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Kinematics of finite-dimensional mechanical systems on Galilean manifolds

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In a coordinate-free description of time-independent finite-dimensional mechanical systems the configuration manifold plays a central role. In the case of time-dependent mechanical systems, time needs to be included in the space on which the related physical theory is formulated. In this respect, we show that a so-called Galilean manifold not only provides a 'generalized space-time' but that it allows the coordinate-free presentation of a physical theory for time-dependent finite-dimensional mechanical systems. The motion of a mechanical system is interpreted as an integral curve of a second-order vector field on the state space related to the Galilean manifold of the system. Second-order vector fields, which are the coordinate-free equivalent of second-order differential equations, are in one-to-one correspondence with the action forms introduced by Loos [4, 5]. Because of this bijective relation, the kinetic part of the theory can be formulated by postulating the action form governing the motion of a finite-dimensional mechanical system.

In this work, we show that the concept of a Galilean manifold introduced by [2] is a suitable space to formulate a theory for the description of time-dependent finite-dimensional mechanical systems in the language of contemporary differential geometry (see [3]). For a mechanical system with n degrees of freedom, we consider an $(n+1)$-dimensional smooth manifold M. We say that a closed, non-vanishing differential one-form ϑ endows the manifold M with a *time structure*. Let (U, ψ) be a chart of M such that

$$\psi\colon M \supseteq U \to \mathbb{R}^{n+1},\ p \mapsto (x^0, \dots, x^n). \tag{1}$$

Following [4], we say that a chart is *adapted* to a time structure ϑ if $\vartheta = \mathrm{d}x^0$ holds. The existence of such charts around any point $p \in M$ is guaranteed by the Poincaré lemma and the fact that ϑ does not vanish. We will often use the letter t instead of x^0 to denote the first coordinate of an adapted chart.

Let $v_p \in T_pM$ be a tangent vector to a smooth manifold M with time structure ϑ. If $\vartheta_p(v_p) = 0$, then v_p is called a *spacelike* vector. If $\vartheta_p(v_p) = 1$, the tangent vector v_p is said to be *time-normalized*. We denote the sets of spacelike and of time-normalized vectors at a point $p \in M$ by

$$A^0_pM \coloneqq \big\{v_p \in T_pM \mid \vartheta_p(v_p) = 0\big\} \quad \text{and} \quad A^1_pM \coloneqq \big\{v_p \in T_pM \mid \vartheta_p(v_p) = 1\big\}, \tag{2}$$

respectively. While the set of spacelike vectors A^0_pM is a vector subspace of the tangent space T_pM, the set of time-normalized vectors A^1_pM is an affine subspace of T_pM. The unions

$$A^0M \coloneqq \bigcup_{p\in M} \{p\} \times A^0_pM \quad \text{and} \quad A^1M \coloneqq \bigcup_{p\in M} \{p\} \times A^1_pM \tag{3}$$

assemble the spaces of spacelike and of time-normalized vectors to the manifold M, respectively. As a vector subbundle of the tangent bundle TM, the bundle A^0M is integrable because ϑ is closed, i.e., $\mathrm{d}\vartheta = 0$. Each leaf of the related foliation consists of synchronous events. A *Galilean metric* on (M, ϑ) is a positive definite fibre metric on the bundle A^0M and the triple (M, ϑ, g) is called a *Galilean manifold*. The Galilean metric models the mass of the mechanical system, i.e., its expression in the local coordinates provided by a chart of M provides the mass matrix of the system. Let

$$\gamma\colon \mathbb{R} \supset I \to M,\ \tau \mapsto \gamma(\tau) \tag{4}$$

be a curve in a manifold with time structure (M, ϑ). The *motion* of a mechanical system can be seen as a *time-parametrized* curve (4) that satisfies $\vartheta(\dot\gamma) = 1$, where $\dot\gamma$ denotes the tangent field along γ.

The bundle A^1M, which we refer to as *state space*, is an affine subbundle of TM. By its definition (3), the affine bundle A^1M comes with a natural projection $\pi\colon A^1M \to M$, $(p, v_p) \mapsto p$. The time structure ϑ of M induces the time structure $\hat\vartheta \coloneqq \pi^*(\vartheta)$ on A^1M by its pullback with the natural projection. An adapted chart (U, ψ) of M induces the *natural chart* on A^1M that is given by

$$\Psi\colon A^1M \supseteq \pi^{-1}(U) \to \mathbb{R}^{2n+1},\ \big(p, v_p = \tfrac{\partial}{\partial t}\big|_p + u^i \tfrac{\partial}{\partial x^i}\big|_p\big) \mapsto (t, x^1, \dots, x^n, u^1, \dots, u^n), \tag{5}$$

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where $\pi^{-1}(U)$ denotes the preimage of the neighbourhood U with the natural projection π. Moreover, we have adopted Einstein's summation convention in (5) which says that a summation from 1 to n is understood over a repeated index that appears once as an upper and once as a lower index.

The tangent field $\dot{\gamma}$ along a motion γ is a curve in the state space A^1M. We regard this curve as being an integral curve of a vector field Z. In general, a curve $\beta\colon \mathbb{R} \supseteq I \to A^1M$, $\tau \mapsto \beta(\tau)$ is the integral curve of a vector field X on A^1M if

$$\dot{\beta}(\tau) = X\big(\beta(\tau)\big), \quad \text{or locally} \quad \begin{bmatrix} \dot{t}(\tau) \\ \dot{\mathbf{x}}(\tau) \\ \dot{\mathbf{u}}(\tau) \end{bmatrix} = \begin{bmatrix} a(t(\tau), \mathbf{x}(\tau), \mathbf{u}(\tau)) \\ \mathbf{A}(t(\tau), \mathbf{x}(\tau), \mathbf{u}(\tau)) \\ \mathbf{B}(t(\tau), \mathbf{x}(\tau), \mathbf{u}(\tau)) \end{bmatrix}, \tag{6}$$

where the coordinates $(t, \mathbf{x}, \mathbf{u}) = (t, x^1, \dots, x^n, u^1, \dots, u^n)$ are provided by the natural chart (5) and the coefficient functions from the local representation of the vector field X as

$$X = a\frac{\partial}{\partial t} + A^i \frac{\partial}{\partial x^i} + B^i \frac{\partial}{\partial u^i} \tag{7}$$

are gathered as $(a, \mathbf{A}, \mathbf{B}) = (a, A^1, \dots, A^n, B^1, \dots, B^n)$. Seen as a curve in A^1M, the motion of a mechanical system is a particular type of curve. In order to describe the motion of a mechanical system, the curve β needs to be time-parametrized and to be the tangent field along its projection to M such that

$$\hat{\vartheta}(\dot{\beta}) = 1 \qquad \text{and} \qquad \beta = (\pi \circ \beta)^{\dot{}}. \tag{8}$$

A curve β which satisfies (8) is referred to as *second-order curve*. The integral curve β from (6) is a second-order curve if and only if the vector field X satisfies the restrictions

$$\hat{\vartheta}(X) = 1 \qquad \text{and} \qquad \mathrm{D}\pi \circ X = \mathrm{id}_W, \tag{9}$$

where $\mathrm{D}\pi\colon T(A^1M) \to TM$ denotes the differential of the natural projection π and $\mathrm{id}_W\colon M \supseteq W \to W$ is the identity map on the neighbourhood $W \subseteq A^1M$ on which the vector field X is defined. A vector field X that satisfies (9) is said to be a *second-order field* because by (6) it represents a second-order differential equation

$$\begin{bmatrix} \dot{t}(\tau) \\ \dot{\mathbf{x}}(\tau) \\ \dot{\mathbf{u}}(\tau) \end{bmatrix} = \begin{bmatrix} 1 \\ \mathbf{u} \\ \mathbf{B}(t(\tau), \mathbf{x}(\tau), \mathbf{u}(\tau)) \end{bmatrix}, \quad \text{i.e.,} \quad X = \frac{\partial}{\partial t} + u^i \frac{\partial}{\partial x^i} + B^i \frac{\partial}{\partial u^i}. \tag{10}$$

The *action form* of a second-order field $Z = \frac{\partial}{\partial t} + u^i \frac{\partial}{\partial x^i} + Z^i \frac{\partial}{\partial u^i}$ on A^1M defined by Loos in [4,5], is the differential two-form Ω locally given by

$$\Omega = g_{ij}\Big(\mathrm{d}u^i - \frac{1}{2}\frac{\partial Z^i}{\partial u^k}\big(\mathrm{d}x^k - u^k\mathrm{d}t\big) - Z^i\mathrm{d}t\Big) \wedge \big(\mathrm{d}x^j - u^j\mathrm{d}t\big), \tag{11}$$

where $g_{ij} = g_{ij}(t, \mathbf{x})$ denote the coefficient functions of the Galilean metric $g = g_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j$. It can be easily seen from the local expressions (7) and (11) that the requirements

$$\hat{\vartheta}(X) = 1 \quad \text{and} \quad \Omega(X, \cdot) = 0$$

imply that $X = Z$ and, thereby, uniquely define the vector field X. For a coordinate-free definition of action forms and a mathematically rigorous establishment of the one-to-one correspondence between second-order fields and action forms the reader is referred to [4,5]. This bijective relation between second-order fields and actions forms can be exploited to formulate a physical theory for the description of finite-dimensional mechanical systems by postulating an action form governing the dynamics of the system. For a rough overview, the reader is referred to the related PAMM contribution 'Dynamics of finite-dimensional mechanical systems on Galilean manifolds'. One important consequence of the presented approach is that a definition of force can be given.

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