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An Intrinsic Geometric Formulation of the Equilibrium Equations in Continuum Mechanics

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The theory of invariant continuum mechanics is based on the concept that forces and stresses are defined as elements of the cotangent bundle of the configuration manifold. While body and physical space are modeled as differentiable manifolds, the infinite dimensional configuration manifold is given by all configurations of the body in the physical space. In this paper a virtual work principle is postulated which leads together with an induced traction stress and Stokes' theorem directly to the local equilibrium equations and the traction boundary conditions.

1 Kinematics

Let the body $\mathcal{B}$ be a compact three-dimensional smooth manifold with boundary and local charts $\theta\colon \mathcal{B} \supset U \to \mathbb{R}^3$. The physical space $\mathcal{S}$ is a three-dimensional smooth manifold without boundary equipped with an affine connection ∇ and local charts $x\colon \mathcal{S} \supset V \to \mathbb{R}^3$. A smooth vector field $\mathbf{u}\colon \mathcal{S} \to T\mathcal{S}$ is a mapping from the manifold $\mathcal{S}$ to the tangent bundle $T\mathcal{S}$ such that $\pi_S \circ \mathbf{u} = \mathrm{Id}_{\mathcal{S}}$, where $\pi_S\colon T\mathcal{S} \to \mathcal{S}$ is the natural projection which maps the ordered pair of base point and tangent vector $(Q, \mathbf{u}) \in T\mathcal{S}$ to the base point Q. An affine connection assigns to a pair of smooth vector fields $\mathbf{u}, \mathbf{v} \in \Gamma(T\mathcal{S})$ another smooth vector field $\nabla_{\mathbf{u}}\mathbf{v} \in \Gamma(T\mathcal{S})$ called covariant derivative of $\mathbf{v}$ along $\mathbf{u}$. For the vector fields of a holonomic basis $\partial_{x^i} \in \Gamma(T\mathcal{S})$ with $i = \{1, 2, 3\}$ to the local chart x, the covariant derivative of the basis vector ∂_{x^k} along another basis vector ∂_{x^m} is expressed as linear combination $\nabla_{\partial_{x^m}}\partial_{x^k} = \Gamma^i_{mk}\partial_{x^i}$, where Γ^i_{mk} are the Christoffel symbols and Einstein's summation convention is applied. Let $\boldsymbol{\omega} \in \Gamma(T^*\mathcal{S})$ be a covector field on $\mathcal{S}$, then the affine connection directly induces a smooth tensor field $\nabla\mathbf{v} \in \Gamma(T\mathcal{S}\otimes T^*\mathcal{S})$ by the relation $\nabla\mathbf{v}(\boldsymbol{\omega}, \mathbf{u}) = \boldsymbol{\omega}\cdot\nabla\mathbf{v}\cdot\mathbf{u} = \boldsymbol{\omega}\cdot\nabla_{\mathbf{u}}\mathbf{v}$, where the dot denotes the duality pairing between the adjacent tensor slots. Locally, the covariant derivative can be expressed as $\nabla\mathbf{v} = v^i{}_{;j}\partial_{x^i}\otimes \mathrm{d}x^j = (v^i{}_{,j} + \Gamma^i_{jk}v^k)\partial_{x^i}\otimes\mathrm{d}x^j$.

We define a configuration of the body in the physical space to be the continuously differentiable map $\kappa\colon \mathcal{B} \to \mathcal{S}$. If additionally the principle of impenetrability and the permanence of matter are claimed, the mapping κ and its differentials $D\kappa(P)\colon T_P\mathcal{B} \to T_{\kappa(P)}\mathcal{S}$ at $P \in \mathcal{B}$ have to be injective. These conditions induce the configuration κ to be a C^1-continuous embedding of the body into the physical space. Hence, the set of C^1-continuous embeddings $\mathrm{Emb}(\mathcal{B}, \mathcal{S})$ constitute the infinite dimensional configuration manifold $\mathcal{Q}$. The pullback tangent bundle over the body $\mathcal{B}$ by κ is the triple $(\kappa^*T\mathcal{S}, \kappa^*\pi_{\mathcal{S}}, \mathcal{B})$ with the total space $\kappa^*T\mathcal{S} := \{(P, \mathbf{v}) \in \mathcal{B} \times T\mathcal{S}\colon \pi_{\mathcal{S}}(\mathbf{v}) = \kappa(P)\}$ and the natural projection $\kappa^*\pi_{\mathcal{S}}\colon \kappa^*T\mathcal{S} \to \mathcal{B}$ defined by $\kappa^*\pi_{\mathcal{S}}(P, \mathbf{v}) = P$. For any smooth vector field $\mathbf{v} \in \Gamma(T\mathcal{S})$ we associate a C^1-continuous section through the pullback tangent bundle by $\delta\boldsymbol{\kappa} = \kappa^*\mathbf{v} = (\cdot, \mathbf{v} \circ \kappa) \in C^1(\kappa^*T\mathcal{S})$ which assigns to every material point $P \in \mathcal{B}$ the material point itself and a tangent vector in the physical space at the current configuration of the material point. The isomorphism between the tangent space $T_\kappa\mathcal{Q}$ and the set of pullback sections $C^1(\kappa^*T\mathcal{S})$ is discussed in Eugster [1]. Consequently, an element $\delta\boldsymbol{\kappa} \in C^1(\kappa^*T\mathcal{S}) \cong T_\kappa\mathcal{Q}$ is called virtual displacement field. By definition of the pullback sections, the holonomic basis vectors $\partial_{x^i} \in \Gamma(T\mathcal{S})$ directly imply a basis $\mathbf{g}_i := \partial_{x^i} \circ \kappa$ for pullback sections $\delta\boldsymbol{\kappa} = (\cdot, \delta\kappa^i\mathbf{g}_i) \in C^1(\kappa^*T\mathcal{S})$. In a similiar way, the affine connection ∇ induces an affine connection $\hat{\nabla}$ on the pullback tangent bundle $\kappa^*T\mathcal{S}$. This enables us to define the tensor field of the total covariant derivative $\hat{\nabla}\delta\boldsymbol{\kappa} \in C^0(\kappa^*T\mathcal{S} \otimes T^*\mathcal{B})$ which is represented locally as

$$\hat{\nabla}\delta\boldsymbol{\kappa} = \delta\kappa^i{}_{;J}\mathbf{g}_i \otimes \mathrm{d}\theta^J = (\delta\kappa^i{}_{,J} + \hat{\Gamma}^i_{Jk}\delta\kappa^k)\mathbf{g}_i \otimes \mathrm{d}\theta^J = (\delta\kappa^i{}_{,J} + F^m_J(\Gamma^i_{mk} \circ \kappa)\delta\kappa^k)\mathbf{g}_i \otimes \mathrm{d}\theta^J \ , \tag{1}$$

where $F^m_J := \frac{\partial\kappa^m}{\partial\theta^J}$ is the coordinate representation of the tangent map between body and physical space.

2 Force Representation

In the sense of analytical mechanics, the space of forces is the set of all real-valued linear functionals on the space of virtual displacements, i.e. $\mathbf{f} \in C^1(\kappa^*T\mathcal{S})^* := \{\mathbf{f}\colon C^1(\kappa^*T\mathcal{S}) \to \mathbb{R} \,|\, \text{linear}\}$. The scalar $\delta W := \mathbf{f}(\delta\boldsymbol{\kappa})$ obtained by the evaluation of a force $\mathbf{f} \in C^1(\kappa^*T\mathcal{S})^*$ acting on a virtual displacement $\delta\boldsymbol{\kappa}$ is called virtual work. According to Segev [2], forces of $C^1(\kappa^*T\mathcal{S})^*$ have a representation by a collection of tensor measures $(\mathbf{f}_0, \mathbf{f}_1) \in C^0(\kappa^*T\mathcal{S})^* \oplus C^0(\kappa^*T\mathcal{S} \otimes T^*\mathcal{B})^*$ with the virtual work contribution

$$\delta W(\delta\boldsymbol{\kappa}) = \int_{\mathcal{B}} \delta\boldsymbol{\kappa}\mathrm{d}\mathbf{f}_0 + \int_{\mathcal{B}} \hat{\nabla}\delta\boldsymbol{\kappa}\mathrm{d}\mathbf{f}_1 \ . \tag{2}$$

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It is convenient to assume the existence of a virtual work density which is of the same form for any subbody $\mathcal{B}' \subset \mathcal{B}$. Then the set of forces can be restricted further to smooth tensor fields of covector-valued volume forms $\Gamma(\kappa^* T^*\mathcal{S} \otimes \Lambda^3 T^*\mathcal{B})$ and tensor-valued volume forms $\Gamma(\kappa^* T^*\mathcal{S} \otimes T\mathcal{B} \otimes \Lambda^3 T^*\mathcal{B})$ in the interior of the body and covector-valued volume forms $\Gamma(\kappa^* T^*\mathcal{S} \otimes \Lambda^2 T^*\partial\mathcal{B})$ on the boundary manifold $\partial\mathcal{B}$. In the sense of Germain [3], the forces acting on the body are classified into internal and external forces, respectively. Whereas internal forces of $\mathcal{B}$ are forces which model the interaction between subbodies of $\mathcal{B}$, external forces are forces from other bodies which interact with $\mathcal{B}$ and have no common material point with $\mathcal{B}$. External forces, represented by the tensor fields of body forces $\boldsymbol{\beta} \in \Gamma(\kappa^* T^*\mathcal{S} \otimes \Lambda^3 T^*\mathcal{B})$ and traction forces $\boldsymbol{\tau} \in \Gamma(\kappa^* T^*\mathcal{S} \otimes \Lambda^2 T^*\partial\mathcal{B})$ on the boundary, model long range forces. Internal forces, represented by the tensor field of variational stresses $\boldsymbol{\pi} \in \Gamma(\kappa^* T^*\mathcal{S} \otimes T\mathcal{B} \otimes \Lambda^3 T^*\mathcal{B})$, model short range forces. The corresponding virtual work contributions are

$$\delta W^{\mathrm{ext}}(\delta\boldsymbol{\kappa}) := \int_{\mathcal{B}} \delta\boldsymbol{\kappa} \cdot \boldsymbol{\beta} + \int_{\partial\mathcal{B}} \delta\boldsymbol{\kappa} \cdot \boldsymbol{\tau} \ , \quad \delta W^{\mathrm{int}}(\delta\boldsymbol{\kappa}) := \int_{\mathcal{B}} -\hat{\nabla}\delta\boldsymbol{\kappa} : \boldsymbol{\pi} \ , \tag{3}$$

where the colon indicates the contraction between the two adjacent tensor slots.

3 Equilibrium Equation

As the fundamental axiom of mechanics, the principle of virtual work states that the virtual work of all forces of the body has to vanish for all virtual displacements, i.e. $\delta W(\delta\boldsymbol{\kappa}) = \delta W^{\mathrm{ext}}(\delta\boldsymbol{\kappa}) + \delta W^{\mathrm{int}}(\delta\boldsymbol{\kappa}) = 0, \forall \delta\boldsymbol{\kappa} \in C^1(\kappa^* T\mathcal{S})$. By a contraction between the second and the third slot in the variational stress $\boldsymbol{\pi}$ a mapping $p_\sigma \colon \Gamma(\kappa^* T^*\mathcal{S} \otimes T\mathcal{B} \otimes \Lambda^3 T^*\mathcal{B}) \to \Gamma(\kappa^* T^*\mathcal{S} \otimes \Lambda^2 T^*\mathcal{B})$ is defined which induces to every variational stress $\boldsymbol{\pi}$ a covector-valued two-form $\boldsymbol{\sigma} = p_\sigma(\boldsymbol{\pi})$ called traction stress. Using a telescopic expansion by $\mathrm{d}(\delta\boldsymbol{\kappa} \cdot \boldsymbol{\sigma}) \in \Gamma(\Lambda^3 T^*\mathcal{B})$ and applying Stokes' theorem, the virtual work of the body is

$$\begin{aligned}
\delta W(\delta\boldsymbol{\kappa}) &= \int_{\mathcal{B}} \Big\{ -\hat{\nabla}\delta\boldsymbol{\kappa} : \boldsymbol{\pi} + \mathrm{d}(\delta\boldsymbol{\kappa} \cdot \boldsymbol{\sigma}) - \mathrm{d}(\delta\boldsymbol{\kappa} \cdot \boldsymbol{\sigma}) + \delta\boldsymbol{\kappa} \cdot \boldsymbol{\beta} \Big\} + \int_{\partial\mathcal{B}} \delta\boldsymbol{\kappa} \cdot \boldsymbol{\tau} \\
&= \int_{\mathcal{B}} \{ -\hat{\nabla}\delta\boldsymbol{\kappa} : \boldsymbol{\pi} + \mathrm{d}(\delta\boldsymbol{\kappa} \cdot \boldsymbol{\sigma}) + \delta\boldsymbol{\kappa} \cdot \boldsymbol{\beta} \} + \int_{\partial\mathcal{B}} \{ \delta\boldsymbol{\kappa} \cdot \boldsymbol{\tau} - \iota^*(\delta\boldsymbol{\kappa} \cdot \boldsymbol{\sigma}) \} \ ,
\end{aligned} \tag{4}$$

where $\iota \colon \partial\mathcal{B} \to \mathcal{B}$, $\iota(P) = P$ is the inclusion of the boundary in $\mathcal{B}$. Furthermore, the divergence of the variational stress is defined by the relation $\delta\boldsymbol{\kappa} \cdot \mathrm{Div}\,\boldsymbol{\pi} := -\hat{\nabla}\delta\boldsymbol{\kappa} : \boldsymbol{\pi} + \mathrm{d}(\delta\boldsymbol{\kappa} \cdot p_\sigma(\boldsymbol{\pi}))$. By straight forward computation

$$\begin{aligned}
-\hat{\nabla}\delta\boldsymbol{\kappa} : \boldsymbol{\pi} + \mathrm{d}(\delta\boldsymbol{\kappa} \cdot p_\sigma(\boldsymbol{\pi})) &= ((\delta\kappa^i \pi^J_{i123})_{,J} - (\delta\kappa^i{}_{,J} + \hat{\Gamma}^i_{Jk}\delta\kappa^k)\pi^J_{i123})\mathrm{d}\theta^1 \wedge \mathrm{d}\theta^2 \wedge \mathrm{d}\theta^3 \\
&= \delta\kappa^i(\pi^J_{i123,J} - \hat{\Gamma}^k_{Ji}\pi^J_{k123})\mathrm{d}\theta^1 \wedge \mathrm{d}\theta^2 \wedge \mathrm{d}\theta^3 = \delta\boldsymbol{\kappa} \cdot \mathrm{Div}\,\boldsymbol{\pi}
\end{aligned} \tag{5}$$

we obtain the coordinate representation of the divergence $\mathrm{Div}\,\boldsymbol{\pi} = (\pi^J_{i123,J} - \hat{\Gamma}^k_{Ji}\pi^J_{k123})\mathbf{g}^i \otimes \mathrm{d}\theta^1 \wedge \mathrm{d}\theta^2 \wedge \mathrm{d}\theta^3$. Using local charts $\lambda \colon \partial\mathcal{B} \supset W \to \mathbb{R}^2$ for the boundary manifold, the pullback of the dual base vector of $T^*\mathcal{B}$ to the boundary can be expressed by $\iota^*(\mathrm{d}\theta^I) = \frac{\partial \iota^I}{\partial \lambda^\alpha}\mathrm{d}\lambda^\alpha$, where greek indices run from 1 to 2 and $\iota^I := \theta^I \circ \iota \circ \lambda^{-1}$. The pullback of the traction stress to the boundary $\iota^*(\boldsymbol{\sigma})$ is defined by the relation $\delta\boldsymbol{\kappa} \cdot \iota^*(p_\sigma(\boldsymbol{\pi})) := \iota^*(\delta\boldsymbol{\kappa} \cdot p_\sigma(\boldsymbol{\pi}))$ which is in local coordinates

$$\begin{aligned}
\iota^*(\delta\boldsymbol{\kappa} \cdot p_\sigma(\boldsymbol{\pi})) &= \iota^*(\delta\kappa^k(\pi^1_{k123}\,\mathrm{d}\theta^2 \wedge \mathrm{d}\theta^3 - \pi^2_{k123}\,\mathrm{d}\theta^1 \wedge \mathrm{d}\theta^3 + \pi^3_{k123}\,\mathrm{d}\theta^1 \wedge \mathrm{d}\theta^2)) \\
&= \delta\kappa^k \left(\pi^J_{k123}\varepsilon_{JKL}\frac{\partial \iota^K}{\partial \lambda^1}\frac{\partial \iota^L}{\partial \lambda^2}\mathrm{d}\lambda^1 \wedge \mathrm{d}\lambda^2 \right) = \delta\kappa^k \left(\pi^J_{k123} N_J\,\mathrm{d}\lambda^1 \wedge \mathrm{d}\lambda^2 \right) = \delta\boldsymbol{\kappa} \cdot \iota^*(p_\sigma(\boldsymbol{\pi})) \ ,
\end{aligned} \tag{6}$$

where we introduced the abbreviation $N_J = \varepsilon_{JKL}\frac{\partial \iota^K}{\partial \lambda^1}\frac{\partial \iota^L}{\partial \lambda^2}$ and the Levi-Civita symbol ε_{JKL}. Thus, the coordinate representation of the traction force on the boundary induced by the traction stress is $\iota^*(p_\sigma(\boldsymbol{\pi})) = \pi^J_{k123}N_J\,\mathbf{g}^k \otimes \mathrm{d}\lambda^1 \wedge \mathrm{d}\lambda^2$. Using (5) and (6) together with (4) leads to the following form of the principle of virtual work

$$\delta W(\delta\boldsymbol{\kappa}) = \int_{\mathcal{B}} \delta\boldsymbol{\kappa} \cdot (\mathrm{Div}(\boldsymbol{\pi}) + \boldsymbol{\beta}) + \int_{\partial\mathcal{B}} \delta\boldsymbol{\kappa} \cdot (\boldsymbol{\tau} - \iota^*(\boldsymbol{\sigma})) = 0 \quad \forall \delta\boldsymbol{\kappa} \in C^1(\kappa^* T\mathcal{S}) \ . \tag{7}$$

The principle of virtual work (7) together with the Fundamental Theorem of Calculus of Variations directly implies the complete boundary value problem with the equilibrium equation $\mathrm{Div}(\boldsymbol{\pi}) + \boldsymbol{\beta} = 0$ in the interior of the body and the traction boundary condition $\boldsymbol{\tau} = \iota^*(\boldsymbol{\sigma})$.

References

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