A nonlinear Timoshenko beam formulation for modeling a tendon-driven compliant neck mechanism

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In this paper, we consider the dynamical behavior of a tendon-driven compliant neck mechanism which is used to actuate the head of a humanoid robot as proposed by [1, 2]. The neck is realized as a silicone block mounted onto the robot's torso. At the top end of the silicone block an aluminum plate interconnects the compliant neck with the head. At the same plate, tendons are attached whose actuation causes the soft and flexible block to deform thereby inducing a motion of the robot's head. For workspace design and control of the head's trajectory, a mechanical model is required which appropriately describes the entire neck-head system. We present a dynamic model of the system in which the silicone block and the head are modeled as a planar nonlinear Timoshenko beam and a rigid body, respectively. The tendon actuations are included as external configuration-dependent forces.

The motion of the system is described in the three-dimensional Euclidean vector space $\mathbb{E}^3$ with origin O and right-handed orthonormal coordinate frame $\mathbf{e}_i^I \in \mathbb{E}^3, i = \{x, y, z\}$ and takes place exclusively in the $\mathbf{e}_x^I$-$\mathbf{e}_z^I$-plane, see Fig. 1b. The cartesian coordinate representation of a vector $\mathbf{a} \in \mathbb{E}^3$ in an arbitrary orthonormal B-system rotated against the I-system is denoted as ${}_B\mathbf{a} = (a_x^B, a_y^B, a_z^B)^\mathrm{T} \in \mathbb{R}^3$ with $\mathbf{a} = a_x^B \mathbf{e}_x^B + a_y^B \mathbf{e}_y^B + a_z^B \mathbf{e}_z^B \in \mathbb{E}^3$. The neck made out of silicone is modeled as a planar nonlinear Timoshenko beam. According to the Timoshenko beam assumptions, the motion of the three-dimensional continuum can be described merely by the motion of a centerline and the rotations of plane rigid cross-sections attached to every point of the centerline. The centerline $\mathbf{r} = \mathbf{r}(s,t) \in \mathbb{E}^3$ is a plane curve at time t parameterized by $s = [0, L] \subset \mathbb{R}$ being the arclength of the undeformed beam with length L. The cross-sections of the beam are represented by the cross-section-fixed frames $\mathbf{e}_i^C = \mathbf{e}_i^C(s,t) \in \mathbb{E}^3, i = \{x, y, z\}$ continuously varying along the centerline and in time. The beam is fixed to the ground such that $\mathbf{r}(0,t) = 0$ and $\mathbf{e}_i^C(0,t) = \mathbf{e}_i^I, i = \{x, y, z\}$. The inertia properties of the beam ρA and ρJ_{yy} are the mass and moment of inertia density per unit arclength s, respectively. On top of the beam at $s = L$ a rigid and massless plate with a width of $2b$ is attached as depicted in Fig. 1c. In P and R two massless tendons are connected to the plate. Both tendons are redirected by a pulley and subjected at their ends to the tensile forces $\lambda_l \geq 0$ and $\lambda_r \geq 0$, respectively. The head is modeled as a rigid body with center of mass (CoM) S, mass m, moment of inertia J and is rigidly connected to the beam in $\mathbf{r}(L,t)$ such that the head-fixed frame $\mathbf{e}_i^H(t) := \mathbf{e}_i^C(L,t), i = \{x, y, z\}$.

The centerline $\mathbf{r}$ and the orthogonal transformation matrix $\mathbf{A}_{IC}$ relating the respective coordinates according to ${}_I\mathbf{a} = \mathbf{A}_{IC\,C}\mathbf{a}$ are determined by the real-valued generalized position functions $x = x(s,t)$, $z = z(s,t)$ and $\theta = \theta(s,t)$, i.e.

$$
{}_I\mathbf{r}(s,t) = \begin{pmatrix} x(s,t) \\ 0 \\ z(s,t) \end{pmatrix} , \quad \mathbf{A}_{IC}(s,t) = \begin{pmatrix} \cos\theta(s,t) & 0 & \sin\theta(s,t) \\ 0 & 1 & 0 \\ -\sin\theta(s,t) & 0 & \cos\theta(s,t) \end{pmatrix} . \tag{1}
$$

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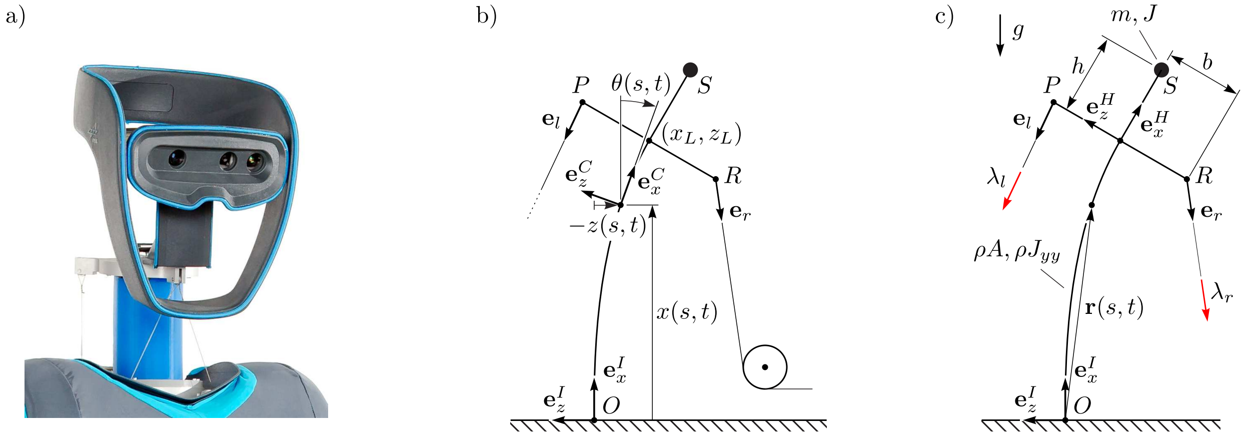

Fig. 1: a) Neck-head system of humanoid presented in [1]. b) Planar kinematics. c) Dimensions, applied forces and inertia properties.

Let $j : \mathbb{R}^3 \to so(3)$ be the bijective map that relates triples with skew-symmetric matrices of $\mathbb{R}^{3\times 3}$. The changes in the orientation of the cross-section along s or when time t passes are described by the material curvature vector $\mathbf{k}_{IC}$ and the angular velocity vector $\boldsymbol{\omega}_{IC}$ which are determined in the I-system by the relations

$$ {}_I\mathbf{k}_{IC}(s,t) = j^{-1}\big(\mathbf{A}'_{IC}(s,t)\mathbf{A}^{\mathrm{T}}_{IC}(s,t)\big)\,, \quad {}_I\boldsymbol{\omega}_{IC}(s) = j^{-1}\big(\dot{\mathbf{A}}_{IC}(s,t)\mathbf{A}^{\mathrm{T}}_{IC}(s,t)\big)\,. \tag{2} $$

Note that $(\bullet)'$ and $(\dot{\bullet})$ denote the derivatives with respect to s and t, respectively. Inserting (1) into (2), straightforward computation leads to the material curvature vector $\mathbf{k}_{IC}(s,t) = \theta'(s,t)\mathbf{e}^I_y$ and the angular velocity vector $\boldsymbol{\omega}_{IC}(s,t) = \dot{\theta}(s,t)\mathbf{e}^I_y$.

Consider $\hat{x} = \hat{x}(s,t,\varepsilon)$, $\hat{z} = \hat{z}(s,t,\varepsilon)$ and $\hat{\theta} = \hat{\theta}(s,t,\varepsilon)$ to be differentiable parametrizations with respect to a parameter $\varepsilon \in \mathbb{R}$ such that the actual positions are embedded in the parametrization and are obtained for $\varepsilon = \varepsilon_0$. Replacing the corresponding functions in (1), the variational families $\hat{\mathbf{r}} = \hat{\mathbf{r}}(s,t,\varepsilon)$ and $\hat{\mathbf{A}}_{IC} = \hat{\mathbf{A}}_{IC}(s,t,\varepsilon)$ are obtained. Introducing $\delta x = \partial\hat{x}/\partial\varepsilon|_{\varepsilon=\varepsilon_0}\delta\varepsilon$, $\delta z = \partial\hat{z}/\partial\varepsilon|_{\varepsilon=\varepsilon_0}\delta\varepsilon$ and $\delta\theta = \partial\hat{\theta}/\partial\varepsilon|_{\varepsilon=\varepsilon_0}\delta\varepsilon$, the virtual displacement of the centerline $\delta\mathbf{r} = \delta\mathbf{r}(s,t)$ and the virtual rotations of the cross-sections $\delta\boldsymbol{\phi}_{IC} = \delta\boldsymbol{\phi}_{IC}(s,t)$ are defined by

$$ {}_I\delta\mathbf{r} = \frac{\partial_I\hat{\mathbf{r}}}{\partial\varepsilon}\bigg|_{\varepsilon=\varepsilon_0}\delta\varepsilon = (\delta x, 0, \delta z)^{\mathrm{T}}\,, \quad {}_I\delta\boldsymbol{\phi}_{IC} = j^{-1}\bigg(\frac{\partial\hat{\mathbf{A}}_{IC}}{\partial\varepsilon}\bigg|_{\varepsilon=\varepsilon_0}\delta\varepsilon\;\mathbf{A}^{\mathrm{T}}_{IC}\bigg) = (0,\delta\theta,0)^{\mathrm{T}}\,. \tag{3} $$

To establish the virtual work of the external forces acting on the system, we will need the virtual displacement of the head's CoM and the virtual displacements of the tendon connection points P and R. Using the abbreviations $\delta\theta_L(t) = \delta\theta(L,t)$ and $\delta\mathbf{r}_L(t) = \delta\mathbf{r}(L,t)$ together with the dimensions of Fig. 1c, the required virtual displacements are

$$ \delta\mathbf{r}_S = \delta\mathbf{r}_L - h\delta\theta_L\mathbf{e}^H_z\,, \quad \delta\mathbf{r}_P = \delta\mathbf{r}_L + b\delta\theta_L\mathbf{e}^H_x\,, \quad \delta\mathbf{r}_R = \delta\mathbf{r}_L - b\delta\theta_L\mathbf{e}^H_x\,. \tag{4} $$

The axial strain $\gamma_x = \gamma_x(s,t)$, the shear strain $\gamma_z = \gamma_z(s,t)$ and the material curvature $\kappa = \kappa(s,t)$ capturing the deformation of the beam are defined by the relations

$$ (\gamma_x, 0, \gamma_z)^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}}_{IC\,I}\mathbf{r}' = (x'\cos\theta - y'\sin\theta, 0, x'\sin\theta + y'\cos\theta)^{\mathrm{T}}\,, \quad (0,\kappa,0)^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}}_{IC\,I}\mathbf{k}_{IC} = (0,\theta',0)^{\mathrm{T}}\,. \tag{5} $$

The hyperelastic constitutive behavior of the beam is captured by the specific strain energy function

$$ \Psi(\gamma_x,\gamma_z,\kappa) = \frac{EA}{3}\Big(\frac{\gamma_x^2}{2} + \gamma_x^{-1}\Big) + \frac{GA}{2}\gamma_z^2 + \frac{EI}{2}\kappa^2 \tag{6} $$

in which EA, GA, and EI denote the axial stiffness, the shear stiffness and the bending stiffness of the beam, respectively. The corresponding virtual work contribution of the internal forces is then

$$ \delta W^{\mathrm{int}} = -\int_0^L \delta\Psi(\gamma_x,\gamma_z,\kappa)\mathrm{d}s = -\int_0^L \bigg\{\frac{\partial\Psi}{\partial\gamma_x}\delta\gamma_x + \frac{\partial\Psi}{\partial\gamma_z}\delta\gamma_z + \frac{\partial\Psi}{\partial\kappa}\delta\kappa\bigg\}\,\mathrm{d}s\,. \tag{7} $$

The virtual work contribution of the external forces of the system

$$ \delta W^{\mathrm{ext}} = \int_0^L -\rho A g\mathbf{e}^I_x\cdot\delta\mathbf{r}\,\mathrm{d}s - mg\mathbf{e}^I_x\cdot\delta\mathbf{r}_S + \lambda_l\mathbf{e}_l\cdot\delta\mathbf{r}_P + \lambda_r\mathbf{e}_r\cdot\delta\mathbf{r}_R \tag{8} $$

is due to gravity with gravity constant g and due to the tendon actuation with tendon forces $\lambda_l\mathbf{e}_l$ and $\lambda_r\mathbf{e}_r$. Note that the unit tendon direction vectors for the left $\mathbf{e}_l \in \mathbb{E}^3$ and right $\mathbf{e}_r \in \mathbb{E}^3$ tendon as depicted in Fig. 1c are nonlinear functions of the centers of the pulleys and the position vectors of P and R. Introducing the abbreviations $x_L(t) = x(L,t)$, $z_L(t) = z(L,t)$ and $\theta_L(t) = \theta(L,t)$ together with the acceleration of the head's CoM ${}_I\mathbf{a}_S = (\ddot{x}_L - h\sin\theta_L\ddot{\theta}_L - h\cos\theta_L\dot{\theta}_L^2, 0, \ddot{z}_L - h\cos\theta_L\ddot{\theta}_L + h\sin\theta_L\dot{\theta}_L^2)^{\mathrm{T}}$, the virtual work expression due to inertia is

$$ \delta W^{\mathrm{dyn}} = -\int_0^L \{\rho A\ddot{\mathbf{r}}\cdot\delta\mathbf{r} + \rho J_{yy}\ddot{\theta}\delta\theta\}\mathrm{d}s - m\mathbf{a}_S\cdot\delta\mathbf{r}_S - J\ddot{\theta}_L\delta\theta_L\,. \tag{9} $$

To determine the dynamical equations of the system, we postulate the principle of virtual work that assumes the total virtual work $\delta W^{\mathrm{tot}} = \delta W^{\mathrm{int}} + \delta W^{\mathrm{ext}} + \delta W^{\mathrm{dyn}}$ of the system to vanish for all admissible virtual displacements. The principle of virtual work with the contributions (7), (8) and (9) corresponds to the weak variational expression of a nonlinear partial differential equation which, together with its associated boundary conditions, can be obtained by integration by parts of the contributions in (7). Since this PDE cannot be solved analytically, in [3] a finite element approximation in space is applied which discretizes the total virtual work by suitable shape functions for $x(s,t)$, $z(s,t)$ and $\theta(s,t)$. For the dynamical case considered at hand, this discretization would lead directly to a set of ODE's which can be solved numerically by standard ODE-solvers.

References

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