{\beta}^\alpha \otimes \boldsymbol{\varepsilon}^i \otimes \mathbf{e}_j)(\mathbf{u}, \mathbf{v}, \boldsymbol{\omega}) \\ &\overset{(A.8)}{=} F^j_{\alpha i}\, \boldsymbol{\beta}^\alpha(u^\beta \mathbf{b}_\beta)\, \boldsymbol{\varepsilon}^i(v^m \mathbf{e}_m)\, \mathbf{e}_j(\omega_n \boldsymbol{\varepsilon}^n) \\ &\overset{(A.3)}{=} F^j_{\alpha i}\, u^\beta v^m \omega_n\, \delta^\alpha_\beta\, \delta^i_m\, \delta^n_j \overset{(A.4)}{=} F^j_{\alpha i}\, u^\alpha v^i\, \omega_j \ . \end{aligned} \tag{A.11}$$

Let $F^j_{\alpha i} = \mathbf{F}(\mathbf{b}_\alpha, \mathbf{e}_i, \boldsymbol{\varepsilon}^j)$. Then comparison of (A.10) and (A.11) proves that $\mathcal{B}$ spans $L(U, V, V^*)$.

To show that $\mathcal{B}$ is linearly independent, suppose some linear combination equals zero:

$$\mathbf{F} = F^j_{\alpha i}\, \boldsymbol{\beta}^\alpha \otimes \boldsymbol{\varepsilon}^i \otimes \mathbf{e}_j = 0$$

Applying the multilinear form $\mathbf{F}$ to arbitrary vectors $\mathbf{u} \in U$, $\mathbf{v} \in V$ and $\boldsymbol{\omega} \in V^*$, it follows by the computation of (A.11) that

$$\forall \mathbf{u} \in U, \mathbf{v} \in V, \boldsymbol{\omega} \in V^* \colon \mathbf{F}(\mathbf{u}, \mathbf{v}, \boldsymbol{\omega}) = 0 \ \Rightarrow\ \forall\, u^\alpha, v^i, \omega_j \in \mathbb{R} \colon F^j_{\alpha i} u^\alpha v^i \omega_j = 0 \ \Rightarrow\ F^j_{\alpha i} = 0 \ .$$

That means, the only linear combination of elements of $\mathcal{B}$ that sums to zero is the trivial one. $\square$

Lead by the basis representation of a multilinear form (A.9) and by Lee (2012), Prop. 12.10, we use the notation $U^* \otimes V^* \otimes V$ to denote the space of multilinear forms $L(U, V, V^*; \mathbb{R})$. For more general spaces of multilinear forms the notation works analogously.

Definition A.4 (Covariant k-Tensor)**.** For a positive integer k, we define the space of *covariant k-tensors on V* to be the vector space

$$T^k(V^*) := \underbrace{V^* \otimes \cdots \otimes V^*}_{k \text{ copies}} \cong L(\underbrace{V, \ldots, V}_{k \text{ copies}}; \mathbb{R}) \ .$$

The number k is called the *rank of the tensor*.

Definition A.5 (Contravariant k-Tensor)**.** For a positive integer k, we define the space of *contravariant k-tensors on V* to be the vector space

$$T^k(V) := \underbrace{V \otimes \cdots \otimes V}_{k \text{ copies}} \cong L(\underbrace{V^*, \ldots, V^*}_{k \text{ copies}}; \mathbb{R}) \ .$$

The number k is called the *rank of the tensor*.

A 0-tensor is, by convention, just a real number. The tensor product between a 0-tensor and a k-tensor corresponds to a scalar multiplication.

Definition A.6 (Mixed (k,l)-Tensor)**.** For a positive integers k, l, we define the space of *mixed (k,l)-tensors on V* to be the vector space

$$T^{(k,l)}(V) := \underbrace{V \otimes \cdots \otimes V}_{k \text{ copies}} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{l \text{ copies}} \cong L(\underbrace{V^*, \ldots, V^*}_{k \text{ copies}}, \underbrace{V, \ldots, V}_{l \text{ copies}}; \mathbb{R}) \ .$$

Example A.4. Some of the defined tensor spaces are identical, i.e.

$$\begin{aligned}
T^{(0,0)}(V) &= T^0(V^*) = T^0(V) = \mathbb{R} \ , \\
T^{(0,1)}(V) &= T^1(V^*) = V^* \ , \\
T^{(1,0)}(V) &= T^1(V) = V \ , \\
T^{(0,k)}(V) &= T^k(V^*) \ , \\
T^{(k,0)}(V) &= T^k(V) \ .
\end{aligned}$$

A.3 Alternating Tensors

For a positive integer $k \in \mathbb{N}$, let S_k denote the *symmetric group on k elements*, i.e. the group of all bijective maps $s\colon \{1, \ldots, k\} \to \{1, \ldots, k\}$. An element of S_k is called a *permutation*. Explicitly, $s \in S_k$ is represented in the form

$$\begin{pmatrix} 1 & 2 & \cdots & k \\ s(1) & s(2) & \cdots & s(k) \end{pmatrix} \ .$$

A *transposition* is a permutation which exchanges two elements and keeps all others fixed. Any permutation can be expressed as a non-unique composition of transpositions. There exists an invariant in the representation of a permutation s by transpositions, which is the number of transpositions n modulo 2, denoted by $\operatorname{sgn}(s) = (-1)^n$. This invariant is called the *sign of the permutation* s.

Let $s, t \in S_k$ be two permutations. Since the composition of two odd $\operatorname{sgn}(1) = -1$ or two even $\operatorname{sgn}(s) = 1$ permutations, respectively, is an even permutation and the composition of an even and an odd permutation is an odd permutation, the sign of the composition $t \circ s$ is given by

$$\operatorname{sgn}(t \circ s) = \operatorname{sgn}(t) \operatorname{sgn}(s) \ . \tag{A.12}$$

Example A.5. Let $s \in S_3$ be a permutation defined by

$$s = \begin{pmatrix} 1 & 2 & 3 \\ s(1) = 1 & s(2) = 3 & s(3) = 2 \end{pmatrix} \ . \tag{A.13}$$

The permutation is expressible as a transposition between 2 and 3 which is an odd number of transpositions. Hence, the sign of the permutation is $\operatorname{sgn}(s) = -1$. Any possible permutations $s, v, t \in S_3$ with a positive sign are

$$s = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \ , \quad v = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \ , \quad t = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \ .$$

Any possible permutations $s, v, t \in S_3$ with a negative sign are

$$s = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \ , \quad v = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \ , \quad t = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \ .$$

Definition A.7 (Action of a Permutation)**.** We define the *action of a permutation* $s \in S_k$ on a covariant k-tensor[2] $\mathbf{F} \in T^k(V^*)$ as follows:

$$s\mathbf{F} \colon (\mathbf{v}_1, \ldots, \mathbf{v}_k) \mapsto \mathbf{F}(\mathbf{v}_{s(1)}, \ldots, \mathbf{v}_{s(k)}) \ . \tag{A.14}$$

Example A.6. Assume the permutation $s = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$. The action of s on a tensor $\mathbf{F} \in T^4(V^*)$ is defined by its application on $\mathbf{v}_1, \ldots, \mathbf{v}_4 \in V$ as

$$s\mathbf{F}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) = \mathbf{F}(\mathbf{v}_2, \mathbf{v}_1, \mathbf{v}_4, \mathbf{v}_3) \ .$$

Definition A.8 (*Alternating Tensor*)**.** An *alternating covariant k-tensor* (*alternating multilinear form* or a *k-form*) is a tensor $\mathbf{A} \in T^k(V^*)$ for which

$$\forall s \in S_k \quad s\mathbf{A} = \operatorname{sgn}(s)\mathbf{A} \ , \tag{A.15}$$

holds.

[2]Similar, we could define the action of a permutation on a contravariant k-tensor. For a mixed tensor the action of a permutation is meaningless.

That means, whenever two arguments of an alternating tensor are interchanged, then its sign changes. We denote the set of all alternating tensors by

$$\Lambda^k(V^*) \coloneqq \left\{ \mathbf{A} \in T^k(V^*) \,|\, s\mathbf{A} = \operatorname{sgn}(s)\mathbf{A},\, \forall s \in S_k \right\} \;.$$

Obviously, the set of alternating k-tensors is a subset of k-tensors, i.e. $\Lambda^k(V^*) \subset T^k(V^*)$. Since the action of a permutation can also be defined for a contravariant tensors, it is also possible to introduce alternating contravariant k-tensors.

Example A.7. Let $\mathbf{B} \in \Lambda^3(V^*)$ be a 3-form. For $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in V$ it holds:

$$\begin{aligned} \mathbf{B}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = -\mathbf{B}(\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_2) = \mathbf{B}(\mathbf{v}_3, \mathbf{v}_1, \mathbf{v}_2) = \\ -\mathbf{B}(\mathbf{v}_3, \mathbf{v}_2, \mathbf{v}_1) = \mathbf{B}(\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_1) = -\mathbf{B}(\mathbf{v}_2, \mathbf{v}_1, \mathbf{v}_3) \;. \end{aligned}$$

Lemma A.1 (Lee (2012), Lem. 14.1)**.** *Let* $\mathbf{A}$ *be a covariant* k*-tensor on a vector space* V*. Then the following statements are equivalent:*

(a) $\mathbf{A}$ *is alternating, i.e.* $\mathbf{A} \in \Lambda^k(V^*)$.

(b) $\mathbf{A}(\mathbf{v}_1, \dots, \mathbf{v}_k) = 0$*, whenever the* k*-tuple* $(\mathbf{v}_1, \dots, \mathbf{v}_k)$ *is linearly dependent.*

(c) $\mathbf{A}$ *gives the value zero whenever two of its arguments are equal:*

$$\mathbf{A}(\mathbf{v}_1, \dots, \mathbf{w}, \dots, \mathbf{w}, \dots, \mathbf{v}_k) = 0 \;.$$

Proof. For the proof, we refer to Lee (2012). □

Definition A.9 (Alternation)**.** We define the function $\operatorname{Alt}\colon T^k(V^*) \to \Lambda^k(V^*)$, called *alternation*, as follows:

$$\operatorname{Alt} \mathbf{F} = \frac{1}{k!} \sum_{s \in S_k} \operatorname{sgn}(s)\, s\mathbf{F} \;, \tag{A.16}$$

where S_k is the symmetric group on k elements and $s\mathbf{F}$ denotes the action of a permutation s on the tensor $\mathbf{F}$. More explicitly, this means for $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$

$$\operatorname{Alt} \mathbf{F}(\mathbf{v}_1, \dots, \mathbf{v}_k) = \frac{1}{k!} \sum_{s \in S_k} \operatorname{sgn}(s)\; \mathbf{F}(\mathbf{v}_{s(1)}, \dots, \mathbf{v}_{s(k)}) \;. \tag{A.17}$$

The linearity of the alternation follows directly by the linearity of the summation and can be shown by straight forward computation.

Proposition A.3. *Let* $\mathbf{A}$ *be a covariant* k*-tensor on a vector space.*

(a) $\operatorname{Alt} \mathbf{A}$ *is alternating.*

(b) $\operatorname{Alt} \mathbf{A} = \mathbf{A}$ *if and only if* $\mathbf{A}$ *is alternating.*

Proof. (a) First we have to prove, that $\operatorname{Alt}\mathbf{A}$ satisfies condition (A.15). To this end, let $s,t \in S_k$ and $\mathbf{A} \in T^k(V^*)$. The composition of the two permutations is another permutation $r = t \circ s \in S_k$. The action of the permutation t on $\operatorname{Alt}\mathbf{A}$ is computed as follows:

$$\begin{aligned} t(\operatorname{Alt}\mathbf{A}) &\overset{(A.16)}{=} \frac{1}{k!}\sum_{s\in S_k} \operatorname{sgn}(s)\,(t\circ s)\mathbf{A} = \frac{1}{k!}\sum_{s\in S_k} \operatorname{sgn}(t)\operatorname{sgn}(t)\operatorname{sgn}(s)\,(t\circ s)\mathbf{A} \\ &\overset{(A.12)}{=} \frac{1}{k!}\sum_{s\in S_k} \operatorname{sgn}(t)\operatorname{sgn}(t\circ s)\,(t\circ s)\mathbf{A} = \frac{1}{k!}\sum_{r\in S_k} \operatorname{sgn}(t)\operatorname{sgn}(r)\,r\mathbf{A} \\ &\overset{(A.16)}{=} \operatorname{sgn}(t)(\operatorname{Alt}\mathbf{A})\ . \end{aligned}$$

This demonstrates that $\operatorname{Alt}\mathbf{A}$ is alternating and proves the first claim.

(b) Let $\mathbf{A} \in \Lambda^k(V^*)$ be an alternating tensor. Due to Definition A.8, we have $s\mathbf{A} = \operatorname{sgn}(s)\mathbf{A}$ for each $s \in S_k$ and consequently

$$\operatorname{sgn}(s)s\mathbf{A} \overset{(A.15)}{=} (\operatorname{sgn}(s))^2\mathbf{A} = \mathbf{A}\ . \tag{A.18}$$

Since there are $k!$ permutations $s \in S_k$, $\sum_{s\in S_k} 1 = k!$ and

$$\operatorname{Alt}\mathbf{A} \overset{(A.16)}{=} \frac{1}{k!}\sum_{s\in S_k} \operatorname{sgn}(s)\ s\mathbf{A} \overset{(A.18)}{=} \frac{1}{k!}\sum_{s\in S_k} \mathbf{A} = \frac{1}{k!}k!\ \mathbf{A}\ = \mathbf{A}\ ,$$

which finishes the proof. □

Example A.8. Let $\mathbf{G} \in T^2(V^*)$ and $\mathbf{H} \in T^3(V^*)$ be two covariant tensors. The alternation of $\mathbf{G}$ is computed as

$$\operatorname{Alt}\mathbf{G}(\mathbf{v}_1,\mathbf{v}_2) = \frac{1}{2}(\mathbf{G}(\mathbf{v}_1,\mathbf{v}_2) - \mathbf{G}(\mathbf{v}_2,\mathbf{v}_1))\ ,$$

and the alternation of $\mathbf{H}$ as

$$\begin{aligned} \operatorname{Alt}\mathbf{H}(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3) = \frac{1}{6}(&\mathbf{H}(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3) + \mathbf{H}(\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_1) + \mathbf{H}(\mathbf{v}_3,\mathbf{v}_1,\mathbf{v}_2) \\ &- \mathbf{H}(\mathbf{v}_1,\mathbf{v}_3,\mathbf{v}_2) - \mathbf{H}(\mathbf{v}_2,\mathbf{v}_1,\mathbf{v}_3) - \mathbf{H}(\mathbf{v}_3,\mathbf{v}_2,\mathbf{v}_1))\ . \end{aligned}$$

Example A.9. Let $\mathbf{G} \in T^2(V^*)$ be a 2-tensor, then

$$\begin{aligned} \operatorname{Alt}\mathbf{G}(\mathbf{v}_2,\mathbf{v}_1) &= \frac{1}{2}(\mathbf{G}(\mathbf{v}_2,\mathbf{v}_1) - \mathbf{G}(\mathbf{v}_1,\mathbf{v}_2)) \\ &= -\frac{1}{2}(\mathbf{G}(\mathbf{v}_1,\mathbf{v}_2) - \mathbf{G}(\mathbf{v}_2,\mathbf{v}_1)) = -\operatorname{Alt}\mathbf{G}(\mathbf{v}_1,\mathbf{v}_2)\ . \end{aligned}$$

The next Lemma, allows to formulate the proof of the upcoming Lemma A.3 by straight forward computation.

Lemma A.2. *Let $\mathbf{F} \in T^k(V^*)$ and $\mathbf{G} \in T^l(V^*)$ be covariant tensors of rank k and l, respectively, then*

$$\operatorname{Alt}(\operatorname{Alt}\mathbf{F} \otimes \operatorname{Alt}\mathbf{G}) = \operatorname{Alt}(\mathbf{F}\otimes\mathbf{G})\ . \tag{A.19}$$

Proof. By definition of the alternation (A.16), we can write (A.19) as

$$\mathrm{Alt}(\mathrm{Alt}\mathbf{F} \otimes \mathrm{Alt}\mathbf{G}) = \frac{1}{k!\,l!}\frac{1}{(k+l)!}\sum_{r\in S_{k+l}}\sum_{s\in S_k}\sum_{q\in S_l}\mathrm{sgn}(r)\mathrm{sgn}(s)\mathrm{sgn}(q)\, r(s\mathbf{F}\otimes q\mathbf{G})\ . \quad (A.20)$$

The permutations $s \in S_k$ and $q \in S_l$ can be embedded in the set of permutations S_{k+l}. Let us define two new permutations $s' \in S'_k \subset S_{k+l}$ and $q'' \in S''_l \subset S_{k+l}$, such that s' acts on the first k elements and q'' acts on the last l elements of total $k+l$ elements, i.e.

$$s'(i) = \begin{cases} s(i) & \text{for } i \le k \\ i & \text{for } i > k \end{cases}\ , \qquad q''(i) = \begin{cases} i & \text{for } i \le k \\ q(i-k)+k & \text{for } i > k\ . \end{cases}$$

We calculate:

$$\begin{aligned}
&\mathrm{Alt}(\mathrm{Alt}\,\mathbf{F}\otimes\mathrm{Alt}\,\mathbf{G}) = \\
&\overset{(A.20)}{=} \frac{1}{k!\,l!}\frac{1}{(k+l)!}\sum_{r\in S_{k+l}}\sum_{s'\in S'_k}\sum_{q''\in S''_l}\mathrm{sgn}(r)\mathrm{sgn}(s')\mathrm{sgn}(q'')\ (r\circ s'\circ q'')(\mathbf{F}\otimes\mathbf{G}) \\
&\overset{(A.12)}{=} \frac{1}{k!\,l!}\frac{1}{(k+l)!}\sum_{r\in S_{k+l}}\sum_{s'\in S'_k}\sum_{q''\in S''_l}\mathrm{sgn}(r)\mathrm{sgn}(s'\circ q'')\ r\circ(s'\circ q'')(\mathbf{F}\otimes\mathbf{G})
\end{aligned}$$

Since we sum over all permutations S_{k+l}, we can interchange the order of the permutations as follows

$$\begin{aligned}
&\mathrm{Alt}(\mathrm{Alt}\mathbf{F}\otimes\mathrm{Alt}\mathbf{G}) = \\
&= \frac{1}{k!\,l!}\sum_{s'\in S'_k}\sum_{q''\in S''_l}\mathrm{sgn}(s'\circ q'')(s'\circ q'')\left(\frac{1}{(k+l)!}\sum_{r\in S_{k+l}}\mathrm{sgn}(r)\, r(\mathbf{F}\otimes\mathbf{G})\right) \\
&\overset{(A.16)}{=} \frac{1}{k!\,l!}\sum_{s'\in S'_k}\sum_{q''\in S''_l}\mathrm{sgn}(s'\circ q'')(s'\circ q'')\mathrm{Alt}(\mathbf{F}\otimes\mathbf{G}) \\
&\overset{(A.18)}{=} \frac{1}{k!\,l!}\sum_{s'\in S'_k}\sum_{q''\in S''_l}\mathrm{Alt}(\mathbf{F}\otimes\mathbf{G}) = \frac{1}{k!\,l!}\mathrm{Alt}(\mathbf{F}\otimes\mathbf{G})\sum_{s'\in S'_k}\sum_{q''\in S''_l}1 = \mathrm{Alt}(\mathbf{F}\otimes\mathbf{G})\ .
\end{aligned}$$

In the last step we used, that S'_k and S''_l are described by $k!$ and $l!$ numbers of permutations. Thus, we have $k!l!$ compositions $s'\circ q''$, $s'\in S'_k$, $q''\in S''_l$. □

Definition A.10 (Elementary Alternating Tensor)**.** Let $I = (i_1, \dots, i_k)$ be a *multi-index of length* k, i.e. a k-tuple of positive integers, and $(\boldsymbol{\varepsilon}^1, \dots, \boldsymbol{\varepsilon}^n)$ be a basis of V^*. We define the *elementary alternating tensor* (or *elementary k-covector*) $\boldsymbol{\varepsilon}^I \in \Lambda^k(V^*)$ as

$$\boldsymbol{\varepsilon}^I \ := \ k!\,\mathrm{Alt}(\boldsymbol{\varepsilon}^{i_1}\otimes\cdots\otimes\boldsymbol{\varepsilon}^{i_k}) = \sum_{s\in S_k}\mathrm{sgn}(s)\,\boldsymbol{\varepsilon}^{i_{s(1)}}\otimes\cdots\otimes\boldsymbol{\varepsilon}^{i_{s(k)}}\ . \quad (A.21)$$

According to Proposition A.3, the elementary alternating tensor $\boldsymbol{\varepsilon}^I$ is by definition an alternating tensor. Interchanging two indices in the multi-index I consequently changes the sign of $\boldsymbol{\varepsilon}^I$.

Example A.10. Let $I = (1, 3)$. Then the elementary alternating tensor is obtained as

$$\boldsymbol{\varepsilon}^{13} = \boldsymbol{\varepsilon}^1 \otimes \boldsymbol{\varepsilon}^3 - \boldsymbol{\varepsilon}^3 \otimes \boldsymbol{\varepsilon}^1 = -\boldsymbol{\varepsilon}^{31} \ .$$

We want to remark the slight abuse of notation for the explicit use of multi-indices. For a multi-index $I = (3, 2, 5)$ the elementary alternating tensor is computed as

$$\begin{aligned}\boldsymbol{\varepsilon}^{325} &= \boldsymbol{\varepsilon}^3 \otimes \boldsymbol{\varepsilon}^2 \otimes \boldsymbol{\varepsilon}^5 - \boldsymbol{\varepsilon}^3 \otimes \boldsymbol{\varepsilon}^5 \otimes \boldsymbol{\varepsilon}^2 + \boldsymbol{\varepsilon}^2 \otimes \boldsymbol{\varepsilon}^5 \otimes \boldsymbol{\varepsilon}^3 \\ &\quad - \boldsymbol{\varepsilon}^2 \otimes \boldsymbol{\varepsilon}^3 \otimes \boldsymbol{\varepsilon}^5 + \boldsymbol{\varepsilon}^5 \otimes \boldsymbol{\varepsilon}^3 \otimes \boldsymbol{\varepsilon}^2 - \boldsymbol{\varepsilon}^5 \otimes \boldsymbol{\varepsilon}^2 \otimes \boldsymbol{\varepsilon}^3 \ .\end{aligned}$$

Definition A.11 (Determinant)**.** Let $[F]$ be an $(n \times n)$-matrix with components F^i_j. The *determinant* of the matrix $[F]$ is defined as

$$\det([F]) \coloneqq \sum_{s \in S_n} \operatorname{sgn}(s) F_1^{s(1)} \cdots F_n^{s(n)} \ . \tag{A.22}$$

Since the multiplication of scalars commute, by relabeling, we are allowed to rewrite the determinant as

$$\det([F]) = \sum_{s \in S_n} \operatorname{sgn}(s) F^1_{s(1)} \cdots F^n_{s(n)} \ .$$

While the first version can be associated with the expansion of the determinant along the columns, the second version corresponds to the expansion along the row.

Let $\boldsymbol{\varepsilon}^I$ be the alternating k-tensor of (A.21) with a multi-index $I = (i_1, \ldots, i_k)$ and $\mathbf{v}_1, \ldots, \mathbf{v}_k \in V$. We denote the $(n \times k)$-matrix of the component description of the vectors in the basis $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ by $[v] = ([\mathbf{v}_1], \ldots, [\mathbf{v}_k])$. Choosing only the rows $i_1, \ldots, i_k$ of $[v]$, we obtain the submatrix $[v^I]$. If we apply the alternating tensor $\boldsymbol{\varepsilon}^I$ on the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k \in V$, we can express the result using the determinant of the submatrix $[v^I]$, i.e.

$$\begin{aligned}\boldsymbol{\varepsilon}^I(\mathbf{v}_1, \ldots, \mathbf{v}_k) &= \sum_{s \in S_k} \operatorname{sgn}(s)\ \boldsymbol{\varepsilon}^{i_{s(1)}} \otimes \cdots \otimes \boldsymbol{\varepsilon}^{i_{s(k)}}(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \sum_{s \in S_k} \operatorname{sgn}(s)\ v_1^{i_{s(1)}} \cdots v_k^{i_{s(k)}} \\ &\overset{(A.22)}{=} \det \begin{pmatrix} v_1^{i_1} & \ldots & v_k^{i_1} \\ \vdots & \ddots & \vdots \\ v_1^{i_k} & \ldots & v_k^{i_k} \end{pmatrix} = \det([v^I]) \ .\end{aligned}$$

Example A.11. In the previous setting, let $\mathbf{v}_1, \mathbf{v}_2 \in V$. Then

$$\begin{aligned}\boldsymbol{\varepsilon}^{13}(\mathbf{v}_1, \mathbf{v}_2) &= (\boldsymbol{\varepsilon}^1 \otimes \boldsymbol{\varepsilon}^3 - \boldsymbol{\varepsilon}^3 \otimes \boldsymbol{\varepsilon}^1)(\mathbf{v}_1, \mathbf{v}_2) = \boldsymbol{\varepsilon}^1(\mathbf{v}_1)\boldsymbol{\varepsilon}^3(\mathbf{v}_2) - \boldsymbol{\varepsilon}^3(\mathbf{v}_1)\boldsymbol{\varepsilon}^1(\mathbf{v}_2) \\ &= v_1^1 v_2^3 - v_1^3 v_2^1 = \det \begin{pmatrix} v_1^1 & v_2^1 \\ v_1^3 & v_2^3 \end{pmatrix} \ .\end{aligned}$$

Definition A.12 (Multi-Index Kronecker Delta)**.** Let $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_k)$ be two multi-indices and $(\boldsymbol{\varepsilon}^1, \ldots, \boldsymbol{\varepsilon}^n)$ be a dual basis of the basis $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ on V. By applying the elementary alternating tensor $\boldsymbol{\varepsilon}^I$ to the set of basis vectors $(\mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_k})$, we define a multi-index Kronecker delta as follows:

$$\delta^I_J \coloneqq \boldsymbol{\varepsilon}^I(\mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_k}) = \det \begin{pmatrix} \delta^{i_1}_{j_1} & \ldots & \delta^{i_1}_{j_k} \\ \vdots & \ddots & \vdots \\ \delta^{i_k}_{j_1} & \ldots & \delta^{i_k}_{j_k} \end{pmatrix} \ . \tag{A.23}$$

It is easily shown, that the multi-index Kronecker delta is characterized by the following function:

$$\delta^I_J = \begin{cases} \operatorname{sgn}(s), & \text{if neither } I \text{ nor } J \text{ has a repeated index and } J = s(I) \text{ for some } s \in S_k \,, \\ 0, & \text{if } I \text{ or } J \text{ has a repeated index or } J \text{ is not a permutation of } I \,. \end{cases} \tag{A.24}$$

Example A.12. Let $I = (1, 2)$ and $J = (2, 3)$ be two multi-indices. Then

$$\delta^{12}_{23} = \det \begin{pmatrix} \delta^1_2 & \delta^1_3 \\ \delta^2_2 & \delta^2_3 \end{pmatrix} = \det \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0 \,.$$

Using the argumentation with the function (A.24), the multi-index Kronecker delta is zero, since J is not a permutation of I. In the case that $J = (2, 1)$, the multi-index J is obtained by one transposition of the multi-index I, which is an odd permutation of sign -1 and consequently $\delta^I_J = -1$. Again we verify the result using the definition of the multi-index Kronecker delta, i.e.

$$\delta^{12}_{21} = \det \begin{pmatrix} \delta^1_2 & \delta^1_1 \\ \delta^2_2 & \delta^2_1 \end{pmatrix} = \det \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1 \,.$$

Proposition A.4. *Let $(\boldsymbol{\varepsilon}^1, \ldots, \boldsymbol{\varepsilon}^n)$ be a basis of V^*. Then for each $k \in \{1, \ldots, n\}$, the set*

$$\mathcal{E} = \{\boldsymbol{\varepsilon}^I \mid I \textit{ is an increasing multi-index of length } k\}$$

is a basis for $\Lambda^k(V^*)$. Therefore,

$$\dim \Lambda^k(V^*) = \binom{n}{k} = \frac{n!}{k!\,(n-k)!} \,. \tag{A.25}$$

If $k > n$, then $\dim \Lambda^k(V^*) = 0$.

According to Proposition A.4, we can write a k-form $\mathbf{A} \in \Lambda^k(V^*)$ as a linear combination

$$\mathbf{A} = \sum_{\{I \colon i_1 < \cdots < i_k\}} A_I \boldsymbol{\varepsilon}^I =: \sum_I{}' A_I \boldsymbol{\varepsilon}^I \,, \tag{A.26}$$

where the primed sum denotes the summation over increasing multi-indices of length k.

Proof. The fact that $\Lambda^k(V^*)$ is the trivial vector space when $k > n$ follows immediately from Lemma A.1(b), since every k-tuple of vectors is linearly dependent in that case. For the case $k \leq n$, we need to show that the set $\mathcal{E}$ spans $\Lambda^k(V^*)$ and is linearly independent. Let $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ be the basis for V dual to $(\boldsymbol{\varepsilon}^1, \ldots, \boldsymbol{\varepsilon}^n)$. To show that $\mathcal{E}$ spans $\Lambda^k(V^*)$, let $\mathbf{A} \in \Lambda^k(V^*)$ and $J = (j_1, \ldots, j_k)$ be a multi-index. Since $\mathbf{A} \in \Lambda^k(V^*) \subset T^k(V^*)$, an alternating tensor is spanned with a basis of $T^k(V^*)$ by $\mathbf{A} = A_J \boldsymbol{\varepsilon}^{j_1} \otimes \cdots \otimes \boldsymbol{\varepsilon}^{j_k}$. According to

Proposition A.3 it holds for any alternating tensor, that $\mathbf{A} = \operatorname{Alt}\mathbf{A}$. Using the definition of the alternation (A.16) and its linearity, we can write

$$\begin{aligned}\mathbf{A} &= \operatorname{Alt}(A_J \boldsymbol{\varepsilon}^{j_1} \otimes \cdots \otimes \boldsymbol{\varepsilon}^{j_k}) \overset{(A.16)}{=} \frac{1}{k!} \sum_{s \in S_k} \operatorname{sgn}(s)\, s(A_J\, \boldsymbol{\varepsilon}^{j_1} \otimes \cdots \otimes \boldsymbol{\varepsilon}^{j_k}) \\ &= A_J \frac{1}{k!} \sum_{s \in S_k} \operatorname{sgn}(s)\, s(\boldsymbol{\varepsilon}^{j_1} \otimes \cdots \otimes \boldsymbol{\varepsilon}^{j_k}) \overset{(A.21)}{=} \frac{1}{k!} A_J \boldsymbol{\varepsilon}^J \,.\end{aligned} \tag{A.27}$$

Let I be any increasing multi-index of length k. Since $\boldsymbol{\varepsilon}^J$ is alternating, we can rewrite it by an elementary alternating tensor $\boldsymbol{\varepsilon}^I$ of increasing multi-indices as $\boldsymbol{\varepsilon}^J = \operatorname{sgn}(s)\boldsymbol{\varepsilon}^I$. Since $\mathbf{A}$ is an alternating tensor the components A_J are connected to the Ith components of the increasing multi-index by $A_J = \operatorname{sgn}(s)A_I$. Suppressing the summation convention it holds that

$$A_J \boldsymbol{\varepsilon}^J = \operatorname{sgn}(s)^2 A_I \boldsymbol{\varepsilon}^I = A_I \boldsymbol{\varepsilon}^I \,, \quad \text{no summation.}$$

There exist $k!$ permutations of each increasing multi-index I. Hence, the alternating k-form (A.27) is transformed further to

$$\mathbf{A} = \frac{1}{k!} A_J \boldsymbol{\varepsilon}^J = {\sum_I}' A_I \boldsymbol{\varepsilon}^I \,, \tag{A.28}$$

which proves that $\mathcal{E}$ spans $\Lambda^k(V^*)$.

To show that $\mathcal{E}$ is a linearly independent set, suppose the identity ${\sum_I}' A_I \boldsymbol{\varepsilon}^I = 0$ holds for some coefficients A_I. Let J be any increasing multi-index. Applying both sides of the identity to the vectors $(\mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_k})$ and using (A.23), we get

$$0 = {\sum_I}' A_I \boldsymbol{\varepsilon}^I(\mathbf{e}_{j_1}, \ldots, \mathbf{e}_{j_k}) = {\sum_I}' A_I \delta^I_J = A_J \,.$$

Thus each coefficient A_J is zero, what shows the linear independency of $\mathcal{E}$. □

Example A.13. Let V be a vector space of dimension $n = 3$. The space of alternating 2-tensors $\Lambda^2(V^*)$ has the dimension

$$\dim \Lambda^2(V^*) = \binom{3}{2} = \frac{3!}{2!(3-2)!} = 3 \,.$$

The basis of $\Lambda^2(V^*)$ is given by all three elementary alternating tensors $\boldsymbol{\varepsilon}^I$ with an increasing multi-index I of length 2, i.e.

$$\begin{aligned}\boldsymbol{\varepsilon}^{12} &= \boldsymbol{\varepsilon}^1 \otimes \boldsymbol{\varepsilon}^2 - \boldsymbol{\varepsilon}^2 \otimes \boldsymbol{\varepsilon}^1 \\ \boldsymbol{\varepsilon}^{13} &= \boldsymbol{\varepsilon}^1 \otimes \boldsymbol{\varepsilon}^3 - \boldsymbol{\varepsilon}^3 \otimes \boldsymbol{\varepsilon}^1 \\ \boldsymbol{\varepsilon}^{23} &= \boldsymbol{\varepsilon}^2 \otimes \boldsymbol{\varepsilon}^3 - \boldsymbol{\varepsilon}^3 \otimes \boldsymbol{\varepsilon}^2\end{aligned}$$

Hence, we represent any 2-form $\mathbf{A} \in \Lambda^2(V^*)$ as

$$\mathbf{A} \overset{(A.26)}{=} {\sum_I}' \mathbf{A}_I \boldsymbol{\varepsilon}^I = A_{12}\boldsymbol{\varepsilon}^{12} + A_{13}\boldsymbol{\varepsilon}^{13} + A_{23}\boldsymbol{\varepsilon}^{23} \,.$$

According to (A.28), it is also possible to span the alternating tensor with all elementary alternating tensors. Since the components of an alternating tensor of rank 2 satisfy $A_{ij} = -A_{ji}$, we compute

$$\begin{aligned}
\mathbf{A} &= \frac{1}{2!} A_J \boldsymbol{\varepsilon}^J = \frac{1}{2}(A_{12}\boldsymbol{\varepsilon}^{12} + A_{21}\boldsymbol{\varepsilon}^{21} + A_{13}\boldsymbol{\varepsilon}^{13} + A_{31}\boldsymbol{\varepsilon}^{31} + A_{23}\boldsymbol{\varepsilon}^{23} + A_{32}\boldsymbol{\varepsilon}^{32}) \\
&= \frac{1}{2}\big((A_{12} - A_{21})\boldsymbol{\varepsilon}^{12} + (A_{13} - A_{31})\boldsymbol{\varepsilon}^{13} + (A_{23} - A_{32})\boldsymbol{\varepsilon}^{23}\big) \\
&= A_{12}\boldsymbol{\varepsilon}^{12} + A_{13}\boldsymbol{\varepsilon}^{13} + A_{23}\boldsymbol{\varepsilon}^{23} = \sum{}' A_I \boldsymbol{\varepsilon}^I \ .
\end{aligned}$$

A.4 The Wedge Product

Definition A.13 (Wedge Product)**.** Given $\mathbf{A} \in \Lambda^k(V^*)$ and $\mathbf{B} \in \Lambda^l(V^*)$, we define their *wedge product* (or *exterior product*) as

$$\mathbf{A} \wedge \mathbf{B} = \frac{(k+l)!}{k!\,l!}\mathrm{Alt}(\mathbf{A} \otimes \mathbf{B}) = \frac{1}{k!\,l!} \sum_{s \in S_{k+l}} \mathrm{sgn}(s)\, s(\mathbf{A} \otimes \mathbf{B}) \ . \tag{A.29}$$

Example A.14. Let $\mathbf{A} \in \Lambda^1(V^*) = T^1(V^*)$ und $\mathbf{B} \in \Lambda^1(V^*) = T^1(V^*)$, then their wedge product is

$$\begin{aligned}
(\mathbf{A} \wedge \mathbf{B})(\mathbf{v}_1, \mathbf{v}_2) &\overset{(A.29)}{=} (\mathbf{A} \otimes \mathbf{B})(\mathbf{v}_1, \mathbf{v}_2) - (\mathbf{A} \otimes \mathbf{B})(\mathbf{v}_2, \mathbf{v}_1) \\
&= (\mathbf{A} \otimes \mathbf{B} - \mathbf{B} \otimes \mathbf{A})(\mathbf{v}_1, \mathbf{v}_2) \ .
\end{aligned}$$

Lemma A.3. *Let $(\boldsymbol{\varepsilon}^1, \ldots, \boldsymbol{\varepsilon}^n)$ be a basis for V^*. For multi-indices $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_l)$, the wedge product of the two associated elementary alternating tensors $\boldsymbol{\varepsilon}^I$ and $\boldsymbol{\varepsilon}^J$ satisfy*

$$\boldsymbol{\varepsilon}^I \wedge \boldsymbol{\varepsilon}^J = \boldsymbol{\varepsilon}^{IJ} \ ,$$

where $IJ = (i_1, \ldots, i_k, j_1, \ldots, j_l)$ is obtained by concatenating I and J.

Proof. We calculate

$$\begin{aligned}
\boldsymbol{\varepsilon}^I \wedge \boldsymbol{\varepsilon}^J &\overset{(A.29)}{=} \frac{(k+l)!}{k!\,l!}\mathrm{Alt}(\boldsymbol{\varepsilon}^I \otimes \boldsymbol{\varepsilon}^J) \\
&\overset{(A.21)}{=} \frac{(k+l)!}{k!\,l!}\mathrm{Alt}(k!\,\mathrm{Alt}(\boldsymbol{\varepsilon}^{i_1} \otimes \cdots \otimes \boldsymbol{\varepsilon}^{i_k}) \otimes l!\,\mathrm{Alt}(\boldsymbol{\varepsilon}^{j_1} \otimes \cdots \otimes \boldsymbol{\varepsilon}^{j_l})) \\
&\overset{(A.19)}{=} (k+l)!\mathrm{Alt}((\boldsymbol{\varepsilon}^{i_1} \otimes \cdots \otimes \boldsymbol{\varepsilon}^{i_k}) \otimes (\boldsymbol{\varepsilon}^{j_1} \otimes \cdots \otimes \boldsymbol{\varepsilon}^{j_l})) \\
&\overset{(A.21)}{=} \boldsymbol{\varepsilon}^{IJ} \ .
\end{aligned}$$

□

The algebraic properties of the wedge product are summarized in the following proposition.

Proposition A.5 (Lee (2012), Prop. 14.11)**.** *Suppose* $\mathbf{A}, \mathbf{B}$*, and* $\mathbf{C}$ *are multicovectors on a vector space* V*.*

a) BILINEARITY: *For* $a, b \in \mathbb{R}$*,*

$$\begin{aligned} (a\mathbf{A} + b\mathbf{B}) \wedge \mathbf{C} &= a(\mathbf{A} \wedge \mathbf{C}) + b(\mathbf{B} \wedge \mathbf{C}) \\ \mathbf{C} \wedge (a\mathbf{A} + b\mathbf{B}) &= a(\mathbf{C} \wedge \mathbf{A}) + b(\mathbf{C} \wedge \mathbf{B}) \ . \end{aligned} \tag{A.30}$$

b) ASSOCIATIVITY:

$$\mathbf{A} \wedge (\mathbf{B} \wedge \mathbf{C}) = (\mathbf{A} \wedge \mathbf{B}) \wedge \mathbf{C} \ .$$

c) ANTICOMMUTATIVITY: *For* $\mathbf{A} \in \Lambda^k(V^*)$ *and* $\mathbf{B} \in \Lambda^l(V^*)$*,*

$$\mathbf{A} \wedge \mathbf{B} = (-1)^{kl} \mathbf{B} \wedge \mathbf{A} \ .$$

d) *If* $(\boldsymbol{\varepsilon}^1, \ldots, \boldsymbol{\varepsilon}^n)$ *is a basis of* V^* *and* $I = (i_1, \ldots, i_k)$ *a multi-index, then*

$$\boldsymbol{\varepsilon}^I = \boldsymbol{\varepsilon}^{i_1} \wedge \cdots \wedge \boldsymbol{\varepsilon}^{i_k} \ . \tag{A.31}$$

e) *For any covectors* $\boldsymbol{\omega}^1, \ldots, \boldsymbol{\omega}^k \in V^*$ *and vectors* $\mathbf{v}_1, \ldots, \mathbf{v}_k \in V$*,*

$$\boldsymbol{\omega}^1 \wedge \cdots \wedge \boldsymbol{\omega}^k(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \det([\boldsymbol{\omega}^j(\mathbf{v}_i)]) \ .$$

Proof. For the proof, we refer to Lee (2012). □

Example A.15. Let $(\boldsymbol{\varepsilon}^1, \ldots, \boldsymbol{\varepsilon}^n)$ be a basis of V^*. Then we have

$$\boldsymbol{\varepsilon}^{12} \overset{(A.21)}{=} 2\mathrm{Alt}(\boldsymbol{\varepsilon}^1 \otimes \boldsymbol{\varepsilon}^2) \overset{(A.16)}{=} \boldsymbol{\varepsilon}^1 \otimes \boldsymbol{\varepsilon}^2 - \boldsymbol{\varepsilon}^2 \otimes \boldsymbol{\varepsilon}^1 \overset{(A.29)}{=} \boldsymbol{\varepsilon}^1 \wedge \boldsymbol{\varepsilon}^2 \ .$$

Let $\dim(V) = 3$ and $\mathbf{A} \in \Lambda^2(V^*)$ be an arbitrary 2-form. According to Propositions A.4 and A.5, the 2-form can be written in component form as

$$\begin{aligned} \mathbf{A} &= A_{12}\boldsymbol{\varepsilon}^{12} + A_{13}\boldsymbol{\varepsilon}^{13} + A_{23}\boldsymbol{\varepsilon}^{23} \\ &= A_{12}(\boldsymbol{\varepsilon}^1 \wedge \boldsymbol{\varepsilon}^2) + A_{13}(\boldsymbol{\varepsilon}^1 \wedge \boldsymbol{\varepsilon}^3) + A_{23}(\boldsymbol{\varepsilon}^2 \wedge \boldsymbol{\varepsilon}^3) \ . \end{aligned}$$

The wedge product of the 2-form $\mathbf{A} \in \Lambda^2(V^*)$ and a 1-form $\mathbf{B} \in \Lambda^1(V^*)$ is written in component form as

$$\begin{aligned} \mathbf{A} \wedge \mathbf{B} &= (A_{12}\boldsymbol{\varepsilon}^{12} + A_{13}\boldsymbol{\varepsilon}^{13} + A_{23}\boldsymbol{\varepsilon}^{23}) \wedge (B_1\boldsymbol{\varepsilon}^1 + B_2\boldsymbol{\varepsilon}^2 + B_3\boldsymbol{\varepsilon}^3) \\ &\overset{(A.30)}{=} A_{12}B_3\boldsymbol{\varepsilon}^{123} + A_{13}B_2\boldsymbol{\varepsilon}^{132} + A_{23}B_1\boldsymbol{\varepsilon}^{231} \\ &\overset{(A.15)}{=} (A_{12}B_3 - A_{13}B_2 + A_{23}B_1)\boldsymbol{\varepsilon}^{123} \\ &\overset{(A.31)}{=} (A_{12}B_3 - A_{13}B_2 + A_{23}B_1)\boldsymbol{\varepsilon}^1 \wedge \boldsymbol{\varepsilon}^2 \wedge \boldsymbol{\varepsilon}^3 \ . \end{aligned}$$

A.5 In Eight Steps to the Wedge Product

1. Let $\{1, \dots k\}$ be a set of k positive integers. A permutation $s \in S_k$ is the bijective map $s : \{1, \dots k\} \to \{1, \dots k\}, (1, \dots, k) \mapsto (s(1), \dots s(k))$. The sign of a permutation $s \in S_k$ is defined by
$$\operatorname{sgn}(s) := \begin{cases} +1 & \text{if } s \text{ is even ,} \\ -1 & \text{if } s \text{ is odd .} \end{cases}$$

2. Action of a permutation $s \in S_k$ on a covariant k-tensor $\mathbf{F} \in T^k(V^*)$. Let $\mathbf{v}_1, \dots, \mathbf{v}_k \in V$, then
$$s\mathbf{F} \colon (\mathbf{v}_1, \dots, \mathbf{v}_k) \mapsto \mathbf{F}(\mathbf{v}_{s(1)}, \dots, \mathbf{v}_{s(k)}) \ .$$

3. An alternating covariant k-tensor $\mathbf{A} \in \Lambda^k(V^*)$ satisfies $\forall s \in S_k$
$$s\mathbf{A} = \operatorname{sgn}(s)\mathbf{A} \ .$$

4. The alternation projection Alt : $T^k(V^*) \to \Lambda^k(V^*)$ is defined as
$$\operatorname{Alt}\mathbf{F} := \frac{1}{k!}\sum_{s \in S_k} \operatorname{sgn}(s)\, s\mathbf{F} \ .$$

5. Let $\boldsymbol{\varepsilon}^1, \dots, \boldsymbol{\varepsilon}^n$ be a basis of V^* and $I = (i_1, \dots, i_k)$ be a multi-index. The elementary alternating tensor is defined as
$$\boldsymbol{\varepsilon}^I := \ k! \operatorname{Alt}(\boldsymbol{\varepsilon}^{i_1} \otimes \cdots \otimes \boldsymbol{\varepsilon}^{i_k}) \ .$$

6. Let J be an arbitrary and I be an increasing multi-index of length k. The component description of $\mathbf{A} \in \Lambda^k(V^*)$ with $n = \dim V$ and $\dim \Lambda^k(V^*) = \binom{n}{k}$ is given by
$$\mathbf{A} = \frac{1}{k!} A_J \boldsymbol{\varepsilon}^J = {\sum_I}' A_I \boldsymbol{\varepsilon}^I = \sum_{\{I : i_1 < \cdots < i_k\}} A_I \boldsymbol{\varepsilon}^I \ .$$

7. Let $\mathbf{A} \in \Lambda^k(V^*)$ and $\mathbf{B} \in \Lambda^l(V^*)$. The wedge product is defined as
$$\mathbf{A} \wedge \mathbf{B} = \frac{(k+l)!}{k!\, l!} \operatorname{Alt}(\mathbf{A} \otimes \mathbf{B}) = \frac{1}{k!\, l!} \sum_{s \in S_{k+l}} \operatorname{sgn}(s)\, s(\mathbf{A} \otimes \mathbf{B}) \ .$$

8. Let $I = (i_1, \dots, i_k)$, $J = (j_1, \dots, j_l)$ and $IJ = (i_1, \dots, i_k, j_1, \dots, j_l)$ be multi-indices. The wedge product of elementary alternating tensors satisfies
$$\boldsymbol{\varepsilon}^I \wedge \boldsymbol{\varepsilon}^J = \boldsymbol{\varepsilon}^{IJ} \ .$$
By induction, we have $\boldsymbol{\varepsilon}^I = \boldsymbol{\varepsilon}^{i_1} \wedge \cdots \wedge \boldsymbol{\varepsilon}^{i_k}$.

Appendix B

Properties of the Cross Product

The cross product $\times$ as a skew-symmetric operator on $\mathbb{R}^3$ has some useful identities which are frequently used in this paper. In $\mathbb{R}^3$ the cross product fulfills the Jacobi identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = 0 \qquad \forall \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3 \, . \tag{B.1}$$

The triple product is invariant with respect to even permutation, i.e.

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) \qquad \forall \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3 \, . \tag{B.2}$$

The vector triple product satisfies Grassmann's identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \qquad \forall \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3 \, . \tag{B.3}$$

The quadruple product

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}) \qquad \forall \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{R}^3 \, , \tag{B.4}$$

and another useful identity, where the tilde denotes the skew-symmetric tensor to an associated axial vector, is

$$\tilde{\mathbf{a}}\tilde{\mathbf{b}}\tilde{\mathbf{b}}\mathbf{a} = \mathbf{a} \times (\mathbf{b} \times (\mathbf{b} \times \mathbf{a})) = -\mathbf{b} \times (\mathbf{a} \times (\mathbf{a} \times \mathbf{b})) = -\tilde{\mathbf{b}}\tilde{\mathbf{a}}\tilde{\mathbf{a}}\mathbf{b} \qquad \forall \mathbf{a}, \mathbf{b} \in \mathbb{R}^3 \, . \tag{B.5}$$

Bibliography

R. Abraham and J. E. Marsden. *Foundations of Mechanics*. Addison-Wesley, 2nd edition, 1978.

R. Abraham and S. Smale. *Lectures of Smale on Differential Topology*. Columbia University, 1963.

R. Abraham, J. E. Marsden, and T. S. Ratiu. *Manifolds, Tensor Analysis, and Applications*, volume 75 of *Applied Mathematical Sciences*. Springer, 1988.

S. S. Antman. Ordinary differential equations of non-linear elasticity I: Foundations of the theories of non-linearly elastic rods and shells. *Archive for Rational Mechanics and Analysis*, 61:307–351, 1976.

S. S. Antman. *Nonlinear Problems of Elasticity*, volume 107 of *Applied Mathematical Sciences*. Springer, 2nd edition, 2005.

S. S. Antman and J. E. Osborn. The principle of virtual work and integral laws of motion. *Archive for Rational Mechanics and Analysis*, 63:231–262, 1979.

V. I. Arnold. *Mathematical Methods of Classical Mechanics*, volume 60 of *Graduate Texts in Mathematics*. Springer, 2nd edition, 1989.

T. Aubin. *A Course in Differential Geometry*, volume 27 of *Graduate Studies in Mathematics*. American Mathematical Society, 2001.

D. Aubram. *Differential Geometry Applied to Continuum Mechanics*. Shaker Verlag, 2009.

F. Auricchio, P. Carotenuto, and A. Reali. On the geometrically exact beam model: A consistent, effective and simple derivation from three-dimensional finite-elasticity. *International Journal of Solids and Structures*, 45(17):4766–4781, 2008.

Y. Başar and D. Weichert. *Nonlinear Continuum Mechanics of Solids: Fundamental Mathematical and Physical Concepts*. Springer, 2000.

P. Ballard and A. Millard. *Poutres et Arcs Élastiques*. Les Éditions de l'École Polytechnique, 2009.

O. A. Bauchau and J. I. Craig. *Structural Analysis: With Applications to Aerospace Structures*, volume 163 of *Solid Mechanics and its Applications*. Springer, 2009.

O. A. Bauchau and S. Han. Three-dimensional beam theory for flexible multibody dynamics. *Journal of Computational and Nonlinear Dynamics*, 2014. doi: 10.1115/1.4025820. 12 pages.

E. Binz and H. R. Fischer. The manifold of embeddings of a closed manifold. In *Differential Geometric Methods in Mathematical Physics*, pages 310–325. Springer, 1981.

E. Binz, J. Sniatycki, and H. R. Fischer. *Geometry of Classical Fields*, volume 154 of *North-Holland Mathematics Studies*. Elsevier Science, 1988.

M. A. Biot. Non-linear thermoelasticity, irreversible thermodynamics and elastic instability. *Indiana University Mathematics Journal*, 23:309–335, 1974.

H. Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Universitext. Springer, 2010.

A. Cauchy. De la pression ou tension dans un corps solide. In *Œuvres Complètes*, volume 7 of *2*, pages 60–78. Gauthier-Villars et Fils, 1827.

A. Cauchy. Sur les équations qui expriment les conditions d'équilibre ou les lois du mouvement intérieur d'un corps solide, élastique ou non élastique. In *Œuvres Complètes*, volume 8 of *2*, pages 195–226. Gauthier-Villars et Fils, 1828.

P. Chadwick. *Continuum Mechanics: Concise Theory and Problems*. Dover Books on Physics. Dover Publications, 1999.

P. G. Ciarlet. *Mathematical Elasticity: Three-Dimensional Elasticity*, volume 20 of *Studies in Mathematics and its Applications*. North-Holland, 1988.

M. Clerici. *Finite-Elemente-Modellierung und Simulation von geometrisch exakten Timoshenko-Balken*. PhD thesis, ETH Zurich, 2001.

H. Cohen. A non-linear theory of elastic directed curves. *International Journal of Engineering Sciences*, 4:511–524, 1966.

E. Cosserat and F. Cosserat. *Théorie des Corps déformables*. Librairie Scientifique A. Hermann et Fils, 1909.

D. A. Danielson and D. H. Hodges. Nonlinear beam kinematics by decomposition of the rotation tensor. *Journal of Applied Mechanics*, 54:258–262, 1987.

D. A. Danielson and D. H. Hodges. A beam theory for large global rotation, moderate local rotation, and small strain. *Journal of Applied Mechanics*, 55:179–184, 1988.

G. Del Piero. On the method of virtual power in continuum mechanics. *Journal of Mechanics of Materials and Structures*, 4:281–292, 2009.

E. H. Dill. Kirchhoff's theory of rods. *Archive for History of Exact Sciences*, 44:1–23, 1992.

E. H. Dowell, J. Traybar, and D. H. Hodges. An experimental-theoretical correlation study of non-linear bending and torsion deformations of a cantilever beam. *Journal of Sound and Vibration*, 50(4):533–544, 1977.

T. C. Doyle and J. L. Ericksen. Nonlinear elasticity. *Advances in Applied Mechanics*, 4: 53–115, 1956.

E. N. Dvorkin and M. B. Goldschmit. *Nonlinear Continua.* Computational Fluid and Solid Mechanics. Springer, 2006.

D. G. Ebin and J. E. Marsden. Groups of diffeomorphisms and the solution of the classical Euler equations for a perfect fluid. *Bulletin of the American Mathematical Society*, 75 (5):962–967, 09 1969.

J. Eells, Jr. On the geometry of function spaces. In *Symposium internacional de topología algebraica International symposium on algebraic topology*, pages 303–308. Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958.

J. Eells, Jr. A setting for global analysis. *Bulletin of the American Mathematical Society*, 72(5):751–807, 1966.

H. I. Eliasson. Geometry of manifolds of maps. *Journal of Differential Geometry*, 1(1-2): 169–194, 1967.

M. Epstein. Thin-walled beams as directed curves. *Acta Mechanica*, 33(3):229–242, 1979.

M. Epstein. *The Geometrical Language of Continuum Mechanics.* Cambridge University Press, 2010.

M. Epstein and M. Elzanowski. *Material Inhomogeneities and their Evolution: A Geometric Approach.* Springer, 2007.

M. Epstein and D. W. Murray. Large deformation in-plane analysis of elastic beams. *Computers & Structures*, 6:1–9, 1976a.

M. Epstein and D. W. Murray. Three-dimensional large deformation analysis of thin walled beams. *International Journal of Solids and Structures*, 12(12):867–876, 1976b.

M. Epstein and R. Segev. Differentiable manifolds and the principle of virtual work in continuum mechanics. *Journal of Mathematical Physics*, 21:1243–1245, 1980.

S. R. Eugster, C. Hesch, P. Betsch, and Ch. Glocker. Director-based beam finite elements relying on the geometrically exact beam theory formulated in skew coordinates. *International Journal for Numerical Methods in Engineering*, 97(2):111–129, 2014.

L. Euler. *Methodus Inveniendi Lineas Curvas Maximi Minimive Proprietate Gaudentes*, chapter Additamentum De Curvis Elasticis, pages 245–310. Lausanne & Geneva, 1744.

P. Germain. Sur l'application de la méthode des puissances virtuelles en mécanique des milieux. *Comptes Rendus de l'Académie des Sciences Paris*, A:1051–1055, 1972.

P. Germain. The method of virtual power in continuum mechanics. part 2: Microstructure. *SIAM Journal on Applied Mathematics*, 25:556–575, 1973a.

P. Germain. La méthodes des puissances virtuelles en mécanique des milieux continus - 1ère partie, théorie du second gradient. *Journal de Mécanique*, 12:235–274, 1973b.

P. Germain. *Mécanique*. Number 1 in Mécanique. École Polytechnique, 1986.

Ch. Glocker. *Set-Valued Force Laws, Dynamics of Non-Smooth Systems*, volume 1 of *Lecture Notes in Applied Mechanics*. Springer, 2001.

M. Göckeler and T. Schücker. *Differential Geometry, Gauge Theories, and Gravity*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 1989.

A. E. Green and R. S. Rivlin. On Cauchy's equations of motion. *Zeitschrift für Angewandte Mathematik und Physik*, 15:290–292, 1964.

D. Gross, W. Hauger, J. Schröder, and W. A. Wall. *Technische Mechanik 1*. Springer-Lehrbuch. Springer, 2011.

M. E. Gurtin. *An Introduction to Continuum Mechanics*. Academic Press, 1981.

G. Hamel. *Theoretische Mechanik: Eine einheitliche Einführung in die gesamte Mechanik*. Springer, 1967.

E. Hellinger. Die allgemeinen Ansätze der Mechanik der Kontinua. In *Mechanik*, volume IV/4 of *Enzyklopädie der Mathematischen Wissenschaften mit Einschluss ihrer Anwendungen*, pages 602–694. Teubner, 1914.

K. D. Hjelmstad. *Fundamentals of Structural Mechanics*. Springer, 2nd edition, 2005.

D. H. Hodges. Nonlinear beam kinematics for small strains and finite rotations. *Vertica*, 11:573–589, 1987.

D. H. Hodges. *Nonlinear Composite Beam Theory*, volume 2013 of *Progress in Astronautics and Aeronautics*. American Institute of Aeronautics and Astronautics, 2006.

D. H. Hodges and E. H. Dowell. Non-linear equations of motion for the elastic bending and torsion of twist non-uniform rotor blades. AMS Reports 1194 and 1257, Princeton University, January and December 1975.

D. H. Hodges, R. A. Ormiston, and D. A. Peters. On the nonlinear deformation geometry of Euler–Bernoulli beams. Technical report, NASA TP-1566, 1980.

G. A. Holzapfel. *Nonlinear Solid Mechanics, A Continuum Approach for Engineering.* Wiley, 2001.

D. Husemöller. *Fibre Bundles*, volume 20 of *Graduate Texts in Mathematics.* Springer, 1994.

E. Kanso, M. Arroyo, Y. Tong, A. Yavary, J. E. Marsden, and M. Desbrun. On the geometric character of stress in continuum mechanics. *Zeitschrift für Angewandte Mathematik und Physik*, 58:843–856, 2007.

G. R. Kirchhoff. *Vorlesungen über mathematische Physik: Mechanik I.* B.G. Teubner, 1876.

A. Kriegl and P. W. Michor. *The Convenient Setting of Global Analysis*, volume 53 of *Mathematical Surveys and Monographs.* American Mathematical Society, 1997.

W. Kühnel. *Differentialgeometrie: Kurven – Flächen – Mannigfaltigkeiten.* Aufbaukurs Mathematik. Springer Spektrum, 6th edition, 2013.

W. Lacarbonara. *Nonlinear Structural Mechanics: Theory, Dynamical Phenomena and Modeling.* Springer, 2013.

L. D. Landau and E. M. Lifshitz. *Theory of Elasticity*, volume 7 of *Course of Theoretical Physics.* Butterworth-Heinemann, 1986.

J. M. Lee. *Introduction to Smooth Manifolds*, volume 218 of *Graduate Texts in Mathematics.* Springer New York, 2nd edition, 2012.

I. S. Liu. *Continuum Mechanics.* Springer, 2002.

A. E. H. Love. *A Treatise on the Mathematical Theory of Elasticity.* Dover Books on Engineering Series. Dover Publications, 1944.

L. E. Malvern. *Introduction to the Mechanics of a Continuous Medium.* Prentice-Hall Series in Engineering of the Physical Sciences. Prentice-Hall, 1965.

J. E. Marsden. *Applications of Global Analysis in Mathematical Physics*, volume 2 of *Mathematical Lecture Series.* Publish or Perish, Inc., 1974.

J. E. Marsden and T. J. R. Hughes. *Mathematical Foundations of Elasticity.* Dover Publications, 1983.

G. A. Maugin. The method of virtual power in continuum mechanics: Application to coupled fields. *Acta Mechanica*, 35:1–70, 1980.

G. A. Maugin. *Continuum Mechanics Through the Twentieth Century: A Concise Historical Perspective.* Solid Mechanics and Its Applications. Springer, 2013.

P. W. Michor. *Manifolds of Differentiable Mappings.* Shiva Publications, 1980.

P. W. Michor. *Topics in Differential Geometry*, volume 93 of *Graduate Studies in Mathematics*. American Mathematical Society, 2008.

J. J. Moreau. *Fonctionnelles convexes, Séminaire sur les Équations aux Dérivées Partielles, Collège de France, 1966, et Edizioni del Dipartimento di Ingeneria Civile dell'Università di Roma Tor Vergata, Roma.* Séminaire Jean Leray, 1966.

J. J. Moreau. Duality characterization of strain tensor distributions in an arbitrary open set. *Journal of Mathematical Analysis and Applications*, 72(2):760–770, 1979.

J. R. Munkres. *Topology.* Prentice Hall, 2nd edition, 2000.

F. D. Murnaghan. Finite deformations of an elastic solid. *American Journal of Mathematics*, 59(2):235–260, 1937.

P. M. Naghdi. Finite deformation of elastic rods and shells. In *Proceedings of the IUTAM Symposium on Finite Elasticity*, 1980.

P. M. Naghdi and M. B. Rubin. Constrained theories of rods. *Journal of Elasticity*, 14: 343–361, 1984.

M. B. Nayroles. Opérations algébriques en mécanique des structures. *Comptes Rendus de l'Académie des Sciences Paris*, 273:1075–1078, 1971.

W. Noll. A mathematical theory of the mechanical behavior of continuous media. *Archive for Rational Mechanics and Analysis*, 2(1):197–226, 1958.

W. Noll. The foundations of classical mechanics in the light of recent advances in continuum mechanics. In *The Axiomatic Method, with Special Reference to Geometry and Physics*, pages 266–281, 1959.

R. W. Ogden. *Non-linear Elastic Deformations.* Dover Publications, 1997.

O. M. O'Reilly. On constitutive relations for elastic rods. *International Journal of Solids and Structures*, 35(11):1009–1024, 1998.

R. S. Palais. *Foundations of Global Analysis.* W. A. Benjamin, 1968.

O. Papes. *Nonlinear Continuum Mechanics in Modern Engineering Applications.* PhD thesis, ETH Zurich, 2012.

G. Piola. *La Meccanica de' Corpi Naturalmente Estesi: Trattata col Calcolo delle Variazioni.* Presso Paolo Emilio Giusti, 1833.

E. Reissner. On a variational theorem for finite elastic deformations. *Journal of Mathematics and Physics*, 32:129–135, 1953.

E. Reissner. On one-dimensional finite-strain beam theory: The plane problem. *Zeitschrift für Angewandte Mathematik und Physik*, 23:795–804, 1972.

E. Reissner. On finite deformations of space-curved beams. *Zeitschrift für Angewandte Mathematik und Physik*, 32:734–744, 1981.

R. T. Rockafellar. *Convex Analysis*. Princeton Mathematical Series. Princeton University Press, 1970.

M. B. Rubin. Restrictions on nonlinear constitutive equations for elastic rods. *Journal of Elasticity*, 44:9–36, 1996.

M. B. Rubin. *Cosserat Theories: Shells, Rods, and Points*. Solid Mechanics and its Applications. Kluwer Academic Publishers, Dordrecht, 2000.

W. Rudin. *Real and Complex Analysis*. Tata McGraw-Hill, 1987.

J. Salençon. *Handbook of Continuum Mechanics: General Concepts, Thermoelasticity*. Physics and Astronomy Online Library. Springer, 2001.

D. J. Saunders. *The Geometry of Jet Bundles*. Cambridge University Press, 1989.

R. Segev. *Differentiable Manifolds and Some Basic Notions of Continuum Mechanics*. PhD thesis, University of Calgary, 1981.

R. Segev. On the definition of forces in continuum mechanics. In *Mathematical Theory of Dynamical Systems and Microphysics*, pages 341–357, 1984.

R. Segev. On the existence of stresses in continuum mechanics. *Israel Journal of Technology*, 23:75–78, 1986a.

R. Segev. Forces and the existence of stresses in invariant continuum mechanics. *Journal of Mathematical Physics*, 27(1):163–170, 1986b.

R. Segev. A geometrical framework for the statics of materials with microstructure. *Mathematical Models and Methods in Applied Sciences*, 6:871–897, 1994.

R. Segev. Notes on stresses for manifolds. *Universitae Politecnico di Torino*, 58:199–205, 2000.

R. Segev. Metric-independent analysis of the stress-energy tensor. *Journal of Mathematical Physics*, 43(6):3220–3231, 2002.

R. Segev. Notes on metric independent analysis of classical fields. *Mathematical Methods in the Applied Sciences*, 36(5):497–566, 2013.

R. Segev and G. Rodnay. The divergence of stress and the principle of virtual power on manifolds. *Technische Mechanik*, 20(2):129–136, 2000.

J. C. Simo. A finite strain beam formulation. the three-dimensional dynamic problem. part I. *Computer Methods in Applied Mechanics and Engineering*, 49:55–70, 1985.

J. C. Simo and L. Vu-Quoc. A geometrically-exact rod model incorporating shear and torsion-warping deformation. *International Journal of Solids and Structures*, 27(3): 371–393, 1991.

J. C. Simo, J. E. Marsden, and P. S. Krishnaprasad. The Hamiltonian structure of nonlinear elasticity: The material and convective representations of solids, rods, and plates. *Archive for Rational Mechanics and Analysis*, 104(2):125–183, 1988.

I. S. Sokolnikoff. *Mathematical Theory of Elasticity.* McGraw-Hill, 1946.

A. J. M. Spencer. *Continuum Mechanics.* Dover Publications, 2004.

W. Thomson and P. G. Tait. *Treatise on natural philosophy*, volume 1. Clarendon Press, 1867.

S. P. Timoshenko. LXVI. On the correction for shear of the differential equation for transverse vibrations of prismatic bars. *Philosophical Magazine Series 6*, 41(245):744–746, 1921.

S. P. Timoshenko. X. On the transverse vibrations of bars of uniform cross-section. *Philosophical Magazine Series 6*, 43(253):125–131, 1922.

C. Truesdell. Die Entwicklung des Drallsatzes. *Zeitschrift für Angewandte Mathematik und Mechanik*, 44(4-5):149–158, 1964.

C. Truesdell. *A First Course in Rational Continuum Mechanics.* Academic Press, 1977.

C. Truesdell and W. Noll. The non-linear field theories of mechanics. In S. Flügge, editor, *The non-linear field theories of mechanics*, volume III/3 of *Encyclopedia of Physics.* Springer, 1965.

C. Truesdell and R. Toupin. The classical field theories. In S. Flügge, editor, *Principles of Classical Mechanics and Field Theory*, volume III/1 of *Encyclopedia of Physics.* Springer, 1960.

P. Villaggio. *Mathematical Models for Elastic Structures.* Cambridge University Press, 2005.

G. Wempner. *Mechanics of Solids with Applications to Thin Bodies.* McGraw-Hill, 1973.

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Publications

[J1] Eugster, S. R., Hesch, C., Betsch, P., Glocker, Ch.: Director-based beam finite elements relying on the geometrically exact beam theory formulated in skew coordinates. *International Journal for Numerical Methods in Engineering*, 97(2):111–129, 2014.

[J2] Eugster, S. R., Glocker, Ch.: Constraints in structural and rigid body mechanics: A frictional contact problem. *Annals of Solid and Structural Mechanics.* 5(1-2):1–13, 2013.

[P1] Eugster, S. R., Glocker, Ch.: Determination of the transverse shear stress in an Euler–Bernoulli beam using non-admissible virtual displacements. In *Proceedings in Applied Mathematics and Mechanics.* 2 pages, to appear.

[P2] Eugster, S. R., Glocker, Ch.: Dynamical behavior of a nonlinear elastic catenary with a rigid disk – a nonlinear finite element formulation. In *Proceedings of the Multibody Dynamics 2011 ECCOMAS Thematic Conference*, 2011.

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Presentations

- Eugster, S. R., Glocker, Ch.: *Determination of the Transverse Shear Stress in an Euler–Bernoulli Beam using Non-Admissible Virtual Displacements.* GAMM Annual Meeting 2014, Erlangen, Germany, 10 – 14 March 2014.

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