Determination of the Transverse Shear Stress in an Euler-Bernoulli Beam Using Non-Admissible Virtual Displacements

Simon Eugster[1,*] and **Christoph Glocker**[1]

[1] Department of Mechanical and Process Engineering, Center of Mechanics, Institute for Mechanical Systems, ETH Zürich, Tannenstrasse 3, 8092 Zürich, Switzerland

The point of view that a beam can be considered as a three-dimensional continuum with a constrained position field (cf. [1]) together with the virtual work principle and the concept of perfect constraint stresses, leads to a systematic way to reduce the equilibrium equations of the continuous body to an ordinary differential equation describing the constrained displacement field of the beam. Using virtual displacements, being non-admissible with respect to the constrained beam kinematics, together with the solution of the boundary value problem, allows us to analytically determine the constraint stresses and consequently the total stresses of a beam up to a certain indeterminacy.

1 Constrained Displacement Fields in Linear Elasticity

Let the body $\bar{B}$ be a closed subset of the Euclidean three-space $\mathbb{E}^3$ which is parameterized by cartesian coordinates (x, y, z) induced by the basis $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$. The displacement field and the virtual displacement field of the body are the vector valued functions $\mathbf{u}\colon \bar{B} \to \mathbb{E}^3$ and $\delta\mathbf{u}\colon \bar{B} \to \mathbb{E}^3$. The linear strain is the tensor valued function $\boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}\left((\nabla\mathbf{u})^{\mathrm{T}} + \nabla\mathbf{u}\right)$, where ∇ denotes the gradient in $\mathbb{E}^3$. The equilibrium conditions of a continuous body undergoing an infinitesimal displacement field are determined by the principle of virtual work stating that

$$\forall\,\delta\mathbf{u}\ , \quad \delta W(\delta\mathbf{u}) = \delta W^{\mathrm{int}}(\delta\mathbf{u}) - \delta W^{\mathrm{ext}}(\delta\mathbf{u}) = \int_{\bar{B}} \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\delta\mathbf{u})\,\mathrm{d}V - \int_{\bar{B}} \delta\mathbf{u}\cdot\mathrm{d}\mathbf{f} = 0\ , \tag{1}$$

where $\boldsymbol{\sigma}$ denotes the total stress field, $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ and $\mathrm{d}\mathbf{f}$ is a measure which allows for Dirac-contribution as well. For an unconstrained continuous body, the total stress field consists of an impressed stress field $\boldsymbol{\sigma} = \boldsymbol{\sigma}_I$ only. It is possible to restrict the kinematics of the body to a constrained displacement field $\mathbf{u}_{\mathrm{a}}$, being an element of a submanifold $\mathcal{C}$ of the manifold of all piecewise continuous displacement fields. The restriction of the displacements of a continuous body to the submanifold $\mathcal{C}$ is guaranteed by a constraint stress field $\boldsymbol{\sigma}_C$. Hence, the total stress field of a continuous body with a constrained displacement field is $\boldsymbol{\sigma} = \boldsymbol{\sigma}_I + \boldsymbol{\sigma}_C$. We call an element of the tangent space $T_{\mathbf{u}}\mathcal{C}$ of the submanifold $\mathcal{C}$ at $\mathbf{u}$ admissible virtual displacement field. The constraint stress field $\boldsymbol{\sigma}_C$ is called perfect, if the virtual work contribution

$$\delta W_C^{\mathrm{int}}(\delta\mathbf{u}) = \int_{\bar{B}} \boldsymbol{\sigma}_C : \boldsymbol{\varepsilon}(\delta\mathbf{u})\,\mathrm{d}V = 0 \quad \forall\,\delta\mathbf{u}\in T_{\mathbf{u}}\mathcal{C}\ . \tag{2}$$

Let E and G denote Young's and shear modulus, respectively, and $i, j = \{1, 2, 3\}$. Then, the material law of the impressed stresses is for the normal stresses $\sigma_{Iii} = E\varepsilon_{ii}$ and for the shear stresses $\sigma_{Iij} = 2G\varepsilon_{ij}$, where $i \neq j$.

2 The Plane Linear Euler-Bernoulli Beam

Consider a clamped beam of length l, as depicted in Fig. 1, with constant cross section area A, Young's modulus E and shear modulus G. The line of centroids is parallel to the $\mathbf{e}_x$-direction and passes through the origin O. At $x = l$, at the centroid of the cross section, a concentrated force $P\mathbf{e}_z$ is applied. Let $\mathbf{q}(x) = (u(x), w(x))^{\mathrm{T}}$ be the generalized displacement functions. Denoting derivative with respect to x by a superposed prime $(\cdot)'$, we assume the beam, as a continuous body, to follow the constrained displacement field $\mathbf{u} = \mathbf{u}_{\mathrm{a}}(\mathbf{q}, \mathbf{q}')$ defined pointwise for any $(x, y, z) \in \bar{B}$ as

$$\mathbf{u}(x, y, z) = \mathbf{u}_{\mathrm{a}}(\mathbf{q}, \mathbf{q}')(x, y, z) = (u(x) - w'(x)z, 0, w(x))^{\mathrm{T}}\ , \quad u(0) = w(0) = w'(0) = 0\ . \tag{3}$$

The generalized displacement functions u and w, being the new unknowns of the constrained body, describe the longitudinal and transversal displacements, respectively. The admissible virtual displacements, i.e. $\delta\mathbf{u} \in T_{\mathbf{u}}\mathcal{C}$ are induced by the variation of the constrained displacement field (3) as $\delta\mathbf{u}_{\mathrm{a}} = (\delta u - \delta w' z, 0, \delta w)^{\mathrm{T}} \in T_{\mathbf{u}}\mathcal{C}$ with $\delta u(0) = \delta w(0) = \delta w'(0) = 0$.

At first, the virtual work of the continuous body (1) is evaluated for admissible virtual displacements $\delta\mathbf{u}_{\mathrm{a}}$. Since the constraint stresses due to the kinematical restrictions (3) are assumed to be perfect, their virtual work contribution (2) vanishes

* Corresponding author: e-mail eugster@imes.mavt.ethz.ch, phone +41 44 632 7754, fax +41 44 632 1145

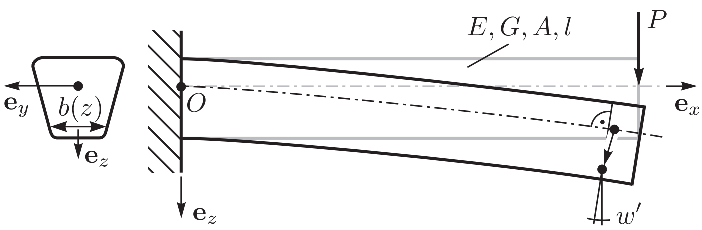

Fig. 1: Plane linearized Euler-Bernoulli beam.

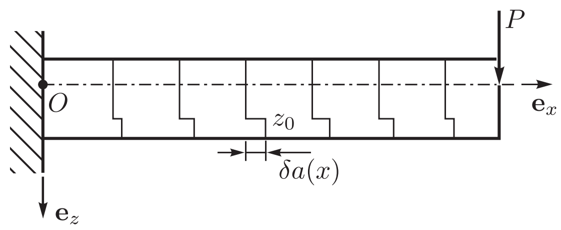

Fig. 2: Non-admissible virtual displacement field.

by definition. The admissible virtual strains are obtained as $\boldsymbol{\varepsilon}(\delta\mathbf{u}_\mathrm{a}) = (\delta u' - \delta w'' z)\mathbf{e}_x \otimes \mathbf{e}_x$. We denote $N_I(x) := \int_A \sigma_{Ixx} \mathrm{d}A$ and $M_I(x) := \int_A \sigma_{Ixx} \mathrm{d}A$ as the resultant contact normal forces and the resultant contact couples, respectively. Defining $A := \int_A \mathrm{d}A$ and $I := \int_A z^2 \mathrm{d}A$ together with the property of the line of centroids, i.e. $\int_A z \mathrm{d}A = 0$, the virtual work principle (1) for admissible virtual displacements with the previously introduced material law of the impressed stresses $\boldsymbol{\sigma}_I$, takes the form

$$\begin{aligned} \delta W(\delta\mathbf{u}_\mathrm{a}) &= \int_0^l \left(\delta u' \int_A \sigma_{Ixx} \mathrm{d}A - \delta w'' \int_A z\sigma_{Ixx} \mathrm{d}A \right) \mathrm{d}x - P\delta w(l) = \int_0^l (\delta u' N_I - \delta w'' M_I)\, \mathrm{d}x - P\delta w(l) \\ &= \int_0^l (\delta u' EAu' + \delta w'' EIw'')\, \mathrm{d}x - \delta w(l) P = 0 \quad \forall\, \delta u, \delta w\ , \end{aligned} \tag{4}$$

where $\delta u(0) = \delta w(0) = \delta w'(0) = 0$ in order to satisfy the clamping boundary condition. The generalized constitutive laws $N_I = EAu'$ and $M_I = -EIw''$ follow directly from (4). Using the material law of $\boldsymbol{\sigma}_I$ for the constrained displacement field $\mathbf{u}_\mathrm{a}$ together with the generalized constitutive laws, the relations $\sigma_{Ixx} = E(u' - zw'') = \frac{N_I}{A} + \frac{M_I}{I} z$ and $\sigma_{Ixy} = \sigma_{Ixz} = \sigma_{Iyy} = \sigma_{Iyz} = \sigma_{Izz} = 0$ are obtained. By applying integration by parts to (4), once for the $\delta u'$-terms and twice for the $\delta w''$ terms, the strong variational form of the plane Euler-Bernoulli beam can be derived. The solution of the corresponding boundary value problem is obtained as $N_I(x) = 0$ and $M_I(x) = P(x - l)$.

3 Transverse Shear Stress in the Euler-Bernoulli Beam

To determine the constraint stresses and consequently the total stresses of the beam as a continuous body, the virtual work principle (1) has to be evaluated for non-admissible virtual displacements. Such a non-admissible virtual displacement is $\delta\mathbf{u}_\mathrm{na} = \delta a h(x - x_0)\mathbf{e}_x$, where $h(x)$ denotes the Heaviside function and $\delta a \in \mathbb{R}$. Using the non-admissible virtual strains $\boldsymbol{\varepsilon}(\delta\mathbf{u}_\mathrm{na}) = \delta a\, \delta(x_0)\, \mathbf{e}_x \otimes \mathbf{e}_x$, where $\delta(x)$ denotes the Delta-Dirac distribution, together with the solution of the boundary value problem, the principle of virtual work

$$\delta W(\delta\mathbf{u}_\mathrm{na}) = \delta a \int_A (\sigma_{Cxx} + \sigma_{Ixx})|_{x_0} \, \mathrm{d}A = \delta a \int_A \left(\sigma_{Cxx} + \frac{M_I}{I} z \right)\Big|_{x_0} \mathrm{d}A = \delta a \int_A \sigma_{Cxx} \mathrm{d}A = 0 \quad \forall\, \delta a$$

demand the integral of the normal constraint stresses to vanish. This condition cannot be satisfied uniquely. Here, the trivial solution $\sigma_{Cxx} = 0$ is chosen. As sketched in Fig. 2, to extract the shear constraint stresses, we assume the non-admissible virtual displacements $\delta\mathbf{u}_\mathrm{na} = \delta a(x) h(z - z_0)\mathbf{e}_x$, with $\delta a(x)$ being a smooth function satisfying $\delta a(0) = \delta a(l) = 0$. This leads to the non-admissible virtual strains $\boldsymbol{\varepsilon}(\delta\mathbf{u}_\mathrm{na}) = \delta a' h(z - z_0)\mathbf{e}_x \otimes \mathbf{e}_x + \frac{1}{2}\delta a \delta(z_0)(\mathbf{e}_x \otimes \mathbf{e}_z + \mathbf{e}_z \otimes \mathbf{e}_x)$. Assume the constraint stresses σ_{Cxz} to be constant in $\mathbf{e}_y$-direction and define the first moment of area as $S(z_0) := \int_A h(z - z_0) z \mathrm{d}A$. Then, by applying integration by parts on the $\delta a'$-term and introducing the thickness $b(z_0)$ at z_0, the virtual work principle is

$$\begin{aligned} \delta W(\delta\mathbf{u}_\mathrm{na}) &= \int_0^l \int_A (\sigma_{Ixx} \delta a' h(z - z_0) + \sigma_{Cxz} \delta a \delta(z_0))\, \mathrm{d}A\mathrm{d}x = \int_0^l \left(\delta a' \frac{M_I}{I} S(z_0) + \sigma_{Cxz} \delta a b(z_0) \right) \mathrm{d}x \\ &= \int_0^l \delta a \left(-\frac{M_I' S(z_0)}{I} + \sigma_{Cxz} b(z_0) \right) \mathrm{d}x = 0 \quad \forall\, \delta a\ , \end{aligned} \tag{5}$$

where $\delta a(0) = \delta a(l) = 0$. The bracket in the last line of (5) has to vanish pointwise, which leads to

$$\sigma_{xz}(z) = \sigma_{Cxz}(z) = \frac{M_I' S(z)}{Ib(z)} = \frac{PS(z)}{Ib(z)}$$

which is the commonly known formula for the transverse shear stress in an Euler-Bernoulli beam, cf. [2].

References

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