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On different geometric approaches to the dynamics of finite-dimensional mechanical systems

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The language of differential geometry allows a coordinate-free representation of physical quantities. This led to the development of several geometric theories for the description of finite-dimensional mechanical systems. These approaches differ in the mathematical concepts they invoke and in the classes of mechanical systems they can describe. This short note aims to give an overview on the following three popular approaches, all of which are limited to time-independent mechanical systems. While in the first approach, the motion of the mechanical system is considered as a curve in the system's configuration manifold, in the latter two, the corresponding motions are interpreted as curves in the tangent or the cotangent bundle of the configuration manifold.

For a mechanical system with n degrees of freedom, we consider the *configuration manifold* as an n-dimensional smooth manifold Q representing all possible configurations of the system. Locally, there are charts

$$\phi\colon Q \supseteq U \to \mathbb{R}^n,\, q \mapsto \phi(q) = (q^1, \dots, q^n) \tag{1}$$

commonly referred to as generalized coordinates of the system. The configuration manifold is equipped with a Riemannian metric g, locally written as $g{=}g_{ij}\mathrm{d}q^i{\otimes}\mathrm{d}q^j$.[1] The Riemannian metric endows each tangent space T_qQ with an inner product and models the mass of the mechanical system, i.e., the component matrix $[g_{ij}]$ corresponds to the mass matrix of the system. The pair (Q, g) is a *Riemannian manifold* on which the dynamics is formulated in this first approach. The *motion* of the system can be considered as a curve $\gamma\colon \mathbb{R} \supset I \to Q$. The tangent field along γ is denoted by $\dot\gamma(t) = (\gamma(t), \dot\gamma_{\gamma(t)})$ and associates with each time instant $t \in I$ a position $\gamma(t) \in Q$ and a velocity $\dot\gamma_{\gamma(t)} \in T_{\gamma(t)}Q$ of the mechanical system. Denoting the i-th coordinate function of the chart ϕ by $\phi^i\colon U \to \mathbb{R}$, both the positions and velocities can be expressed locally as

$$\big(q^1(t), \dots, q^n(t)\big) = \big(\phi^1 \circ \gamma(t), \dots, \phi^n \circ \gamma(t)\big)\,, \quad \text{and } \dot\gamma_{\gamma(t)} = \dot q^i(t)\, \frac{\partial}{\partial q^i}\bigg|_{\gamma(t)}\,, \tag{2}$$

where the dot on the right hand side of the second equation stands for the derivative with respect to time t. On a Riemannian manifold, there exists a unique metric compatible and torsion free connection, called Levi-Civita connection, whose Christoffel symbols Γ^k_{lm} are determined by the metric coefficients, i.e., $2\Gamma^k_{lm} = g^{kr}(\partial g_{mr}/\partial q^l + \partial g_{rl}/\partial q^m - \partial g_{lm}/\partial q^r)$ with g^{kr} such that $g^{kr}g_{rm} = \delta^k_m$. Then, the covariant derivative of the Levi-Civita connection allows to define the *acceleration* at time t by

$$(\nabla_{\dot\gamma}\dot\gamma)_{\gamma(t)} = \big(\ddot q^k(t) + \Gamma^k_{lm}(\gamma(t))\dot q^l(t)\dot q^m(t)\big)\, \frac{\partial}{\partial q^k}\bigg|_{\gamma(t)}\,. \tag{3}$$

The *nonpotential forces* $f^{\mathrm{np}} = f^{\mathrm{np}}_i \mathrm{d}q^i \in \Omega^1(Q)$ are modeled as one-forms on the configuration manifold Q. The *potential forces* $f^{\mathrm{p}}{=}{-}\mathrm{d}V \in \Omega^1(Q)$ are those one-forms that can be written as the exterior derivative of a *potential* which is a smooth real-valued function $V\colon Q \to \mathbb{R}$. In the first approach, see Prop. 3.7.4. in [1], we *postulate* that the motion γ is the curve determined by $g_{\gamma(t)}\big(\,\cdot\,, (\nabla_{\dot\gamma}\dot\gamma)_{\gamma(t)}\big) = -\mathrm{d}V_{\gamma(t)}(\cdot) + f^{\mathrm{np}}_{\gamma(t)}(\cdot)$, which locally can be expressed as

$$g_{ik}(\gamma(t))\,\big[\ddot q^k(t) + \Gamma^k_{lm}(\gamma(t))\dot q^l(t)\dot q^m(t)\big]\,\mathrm{d}q^i|_{\gamma(t)} = \left[\left(-\frac{\partial V}{\partial q^i} + f^{\mathrm{np}}_i\right) \circ \gamma(t)\right]\mathrm{d}q^i|_{\gamma(t)}\,, \tag{4}$$

which are the equations of motion of a finite-dimensional mechanical system. Moreover, equation (4) can be interpreted as a generalization of Newton's 'mass × acceleration = applied forces'. The drawback of working on the configuration manifold is that neither velocity- nor time-dependent forces can be considered.

Velocity-dependent forces can be defined on a space comprising positions and velocities. Therefore, the second approach (see [3]) describes the dynamics on the $2n$-dimensional tangent bundle TQ of the configuration manifold (Q, g). The tangent bundle with the natural projection $\pi\colon TQ \to Q$ is called the state space of the system. Indeed, a point $(q, v_q) \in TQ$ represents the system's state, where $q \in Q$ and $v_q \in T_qQ$ represent the position and velocity, respectively. The chart (1) of Q induces the natural chart $\Phi\colon TQ \supseteq \pi^{-1}(U) \to \mathbb{R}^{2n}$, $(q, v_q = u^i\partial/\partial q^i|_q) \mapsto (q^1, \dots, q^n, u^1, \dots, u^n)$ of the tangent bundle TQ, where $\pi^{-1}(U)$ denotes the preimage of the neighborhood U with the natural projection π. The *motion* η of the system is a curve in the

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[1] Here and in what follows, we make use of Einstein's summation convention, which implies summation over repeated indices from 1 to n.

state space TQ and is considered to be an integral curve of the *Lagrangian vector field* $X{=}A^i\partial/\partial q^i{+}B^i\partial/\partial u^i \in \mathrm{Vect}(TQ)$, i.e., $\dot\eta(t) = X(\eta(t))$. The tangent field $\dot\eta(t) = (\eta(t), \dot\eta_{\eta(t)})$ along η associates with each time instant $t \in I$ a position and velocity $\eta(t) \in TQ$ as well as a velocity and acceleration $\dot\eta_{\eta(t)} \in T_{\eta(t)}(TQ)$ of the mechanical system. Locally, this can be written as

$$\big(q^1(t),\ldots,q^n(t),u^1(t),\ldots,u^n(t)\big) = \big(\Phi^1\circ\eta(t),\ldots,\Phi^{2n}\circ\eta(t)\big)\ ,\quad \dot\eta_{\eta(t)} = \dot q^i(t)\,\frac{\partial}{\partial q^i}\bigg|_{\eta(t)} + \dot u^i(t)\,\frac{\partial}{\partial u^i}\bigg|_{\eta(t)} . \tag{5}$$

Note that it will be the role of the Lagrangian vector field to relate the time derivatives $\dot q^i(t)$ with the velocity coordinates $u^i(t)$. The *Lagrangian* $L\colon TQ \to \mathbb{R}$ of a time-independent mechanical system classically is of the form $L(q,v_q) = \frac{1}{2}g_q(v_q,v_q) - V(q)$, where V is the potential modeling the potential forces similar to the first paragraph. The Lagrangian can be used to define the *energy function* $E\colon TQ \to \mathbb{R}$ given by $E = \frac{\partial L}{\partial u^j}u^j - L$ as well as the *Cartan one-form* $\theta_L = \frac{\partial L}{\partial u^i}\mathrm{d}q^i \in \Omega^1(TQ)$. The *Lagrangian two-form*

$$\Omega_L = -\mathrm{d}\theta_L = \frac{\partial^2 L}{\partial q^j\partial u^i}\mathrm{d}q^i\wedge\mathrm{d}q^j + \frac{\partial^2 L}{\partial u^j\partial u^i}\mathrm{d}q^i\wedge\mathrm{d}u^j \in \Omega^2(TQ)\ , \tag{6}$$

obtained by taking the exterior derivative of the Cartan one-form is a symplectic form. The *nonpotential forces* are modeled as semi-basic one-forms $f^{\mathrm{np}} = f_i^{\mathrm{np}}\mathrm{d}q^i \in \Omega^1(TQ)$, see [2]. In this approach, we *postulate* that the Lagrangian vector field X, of which the motion is an integral curve, is determined by $\Omega_L(X,\cdot) = \mathrm{d}E(\cdot) - f^{\mathrm{np}}(\cdot)$. Evaluating these conditions along η, we locally get

$$\begin{aligned} &\dot q^i(t)\,\frac{\partial}{\partial q^i}\bigg|_{\eta(t)} + \dot u^i(t)\,\frac{\partial}{\partial u^i}\bigg|_{\eta(t)} = A^i\circ\eta(t)\,\frac{\partial}{\partial q^i}\bigg|_{\eta(t)} + B^i\circ\eta(t)\,\frac{\partial}{\partial u^i}\bigg|_{\eta(t)}\ ,\\ &0 = \left[\frac{\partial^2 L}{\partial u^j\partial u^i}B^j + \frac{\partial^2 L}{\partial q^j\partial u^i}A^j - \frac{\partial L}{\partial q^i} - f_i^{\mathrm{np}} + \frac{\partial^2 L}{\partial q^i\partial u^j}(u^j - A^j)\right]_{\eta(t)}\mathrm{d}q^i|_{\eta(t)} + \left[\frac{\partial^2 L}{\partial u^i\partial u^j}(u^j - A^j)\right]_{\eta(t)}\mathrm{d}u^i|_{\eta(t)}\ . \end{aligned} \tag{7}$$

The vanishing of the $\mathrm{d}u^i$-components implies the kinematic equations that relate the time derivatives $\dot q^i(t)$ with the velocity coordinates $u^i(t)$. The vanishing $\mathrm{d}q^i$-components provide Lagrange's equations of the second kind for mechanical systems with position- and velocity-dependent forces.

Analogously, the dynamics can be formulated on the phase space (see Prop. 3.3.2. in [1]), i.e., on the cotangent bundle T^*Q of the configuration manifold (Q,g). Here, each state $(q,p_q) \in T^*Q$ corresponds to a position $q \in Q$ and a generalized momentum $p_q \in T_q^*Q$ which locally can be expressed by charts $\Psi\colon T^*Q \supseteq \pi^{-1}(U) \to \mathbb{R}^{2n}$, $(q, p_q = p_i\mathrm{d}q^i|_q) \mapsto (q^1,\ldots,q^n,p_1,\ldots,p_n)$. The *motion* ζ of the system is now seen as a curve in the phase space T^*Q assumed to be an integral curve of the *Hamiltonian vector field* $Y{=}A^i\partial/\partial q^i{+}C_i\partial/\partial p_i \in \mathrm{Vect}(T^*Q)$, i.e., $\dot\zeta(t) = Y(\zeta(t))$. The cotangent bundle comes with the *canonical symplectic form* $\Omega_H{=}{-}\mathrm{d}\theta_H{=}\mathrm{d}q^i{\wedge}\mathrm{d}p_i \in \Omega^2(T^*Q)$, where $\theta_H{=}p_i\mathrm{d}q^i \in \Omega^1(T^*Q)$ is the *canonical one-form* of T^*Q. The *Hamiltonian* $H\colon T^*Q \to \mathbb{R}$ of a time-independent mechanical system classically is of the form $H(q,p_q) = \frac{1}{2}g_q^{-1}(p_q,p_q) + V(q)$, where V is the potential and g^{-1} is the bundle metric of the cotangent bundle induced by the Riemannian metric g. The *nonpotential forces* are modeled as semi-basic one-forms $f^{\mathrm{np}} = f_i^{\mathrm{np}}\mathrm{d}q^i$. In this last approach, we *postulate* that the Hamiltonian vector field Y, of which the motion is an integral curve, is determined by $\Omega_H(Y,\cdot) = \mathrm{d}H(\cdot) - f^{\mathrm{np}}(\cdot)$. Evaluating these conditions along ζ leads to

$$\begin{aligned} &\dot q^i(t)\,\frac{\partial}{\partial q^i}\bigg|_{\zeta(t)} + \dot p_i(t)\,\frac{\partial}{\partial p_i}\bigg|_{\zeta(t)} = A^i\circ\zeta(t)\,\frac{\partial}{\partial q^i}\bigg|_{\zeta(t)} + C_i\circ\zeta(t)\,\frac{\partial}{\partial p_i}\bigg|_{\zeta(t)}\ ,\\ &0 = \left[\frac{\partial H}{\partial q^i} - f_i^{\mathrm{np}} + C_i\right]_{\zeta(t)}\mathrm{d}q^i|_{\zeta(t)} + \left[\frac{\partial H}{\partial p_i} - A^i\right]_{\zeta(t)}\mathrm{d}p_i|_{\zeta(t)} \end{aligned} \tag{8}$$

The vanishing of both the $\mathrm{d}q^i$- and $\mathrm{d}p_i$-components implies Hamilton's equations for a mechanical system with position- and momentum-dependent forces. For the geometric treatment of time-dependent mechanical systems, where also time-dependent forces can be treated, the reader is referred to the related PAMM contributions 'Kinematics of finite-dimensional mechanical systems on Galilean manifolds' as well as 'Dynamics of finite-dimensional mechanical systems on Galilean manifolds'.

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