A geometric view on the kinematics of finite-dimensional mechanical systems

Tom Winandy[1,*], **Giuseppe Capobianco**[1], and **Simon R. Eugster**[1]

[1] Institute for Nonlinear Mechanics, University of Stuttgart, Stuttgart, Germany

Finite-dimensional mechanical systems can be described in terms of a set of generalized coordinates and their time-derivatives. In this case, the Lagrange equations of the second kind provide the equations of motion of these systems. The Volterra–Hamel–Boltzmann equations generalize the Lagrange equations of the second kind in the sense that they allow for more general velocity parametrizations. In this work, we show in the context of scleronomous finite-dimensional mechanical systems that both sets of equations can be interpreted as being different chart representations of the intrinsic Euler–Lagrange equations on the tangent bundle over the configuration manifold of the mechanical system.

A common geometric approach (see e.g. [1–3]) to deal with the dynamics of scleronomous finite-dimensional mechanical systems is to consider a smooth parametrized curve

$$\gamma\colon \mathbb{R} \supseteq I \to Q,\ t \mapsto \gamma(t) \tag{1}$$

in an n-dimensional differentiable manifold Q as the motion of the system. For an introduction to differential geometry, we refer to [4, 5]. The so-called configuration manifold Q is the space of positions of the mechanical system and its dimension n corresponds to its number of degrees of freedom. The curve (1) induces the curve

$$\eta\colon \mathbb{R} \supseteq I \to TQ,\ t \mapsto \eta(t) = \big(\gamma(t), \dot\gamma(t)\big) \tag{2}$$

in the tangent bundle TQ of the configuration manifold Q. For fixed values of the parameter t, $\dot\gamma(t) \in T_{\gamma(t)}Q$ denotes the tangent vector to the curve γ at the point $\gamma(t)$ that can be interpreted as the velocity of the mechanical system. For a variable parameter t, $\dot\gamma$ is a vector field along the curve γ. The tangent bundle TQ is the space of positions and velocities. Solving the equations of motions of a mechanical system boils down to finding an integral curve of a second-order (see eq. (8)) vector field Γ on the tangent bundle TQ such that

$$\dot\eta(t) = \Gamma\big(\eta(t)\big). \tag{3}$$

The choice of a local chart (U, ϕ) of the manifold Q provides us with a set of generalized coordinates

$$\phi\colon Q \supseteq U \to \mathbb{R}^n,\ p \mapsto \phi(p) = (q^1, \dots, q^n) = \mathbf{q}. \tag{4}$$

Moreover the chart (4) induces bases on the tangent spaces T_pQ with $p \in U$ such that a tangent vector $v_p \in T_pQ$ can be written as

$$v_p = u^i \frac{\partial}{\partial q^i}\bigg|_p, \tag{5}$$

where a summation from 1 to n is understood over repeated indices appearing once as a lower and once as an upper index. Using (4) and (5), we can define the natural chart $(\pi^{-1}(U), \Phi)$ of the tangent bundle as

$$\Phi\colon TQ \supseteq \pi^{-1}(U) \to \mathbb{R}^{2n},\ (p, v_p = u^i\partial/\partial q^i) \mapsto (\phi(p), u^1, \dots, u^n) = (\mathbf{q}, \mathbf{u}), \tag{6}$$

where $\pi^{-1}(U)$ denotes the pre-image of the open neighbourhood $U \subseteq Q$ with the natural projection $\pi\colon TQ \to Q$ of the tangent bundle. The chart representation of the curve η from (2) is thus given by

$$\Phi \circ \eta(t) = (q^1(t), \dots, q^n(t), u^1(t), \dots, u^n(t)) \tag{7}$$

with $u^i(t) = \dot q^i(t)$. Therefore, the vector field Γ needs to have the local form

$$\Gamma = u^i \frac{\partial}{\partial q^i} + A^i(\mathbf{q}, \mathbf{u}) \frac{\partial}{\partial u^i}. \tag{8}$$

* Corresponding author: winandy@inm.uni-stuttgart.de

Vector fields of the form (8) are called second-order fields, because their integral curves are determined by solving second-order ordinary differential equations.

A Lagrangian $\mathcal{L}\colon TQ \to \mathbb{R}$ is a scalar-valued function on the tangent bundle. The Euler–Lagrange equations related to the principle of Hamilton

$$\int_{t_0}^{t_1} \mathcal{L}(\eta(t))\,\mathrm{d}t \to \text{stationary} \tag{9}$$

can be written in the geometric form [1]

$$L_\Gamma \theta_{\mathcal{L}} - \mathrm{d}\mathcal{L} = 0, \quad \text{with} \quad \theta_{\mathcal{L}} = \big(\partial\mathcal{L}/\partial u^i\big)\mathrm{d}q^i. \tag{10}$$

In eq. (10), L_Γ denotes the Lie derivative along the vector field Γ and d denotes the exterior derivative. Cartan's magic formula, the use of the natural chart and of eq. (8) allow us to write eq. (10) as

$$\left(L_\Gamma\left(\frac{\partial\mathcal{L}}{\partial u^i}\right) - \frac{\partial\mathcal{L}}{\partial q^i}\right)\mathrm{d}q^i = 0. \tag{11}$$

Evaluating eq. (11) along the curve $\eta(t)$ and using eq. (3), we obtain the Lagrange equations of the second kind

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}\circ\Phi^{-1}}{\partial u^i}\right) - \frac{\partial\mathcal{L}\circ\Phi^{-1}}{\partial q^i} = 0. \tag{12}$$

Next, we want to use another set of velocity parameters $(v^1,\dots,v^n)$ instead of $(u^1,\dots,u^n)$. Therefore, we define a set of basis fields of the tangent spaces T_pQ at all points $p \in U$

$$\left.\frac{\partial}{\partial s^i}\right|_p = B_i^j(p)\left.\frac{\partial}{\partial q^j}\right|_p, \quad \text{such that} \quad v_p = v^i\left.\frac{\partial}{\partial s^i}\right|_p = B_i^j(p)v^i\left.\frac{\partial}{\partial q^j}\right|_p, \tag{13}$$

with $u^j = B_i^j(p)v^i$ and where the matrices $\big(B_i^j(p)\big)$ are assumed to be regular for all $p \in U$. This allows us to define the chart $(\pi^{-1}(U), \Psi)$ of the tangent bundle TQ as

$$\Psi\colon TQ \supseteq \pi^{-1}(U) \to \mathbb{R}^{2n},\ (p, v_p = v^i\partial/\partial s^i) \mapsto (\phi(p), v^1,\dots,v^n) = (\mathbf{y},\mathbf{v}). \tag{14}$$

Note that we have used $\mathbf{y}$ instead of $\mathbf{q}$ to designate the n-tuple $\phi(p)$ in order to avoid mixing up the charts $(\pi^{-1}(U),\Phi)$ and $(\pi^{-1}(U),\Psi)$ of the tangent bundle TQ. The chart transition map $\Psi\circ\Phi^{-1}$ and the relation between the induced basis vectors are given by

$$y^i = q^i, \quad v^i = A_j^i u^j, \tag{15}$$

and

$$\frac{\partial}{\partial y^i} = \frac{\partial}{\partial q^i} + v^k\frac{\partial B_k^j}{\partial y^i}\frac{\partial}{\partial u^j}, \quad \frac{\partial}{\partial v^i} = B_i^j\frac{\partial}{\partial u^j}, \tag{16}$$

respectively. It holds that $A_j^i B_k^j = \delta_k^i$, where δ_k^i is the Kronecker delta that is one if $i = k$ and zero, otherwise. Now, we can express eqs. (10)–(12) in the chart $(\pi^{-1}(U),\Psi)$. After some calculations, we obtain that

$$A_i^j\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}\circ\Psi^{-1}}{\partial v^j}\right) - B_j^k\frac{\partial\mathcal{L}\circ\Psi^{-1}}{\partial y^k} + B_j^k B_r^s v^r\left(\frac{\partial A_k^l}{\partial y^s} - \frac{\partial A_s^l}{\partial y^k}\right)\frac{\partial\mathcal{L}\circ\Psi^{-1}}{\partial v^l}\right) = 0, \tag{17}$$

which are the Volterra–Hamel–Boltzmann equations as they can be found in Hamel's paper from 1904 [6]. Similar equations can be found in the work of Boltzmann from 1902 [7] and in the one of Volterra from 1898 [8]. In [9], Bremer refers to the equations as Hamel–Boltzmann equations. We added Volterra to the designation because it seems to us that his contribution has been missed so far.

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