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Dynamics of finite-dimensional mechanical systems on Galilean manifolds

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As presented in the related PAMM contribution 'Kinematics of finite-dimensional mechanical systems on Galilean manifolds', the state space of a time-dependent finite-dimensional mechanical system is defined as an affine subbundle of the tangent bundle to the Galilean manifold modeling the generalized space-time of the system. The second-order vector field on the state space that describes the system's motion can be associated with a differential two-form called the action form of the mechanical system. In this paper, we postulate the action form for time-dependent finite-dimensional mechanical systems. Moreover, we show that Lagrange's equations of the second kind can be derived as a chart representation of the conditions that define the second-order vector field which describes the motion.

We model the generalized space-time of an n-dimensional time-dependent finite-dimensional mechanical system as an $n{+}1$-dimensional Galilean manifold M with time-structure $\vartheta \in \Omega^1(M)$ and Galilean metric g. The non-vanishing closed one-form ϑ allows the definition of the spacelike bundle A^0M and the state space A^1M of the system as subbundles of the tangent bundle TM to the generalized space-time M. These subbundles result from TM by the pointwise restrictions

$$A^0_pM = \ker \vartheta_p = \{v_p \in T_pM \,|\, \vartheta_p(v_p) = 0\} \quad \text{and} \quad A^1_pM = \{v_p \in T_pM \,|\, \vartheta_p(v_p) = 1\} \tag{1}$$

of the tangent space T_pM in every $p \in M$. Let Ψ be a (local) natural chart of A^1M which is adapted in the sense, that with $\Psi(p, v_p){=}(t, x^1, \dots, x^n, u^1, \dots, u^n)$ the time coordinate t is such that $\hat{\vartheta}{=}\mathrm{d}t$, where $\hat{\vartheta} = \pi^*\vartheta$ is the pullback of the time-structure with the natural projection $\pi : A^1M \to M, (p, v_p) \mapsto p$. We choose the velocity coordinates u^i to be the natural coordinates to the generalized coordinates x^i of the system.

The motion β of the system is a curve in the state space A^1M and is considered to be an integral curve of a second-order (vector) field $X{\in}\mathrm{Vect}(A^1M)$, i.e., the motion is a solution of the first-order ordinary differential equation $\dot{\beta}(\tau){=}X(\beta(\tau))$, which is the first-order form of a second-order differential equation. It is shown by [1] that on a Galilean manifold to every second-order field Z there is an action form $\Omega \in \Omega^2(A^1M)$. Moreover, if any arbitrary vector field X satisfies

$$\hat{\vartheta}(X) = 1 \quad \text{and} \quad \Omega(X, \cdot\,) = 0\,, \tag{2}$$

it follows that $X = Z$. Therefore, the modeling process for finite-dimensional mechanical systems consists in specifying the action form Ω of the system, which by (2) defines the second-order vector field X having the motion as its integral curve. For more details on the concepts introduced so far, we refer to the related PAMM contribution 'Kinematics of finite-dimensional mechanical systems on Galilean manifolds' and references therein.

We assume that the action form of the system $\Omega = \Omega_R{+}\Phi_R$ is the sum of the action form of the force-free motion Ω_R, which models the inertia of the system and is related to the kinetic energy, and the force two-form Φ_R related to the forces acting on the system. In order to define the kinetic energy T_R of the system, we choose a section R of A^1M, which we call reference field, and define $T_R : A^1M \to \mathbb{R},\ (p, v_p) \mapsto \frac{1}{2}g_p(v_p{-}R_p, v_p{-}R_p)$ using the bundle metric g on A^0M, which models the mass of the system. We postulate the action form of the force-free motion with respect to the reference field R as

$$\Omega_R = \mathrm{d}(T_R\hat{\vartheta} + \partial T_R)\,, \tag{3}$$

where d denotes the exterior derivative and ∂ is the differential operator defined in [2]. Like d, the operator ∂ is an anti-derivation. Let f be a smooth function on A^1M, then the following holds in the adapted natural chart Ψ

$$\partial f = \frac{\partial f}{\partial u^i}(\mathrm{d}x^i - u^i\mathrm{d}t), \quad \partial(\mathrm{d}t) = \partial(\mathrm{d}x^i) = 0, \quad \partial(\mathrm{d}u^i) = \mathrm{d}u^i \wedge \mathrm{d}t\,. \tag{4}$$

Note, that we use Einstein's summation convention, which implies summation over repeated indices.

The differential $\mathrm{D}\pi$ of the natural projection defines the vertical bundle $\mathrm{Ver}(A^1M)$ as the subbundle of the tangent bundle $T(A^1M)$. The vertical bundle is pointwise defined by the subspaces of $T_a(A^1M)$ given by the kernel of $\mathrm{D}\pi_a$ in $a \in A^1M$, i.e.,

$$\mathrm{Ver}(A^1M) = \bigcup_{a \in \pi^{-1}(U)} \{a\} \times \mathrm{span}\left\{ \left.\frac{\partial}{\partial u^1}\right|_a, \dots, \left.\frac{\partial}{\partial u^n}\right|_a \right\}, \tag{5}$$

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where $\pi^{-1}(U)$ is the domain of the chart Ψ. Let $\mathrm{Ver}^*(A^1M)$ be the dual bundle of $\mathrm{Ver}(A^1M)$. We call a section F of the bundle $\mathrm{Ver}^*(A^1M)$ a force. Therefore, it has the local representation $F{=}F_i\mathrm{d}u^i$. As proven in [1], there is a bijection between any force F and the force two-form

$$\Phi = F_i\,\mathrm{d}x^i\wedge\mathrm{d}t + \frac{1}{2}\frac{\partial F_i}{\partial u^j}(\mathrm{d}x^i - u^i\mathrm{d}t)\wedge(\mathrm{d}x^j - u^j\mathrm{d}t)\;. \tag{6}$$

We say that a force F^{p} is a potential force if its related force two-form Φ^{p} is closed, i.e., $\mathrm{d}\Phi^{\mathrm{p}}=0$. Let $\Phi_R=\Phi_R^{\mathrm{p}}+\Phi_R^{\mathrm{np}}$ denote the force two-form related to the sum of all forces that act on the mechanical system, which we split into the sum of a potential force two-form Φ_R^{p} with $\mathrm{d}\Phi_R^{\mathrm{p}}=0$ and a nonpotential force two-form Φ_R^{np}. By the Poincaré lemma, there is a one-form α_R such that the potential force two-form is locally given by $\Phi_R^{\mathrm{p}}{=}\mathrm{d}\alpha_R$. We assume, that the potential forces are velocity-independent. In that case, it can be shown that $\alpha_R{=}-V_R\mathrm{d}t{=}-V_R\hat{\vartheta}$, where the velocity-independent coefficient function V_R is called the potential energy of the system. The action form of the mechanical system is given by $\Omega{=}\Omega_R{+}\Phi_R{=}\Omega_R{+}\mathrm{d}\alpha_R{+}\Phi_R^{\mathrm{np}}$. As $\partial V_R=0$, we have $\mathrm{d}\alpha_R=\mathrm{d}(-V_R\hat{\vartheta}+\partial(-V_R))$, which together with (3) leads to

$$\Omega = \mathrm{d}(L_R\hat{\vartheta}+\partial L_R)+\Phi_R^{\mathrm{np}} = \mathrm{d}\omega_R+\Phi_R^{\mathrm{np}} \quad \text{with} \quad \omega_R = L_R\hat{\vartheta}+\partial L_R\;, \tag{7}$$

where we have introduced the Lagrangian $L_R=T_R-V_R$ and the Cartan one-form ω_R of the system.

Summing up, we *postulate* that the vector field X, of which the motion β of the finite-dimensional mechanical system is an integral curve, is defined by (2), where the action form Ω is given by (7).

To gain more insight, we state the above postulate in the local coordinates given by the natural chart Ψ. Dropping the subscript R for the Lagrangian, the vector field X with $\hat{\vartheta}(X){=}1$ and the Cartan one-form can be locally written as

$$X = \frac{\partial}{\partial t} + A^i\frac{\partial}{\partial x^i} + B^i\frac{\partial}{\partial u^i} \quad \text{and} \quad \omega_R = L\,\mathrm{d}t + \frac{\partial L}{\partial u^i}(\mathrm{d}x^i - u^i\mathrm{d}t)\;. \tag{8}$$

Let F_i denote the coefficients of the nonpotential force two-form Φ_R^{np}, which has the form (6). Using this and evaluating the exterior derivative of ω_R given by (8) leads to the expression

$$\Omega = \mathrm{d}L\wedge\mathrm{d}t + \mathrm{d}\left(\frac{\partial L}{\partial u^i}\right)\wedge(\mathrm{d}x^i - u^i\mathrm{d}t) - \frac{\partial L}{\partial u^i}\mathrm{d}u^i\wedge\mathrm{d}t + F_i\,\mathrm{d}x^i\wedge\mathrm{d}t + \frac{1}{2}\frac{\partial F_i}{\partial u^j}(\mathrm{d}x^i - u^i\mathrm{d}t)\wedge(\mathrm{d}x^j - u^j\mathrm{d}t) \tag{9}$$

for the action form (7). By (2), the one-form $\varphi=\Omega(X,\,\cdot\,)\in\Omega^1(A^1M)$ has to be zero, which is equivalent to the vanishing of all of its components with respect to the basis $\{\mathrm{d}t,\mathrm{d}x^i,\mathrm{d}u^i\}$ of $\Omega^1(A^1M)$. Using (9), the $\mathrm{d}u^i$-components of φ are

$$\varphi\left(\frac{\partial}{\partial u^i}\right) = \Omega\left(X\,,\frac{\partial}{\partial u^i}\right) = \frac{\partial^2 L}{\partial u^i\partial u^j}(u^j - A^j) = g_{ij}(u^j - A^j)\;, \tag{10}$$

where the last equality follows by the definition of the Lagrangian and the g_{ij} denote the coefficients of the Galilean metric g, which has full rank. Therefore, the $\mathrm{d}u^i$-components (10) of φ vanish if and only if $A^j=u^j$ for $j=1,\dots,n$, which makes the vector field X a second-order field. The $\mathrm{d}x^i$-components of φ are

$$\varphi\left(\frac{\partial}{\partial x^i}\right) = \Omega\left(X\,,\frac{\partial}{\partial x^i}\right) = \mathcal{L}_X\left(\frac{\partial L}{\partial u^i}\right) - \frac{\partial L}{\partial x^i} - F_i\;, \tag{11}$$

where we already used the second-order condition $A^j=u^j$ for $j=1,\dots,n$ and $\mathcal{L}_X$ denotes the Lie derivative with respect to X. The vanishing of the $\mathrm{d}x^i$-components (11) of φ implies Lagrange's equations of the second kind for mechanical systems with position-, velocity- and time-dependent forces. To bring this in a more common form, we note, that the $\mathrm{d}x^i$-components have to vanish for every point of the state space, this is true especially for points along the motion β. Using this together with the definition of the Lie derivative and the fact that the motion β is an integral curve of the second-order vector field X, the vanishing of the $\mathrm{d}x^i$-components (11) of φ along the motion implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial u^i}\circ\beta(t)\right) - \frac{\partial L}{\partial x^i}\circ\beta(t) - F_i\circ\beta(t) = 0\;. \tag{12}$$

Note, that by the conditions $A^j{=}u^j$ and $\hat{\vartheta}(X){=}1$, the motion has the chart representation $\Psi\circ\beta(t) = (t,x^i(t),\dot{x}^i(t))$. Moreover, it can be shown, that the $\mathrm{d}t$-component of φ is linearly dependent on the other components and therefore vanishes if the others vanish, which gives no new conditions besides (10) and (11). An important aspect of the presented approach is that physical quantities such as the kinetic energy, the forces and the action form are defined as chart-independent objects. This culminates in the fact that the classical equations of motion, such as Lagrange's and Hamilton's equations, are unified as chart representations of the above postulate, see [1] for Hamilton's equations.

References

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