# Comparison of Moreau-type integrators based on the time finite element discretization of the virtual action

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*Summary*. In this paper, we derive and compare three integrators for nonsmooth finite dimensional mechanical systems by discretizing the principle of virtual action with finite elements in time. As shape functions, linear Lagrangian polynomials are used. The different integrators are derived by applying different quadrature rules for the discretization of the strong or the weak variational form of the virtual action. After the discretization, the constitutive laws for the contact forces are introduced, resulting in Moreau's time-stepping scheme and two other schemes. Several examples are used to compare these integrators in terms of longterm performance.

Consider the motion of a finite dimensional mechanical system $\mathcal{S}$ during the time interval $\mathcal{I} = [0,\, T]$, where the system is parametrized by the time $t$ and a set of generalized coordinates $\mathbf{q}(t) \in \mathbb{R}^n$. Let $\dot{\mathbf{q}}(t) \in \mathbb{R}^n$ denote the corresponding generalized velocities of $\mathcal{S}$. Furthermore, the mechanical system $\mathcal{S}$ is subjected to scleronomic geometric unilateral constraints. The weak variational form (weak with respect to time) of the virtual action of the system is

$$\delta A = \int_0^T \{\delta T(\mathbf{q}, \dot{\mathbf{q}}) + \delta\mathbf{q}^{\mathrm{T}}(\mathbf{f} + \mathbf{W}\boldsymbol{\lambda})\}\mathrm{d}t + \int_0^T \delta\mathbf{q}^{\mathrm{T}}\mathbf{W}\mathrm{d}\boldsymbol{\Lambda} + \delta\mathbf{q}(0)^{\mathrm{T}}\mathbf{p}_0 - \delta\mathbf{q}(T)^{\mathrm{T}}\mathbf{p}_T\ ,$$

which includes the virtual action of the impulsive and nonimpulsive contact forces $\mathrm{d}\boldsymbol{\Lambda}$ and $\boldsymbol{\lambda}$, respectively. The nonimpulsive forces $\boldsymbol{\lambda}$ model the contact interactions during the impact-free motion. The impulsive forces $\mathrm{d}\boldsymbol{\Lambda}$, given by a sum of Dirac point measures, [1], model the contact interactions during impacts. The matrix $\mathbf{W}$ is composed of the generalized force directions of each contact and the kinetic energy $T = \frac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{M}(\mathbf{q})\dot{\mathbf{q}}$ with symmetric mass matrix $\mathbf{M}$ models the inertia of the system. Integrating the inertia term $\delta\dot{\mathbf{q}}^{\mathrm{T}}\,\mathbf{M}\dot{\mathbf{q}}$ arising in the variation of the kinetic energy by parts leads to the strong variational form of the virtual action $\delta A$ of the mechanical system. The principle of virtual action states that $\delta A$ vanishes for all virtual displacement fields $\delta\mathbf{q}$. A temporal finite element discretization of this variational principle by choosing compatible virtual displacement fields allows to deduce time-stepping schemes for nonsmooth mechanical systems.

Three integrators are derived by the discretization of either the weak or the strong variational form of the principle of virtual action with linear Lagrangian elements using different quadrature rules for the integral over time $t$. These integrators are stepping schemes of the form $(\mathbf{q}_{k-1}, \mathbf{q}_k) \mapsto (\mathbf{q}_k, \mathbf{q}_{k+1})$, where $\mathbf{q}_k \approx \mathbf{q}(t_k)$ approximates the generalized coordinate at time $t_k$. A constant time step $\Delta t = t_{k+1} - t_k$ is used for each step $k$.

Using a rectangle rule as quadrature for the integral over time in the discretized strong variational form of the virtual action leads to Moreau's time-stepping scheme [2, 4], which is given by

$$\begin{aligned} &\mathbf{M}(\mathbf{q}_k, t_k)\big(\mathbf{u}_k - \mathbf{u}_{k-1}\big) - \mathbf{h}(\mathbf{q}_k, \mathbf{u}_{k-1}, t_k)\Delta t = \mathbf{W}(\mathbf{q}_k)\mathbf{P}_k \\ &\text{with} \quad \mathbf{u}_k = \frac{1}{\Delta t}(\mathbf{q}_{k+1} - \mathbf{q}_k), \qquad \mathbf{u}_{k-1} = \frac{1}{\Delta t}(\mathbf{q}_k - \mathbf{q}_{k-1})\ , \end{aligned} \tag{1}$$

where $\mathbf{u}_k$ can be interpreted as an approximation of the velocity $\dot{\mathbf{q}}(t)$ for $t \in (t_k, t_{k+1})$. Furthermore, we refer to [1] for details on the force vector $\mathbf{h}$. A symmetric version of Moreau's scheme, which we call symmetric Moreau-type scheme, is derived by choosing the trapezoidal rule instead. This leads to the stepping scheme

$$\begin{aligned} &\mathbf{M}(\mathbf{q}_k, t_k)\big(\mathbf{u}_k - \mathbf{u}_{k-1}\big) - \frac{\Delta t}{2}\Big(\mathbf{h}(\mathbf{q}_k, \mathbf{u}_{k-1}, t_k) + \mathbf{h}(\mathbf{q}_k, \mathbf{u}_k, t_k)\Big) = \mathbf{W}(\mathbf{q}_k)\mathbf{P}_k \\ &\text{with} \quad \mathbf{u}_k = \frac{1}{\Delta t}(\mathbf{q}_{k+1} - \mathbf{q}_k), \qquad \mathbf{u}_{k-1} = \frac{1}{\Delta t}(\mathbf{q}_k - \mathbf{q}_{k-1})\ . \end{aligned} \tag{2}$$

The discretization of the strong variational form is performed using similar ideas as in [5]. The main difference is that a discontinuous Galerkin approach is used in [5] whereas continuous shape functions for the virtual displacements are used here.

The discretization of the weak variational form of the principle of virtual action together with the trapezoidal rule leads to the variational Moreau-type stepping scheme derived in [3], which is given by

$$\begin{aligned} &\Bigg[\frac{1}{2}\Big(-\mathbf{M}(\mathbf{q}_{k+1}, t_{k+1})\mathbf{u}_k - \mathbf{M}(\mathbf{q}_k, t_k)\big(\mathbf{u}_k - \mathbf{u}_{k-1}\big) + \mathbf{M}(\mathbf{q}_{k-1}, t_{k-1})\mathbf{u}_{k-1}\Big) \\ &\qquad + \frac{1}{2}\Big(\mathbf{b}(\mathbf{q}_k, \mathbf{u}_{k-1}, t_k) + \mathbf{b}(\mathbf{q}_k, \mathbf{u}_k, t_k)\Big)\Delta t + \mathbf{W}(\mathbf{q}_k)\mathbf{P}_k\Bigg] = 0 \\ &\text{with} \quad \mathbf{u}_k = \frac{1}{\Delta t}(\mathbf{q}_{k+1} - \mathbf{q}_k), \qquad \mathbf{u}_{k-1} = \frac{1}{\Delta t}(\mathbf{q}_k - \mathbf{q}_{k-1})\ , \end{aligned} \tag{3}$$

where the force vector $\mathbf{b}$ denotes the sum $\mathbf{b} = \left(\frac{\partial T}{\partial \mathbf{q}}\right)^{\mathrm{T}} + \mathbf{f}$. The discretization of the weak variational form of the virtual action is a common way to obtain variational integrators in the framework of discrete mechanics, cf. [6]. Therefore, this integrator can be regarded as a variational integrator for nonsmooth mechanical systems.
The discrete constitutive laws for the contact forces are introduced as normal cone inclusions between the percussion $\mathbf{P}$, which combines the effects of both $\mathrm{d}\boldsymbol{\Lambda}$ and $\boldsymbol{\lambda}$ during a temporal element, and a kinematic quantity $\boldsymbol{\xi}$, depending on is given by pre- and post-impact velocities. Introduction of the gap functions $g_i(\mathbf{q})$ which indicate if the $i^{\mathrm{th}}$ contact is open ($g_i(\mathbf{q}) > 0$), closed ($g_i(\mathbf{q}) = 0$) or penetrated ($g_i(\mathbf{q}) < 0$) allows to formulate the discrete contact law as

$$\begin{aligned} &\forall i \in \mathcal{J} = \{i \,|\, g_i(\mathbf{q}_k) \leq 0\} : \\ &-\xi_{Ni,k} \in \mathcal{N}_{\mathbb{R}_0^+}(P_{Ni,k}) \quad \text{with} \quad \xi_{Ni,k} = \mathbf{w}_{Ni}^{\mathrm{T}}(\mathbf{q}_k)\big(\mathbf{u}_k + e_{Ni}\mathbf{u}_{k-1}\big) \\ &-\boldsymbol{\xi}_{Ti,k} \in \mathcal{N}_{C_{Ti}(P_{Ni,k})}(\mathbf{P}_{Ti,k}) \quad \text{with} \quad \boldsymbol{\xi}_{Ti,k} = \mathbf{W}_{Ti}^{\mathrm{T}}(\mathbf{q}_k)\big(\mathbf{u}_k + e_{Ti}\mathbf{u}_{k-1}\big) \ , \end{aligned} \tag{4}$$

where $\mathbf{w}_{Ni}$ and $\mathbf{W}_{Ti}$ are the generalized force directions in normal and tangential direction with respect to the contact plane. The set $C_{Ti}$ is the set of allowed friction forces of the $i^{\mathrm{th}}$ contact and the parameters $e_{Ni/Ti}$ denote the restitution coefficients of the underlying Newton impact law.
The above schemes share many advantageous properties with the one of Moreau. For instance, these schemes allow for multiple, simultaneous contacts and can overcome accumulation points. Benchmarks to show these properties are the bouncing ball and the woodpecker toy [7]. As the three integrators coincide for systems with constant mass matrix and forces which are independent of the system's velocity, for this type of systems all results about Moreau's scheme are valid also for the other schemes. However, the other schemes show better longterm simulation behavior than Moreau's scheme. The constrained two dimensional spring pendulum, depicted in Figure 1, exemplarily shows the good longterm energy behavior of the integrators, cf. Figure 2.

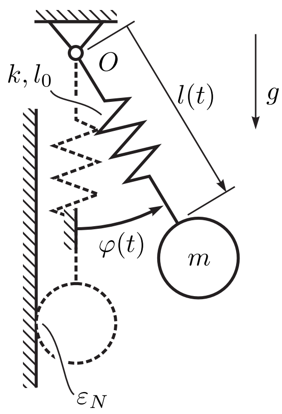


Figure 1: The spring pendulum [3].

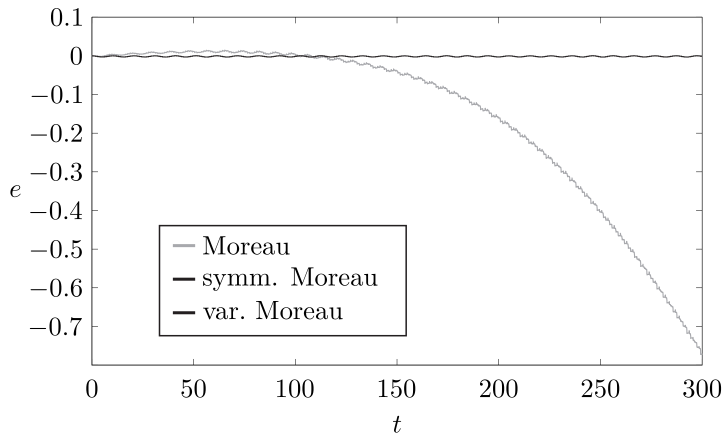


Figure 2: Energy error $e$ of the conservative spring pendulum plotted over time $t$.

The symmetric and variational Moreau-type integrators exhibit good longterm conservation properties due to their symmetric or variational nature. This structure of the integrators is often easier to achieve by a derivation from a variational formulation, than by derivation from a measure differential inclusion, which is also used to describe the motion of a nonsmooth mechanical system. In fact, the discretization of a weak variational form with finite elements in time always leads to variational integrators, which show good structure preserving properties, cf. [6]. Moreover, the choice of the quadrature rule directly influences the symmetry of the scheme, which for time reversible systems guarantees a good longterm behavior for first integrals of the mechanical system, cf. [8].

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