Kinematics of finite-dimensional mechanical systems on Galilean manifolds

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In a coordinate-free description of time-independent finite-dimensional mechanical systems the configuration manifold plays a central role. In the case of time-dependent mechanical systems, time needs to be included in the space on which the related physical theory is formulated. In this respect, we show that a so-called Galilean manifold not only provides a "generalized space-time" but that it allows the coordinate-free presentation of a physical theory for time-dependent finite-dimensional mechanical systems. The motion of a mechanical system is interpreted as an integral curve of a second-order vector field on the state space related to the Galilean manifold of the system. Second-order vector fields, which are the coordinate-free equivalent of second-order differential equations, are in one-to-one correspondence with the action forms introduced by Loos [4, 5]. Because of this bijective relation, the kinetic part of the theory can be formulated by postulating the action form governing the motion of a finite-dimensional mechanical system.

In this work, we show that the concept of a Galilean manifold introduced by [2] is a suitable space to formulate a theory for the description of time-dependent finite-dimensional mechanical systems in the language of contemporary differential geometry (see [3]). For a mechanical system with \( n \) degrees of freedom, we consider an \((n + 1)\)-dimensional smooth manifold \( M \). We say that a closed, non-vanishing differential one-form \( \vartheta \) endows the manifold \( M \) with a time structure. Let \((U, \psi)\) be a chart of \( M \) such that

\[
\psi: M \supseteq U \rightarrow \mathbb{R}^{n+1}, \; p \mapsto (x^0, \ldots, x^n).
\]

Following [4], we say that a chart is adapted to a time structure \( \vartheta \) if \( \vartheta = dx^0 \) holds. The existence of such charts around any point \( p \in M \) is guaranteed by the Poincaré lemma and the fact that \( \vartheta \) does not vanish. We will often use the letter \( t \) instead of \( x^0 \) to denote the first coordinate of an adapted chart.

Let \( v_p \in T_p M \) be a tangent vector to a smooth manifold \( M \) with time structure \( \vartheta \). If \( \vartheta_p(v_p) = 0 \), then \( v_p \) is called a spacelike vector. If \( \vartheta_p(v_p) = 1 \), the tangent vector \( v_p \) is said to be time-normalized. We denote the sets of spacelike and of time-normalized vectors at a point \( p \in M \) by

\[
A^0_p M := \{ v_p \in T_p M \mid \vartheta_p(v_p) = 0 \} \quad \text{and} \quad A^1_p M := \{ v_p \in T_p M \mid \vartheta_p(v_p) = 1 \},
\]

respectively. While the set of spacelike vectors \( A^0_p M \) is a vector subspace of the tangent space \( T_p M \), the set of time-normalized vectors \( A^1_p M \) is an affine subspace of \( T_p M \). The unions

\[
A^0 M := \bigcup_{p \in M} \{ p \} \times A^0_p M \quad \text{and} \quad A^1 M := \bigcup_{p \in M} \{ p \} \times A^1_p M
\]

assemble the spaces of spacelike and of time-normalized vectors to the manifold \( M \), respectively. As a vector subbundle of the tangent bundle \( TM \), the bundle \( A^0 M \) is integrable because \( \vartheta \) is closed, i.e., \( d \vartheta = 0 \). Each leaf of the related foliation consists of synchronous events. A Galilean metric on \((M, \vartheta)\) is a positive definite fibre metric on the bundle \( A^0 M \) and the triple \((M, \vartheta, g)\) is called a Galilean manifold. The Galilean metric models the mass of the mechanical system, i.e., its expression in the local coordinates provided by a chart of \( M \) provides the mass matrix of the system. Let

\[
\gamma: \mathbb{R} \supseteq I \rightarrow M, \; \tau \mapsto \gamma(\tau)
\]

be a curve in a manifold with time structure \((M, \vartheta)\). The motion of a mechanical system can be seen as a time-parametrized curve (4) that satisfies \( \vartheta(\dot{\gamma}) = 1 \), where \( \dot{\gamma} \) denotes the tangent field along \( \gamma \).

The bundle \( A^1 M \), which we refer to as state space, is an affine subbundle of \( TM \). By its definition (3), the affine bundle \( A^1 M \) comes with a natural projection \( \pi: A^1 M \rightarrow M, \; (p, v_p) \mapsto p \). The time structure \( \vartheta \) of \( M \) induces the time structure \( \dot{\vartheta} := \pi^*(\vartheta) \) on \( A^1 M \) by its pullback with the natural projection. An adapted chart \((U, \psi)\) of \( M \) induces the natural chart on \( A^1 M \) that is given by

\[
\Psi: A^1 M \supseteq \varpi^{-1}(U) \rightarrow \mathbb{R}^{2n+1}, \; (p, v_p = \frac{\partial}{\partial x^i} |_p + u^i \frac{\partial}{\partial u^i} |_p) \mapsto (t, x^1, \ldots, x^n, u^1, \ldots, u^n),
\]

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where \( \pi^{-1}(U) \) denotes the preimage of the neighbourhood \( U \) with the natural projection \( \pi \). Moreover, we have adopted Einstein’s summation convention in (5) which says that a summation from 1 to \( n \) is understood over a repeated index that appears once as an upper and once as a lower index.

The tangent field \( \dot{\gamma} \) along a motion \( \gamma \) is a curve in the state space \( A^1M \). We regard this curve as being an integral curve of a vector field \( Z \). In general, a curve \( \beta : \mathbb{R} \supset I \to A^1M, \tau \mapsto \beta(\tau) \) is the integral curve of a vector field \( X \) on \( A^1M \) if

\[
\dot{\beta}(\tau) = X(\beta(\tau)), \quad \text{or locally} \quad \begin{bmatrix} \dot{t}(\tau) \\ \dot{x}(\tau) \\ \dot{u}(\tau) \end{bmatrix} = \begin{bmatrix} a(t(\tau), x(\tau), u(\tau)) \\ A(t(\tau), x(\tau), u(\tau)) \\ B(t(\tau), x(\tau), u(\tau)) \end{bmatrix},
\]

where the coordinates \( (t, x, u) = (t, x^1, \ldots, x^n, u^1, \ldots, u^n) \) are provided by the natural chart (5) and the coefficient functions from the local representation of the vector field \( X \) as

\[
X = a \frac{\partial}{\partial t} + A^i \frac{\partial}{\partial x^i} + B^i \frac{\partial}{\partial u^i}
\]

are gathered as \( (a, A, B) = (a, A^1, \ldots, A^n, B^1, \ldots, B^n) \). Seen as a curve in \( A^1M \), the motion of a mechanical system is a particular type of curve. In order to describe the motion of a mechanical system, the curve \( \beta \) needs to be time-parametrized and to be the tangent field along its projection to \( M \) such that

\[
\dot{\beta}(\beta) = 1 \quad \text{and} \quad \beta = (\pi \circ \beta).
\]

A curve \( \beta \) which satisfies (8) is referred to as second-order curve. The integral curve \( \beta \) from (6) is a second-order curve if and only if the vector field \( X \) satisfies the restrictions

\[
\dot{\beta}(X) = 1 \quad \text{and} \quad \delta \pi \circ X = \text{id}_W,
\]

where \( \delta \pi : T(A^1M) \to TM \) denotes the differential of the natural projection \( \pi \) and \( \text{id}_W : M \supset W \to W \) is the identity map on the neighbourhood \( W \subset A^1M \) on which the vector field \( X \) is defined. A vector field \( X \) that satisfies (9) is said to be a second-order field because by (6) it represents a second-order differential equation

\[
\begin{bmatrix} \dot{t}(\tau) \\ \dot{x}(\tau) \\ \dot{u}(\tau) \end{bmatrix} = \begin{bmatrix} 1 \\ u \\ \mathbf{B}(t(\tau), x(\tau), u(\tau)) \end{bmatrix}, \quad \text{i.e.,} \quad X = \frac{\partial}{\partial t} + u^i \frac{\partial}{\partial x^i} + B^i \frac{\partial}{\partial u^i},
\]

The action form of a second-order field \( Z = \frac{\partial}{\partial t} + u^i \frac{\partial}{\partial x^i} + Z^i \frac{\partial}{\partial u^i} \) on \( A^1M \) defined by Loos in [4,5], is the differential two-form \( \Omega \) locally given by

\[
\Omega = g_{ij} \left( du^i - \frac{1}{2} \frac{\partial Z^j}{\partial u^k} (dx^k - u^k dt) - Z^j dt \right) \wedge (dx^j - u^j dt),
\]

where \( g_{ij} = g_{ij}(t, x) \) denote the coefficient functions of the Galilean metric \( g = g_{ij} dx^i \otimes dx^j \). It can be easily seen from the local expressions (7) and (11) that the requirements

\[
\dot{\beta}(X) = 1 \quad \text{and} \quad \Omega(X, \cdot) = 0
\]

imply that \( X = Z \) and, thereby, uniquely define the vector field \( X \). For a coordinate-free definition of action forms and a mathematically rigorous establishment of the one-to-one correspondence between second-order fields and action forms the reader is referred to [4,5]. This bijective relation between second-order fields and actions forms can be exploited to formulate a physical theory for the description of finite-dimensional mechanical systems by postulating an action form governing the dynamics of the system. For a rough overview, the reader is referred to the related PAMM contribution ‘Dynamics of finite-dimensional mechanical systems on Galilean manifolds’. One important consequence of the presented approach is that a definition of force can be given.

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References