An ignored source in the foundations of continuum physics "Die Allgemeinen Ansätze der Mechanik der Kontinua" by E. Hellinger

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In 1913, Ernst Hellinger finished his remarkable article DIE ALLGEMEINEN ANSÄTZE DER MECHANIK DER KONTINUA, which appeared in German language in the "Encyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen". Following his predecessors J.-L. Lagrange, G. Piola and the Cosserat brothers, Hellinger presents in this paper a nonlinear field theory of continuum physics entirely based on a variational principle "the principle of virtual displacements". Even though Hellinger focuses on the fundamentals of continuum mechanics, he presents within the very same variational framework the physics of optics, electrodynamics, thermodynamics and the theory of relativity. Accordingly, Hellinger's paper can be understood as a contribution to continuum physics in general. However, due to the establishment of English as the upcoming scientific language and due to the refusal of a variational formulation of continuum mechanics in the subsequent period, Hellinger's contribution to the foundations of continuum physics has been ignored for decades and has almost fallen into oblivion.

1 Introduction

Ernst Hellinger, who was born in 1883 in Striegau, formerly Germany, enjoyed his scientific education at the Universities of Heidelberg, Breslau and Göttingen. As a pupil of D. Hilbert, he received his doctoral degree from the University of Göttingen in 1907. After two more years in Göttingen, as assistant of Hilbert, he moved to Marburg where he accepted a position as "Privatdozent". During that time at the University of Marburg, Hellinger wrote his masterpiece on the foundations of continuum mechanics – certainly in German language. More than a hundred years had to pass, until recently the authors of this paper have started an exegetic series about Hellinger's encyclopedia article. Besides epistemological and historical commentaries, the series, which starts with [1], contains the first complete translation into English. With the aim to trace the origins of current ideas of mechanical sciences to their original sources, the extraordinary work of Hellinger is now also available for scientists being not acquainted with the German language. In this short essay, we discuss very briefly the basic philosophy of a variational formulation of continuum mechanics and outline the main content treated in Hellinger's work.

2 A variational field theory

Hellinger presents in a unified way all field theories which were already formulated at his times, assuming as fundamental paradigm for physics the concept of field – in contrast to the concept of particles. The unifying mathematical form, in which all the different theories are embedded, is achieved by the postulation of variational principles. To avoid any misunderstanding, Hellinger explains, as paraphrased in the following, how "variational principles" have to be understood. The calculus of variations is a mathematical theory whose aim is to find extrema for functionals, usually expressed by means of integrals. To find these extrema one computes the first variation of the integral operators involved and obtains some linear functionals of the variations of the unknown fields. However, to base continuum mechanics on an extremum principle may be regarded as a too hazardous choice. Therefore, following Lagrange, Hellinger prefers to base the postulation of mechanics by formulating a principle HAVING THE FORM of the necessary criterion for being an extremum. This point is rather abstract, but its implications have a marvelous impact, allowing for a very general postulation of physical theories. This postulation is based on "the principle of virtual displacements" – nowadays referred to as the principle of virtual work or velocities.

Hellinger adopts the Lagrangian scheme and starts his theory with the kinematics of all different continua he is considering:
1. Introduction: In a few lines Hellinger clarifies his philosophical and epistemological point of view. He answers clearly to essentially all objections always repeated against the postulation based on variational principles.
2. The notion of a continuum: The nonlinear kinematics of the three-dimensional continuous body is introduced in terms of placement functions. The virtual displacements are then defined rigorously as the variations of these placement functions, i.e. using the Gâteaux derivative. Additionally, the kinematics of a two-dimensional, one-dimensional and the Cosserat continua are treated.

In the sense of Lagrange, after kinematics, the theory of statics follows. This theory, relying on the principle of virtual displacements, is formulated independently of which classes of force effects influence the continuum in particular:

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I. The foundations of statics
3. The principle of virtual displacements: The virtual work, being the elemental quantity of a variational formulation, is defined as a linear homogeneous function on the space of all possible virtual displacements. Postulating a certain form of the total virtual work of all force effects, the local equilibrium equations are derived. Moreover, the balance of forces and moments for the body and all its sub-bodies in integral form are obtained applying smoothened discontinuous virtual displacements. Similar conclusions are drawn for the one- and two-dimensional continua.
4. Enhancement of the principle of virtual displacements: Second gradient materials as well as one-, two-, and three-dimensional Cosserat media with additional rotational degrees of freedom are discussed. In order to treat kinematical restrictions, the concept of ideal constraints is introduced.
II. The foundations of kinetics
In the numbers **5 a)** to **5 d)**, by adding the virtual work of the inertia forces to the total virtual work, a theory of kinetics is discussed. Starting from this variational principle, Hamilton's principle and also Gauss' principle of least constraint follow directly. Remarkably, Hellinger suggests also principles of general nature, in which he recognizes the momentum as the quantity dual to the time derivative of the virtual displacements.

Once statics and kinetics are introduced independently of any force laws, Part III-A gives the general structure of possible constitutive laws. In Part III-B, Hellinger shows how a (meta-)theory can be applied to supply the formulation of particular "concrete" theories:
III. The forms of constitutive laws – A. Formulation of general classes
6. The classes with dependence of the force effects on the deformation quantities: With the functional relation between force quantities and the placement functions, the most general form of possible constitutive laws is given. The only restriction is due to the formulation of a requirement which nowadays is known as the principle of material-frame indifference.
7. Media with *one* characteristic function: Due to the variational formulation, hyperelastic materials can be treated in this number without the consideration of any thermodynamical theory. Besides the ordinary continuum, this approach is also applied to second gradient materials resulting in surface potentials, to non-local theories and furthermore to media with oriented particles. Moreover, using a canonical transformation of the strain energy function, Hellinger lays the foundations of the Hellinger-Reissner principle.
8. Limit cases of the ordinary three-dimensional continuum: The basic results in asymptotic analysis as applied to reduced order and reduced dimension mechanical models are outlined.
III. The forms of constitutive laws – B. Individualization for particular fields
9. Effective theory of elasticity: A theory of elasticity, which is developed in the fully nonlinear case, taking into account the objectivity requirements of constitutive equations, is given. Also beams and plates are considered.
10. Dynamics of ideal fluids: The dynamics of ideal fluids is presented based on the results of D'Alembert and Piola including the case of incompressible fluids.
11. Internal friction and elastic hysteresis: A theory of internal friction and elastic hysteresis is given by introducing the Rayleigh potential for internal friction. Furthermore, Boltzmann memory effects are considered together with the basic ideas of plasticity and its potential to the description of sandy materials.
12. Capillarity: A theory of capillarity is discussed, including a treatment of the Laplace concept of surface energy with the suggestion of some generalizations which are even nowadays not completely explored.
13. Optics: Optics is presented in terms of a field theory ready to establish its relationship to electrodynamics.
14. Relations to electrodynamics: Electrodynamics is described considering the nature of free ether and the structure of the field theory leading to Maxwell's equations.
15. Introduction of the thermodynamical foundations: Thermodynamics is presented in such a way that one believes to read many parts of the models subsequently developed by Coleman and Noll.
16. Relations to the theory of relativity: A theory of relativity is given, in which a variational formulation of the basic principles of relativity are described in a way which anticipate Noll's papers on Minkowskian chronometry.

3 Conclusions

Hellinger's extraordinary treatise on continuum physics is just another example how a change in the scientific language can lead to a loss of knowledge. However, this effect has also been intensified by American scientists in mechanics which, after World War II, tried to demolish the scientific heritage of Europe by slighting the contents of the earlier contributions or by not even citing the correct references.

References

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