A second gradient continuum formulation for bi-pantographic fabrics

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Bi-pantographic fabrics are materials that are composed of two families of fibers with a micro-structure. The micro-structure of a straight fiber is given by a periodic arrangement of cells that resembles an expanding barrier. The planar behavior of such a fabric is captured at a macroscopic scale by a second gradient continuum. The corresponding strain energy function depends upon the fiber stretches, as well as upon the gradients in fiber direction of the stretches and inclination angles. In this brief communication, we present the equilibrium equations, which must be satisfied in the bulk of the continuum.

For the continuous modeling of the planar mechanical behavior of bi-pantographic fabrics, we work in a two-dimensional physical space modeled by the two-dimensional Euclidean vector space $\mathbb{E}^2$. The placement $\chi : \Omega \to \mathbb{E}^2, X \mapsto x = \chi(X)$ maps points X from the reference configuration $\Omega \subset \mathbb{E}^2$ into the physical space. The image of the placement map $\omega = \chi(\Omega)$ defines the actual configuration of the continuum. We introduce the two orthonormal and positively oriented bases (E_1, E_2) and (e_1, e_2) for objects with image in the reference and in the actual configuration, respectively. Hence, the placement map is written as $x^i e_i = \chi^i(X) e_i$, where Einstein's summation convention is applied here and in what follows for upper- and lower-case roman letters. Let $\hat{\chi} : \mathbb{R} \times \Omega \to \mathbb{E}^2, (\varepsilon, X) \mapsto x = \hat{\chi}(\varepsilon, X)$ be a one-parameter family of mappings such that $\chi(X) = \hat{\chi}(0, X)$. Then the variation of the placement $\delta\chi(X) = \partial\hat{\chi}^i/\partial\varepsilon(0, X) e_i$ defines the virtual displacement field $\delta\chi = \delta\chi(X) \in \mathbb{E}^2$. For the constitutive modeling of the bi-pantograph, we require the first and second gradient of placement, whose components are $F^i_A = \partial\chi^i/\partial X^A$ and $F^i_{AB} = \partial^2\chi^i/\partial X^A \partial X^B$. From the definition of the virtual displacement and the symmetry of the second derivatives, it readily follows that the variation of these gradients correspond respectively with the first and second gradient of the virtual displacement, i.e., $\delta F^i_A = \partial(\delta\chi^i)/\partial X^A$ and $\delta F^i_{AB} = \partial^2(\delta\chi^i)/\partial X^A \partial X^B$. For an elastic second gradient continuum with a strain energy density $W = W(F^i_A, F^i_{AB})$, the internal virtual work functional can be represented as

$$\delta\mathscr{W}^{\mathrm{int}}(\delta\chi) = -\int_\Omega \{P_i^A \delta F^i_A + P_i^{AB} \delta F^i_{AB}\} \mathrm{d}\Omega = -\delta\mathscr{E}^{\mathrm{int}}(\delta\chi) = -\int_\Omega \left\{ \frac{\partial W}{\partial F^i_A} \delta F^i_A + \frac{\partial W}{\partial F^i_{AB}} \delta F^i_{AB} \right\} \mathrm{d}\Omega \,,$$

where $P_i^A = \partial W/\partial F^i_A$ and $P^i_{AB} = \partial W/\partial F^i_{AB}$ denote the components of the Piola–Lagrange stress and double stress, respectively. If the continuum is subjected to an external volume force density per unit reference volume F^Ω, in [1], it is shown that the equilibrium conditions of a planar second gradient continuum are given by the partial differential equations

$$\frac{\partial}{\partial X^A} \left(P_i^A - \frac{\partial P_i^{AB}}{\partial X^B} \right) + F_i^\Omega = 0 \tag{1}$$

together with suitable boundary conditions, which we omit here for the sake of brevity.

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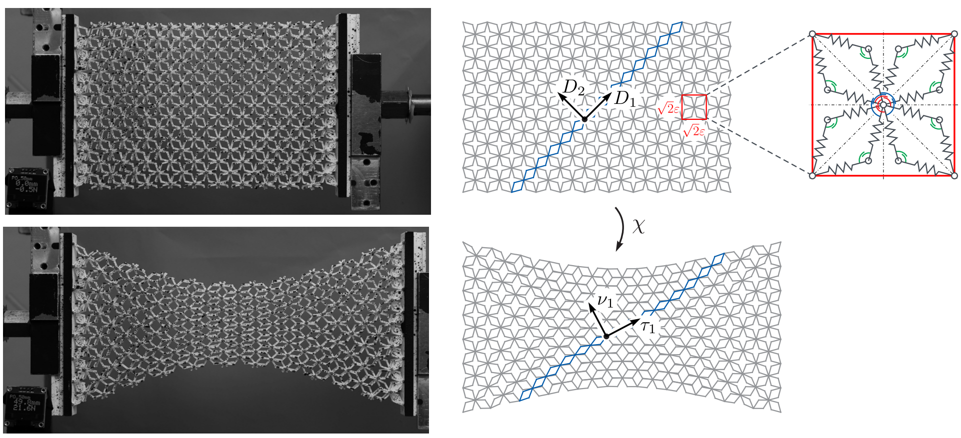

Fig. 1: (left) Tensile test of a bi-pantographic fabric. (right) Continuous fiber kinematics and micro-structure of the discrete model [2, 3].

In [2], the homogenization of a discrete model of the bi-pantograph (Fig. 1) results in a strain energy density of the form

$$W(\rho_\alpha, \rho_{\alpha,\alpha}, \vartheta_{\alpha,\alpha}) = \sum_{\alpha=1}^{2} [f(\rho_\alpha)(\vartheta_{\alpha,\alpha})^2 + g(\rho_\alpha)(\rho_{\alpha,\alpha})^2 + h(\rho_\alpha)] \,, \tag{2}$$

where ρ_α are the fiber stretches, $\rho_{\alpha,\alpha}$ the stretch gradients in fiber direction, and $\vartheta_{\alpha,\alpha}$ the inclination angle gradients in fiber direction, i.e., the fiber curvatures. In the following, we show the dependence of these kinematical quantities on the first and second gradient of placement to derive the equilibrium equations.

As said, the bi-pantograph consists of two fiber families, which in the reference configuration are orthogonal to each other, see Fig. 1. The two families are represented by the unit normal vector fields $D_1 = D_1^A E_A = \sqrt{2}/2(E_1 + E_2)$ and $D_2 = D_2^A E_A = \sqrt{2}/2(-E_1 + E_2)$, which are tangent vectors to the respective fiber curves. Throughout the deformation, these directions are mapped to $d_\alpha = F D_\alpha = \rho_\alpha \tau_\alpha$, where $\rho_\alpha = [F_A^i D_\alpha^A F_{iB} D_\alpha^B]^{1/2}$ denotes the fiber stretch and $\tau_\alpha = \tau_\alpha^i e_i = \tau_i^\alpha e^i$ is the unit tangent vector of the deformed fiber curve (e^i is the dual basis of e_i). The inclination angle with respect to the horizontal e_1-axis can then be computed as $\vartheta_\alpha = \arctan\left(F_A^2 D_\alpha^A / F_A^1 D_\alpha^A\right)$. Defining the fiber derivative of a scalar valued function as $f_{,\alpha} := \partial f / \partial X^A D_\alpha^A$, the variation and the fiber derivative of the stretch can be computed as

$$\delta\rho_\alpha = \tau_i^\alpha \delta F_A^i D_\alpha^A \,, \quad \rho_{\alpha,\alpha} = \tau_i^\alpha F_{AB}^i D_\alpha^A D_\alpha^B \,.$$

Let $R = e^2 \otimes e^1 - e^1 \otimes e^2$ be the linear map defining the normal to the tangent $\nu_\alpha = \nu_i^\alpha e^i = R\tau_\alpha$. The variation and the fiber derivative of the inclination angle, which is the fiber curvature, can then be computed as

$$\delta\vartheta_\alpha = \rho_\alpha^{-1} \nu_i^\alpha \delta F_A^i D_\alpha^A \,, \quad \vartheta_{\alpha,\alpha} = \rho_\alpha^{-1} \nu_i^\alpha F_{AB}^i D_\alpha^A D_\alpha^B \,.$$

Using the identities $\delta_{ij} - \tau_i^\alpha \tau_j^\alpha = \nu_i^\alpha \nu_j^\alpha$ and $R_{ij}\nu_\alpha^j = -\tau_i^\alpha$, the variation of the unit tangent τ_α and its normal ν_α can be written as

$$\delta\tau_i^\alpha = \rho_\alpha^{-1} \nu_i^\alpha \nu_j^\alpha \delta F_A^j D_\alpha^A \,, \quad \delta\nu_i^\alpha = -\rho_\alpha^{-1} \tau_i^\alpha \nu_j^\alpha \delta F_A^j D_\alpha^A \,.$$

Consequently, the variations of the fiber derivative of the stretch and the curvature read as

$$\begin{aligned} \delta\rho_{\alpha,\alpha} &= \delta F_A^i (\vartheta_{\alpha,\alpha} \nu_i^\alpha D_\alpha^A) + \delta F_{AB}^i (\tau_i^\alpha D_\alpha^A D_\alpha^B) \,, \\ \delta\vartheta_{\alpha,\alpha} &= \delta F_A^i (-\rho_\alpha^{-1} \vartheta_{\alpha,\alpha} \tau_i^\alpha D_\alpha^A - \rho_\alpha^{-2} \rho_{\alpha,\alpha} \nu_i^\alpha D_\alpha^A) + \delta F_{AB}^i (\rho_\alpha^{-1} \nu_i^\alpha D_\alpha^A D_\alpha^B) \,. \end{aligned}$$

The Piola–Lagrange stress for the strain energy function of the form (2) is now easily obtained as

$$\begin{aligned} P_i^A &= \sum_{\alpha=1}^{2} \left(\frac{\partial W}{\partial \rho_\alpha} \frac{\partial \rho_\alpha}{\partial F_A^i} + \frac{\partial W}{\partial \rho_{\alpha,\alpha}} \frac{\partial \rho_{\alpha,\alpha}}{\partial F_A^i} + \frac{\partial W}{\partial \vartheta_{\alpha,\alpha}} \frac{\partial \vartheta_{\alpha,\alpha}}{\partial F_A^i} \right) \\ &= \sum_{\alpha=1}^{2} \left(\left[\frac{\partial W}{\partial \rho_\alpha} - \frac{\partial W}{\partial \vartheta_{\alpha,\alpha}} \rho_\alpha^{-1} \vartheta_{\alpha,\alpha} \right] \tau_i^\alpha D_\alpha^A + \left[\frac{\partial W}{\partial \rho_{\alpha,\alpha}} \vartheta_{\alpha,\alpha} - \frac{\partial W}{\partial \vartheta_{\alpha,\alpha}} \rho_\alpha^{-2} \rho_{\alpha,\alpha} \right] \nu_i^\alpha D_\alpha^A \right) . \end{aligned} \tag{3}$$

The Piola–Lagrange double stress and its divergence can be expressed as

$$\begin{aligned} P_i^{AB} &= \sum_{\alpha=1}^{2} \left(\frac{\partial W}{\partial \rho_{\alpha,\alpha}} \frac{\partial \rho_{\alpha,\alpha}}{\partial F_{AB}^i} + \frac{\partial W}{\partial \vartheta_{\alpha,\alpha}} \frac{\partial \vartheta_{\alpha,\alpha}}{\partial F_{AB}^i} \right) = \sum_{\alpha=1}^{2} \left(\frac{\partial W}{\partial \rho_{\alpha,\alpha}} \tau_i^\alpha D_\alpha^A D_\alpha^B + \frac{\partial W}{\partial \vartheta_{\alpha,\alpha}} \rho_\alpha^{-1} \nu_i^\alpha D_\alpha^A D_\alpha^B \right) , \\ \frac{\partial P_i^{AB}}{\partial X^B} &= \sum_{\alpha=1}^{2} \left(\left[\left(\frac{\partial W}{\partial \rho_{\alpha,\alpha}} \right)_{,\alpha} - \frac{\partial W}{\partial \vartheta_{\alpha,\alpha}} \rho_\alpha^{-1} \vartheta_{\alpha,\alpha} \right] \tau_i^\alpha D_\alpha^A + \left[\frac{\partial W}{\partial \rho_{\alpha,\alpha}} \vartheta_{\alpha,\alpha} + \rho_\alpha^{-1} \left(\frac{\partial W}{\partial \vartheta_{\alpha,\alpha}} \right)_{,\alpha} - \frac{\partial W}{\partial \vartheta_{\alpha,\alpha}} \rho_\alpha^{-2} \rho_{\alpha,\alpha} \right] \nu_i^\alpha D_\alpha^A \right) . \end{aligned} \tag{4}$$

Note that we have applied several times the definition of the fiber derivative as well as the identities $\nu_i^\alpha = -\vartheta_{\alpha,\alpha} \tau_i^\alpha$ and $\tau_i^\alpha = \vartheta_{\alpha,\alpha} \nu_i^\alpha$. Finally, inserting the last equalities of (3) and (4) into (1), we obtain the equilibrium conditions for the bi-pantograph as

$$\begin{aligned} \sum_{\alpha=1}^{2} \Bigg\{ \tau_i^\alpha \left(\left[\frac{\partial W}{\partial \rho_\alpha} - \left(\frac{\partial W}{\partial \rho_{\alpha,\alpha}} \right)_{,\alpha} \right]_{,\alpha} + \left(\frac{\partial W}{\partial \vartheta_{\alpha,\alpha}} \right)_{,\alpha} \rho_\alpha^{-1} \vartheta_{\alpha,\alpha} \right) \\ + \nu_i^\alpha \left(\left[\frac{\partial W}{\partial \rho_\alpha} - \left(\frac{\partial W}{\partial \rho_{\alpha,\alpha}} \right)_{,\alpha} \right] \vartheta_{\alpha,\alpha} - \rho_\alpha^{-1} \left(\frac{\partial W}{\partial \vartheta_{\alpha,\alpha}} \right)_{,\alpha\alpha} + \left(\frac{\partial W}{\partial \vartheta_{\alpha,\alpha}} \right)_{,\alpha} \rho_\alpha^{-2} \rho_{\alpha,\alpha} \right) \Bigg\} + F_i^\Omega = 0 \,. \end{aligned}$$

References

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