Moreau-type integrators based on the time finite element discretization of the virtual action

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In this paper, we derive and compare three integrators for nonsmooth mechanical systems by discretizing the principle of virtual action with finite elements in time. The weak as well as the strong variational form of the principle are discretized using a piecewise linear shape function and different quadrature rules. After introducing a suitable constitutive law for the contact forces arising in the discretized system, this approach leads to the well established time-stepping scheme of Moreau [1], the variational Moreau-type scheme derived in [3] and another related scheme, which we call the symmetric Moreau-type scheme. It is shown using a benchmark system that the symmetric and the variatonal Moreau-type schemes, in contrast to Moreau's scheme, show an excellent longterm energy behavior.

1 Introduction

The time-stepping scheme of Moreau [1] is a well established integrator for the simulation of mechanical systems with frictional unilateral constraints. It is known from [2] that this scheme may numerically dissipate energy, which makes it unsuitable for the integration of conservative systems. In this paper, we derive the time-stepping scheme of Moreau by discretizing the strong variational form of the principle of virtual action with finite elements in time. By choosing the same shape functions but different quadrature rules, the discretization leads to a symmetric Moreau-type scheme as well as to the variational Moreau-type scheme of [3]. It is shown using an example that these two schemes show a good longterm energy behavior and are therefore superior to the scheme of Moreau when it comes to the simulation of conservative systems.

2 Discretization of the virtual action

The motion of a finite dimensional mechanical system $\mathcal{S}$, which is assumed to be subjected to scleronomic geometric unilateral constraints, is described by a set of generalized coordinates $\mathbf{q}(t) \in \mathbb{R}^n$ and corresponding generalized velocities $\dot{\mathbf{q}}(t)$, where both depend on time $t \in \mathcal{I} = [0, T]$. The weak variational form (weak with respect to time) of the virtual action of the system is

$$\delta A = \int_0^T \{\delta T(\mathbf{q}, \dot{\mathbf{q}}) + \delta \mathbf{q}^{\mathrm{T}}(\mathbf{f} + \mathbf{W}\boldsymbol{\lambda})\}\mathrm{d}t + \int_0^T \delta \mathbf{q}^{\mathrm{T}}\mathbf{W}\boldsymbol{\Lambda}\mathrm{d}\eta + \delta \mathbf{q}(0)^{\mathrm{T}}\mathbf{p}_0 - \delta \mathbf{q}(T)^{\mathrm{T}}\mathbf{p}_T \ ,$$

which includes the virtual action of the impulsive and nonimpulsive contact forces $\boldsymbol{\Lambda}$ and $\boldsymbol{\lambda}$, respectively. The nonimpulsive forces $\boldsymbol{\lambda}$ model the contact interactions during the impact-free motion. The impulsive forces $\boldsymbol{\Lambda}$ model the contact interactions during impacts and $\mathrm{d}\eta$ is given by a sum of Dirac point measures, cf. [4]. The matrix $\mathbf{W}$ is composed of the generalized force directions of each contact and the kinetic energy $T = \frac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{M}(\mathbf{q})\dot{\mathbf{q}}$ with symmetric mass matrix $\mathbf{M}$ models the inertia of the system. Integration by parts of the inertia term $\delta\dot{\mathbf{q}}^{\mathrm{T}}\,\mathbf{M}\dot{\mathbf{q}}$, which arises in the variation of the kinetic energy, leads to the strong variational form of the virtual action

$$\delta A = \int_{\mathcal{I}} \delta \mathbf{q}^{\mathrm{T}} \left[-\mathbf{M}\ddot{\mathbf{q}} - \dot{\mathbf{M}}\dot{\mathbf{q}} + \left(\frac{\partial T}{\partial \mathbf{q}}\right)^{\mathrm{T}} + \mathbf{f} + \mathbf{W}\boldsymbol{\lambda} \right] \mathrm{d}t + \int_{\mathcal{I}} \delta \mathbf{q}^{\mathrm{T}} \Big[-\mathbf{M}\big(\dot{\mathbf{q}}(t^+) - \dot{\mathbf{q}}(t^-)\big) + \mathbf{W}\boldsymbol{\Lambda} \Big] \mathrm{d}\eta + \mathrm{b.t.} \ ,$$

where $\dot{\mathbf{q}}(t^+)$ and $\dot{\mathbf{q}}(t^-)$ denote the right and left limit of the velocity at t, respectively, and b.t. abbreviates the boundary terms at $t = 0$ and $t = T$. The principle of virtual action now states that δA vanishes for all virtual displacement fields $\delta\mathbf{q}$.

Moreau's time stepping scheme is derived by the discretization of the strong variational form of the principle of virtual action with linear Lagrangian elements using compatible virtual displacement fields and the rectangle quadrature rule for the integral over time t. For details about the scheme we refer to [5]. Using the trapezoidal rule, instead of the rectangle rule, the same discretization procedure leads to the symmetric Moreau-type scheme

$$\mathbf{M}(\mathbf{q}_k)\big(\mathbf{u}_k - \mathbf{u}_{k-1}\big) - \frac{\Delta t}{2}\Big(\mathbf{h}(\mathbf{q}_k, \mathbf{u}_{k-1}, t_k) + \mathbf{h}(\mathbf{q}_k, \mathbf{u}_k, t_k)\Big) = \mathbf{W}(\mathbf{q}_k)\mathbf{P}_k$$

$$\text{with} \quad \mathbf{u}_k = \frac{1}{\Delta t}(\mathbf{q}_{k+1} - \mathbf{q}_k), \qquad \mathbf{u}_{k-1} = \frac{1}{\Delta t}(\mathbf{q}_k - \mathbf{q}_{k-1}), \qquad \mathbf{h} = -\dot{\mathbf{M}}\dot{\mathbf{q}} + \left(\frac{\partial T}{\partial \mathbf{q}}\right)^{\mathrm{T}} + \mathbf{f} \ ,$$

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which together with constitutive contact laws for $\mathbf{P}_k = \boldsymbol{\lambda}_k \Delta t + \boldsymbol{\Lambda}_k$ defines a stepping scheme of the form $(\mathbf{q}_{k-1}, \mathbf{q}_k) \mapsto (\mathbf{q}_k, \mathbf{q}_{k+1})$, where $\mathbf{q}_k \approx \mathbf{q}(t_k)$ approximates the generalized coordinate at time t_k. A constant time step $\Delta t = t_{k+1} - t_k$ is used for each step k.

The discrete constitutive laws for the contact forces are introduced as normal cone inclusions between the percussion $\mathbf{P}_k$ and a kinematic quantity $\boldsymbol{\xi}$, is given by pre- and post-impact velocities. Introduction of the gap functions $g_i(\mathbf{q})$ which indicate if the i^{th} contact is open ($g_i(\mathbf{q}) > 0$), closed ($g_i(\mathbf{q}) = 0$) or penetrated ($g_i(\mathbf{q}) < 0$) allows to formulate the discrete contact law as

$$\begin{aligned} &\forall i \in \mathcal{J} = \{i \,|\, g_i(\mathbf{q}_k) \leq 0\} : \\ &- \xi_{Ni,k} \in \mathcal{N}_{\mathbb{R}_0^+}(P_{Ni,k}) \quad \text{with} \quad \xi_{Ni,k} = \mathbf{w}_{Ni}^{\mathrm{T}}(\mathbf{q}_k)\big(\mathbf{u}_k + e_{Ni}\mathbf{u}_{k-1}\big) \\ &- \boldsymbol{\xi}_{Ti,k} \in \mathcal{N}_{C_{Ti}(P_{Ni,k})}(\mathbf{P}_{Ti,k}) \quad \text{with} \quad \boldsymbol{\xi}_{Ti,k} = \mathbf{W}_{Ti}^{\mathrm{T}}(\mathbf{q}_k)\big(\mathbf{u}_k + e_{Ti}\mathbf{u}_{k-1}\big) \ , \end{aligned} \tag{1}$$

where $\mathbf{w}_{Ni}$ and $\mathbf{W}_{Ti}$ are the generalized force directions in normal and tangential direction with respect to the contact plane. The set C_{Ti} is the set of allowed friction forces of the i^{th} contact and the parameters $e_{Ni/Ti}$ denote the restitution coefficients of the underlying Newton-type impact law.

Finally, the discretization of the weak variational form of the principle of virtual action with linear Lagrangian elements using compatible virtual displacement fields and the trapezoidal rule leads to the variational Moreau-type stepping scheme derived in [3].

The constrained two dimensional spring pendulum, being conservative for a frictionless contact with $e_N = 1$, is depicted in Figure 1 and exemplary demonstrates the longterm energy behavior of the integrators, cf. Figure 2. It clearly shows that

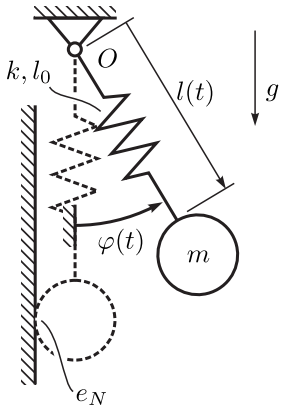

Fig. 1: The spring pendulum [3].

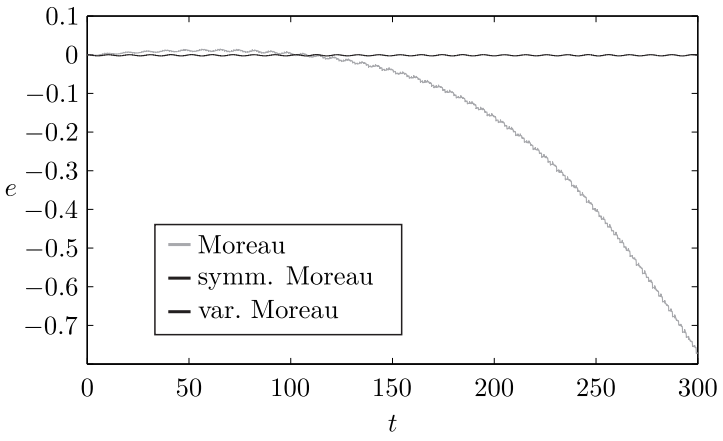

Fig. 2: Energy error e of the conservative spring pendulum plotted over time t.

the scheme of Moreau numerically dissipates energy, whereas both, the symmetric and the variational Moreau-type stepping schemes, show very small energy error. This property is mainly due to their symmetric or variational nature. In fact, the discretization of a weak variational form with finite elements in time always leads to variational integrators, which show good structure preserving properties, cf. [6]. Moreover, the integration of time reversible systems with symmetric schemes is known to exhibit good longterm behavior for first integrals of the mechanical system, cf. [7].

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