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DYNAMICAL BEHAVIOR OF A NONLINEAR ELASTIC CATENARY IN CONTACT WITH A RIGID DISK - A NONLINEAR FINITE ELEMENT FORMULATION

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Abstract. *The main result of the paper is a novel approach for the treatment of hard contacts, i.e. unilateral constraints, within a nonlinear finite element framework. The mechanical description and the corresponding nonlinear finite element formulation allow for large deformations in time and nonlinear elastic material response. The dynamical response of a nonlinear elastic string which comes into contact with a rigid disk is analyzed.*

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1 Introduction

A general theory which treats the dynamics of non-smooth systems in finite degree of freedom mechanics was already developed in Ref. [1]. It emerged that the principle of virtual work, originally stated by Lagrange, may be used even in the extended framework of non-smooth dynamics as a fundamental principle. Recently, Ref. [2] proposed an axiomatic approach where the principle of virtual work is used to derive the equations of motion for generalized continua. Also in Ref. [3], where a thermomechanical problem is considered, the virtual work is the key equation to derive the time evolution of a more general system. In structural mechanics where elements like strings, beams, shells and many more are treated, approximate solutions for the three dimensional problem, especially for the linear theory, are still very present. As shown in Ref. [4], arbitrary structural elements may be deduced by the principle of virtual work and constraint methods. Hence, the authors propose a unified formulation of deformable and rigid bodies at the example of a flexible multibody system including hard contacts, based on the principle of virtual work. In a very natural way the finite element method using absolute nodal description (cf. Ref. [5]) is elaborated. The formulation allows for arbitrary time independent material models and nonlinear deformation.

In Sec. 3 the equations of motion of the discretized system are obtained with the principle of virtual work. General assumptions on force distributions are done and the discretization of the string is shown in general and performed by a nonlinear shape function. In Sec. 4, the constitutive equations for the introduced force distributions are specified. For the contact forces, set-valued force laws are introduced which require an impact law to fully describe the dynamics of the system. In Sec. 5 numerical results of the problem are shown.

2 Problem Description

As an example, the dynamical system depicted in Fig. 1 is analyzed. It consists of a homogenous disk and a string in the vertical plane. The disk is considered as a rigid body and is characterized by its radius R and its density ρ_D. The string which is suspended at A and B, has an undeformed length L, mass density ρ_0 in its reference configuration, stiffness k and is perfectly flexible, i.e. has no bending stiffness. The contact between the disk and the string is modeled as a hard unilateral constraint with Coulomb type friction. In the case of impact, a Newton-type impact law is used. The disk and the string are subjected to gravity.

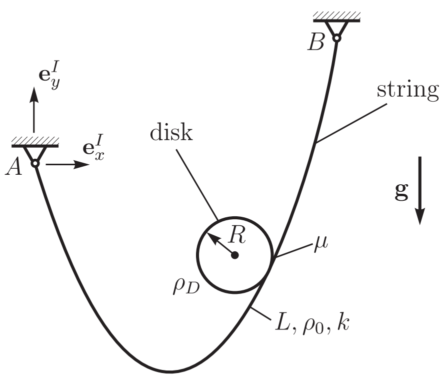

Figure 1: The dynamical system with a rigid disk and a nonlinear elastic string.

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3 Virtual Work of the System

The mechanical system is considered as a set of material points $\mathcal{S}$ each of it placed in a three dimensional Euclidean space by a corresponding time dependent position vector $\boldsymbol{\xi}(\cdot, t)$. Presently $(\cdot)$ stands for a specific material point. The vector field $\boldsymbol{\xi}(\cdot, t)$ defines the motion of the system $\mathcal{S}$. Differentiability with respect to time almost everywhere allows for the definition of a velocity and an acceleration vector field as $\dot{\boldsymbol{\xi}}(\cdot, t) = \partial\boldsymbol{\xi}(\cdot, t)/\partial t$ and $\ddot{\boldsymbol{\xi}}(\cdot, t) = \partial^2\boldsymbol{\xi}(\cdot, t)/\partial t^2$, respectively. The mass $\mathrm{d}m$ of a material point, placed at $\boldsymbol{\xi}$, is subjected to internal and external forces $\mathrm{d}\mathbf{F}$. The principle of virtual work states that if the virtual work

$$\delta W = \int_{\mathcal{S}} \delta\boldsymbol{\xi}^{\mathrm{T}}(\ddot{\boldsymbol{\xi}}\mathrm{d}m - \mathrm{d}\mathbf{F}) = 0 \quad \forall\delta\boldsymbol{\xi}, \forall t \tag{1}$$

vanishes for all variations $\delta\boldsymbol{\xi}$, then the system $\mathcal{S}$ is in dynamical equilibrium.

The system consists of two subsystems: The string $\mathcal{S}_1$ and the disk $\mathcal{S}_2$. Note that the problem is two-dimensional, thus we neglect the third vector component in the further derivation. Subsequently, since the integral is additive, derivations of the kinematics and force distributions are done separately for each body and indexed by $(\bullet)_1$ for the string and $(\bullet)_2$ for the disk.

3.1 Kinematics

The string is modeled as a one-dimensional deformable body which means that each material point of it may be addressed by a parameter $s = [0, L]$. With foresight to the numerical evaluation, the kinematics of the string is constrained in such a way that the corresponding position vector

$$\boldsymbol{\xi}_1(s, t) = \mathbf{r}_1(s, \mathbf{q}_1(t)) \tag{2}$$

can be expressed by introducing finitely many generalized coordinates $\mathbf{q}_1(t)$. Together with the formulation of the systems virtual work this leads naturally to the so called finite element method. We speak of local finite elements if the string is divided in the sense of Ref. [3, p. 324]:

> "The system is divided into a certain number of cells each of which is described by a small number of generalized coordinates, in such a way that interconnections constraints are satisfied."

A cell, commonly called element and indexed by $(\bullet)^e$, is a region of the string $\Omega^e = [n^e, n^{e+1}] \subset [0, L] = \cup_e \Omega^e$. The kinematics of this subset is described by the shape function $\mathbf{r}_1^e(s^e, \mathbf{q}_1^e(t))$. The parameter s^e (cf. Eq. $(3)_2$) is called element coordinate and takes values in the interval $[0, 1]$. The connectivity matrix $\mathbf{C}_1^e$ of an element extracts the small number of generalized coordinates $\mathbf{q}_1^e = \mathbf{C}_1^e\mathbf{q}$ out of the generalized coordinates $\mathbf{q}(t)$ which describe the total system. With a correct choice of shape functions $\mathbf{r}_1^e$ and the use of the characteristic function $\chi_{\Omega^e}(s)$, which is 1 inside and vanishes outside the region Ω^e, the motion of the string

$$\mathbf{r}_1(s, \mathbf{q}_1) = \mathbf{r}_1(s, \mathbf{q}) = \sum_{e=1}^{nel} \chi_{\Omega^e}\mathbf{r}_1^e(s^e, \mathbf{q}_1^e), \qquad s^e = \frac{s - n^e}{n^{e+1} - n^e}, \qquad \mathbf{q}_1^e = \mathbf{C}_1^e\mathbf{q}\ , \tag{3}$$

is discretized by nel number of elements. What does a correct choice of shape functions mean? The local shape functions have to be chosen in such a way that the motion $\mathbf{r}_1$ is at least continuous in s. This condition is asked for in standard, polynomial based, finite element analysis. Since the occurring kinks between the elements may corrupt the contact interaction we

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strengthen this condition and ask for a twice differentiable C^2-function. A shape function satisfying this condition is e.g. a cubic Bézier spline of the form

$$\mathbf{r}_1^e(s^e, \mathbf{q}_1^e) = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + (s^e)^2(3 - 2s^e)\begin{pmatrix} x_4 - x_1 \\ y_4 - y_1 \end{pmatrix} + \\ + 3s^e(1-s^e)^2 r_1 \begin{pmatrix} \cos(\varphi_1) \\ \sin(\varphi_1) \end{pmatrix} - 3(s^e)^2(1-s^e) q_4 \begin{pmatrix} \cos(\varphi_4) \\ \sin(\varphi_4) \end{pmatrix}, \quad s^e \in [0,1] \ , \quad (4)$$

which is depicted in Fig. 2. The element function is neither linear in its generalized coordinates $\mathbf{q}_1^e(t) = (x_1\ y_1\ r_1\ \varphi_1\ x_4\ y_4\ q_4\ \varphi_4)^{\mathrm{T}}$ nor restricted to nodal degrees of freedom as in standard finite element formulations.

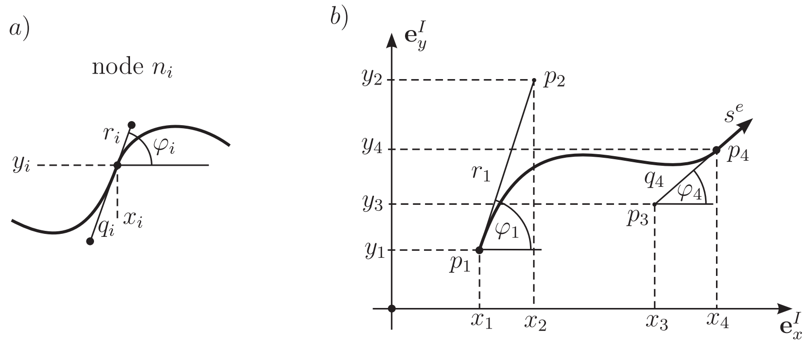

Figure 2: a) The position x_i, y_i and the angle φ_i belong to a node n_i. The weight factors r_i and q_i are degrees of freedom of the element. b) The element on the domain $\Omega^e \subset [0, L]$ can be described by eight generalized coordinates $\mathbf{q}_1^e$.

The disk is modeled as a two-dimensional rigid body. The position of each material point $(r, \vartheta) \in \mathcal{S}_2 = [0, R] \times [0, 2\pi]$, as depicted in Fig. 4, is uniquely described by the position of the center of gravity $\mathbf{r}_{OM} = (x(t)\ y(t))^{\mathrm{T}}$ and the orientation $\varphi(t)$ of the body and can be formulated as

$$\boldsymbol{\xi}_2((r,\vartheta), t) = \mathbf{r}_2((r,\vartheta), \mathbf{q}_2(t)) = \begin{pmatrix} x \\ y \end{pmatrix} + r \begin{pmatrix} \sin(\varphi + \vartheta) \\ -\cos(\varphi + \vartheta) \end{pmatrix}, \qquad \mathbf{q}_2 = \mathbf{C}_2 \mathbf{q}. \qquad (5)$$

Analogous to an element of the string, the generalized coordinates $\mathbf{q}_2(t)$ describing the disk are extracted by its connectivity matrix $\mathbf{C}_2$.

3.2 Force Distributions

Besides the parametrization of the system, the modeling of the occurring forces is equally important and has to be done carefully. In our consideration the string is modeled as perfectly flexible. Thus the stress $\mathbf{t}(s)$ in the string, at least once differentiable in s, will be tangent to the current configuration of the string (cf. Ref. [6, p. 16])

$$\mathbf{t} \times \frac{\partial \mathbf{r}_1}{\partial s} = 0 \Leftrightarrow \mathbf{t} = T \frac{\partial \mathbf{r}_1 / \partial s}{\|\partial \mathbf{r}_1 / \partial s\|} \ , \qquad (6)$$

with a scalar valued function T which contains the force law for a specific material. Using this assumption, we can draw the free body diagram as in Fig. 3 which gives us the following force

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distributions on the string

$$
\begin{aligned}
&\mathrm{d}\mathbf{F}_1 \equiv \tfrac{\partial \mathbf{t}}{\partial s}\mathrm{d}s + (\mathbf{b}_1 + \mathbf{l})\mathrm{d}s + \mathrm{d}\mathbf{z}_1 && \text{for } s \in (0, L) \\
&\mathrm{d}\mathbf{F}_1 \equiv (\mathbf{t} + \mathbf{F}_A)\mathrm{d}\eta && \text{for } s = 0 \\
&\mathrm{d}\mathbf{F}_1 \equiv (-\mathbf{t} + \mathbf{F}_B)\mathrm{d}\eta && \text{for } s = L\ .
\end{aligned}
\tag{7}
$$

The body forces $\mathbf{b}_1$ and the contact force distribution $\mathbf{l}$ are defined per unit line segment $\mathrm{d}s$. The contact force distribution $\mathbf{l}$ models the contact interaction between the string and the disk. A perfect bilateral constraint force distribution $\mathrm{d}\mathbf{z}_1$ guarantees that the string follows the kinematics, dictated by its discretization. The concept of atomic measures $\mathrm{d}\eta$ (cf. Ref. [1, p. 63]) allows us to include concentrated forces at the boundaries.

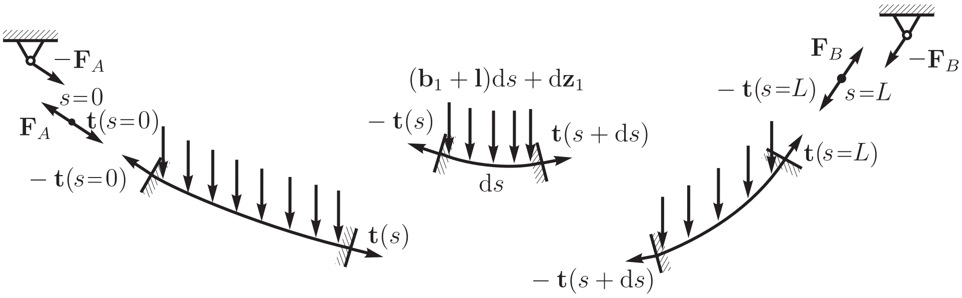

Figure 3: Free body diagram of the string.

The disk is subjected to the body force $\mathbf{b}_2$ defined per unit area segment $r\,\mathrm{d}r\,\mathrm{d}\vartheta$ and the reaction force $-\mathbf{l}$ of the contact force distribution. To fulfill the rigidity conditions a perfect bilateral constraint force distribution $\mathrm{d}\mathbf{z}_2$ is introduced on the interior of the disk and hence

$$
\begin{aligned}
&\mathrm{d}\mathbf{F}_2 \equiv \mathrm{d}\mathbf{z}_2 + \mathbf{b}_2\, r\,\mathrm{d}r\,\mathrm{d}\vartheta && \text{for } (r, \vartheta) \in (0, R) \times (0, 2\pi) \\
&\mathrm{d}\mathbf{F}_2 \equiv -\mathbf{l}R\,\mathrm{d}\vartheta && \text{for } (r, \vartheta) \in \{R\} \times (0, 2\pi)\ .
\end{aligned}
\tag{8}
$$

3.3 Virtual Work for Admissible Virtual Displacements

The principle of virtual work, Eq. (1), holds for all virtual displacements $\delta\boldsymbol{\xi}$. Hence the virtual displacements can be chosen as admissible or as inadmissible with respect to the defined kinematics in Eq. (2) and (5). Virtual displacements admissible to the kinematics are of the form

$$
\delta\boldsymbol{\xi}_i = \frac{\partial \mathbf{r}_i}{\partial \mathbf{q}}\delta\mathbf{q} \qquad \forall \delta\mathbf{q}, \qquad \text{for } i = 1, 2\ . \tag{9}
$$

With respect to the principle of d'Alembert-Lagrange (cf. Ref. [1, p. 48]), the virtual work of the perfect bilateral constraints vanishes for all admissible virtual displacements. Using Eq. (1) together with Eq. (7), (8) and the principle of d'Alembert-Lagrange, the virtual work of $\mathcal{S}$ for admissible virtual displacements, Eq. (9), is obtained as

$$
\begin{aligned}
\delta W = \delta\mathbf{q}^{\mathrm{T}} \Bigg[\Bigg\{ \int_{\mathcal{S}_1} \left(\frac{\partial \mathbf{r}_1}{\partial \mathbf{q}}\right)^{\mathrm{T}} \ddot{\mathbf{r}}_1\,\mathrm{d}m + \int_{\mathcal{S}_2} \left(\frac{\partial \mathbf{r}_2}{\partial \mathbf{q}}\right)^{\mathrm{T}} \ddot{\mathbf{r}}_2\,\mathrm{d}m \Bigg\} + \int_0^L \left(\frac{\partial^2 \mathbf{r}_1}{\partial s \partial \mathbf{q}}\right)^{\mathrm{T}} \mathbf{t}\mathrm{d}s \\
- \Bigg\{ \int_0^L \left(\frac{\partial \mathbf{r}_1}{\partial \mathbf{q}}\right)^{\mathrm{T}} \mathbf{b}_1 \mathrm{d}s + \left(\frac{\partial \mathbf{r}_1}{\partial \mathbf{q}}(s=0)\right)^{\mathrm{T}} \mathbf{F}_A + \left(\frac{\partial \mathbf{r}_1}{\partial \mathbf{q}}(s=L)\right)^{\mathrm{T}} \mathbf{F}_B \\
+ \int_0^{2\pi}\!\! \int_0^R \left(\frac{\partial \mathbf{r}_2}{\partial \mathbf{q}}\right)^{\mathrm{T}} \mathbf{b}_2\, r\,\mathrm{d}r\,\mathrm{d}\vartheta \Bigg\} - \Bigg\{ \int_0^L \left(\frac{\partial \mathbf{r}_1}{\partial \mathbf{q}}\right)^{\mathrm{T}} \mathbf{l}\,\mathrm{d}s - \int_0^{2\pi} \left(\frac{\partial \mathbf{r}_2}{\partial \mathbf{q}}\right)^{\mathrm{T}} \mathbf{l}R\,\mathrm{d}\vartheta \Bigg\} \Bigg]\ ,
\end{aligned}
\tag{10}
$$

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in which integration by parts has been applied. The expressions in the curly brackets are the generalized forces called inertia forces, internal forces, external forces and contact forces. Hence the principle of virtual work can be written in short as

$$\delta W = \delta \mathbf{q}^{\mathrm{T}} \left[\mathbf{f}^{\mathrm{inertia}} + \mathbf{f}^{\mathrm{int}} - \mathbf{f}^{\mathrm{ext}} - \mathbf{f}^{\mathrm{contact}} \right] = 0 \qquad \forall \delta \mathbf{q} \ . \tag{11}$$

Since this equality holds for all virtual displacements $\delta\mathbf{q}$, it relates directly to the strong form of the dynamical equilibrium.

4 Constitutive Equations

To evaluate the system, the generalized forces have to be specified. Therefore constitutive equations for these forces have to be formulated. For several force laws the mass distributions of the two bodies have to be known. The distribution on the boundary vanishes for the particular body. For the string one defines an arbitrary stress free state of the string as its reference state $\mathbf{r}_{1,ref}(s, \mathbf{q}_{1,ref})$. In this state the mass per unit line segment is ρ_0. For the homogenous disk the density ρ_D is defined per unit area segment. Hence the mass distributions of the two bodies are

$$\begin{array}{ll} \mathrm{d}m \equiv \rho_0 \left\| \partial \mathbf{r}_{1,ref} / \partial s \right\| \mathrm{d}s & \text{for } s \in (0, L) \\ \mathrm{d}m \equiv 0 & \text{for } s = \{0, L\} \\ \mathrm{d}m \equiv \rho_D \, r \, \mathrm{d}r \, \mathrm{d}\vartheta & \text{for } (r, \vartheta) \in (0, R) \times (0, 2\pi) \ . \end{array} \tag{12}$$

4.1 Inertia Forces

As stated in Ref. [7, p. 540] or Ref. [8] the following equivalence

$$\begin{aligned} \int_{\mathcal{S}} \left(\frac{\partial \mathbf{r}}{\partial \mathbf{q}} \right)^{\mathrm{T}} \ddot{\mathbf{r}} \mathrm{d}m &= \int_{\mathcal{S}} \mathrm{d}m \left(\frac{\partial \mathbf{r}}{\partial \mathbf{q}} \right)^{\mathrm{T}} \frac{\partial \mathbf{r}}{\partial \mathbf{q}} \ddot{\mathbf{q}} + \int_{\mathcal{S}} \mathrm{d}m \left(\frac{\partial \mathbf{r}}{\partial \mathbf{q}} \right)^{\mathrm{T}} \frac{\partial^2 \mathbf{r}}{\partial \mathbf{q} \otimes \partial \mathbf{q}} : \dot{\mathbf{q}} \otimes \dot{\mathbf{q}} \\ &= \mathbf{M} \ddot{\mathbf{q}} + \boldsymbol{\Gamma} : \dot{\mathbf{q}} \otimes \dot{\mathbf{q}} \end{aligned} \tag{13}$$

holds, where $\mathbf{M}$ is the symmetric and positive definite mass matrix and $\boldsymbol{\Gamma}$ are the Christoffel symbols of the second kind. The colon denotes the double contraction between higher order tensors and $\otimes$ is the tensor product. Evaluated for the two bodies this leads to

$$\begin{aligned} \mathbf{M}_1 &= \sum_e (\mathbf{C}_1^e)^{\mathrm{T}} \left\{ \int_0^1 \rho_0 \left(\frac{\partial \mathbf{r}_1^e}{\partial \mathbf{q}_1^e} \right)^{\mathrm{T}} \left(\frac{\partial \mathbf{r}_1^e}{\partial \mathbf{q}_1^e} \right) \left\| \frac{\partial \mathbf{r}_{1,ref}^e}{\partial s^e} \right\| \mathrm{d}s^e \right\} \mathbf{C}_1^e \\ \mathbf{M}_2 &= (\mathbf{C}_2)^{\mathrm{T}} \left\{ \int_0^{2\pi} \int_0^R \rho_D \left(\frac{\partial \mathbf{r}_2}{\partial \mathbf{q}_2} \right)^{\mathrm{T}} \left(\frac{\partial \mathbf{r}_2}{\partial \mathbf{q}_2} \right) r \, \mathrm{d}r \, \mathrm{d}\vartheta \right\} \mathbf{C}_2 \\ &= (\mathbf{C}_2)^{\mathrm{T}} \left\{ \rho_D \begin{pmatrix} R^2 \pi & 0 & 0 \\ 0 & R^2 \pi & 0 \\ 0 & 0 & \frac{R^4}{2} \pi \end{pmatrix} \right\} \mathbf{C}_2 \\ \boldsymbol{\Gamma}_1 : \dot{\mathbf{q}} \otimes \dot{\mathbf{q}} &= \sum_e (\mathbf{C}_1^e)^{\mathrm{T}} \left\{ \int_0^1 \rho_0 \left(\frac{\partial \mathbf{r}_1^e}{\partial \mathbf{q}_1^e} \right)^{\mathrm{T}} \frac{\partial^2 \mathbf{r}_1^e}{\partial \mathbf{q}_1^e \otimes \partial \mathbf{q}_1^e} : \dot{\mathbf{q}}_1^e \otimes \dot{\mathbf{q}}_1^e \left\| \frac{\partial \mathbf{r}_{1,ref}^e}{\partial s^e} \right\| \mathrm{d}s^e \right\} \\ \boldsymbol{\Gamma}_2 : \dot{\mathbf{q}} \otimes \dot{\mathbf{q}} &= 0 \ , \end{aligned} \tag{14}$$

where $\mathbf{M} = \mathbf{M}_1 + \mathbf{M}_2$, $\boldsymbol{\Gamma} = \boldsymbol{\Gamma}_1 + \boldsymbol{\Gamma}_2$ and $\sum_e$ stands for the summation defined in Eq. (3).

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4.2 Internal Forces

Since the string is a deformable body, the reaction force against deformation, i.e. the stress, is not a constraint force as in a rigid body but contributes to the virtual work of the system. To evaluate the stress a constitutive equation is needed which combines kinematical quantities, e.g. the state of deformation, with force quantities. The measure of deformation compares an actual spatial configuration $\mathbf{r}_1$ with the reference configuration $\mathbf{r}_{1,ref}$, introduced above. An intuitive choice of the measure of deformation is the stretch ν^e, which is the quotient of the incremental lengthes of the actual curve and the curve in the reference configuration

$$\nu^e = \frac{\mathrm{d}L}{\mathrm{d}L_{ref}} = \frac{\|\mathrm{d}\mathbf{r}_1^e\|}{\left\|\mathrm{d}\mathbf{r}_{1,ref}^e\right\|} = \frac{\|\partial\mathbf{r}_1^e/\partial s^e\|}{\left\|\partial\mathbf{r}_{1,ref}^e/\partial s^e\right\|} \,. \tag{15}$$

The internal force vector from Eq. (10) and (11) together with Eq. (6) is

$$\mathbf{f}^{\mathrm{int}} = \sum_e (\mathbf{C}_1^e)^{\mathrm{T}} \int_0^1 \left(\frac{\partial^2\mathbf{r}_1^e}{\partial\mathbf{q}_1^e\partial s^e}\right)^{\mathrm{T}} \mathbf{t}^e \mathrm{d}s^e, \qquad \mathbf{t}^e = T^e(\nu^e)\frac{\partial\mathbf{r}_1^e/\partial s^e}{\|\partial\mathbf{r}_1^e/\partial s^e\|} \,. \tag{16}$$

As an example of a nonlinear material law, a neo-Hookean material is chosen which depends merely on the stretch at the specific material point, i.e.

$$T^e(\nu^e) = \frac{k}{3}\left(\nu^e - \frac{1}{(\nu^e)^2}\right). \tag{17}$$

4.3 External Forces

The external forces of the system are the body forces $\mathbf{b}_1 = \rho_0\|\partial\mathbf{r}_{1,ref}/\partial s\|\mathbf{g}$ and $\mathbf{b}_2 = \rho_D\mathbf{g}$ due to gravity $\mathbf{g} = (0 \;\; -g)^{\mathrm{T}}$ and the bearing reactions $\mathbf{F}_i = \lambda_{Bi}\,\mathbf{e}_i$, for $i = \{A, B\}$. Due to the chosen numerical scheme, the reaction forces are modeled as perfect bilateral constraints with a set-valued force law as depicted in Fig. 5a. The force law is formulated as a normal cone inclusion (cf. Ref. [1, 9]) of the form

$$\mathbf{F}_i = \lambda_{Bi}\,\mathbf{e}_i, \qquad g_{Bi} \in \mathcal{N}_{\mathbb{R}}(-\lambda_{Bi}), \qquad \text{for } i = A, B \tag{18}$$

where g_{Bi} is the gap function between the actual position of the string's endpoints and its desired suspension points. The vector $\mathbf{e}_i$ is the normalized direction vector between these points. Due to Eq. (10) and (11) the external forces can be written as

$$\begin{aligned}\mathbf{f}^{\mathrm{ext}} = &\sum_e (\mathbf{C}_1^e)^{\mathrm{T}} \left\{\int_0^1 \rho_0 \left(\frac{\partial\mathbf{r}_1^e}{\partial\mathbf{q}_1^e}\right)^{\mathrm{T}} \mathbf{g} \left\|\frac{\partial\mathbf{r}_{1,ref}^e}{\partial s^e}\right\| \mathrm{d}s^e\right\} + \left(\mathbf{C}_1^1\right)^{\mathrm{T}} \left\{\left(\frac{\partial\mathbf{r}_1^1(s^e=0)}{\partial\mathbf{q}_1^1}\right)^{\mathrm{T}} \mathbf{e}_A\lambda_{BA}\right\} \\ &+ \left(\mathbf{C}_1^{nel}\right)^{\mathrm{T}} \left\{\left(\frac{\partial\mathbf{r}_1^{nel}(s^e=1)}{\partial\mathbf{q}_1^{nel}}\right)^{\mathrm{T}} \mathbf{e}_B\lambda_{BB}\right\} + (\mathbf{C}_2)^{\mathrm{T}}\left\{\left(0 \;\; -\rho_D R^2\pi g \;\; 0\right)^{\mathrm{T}}\right\} \,.\end{aligned} \tag{19}$$

4.4 Contact Forces

During the motion of the system, the string constraints the disk by surrounding its contour. The force distribution $\mathbf{l}$ as the contact interaction between the two bodies seems to be an appropriate choice for a continuous formulation. But in the kinematically discretized model the contact force distribution will typically degenerate into discrete forces at a limited number of

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points, the positions of which are a priori unknown. A convenient approach is to introduce a dense grid of ncp possible contact points at arbitrarily given material points of the string and to approximate the force distribution $\mathbf{l}$ by introducing discrete contact forces in normal and tangential direction in the sense that

$$\begin{aligned}\mathbf{f}^{\text{contact}} &= \int_0^L \left(\frac{\partial \mathbf{r}_1}{\partial \mathbf{q}}\right)^{\mathrm{T}} \mathbf{l}\mathrm{d}s - \int_0^{2\pi} \left(\frac{\partial \mathbf{r}_2}{\partial \mathbf{q}}\right)^{\mathrm{T}} \mathbf{l}R\mathrm{d}\vartheta \\ &\approx \sum_{i=1}^{ncp} \left[\left(\frac{\partial \mathbf{r}_2}{\partial \mathbf{q}} - \frac{\partial \mathbf{r}_1}{\partial \mathbf{q}}\right)^{\mathrm{T}} (\mathbf{F}_N + \mathbf{F}_T)\right]_i = \sum_{i=1}^{ncp} [\mathbf{w}_N \lambda_N + \mathbf{w}_T \lambda_T]_i \ . \end{aligned} \tag{20}$$

In the following the contact model is developed for an arbitrary point P on the string with material coordinate s_i placed at $\mathbf{r}_{OP} = \mathbf{r}_1(s_i, \mathbf{q}_1)$. The corresponding contact point on the disk at (R, ϑ_i) is placed at $\mathbf{r}_{OQ} = \mathbf{r}_2((R, \vartheta_i), \mathbf{q}_2)$.

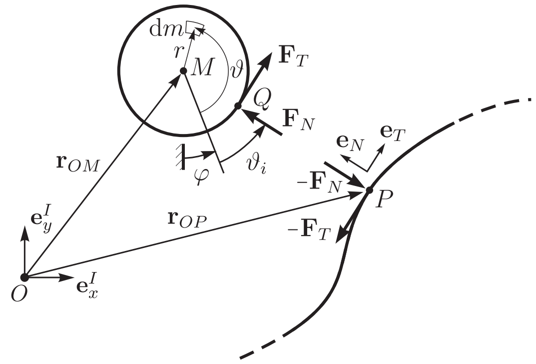

Figure 4: Parametrization of the disk's material points and contact kinematics.

With respect to Fig. (4) the discrete contact forces can be written as $\mathbf{F}_N = \lambda_N \mathbf{e}_N$ and $\mathbf{F}_T = \lambda_T \mathbf{e}_T$ as normal and tangential contact forces, respectively. The normal direction $\mathbf{e}_N$ for a contact point is the normed connection line between the contact point $\mathbf{r}_{OP}$ and the position of the disk's midpoint $\mathbf{r}_{OM} = \mathbf{r}_2((0,0), \mathbf{q}_2)$. The tangential direction $\mathbf{e}_T$ is orthogonal to the normal direction and is introduced as depicted in Fig. (4)

$$\mathbf{e}_N = \frac{\mathbf{r}_{OM} - \mathbf{r}_{OP}}{\|\mathbf{r}_{OM} - \mathbf{r}_{OP}\|} \ , \qquad \mathbf{e}_T = (\mathbf{e}_x^I \otimes \mathbf{e}_y^I - \mathbf{e}_y^I \otimes \mathbf{e}_x^I)\mathbf{e}_N \ . \tag{21}$$

It can be shown easily that for an arbitrary contact point the generalized force directions can be written as

$$\mathbf{w}_N = (\mathbf{C}_2)^{\mathrm{T}} \begin{pmatrix} \mathbf{e}_N \\ 0 \end{pmatrix} - (\mathbf{C}_1^e)^{\mathrm{T}} \left(\frac{\partial \mathbf{r}_1^e}{\partial \mathbf{q}_1^e}\right)^{\mathrm{T}} \mathbf{e}_N, \ \mathbf{w}_T = (\mathbf{C}_2)^{\mathrm{T}} \begin{pmatrix} \mathbf{e}_T \\ R \end{pmatrix} - (\mathbf{C}_1^e)^{\mathrm{T}} \left(\frac{\partial \mathbf{r}_1^e}{\partial \mathbf{q}_1^e}\right)^{\mathrm{T}} \mathbf{e}_T \ . \tag{22}$$

The contact forces contribute to the equations of motion as stated in Eq. (20). For the normal and tangential force law, the contact kinematics in normal and tangential direction are needed. The gap function g_N and its corresponding constraint velocity γ_N are given by

$$g_N = \mathbf{e}_N^{\mathrm{T}} (\mathbf{r}_{OM} - \mathbf{r}_{OP}) - R \tag{23}$$

$$\gamma_N = \mathbf{e}_N^{\mathrm{T}} (\dot{\mathbf{r}}_{OM} - \dot{\mathbf{r}}_{OP}) = \mathbf{e}_N^{\mathrm{T}} \left(\frac{\partial \mathbf{r}_{OM}}{\partial \mathbf{q}} - \frac{\partial \mathbf{r}_{OP}}{\partial \mathbf{q}}\right) \dot{\mathbf{q}} = \mathbf{w}_N^{\mathrm{T}} \dot{\mathbf{q}} \ . \tag{24}$$

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For $g_N > 0$, the contact is open, for a vanishing contact distance, the contact is closed and if $g_N < 0$, then the contact penetrates. The constraint velocity in tangential direction is given as

$$\begin{aligned}\gamma_T &= \mathbf{e}_T^{\mathrm{T}}(\dot{\mathbf{r}}_{OQ} - \dot{\mathbf{r}}_{OP}) = \mathbf{e}_T^{\mathrm{T}}(\dot{\mathbf{r}}_{OM} + R\dot{\varphi}\mathbf{e}_T - \dot{\mathbf{r}}_{OP}) \\ &= \mathbf{e}_T^{\mathrm{T}}\left(R\dot{\varphi}\mathbf{e}_T + \left(\frac{\partial \mathbf{r}_{OM}}{\partial \mathbf{q}} - \frac{\partial \mathbf{r}_{OP}}{\partial \mathbf{q}}\right)\dot{\mathbf{q}}\right) = \mathbf{w}_T^{\mathrm{T}}\dot{\mathbf{q}}\ .\end{aligned} \tag{25}$$

The contact laws are formulated as set-valued force laws which guarantee the impenetrability condition of a contact. The most concise formulation can be attended using the concept of normal cone inclusions. The force law in normal direction is the law for a unilateral constraint

$$\begin{aligned}g_N > 0 \ &\Rightarrow\ -\lambda_N = 0 \\ g_N = 0 \ &\Rightarrow\ \quad \gamma_N \in \mathcal{N}_{\mathbb{R}_0^-}(-\lambda_N)\ .\end{aligned} \tag{26}$$

In tangential direction we introduce a force law for plane Coulomb friction which depends on the normal contact force λ_N, i.e.

$$\begin{aligned}g_N > 0 \ &\Rightarrow\ -\lambda_T = 0 \\ g_N = 0 \ &\Rightarrow\ \quad \gamma_T \in \mathcal{N}_{[-\mu\lambda_N,\mu\lambda_N]}(-\lambda_T).\end{aligned} \tag{27}$$

The two force laws are sketched in Fig. 5b and 5c.

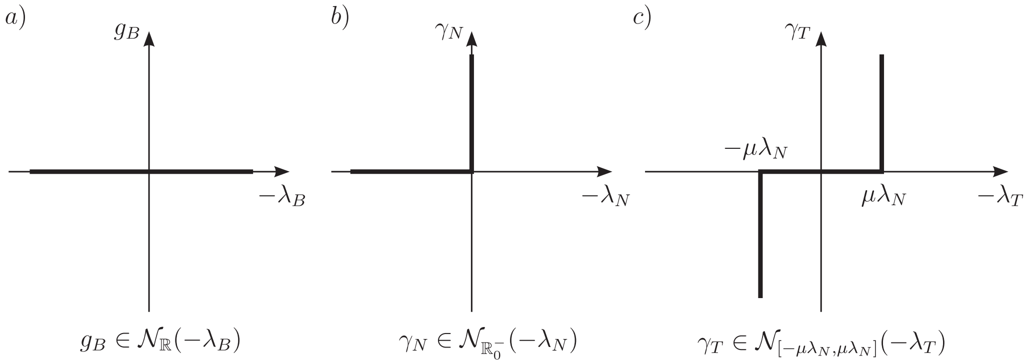

Figure 5: Set-valued force laws. a) Perfect bilateral constraint on displacement level. b) and c) Unilateral frictional constraint on velocity level.

4.5 Impact Laws

So far, the equations of motion in the form of Eq. (11) with their corresponding force laws were formulated. This formulation describes the dynamics of the spatially discretized system where closed contacts remain closed, open contact remain open and stick-slip behavior occurs. Briefly this is called the post impact dynamics of the system. Because of the introduction of set-valued force laws which may e.g. fulfill the impenetrability condition exactly, discontinuities in velocities may occur in addition. For these discontinuities Eq. (11) is not valid anymore. Hence an impact equation and a corresponding impact law is needed for the specific instant of time where the solution jumps. Such impact equations and impact laws are explained concisely in Ref. [9, 10] for a general finite degree of freedom system with contact forces formulated as normal cone inclusions. We chose a Newton type of impact law which connects the pre and post impact velocities by a restitution coefficient ε. Since we deal with elastic structures and due to the argumentation in Ref. [11] the restitution coefficient ε is chosen to be zero for all contact laws.

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Parameter	Value	Parameter	Value
Gravity	$g = 9.81\,\mathrm{kgms^{-2}}$	Length string (undeformed)	$L = 1\,\mathrm{m}$
Distance suspension points	$\overline{AB} = 0.8\,\mathrm{m}$	Density string (undeformed)	$\rho_0 = 0.3142\,\mathrm{kgm^{-1}}$
Friction coefficient	$\mu = 0.2$	Stiffness string	$k = 853.75\,\mathrm{N}$
Radius disk ex.1/ex.2	$R = 0.3/0.1\,\mathrm{m}$	Number of elements	$nel = 2$
Density disk ex.1/ex.2	$\rho_D = 100/0.1\,\mathrm{kgm^{-2}}$	Number of contact pts. ex.1/ex.2	$ncp = 20/60$

Table 1: Simulation parameters for example 1 and 2.

5 Examples

To show the possibilities of our formulation two different examples were simulated. The chosen parameters are listed in Tab. 1. The transient dynamic behavior of the system was evaluated with Moreau's timestepping algorithm as described in Ref. [12], which is a time-discretization on velocity impulse level.

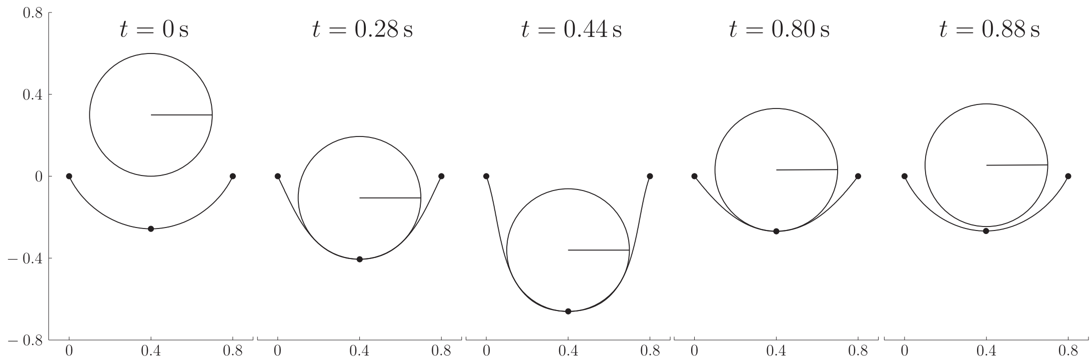

Figure 6: Due to large deformations of the string multiple contact points are closed.

In the first problem (see Fig. 6), a heavy disk falls into the hanging string whereat large deformations of the string occur. During the motion multiple contacts close and the string surrounds the disk. In Fig. 7 one can see how a light disk rolls on the string. The decoupling of the number of contact points from the number of elements allows it, to introduce a dense grid of 60 contact points such that rolling of the disk can be performed. Both problems were simulated with merely two finite elements. Neither convergence nor stability problems occurred.

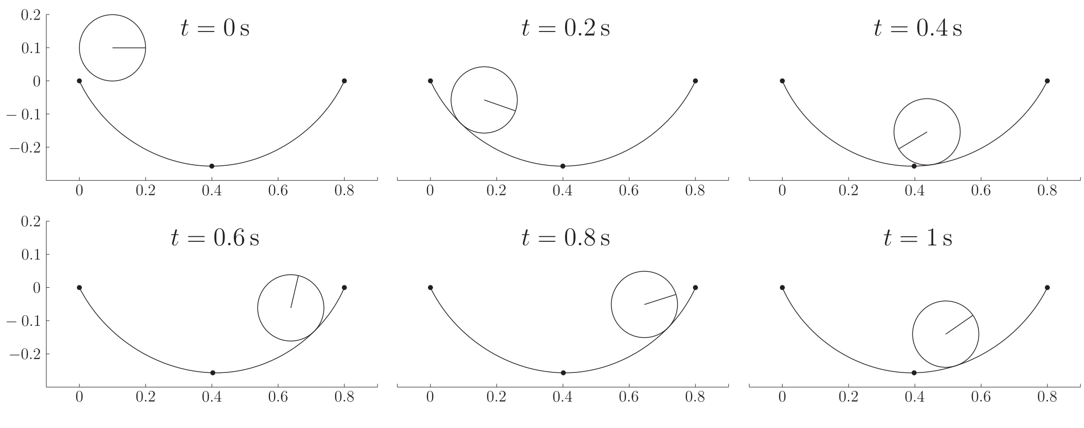

Figure 7: Rolling of the disk is possible due to a dense grid of contact points and frictional contact laws in each contact point.

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6 Conclusions

In this paper we showed a novel approach for the treatment of hard contacts within a nonlinear finite element framework. The fundamental equation for the derivation of the finite element formulation was the virtual work which was stated for the complete system from the very first. For doing that, both the rigid and the deformable body were formulated as continuous bodies. It is obvious that the rigid body sticks together by a constraint force distribution. But we want to mention, that even a deformable body is subjected to a constraint force distribution as soon as we describe its kinematics by finitely many degrees of freedom. Therefore the latter constraint force distribution has to be considered right from the beginning as it was done in Eq. (7). Generally, discrete forces are not included into the concept of classical continuum mechanics. With the introduction of an atomic measure, discrete forces could be applied to the deformable body in the continuous formulation. Even though the boundary of the string is zero-dimensional the boundary forces and so the boundary condition appear explicitly in the virtual work of the system. The contact force distribution as the interaction between the two bodies was approximated by finitely many contact points each of which holds a set-valued force law.

Eventually the nonlinear finite element formulation was obtained as a minor product of the virtual work and the discretization of the deformable body by finitely many generalized coordinates. This natural formulation restricts an element shape function neither to be linear in its generalized coordinates nor to be described only by nodal degrees of freedom. With Bézier splines a non-standard choice of shape function was shown.

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