An alternative perspective on the concept of stress in classical continuum mechanics

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Within a variational formulation of continuum mechanics, as proposed for instance by Germain [1], the internal virtual work contribution of a continuum is postulated as a smooth density integrated over the deformed configuration of the body. In this smooth density the stress field appears as dual quantity to the gradient of the virtual displacement field. Since the mathematical definition of the volume integral naturally provides a macro-micro relation between infinitesimal volume elements and the continuous body, we propose in this paper an alternative definition of stress on the micro level of the infinitesimal volume elements. In particular, the stress is defined as the internal forces of the body that model the mutual force interaction between neighboring volume elements. The existence of the stress tensor on the macro level is then obtained from the summation of all virtual work contributions within the body, followed by a limit process in which the volume elements are sent to zero.

1 Introduction

We start from the concept of the volume integral appearing in the virtual work contribution of a continuous body, as a dissection of the body into volume elements, followed by a limit process in which the elements are refined. We sketch this process for the Riemann integral with a dissection of the deformed configuration Ω into cuboids and an approximation from the inside, cf. Figure 1(a). In this context, we propose a definition of stress on the micro level of the infinitesimal volume elements from which the stress tensor on macro level emerges in the virtual work expression during the limit process related to the volume integral.

Definition 1.1 (Stress) We define *stress* as the internal forces of the body that model the mutual force interactions between neighboring infinitesimal volume elements.

2 The virtual work of stress

For the virtual work expression of the stress with the approximation of the volume integral from the inside, the force interactions between all the *inner* cuboids have to be considered, which are the cuboids in Figure 1(a) colored in grey. The white cuboids can be used formally to evaluate the Riemann sum, but *without* interactions among each other and *without* interactions with the grey cuboids.

In a first step, we consider two neighboring cuboids with cartesian coordinates (x_i, y_j, z_k) and (x_{i+1}, y_j, z_k) aligned in $\mathbf{e}_x^I$-direction, depicted in Figure 1(a) as hatched elements. Figure 1(b) shows a close-up of these two cuboids together with their virtual displacements $\delta\boldsymbol{\xi}$ and their incremental force interactions $\Delta\mathbf{G}_x$, $\Delta\mathbf{H}_x$. For the sake of brevity, we omit in the figures as well as in the upcoming formulas the dependence of $\delta\boldsymbol{\xi}$, $\Delta\mathbf{G}_x$ and $\Delta\mathbf{H}_x$ on the coordinates y_j and z_k, respectively. While $\Delta\mathbf{G}_x$ is the force increment exerted from the right cuboid on the left cuboid, $\Delta\mathbf{H}_x$ is the force increment exerted from the left cuboid on the right cuboid. To relate the force increments $\Delta\mathbf{G}_x$ and $\Delta\mathbf{H}_x$ to each other, we introduce new force increments $\Delta\mathbf{F}_x$ and $\Delta\mathbf{C}_x$ such that $\Delta\mathbf{G}_x(x_i) = \Delta\mathbf{F}_x(x_i) + \Delta\mathbf{C}_x(x_i)$ and $\Delta\mathbf{H}_x(x_i + \Delta x) = -\Delta\mathbf{F}_x(x_i)$.[1] The incremental virtual work contribution of the two adjacent cuboids by the force increments $\Delta\mathbf{G}_x$ und $\Delta\mathbf{H}_x$ is

$$\begin{aligned}
\Delta W^{\mathrm{s}} &= \delta\boldsymbol{\xi}(x_i)\cdot\Delta\mathbf{G}_x(x_i) + \delta\boldsymbol{\xi}(x_i+\Delta x)\cdot\Delta\mathbf{H}_x(x_i+\Delta x)\\
&= \delta\boldsymbol{\xi}(x_i)\cdot\Delta\mathbf{C}_x(x_i) - \Big(\delta\boldsymbol{\xi}(x_i+\Delta x) - \delta\boldsymbol{\xi}(x_i)\Big)\tfrac{1}{\Delta x}\cdot\Delta x\,\Delta\mathbf{F}_x(x_i)\\
&= \left(\delta\boldsymbol{\xi}(x_i)\cdot\frac{\Delta\mathbf{C}_x(x_i)}{\Delta V} - \frac{\delta\boldsymbol{\xi}(x_i+\Delta x)-\delta\boldsymbol{\xi}(x_i)}{\Delta x}\cdot\frac{\Delta\mathbf{F}_x(x_i)}{\Delta A_x}\right)\Delta V\\
&= \delta\boldsymbol{\xi}(x_i)\cdot\mathbf{C}_x(x_i)\,\Delta V - D_x\delta\boldsymbol{\xi}(x_i)\cdot\mathbf{T}_x(x_i)\,\Delta V\ ,
\end{aligned} \tag{1}$$

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[1] Note that we *intentionally* violate here the principle of action and reaction by the force increment $\Delta\mathbf{C}_x$: The principle of action and reaction does not constitute an independent axiom in variational continuum mechanics.

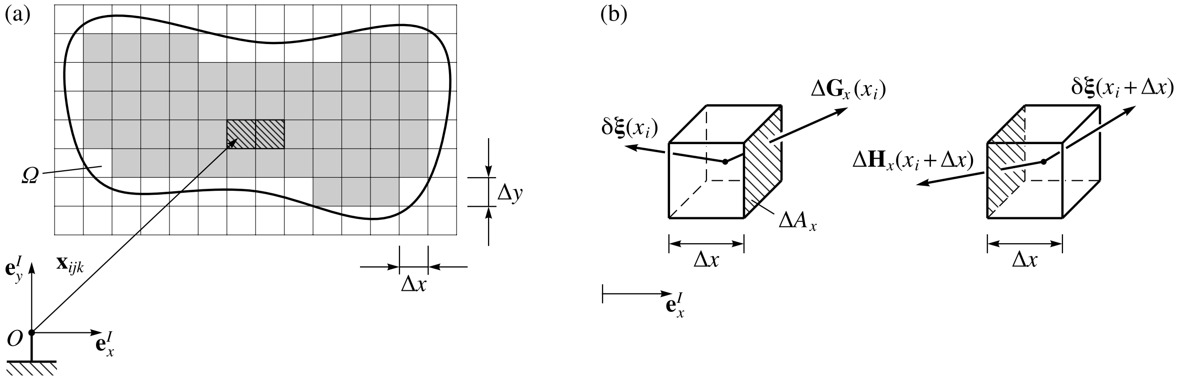

Fig. 1: (a) Dissection of the deformed configuration Ω into cuboids and approximation of the Riemann integral from the inside. The inner cuboids are colored in grey. (b) Force interaction between two neighboring volume elements aligned in $\mathbf{e}_x^I$-direction.

where we denote by $\Delta A_x := \Delta y\,\Delta z$ the incremental surface element that is shared by the adjacent cuboids, and by $\Delta V = \Delta x\,\Delta y\,\Delta z$ the incremental volume element. Furthermore, the abbreviations

$$\mathbf{C}_x(x_i) := \frac{\Delta \mathbf{C}_x(x_i)}{\Delta V}\,, \qquad \mathbf{T}_x(x_i) := \frac{\Delta \mathbf{F}_x(x_i)}{\Delta A_x}\,, \qquad D_x\delta\boldsymbol{\xi}(x_i) := \frac{\delta\boldsymbol{\xi}(x_i+\Delta x) - \delta\boldsymbol{\xi}(x_i)}{\Delta x}\,, \tag{2}$$

have been introduced, where $\mathbf{C}_x$ is the approximation of a volume force, $\mathbf{T}_x$ is the approximation of the stress vector, and $D_x\delta\boldsymbol{\xi}$ is the differential quotient. The force interaction between neighboring cuboids in y- and z-direction contribute in a similar way as (1) with the corresponding force quantities $(\mathbf{C}_y, \mathbf{T}_y)$ and $(\mathbf{C}_z, \mathbf{T}_z)$, respectively. The approximation of the virtual work of *all* mutual force interactions between the inner cuboids is then the sum over all volume elements with their contributions in the three spatial directions

$$\Delta W^{\mathrm{s}} = \sum_{i,j,k} \Big(\delta\boldsymbol{\xi}\cdot\mathbf{C}_x + \delta\boldsymbol{\xi}\cdot\mathbf{C}_y + \delta\boldsymbol{\xi}\cdot\mathbf{C}_z\Big)_{ijk}\Delta V - \Big(D_x\delta\boldsymbol{\xi}\cdot\mathbf{T}_x + D_y\delta\boldsymbol{\xi}\cdot\mathbf{T}_y + D_z\delta\boldsymbol{\xi}\cdot\mathbf{T}_z\Big)_{ijk}\Delta V\,. \tag{3}$$

The virtual work of stress on macro level δW^{s} is then obtained from the approximation (3) by taking the limit $\delta W^{\mathrm{s}} = \lim\limits_{\Delta V\to 0}\Delta W^{\mathrm{s}}$ which requires the limits of the objects in (2) to exist:

$$\mathbf{c}_x({}_I\mathbf{x}) := \lim_{\Delta V\to 0}\mathbf{C}_x(x_i,y_j,z_k)\,, \quad \mathbf{t}_x({}_I\mathbf{x}) := \lim_{\Delta A_x\to 0}\mathbf{T}_x(x_i,y_j,z_k)\,, \quad \delta\boldsymbol{\xi},_x({}_I\mathbf{x}) := \lim_{\Delta x\to 0} D_x\delta\boldsymbol{\xi}(x_i,y_j,z_k)\,. \tag{4}$$

The corresponding contributions in y- and z-direction are denoted accordingly. Using (3), (4) and defining $\mathbf{c} := \mathbf{c}_x + \mathbf{c}_y + \mathbf{c}_z$, we obtain from (3) the virtual work contribution of the stress as

$$\delta W^{\mathrm{s}} = \int_\Omega \Big(\delta\boldsymbol{\xi}\cdot\mathbf{c} - \delta\boldsymbol{\xi},_x\cdot\mathbf{t}_x - \delta\boldsymbol{\xi},_y\cdot\mathbf{t}_y - \delta\boldsymbol{\xi},_z\cdot\mathbf{t}_z\Big)\,\mathrm{d}v\,, \tag{5}$$

where $\mathrm{d}v$ denotes the volume element in the deformed configuration. Using the position vector $\mathbf{x}$ with the cartesian coordinates (x,y,z) in the orthonormal basis $(\mathbf{e}_x^I, \mathbf{e}_y^I, \mathbf{e}_z^I)$, the terms involving the stress vectors $\mathbf{t}_i$ in (5) are reformulated as

$$\delta\boldsymbol{\xi},_x\cdot\mathbf{t}_x + \delta\boldsymbol{\xi},_y\cdot\mathbf{t}_y + \delta\boldsymbol{\xi},_z\cdot\mathbf{t}_z = \left(\frac{\partial\delta\boldsymbol{\xi}}{\partial\mathbf{x}}\cdot\mathbf{e}_x^I\right)\cdot\mathbf{t}_x + \left(\frac{\partial\delta\boldsymbol{\xi}}{\partial\mathbf{x}}\cdot\mathbf{e}_y^I\right)\cdot\mathbf{t}_y + \left(\frac{\partial\delta\boldsymbol{\xi}}{\partial\mathbf{x}}\cdot\mathbf{e}_z^I\right)\cdot\mathbf{t}_z = \frac{\partial\delta\boldsymbol{\xi}}{\partial\mathbf{x}} : \boldsymbol{\sigma}\,, \tag{6}$$

where the *stress tensor* $\boldsymbol{\sigma}$ emerges as being the second order Euclidean tensor field

$$\boldsymbol{\sigma}(\mathbf{x}) := \mathbf{t}_x(\mathbf{x})\otimes\mathbf{e}_x^I + \mathbf{t}_y(\mathbf{x})\otimes\mathbf{e}_y^I + \mathbf{t}_z(\mathbf{x})\otimes\mathbf{e}_z^I\,. \tag{7}$$

The stress tensor field (7) as the tensor field over the deformed configuration Ω is called *Cauchy stress*. For continuous fields $\mathbf{c}$ and $\boldsymbol{\sigma}$, the *axiom of power of internal forces*, cf. [1], induces $\mathbf{c} = 0$ and the symmetry of the stress tensor $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{T}}$. Hence, together with (6), the virtual work of the stress (5) can consequently be written in the form

$$\delta W^{\mathrm{s}} = -\int_\Omega \frac{\partial\delta\boldsymbol{\xi}}{\partial\mathbf{x}} : \boldsymbol{\sigma}\,\mathrm{d}v\,, \quad \text{with} \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{T}}\,,$$

which coincides with the postulated form of the internal virtual work in variational continuum mechanics.

References

[1] P. Germain, SIAM J. appl. math. **25**(3), 556–575 (1973).