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THE PRINCIPLE OF VIRTUAL WORK AND HAMILTON'S PRINCIPLE ON GALILEAN MANIFOLDS

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Abstract. To describe time-dependent finite-dimensional mechanical systems, their generalized space-time is modeled as a Galilean manifold. On this basis, we present a geometric mechanical theory that unifies Lagrangian and Hamiltonian mechanics. Moreover, a general definition of force is given, such that the theory is capable of treating nonpotential forces acting on a mechanical system. Within this theory, we elaborate the interconnections between classical equations known from analytical mechanics such as the principle of virtual work, Lagrange's equations of the second kind, Hamilton's equations, Lagrange's central equation, Hamel's generalized central equation as well as Hamilton's principle.

1. Introduction.

The theory for the coordinate-free description of dynamics of time-dependent finite-dimensional mechanical systems presented in [5, 19, 26] embraces Lagrangian and Hamiltonian mechanics in such a way that both frameworks are mere coordinate representations of the same geometric postulate. Moreover, it gives a precise definition of forces, which may or may not stem from a potential. Hence, this theory perfectly suits the requirements from the classical formulations of analytical mechanics exemplarily represented by [21, 25, 6, 9]. These classical works comprise nonpotential forces and exclusively derive the Hamiltonian from the Lagrangian formalism using a change of coordinates that introduces the generalized momentum coordinates. The fact that the aforementioned geometric theory can cope with nonpotential forces allows to rediscover classical results within this theory. Obviously, the results of classical presentations, e.g. [16, 14], which ignore nonpotential forces can trivially be retrieved by setting them to zero.

The coordinate-free mechanical theory is formulated using a Galilean manifold, which for a mechanical system with n degrees of freedom is an $(n+1)$-dimensional smooth manifold equipped with a time-structure and a Galilean metric. While the time-structure allows to measure the temporal distance between two points on the

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space-time manifold, the Galilean metric captures the inertia of the mechanical system. As a core object, a differential two-form, called the action form, is postulated to characterize the motions of the mechanical system. With this action form, we give a coordinate-free definition of the virtual work needed for the formulation of the principle of virtual work. For this, we define virtual displacement fields as particular types of vector fields on the state space of the mechanical system. In doing so, we break with the common understanding of virtual displacements interpreted as vector fields induced by one-parameter families of configurations. This means also, that we loosen the connection between virtual displacement fields and the calculus of variations. We share the opinion of Hamel, who on page 416 of [8] wrote:[1]

> As fruitful as the connection of mechanics with the calculus of variations has been (Lagrange, Hamilton, Jacobi), it cannot be denied that the above interpretation of the virtual displacements [in the sense of the calculus of variations,] is one-sided and does not fully correspond to their mechanical meaning; that it impedes, in particular, the connection to results that do not lie within the scope of variational calculus. Likewise it appears to me that the roots of the strange, widespread opinion in the literature that the essence of general Lagrangian mechanics can be found in the so-called variational principles lie in that dogma.

The strong bond of analytical mechanics with variational principles manifests for instance in the frequent use of Hamilton's principle. Therefore, we use the calculus of variations on manifolds to state Hamilton's principle within the mechanical theory on Galilean manifolds. To be more precise, we introduce three different, but closely related, action functionals, which are defined on different sets of curves. One of the main goals of this paper is to link the coordinate-free principles, stated within the geometric theory, to equations known from classical textbooks. Following this philosophy, we show how six different classical action functionals of Hamilton's principle can be interpreted as coordinate representations of three action functionals. Moreover, using different types of virtual displacement fields in the principle of virtual work, we derive Lagrange's equations of the second kind, Hamilton's equations, Lagrange's central equation as well as Hamel's generalized central equation.

After a brief introduction of frequently-used notation in Sect. 2, we present a restatement of the mechanical theory on Galilean manifolds according to [19] in Sects. 3–10. For reasons of simplicity, we abstain from elaborating the geometric foundations of the theory in every detail and refer the interested reader to [5], which reviews the original ideas of [19]. In Sect. 11, we introduce the virtual displacement fields to state the principle of virtual work in Sect. 12. Therefrom, we derive Lagrange's equations of the second kind and Hamilton's equations in Sect. 13 as well as the central equation of Lagrange and Hamel's generalized version thereof in Sect. 14. Finally, Hamilton's principle is studied in Sect. 15.

2. **Notation.**

The present paper uses the language of differential geometry as presented for example in [18, 17, 24]. To facilitate reading, we introduce some notation.

Let M be a smooth manifold, then T_pM stands for the tangent space of M in p. The tangent bundle of M is denoted as TM. The set of all smooth sections of a vector bundle E over M is denoted $\Gamma(E)$. Consequently, $\mathrm{Vect}(M) := \Gamma(TM)$ is the set of all vector fields on M. For the evaluation of a vector field $v \in \mathrm{Vect}(M)$ in a

[1] Translation from German by the second author.

point $p \in M$ we use the notation

$$v(p) = (p, v_p), \quad \text{where } v_p \in T_pM\,.$$

To distinguish the evaluation $v(p)$ of a vector field v in a point $p \in M$ from its action as a derivation on a smooth function $f \in C^\infty(M)$, we write the latter as $v[f]$. We denote by $\mathcal{L}_v$ the Lie derivative with respect to the vector field $v \in \mathrm{Vect}(M)$. Using d for the exterior derivative, we have the identity

$$\mathcal{L}_v f = v[f] = \mathrm{d}f(v)$$

for any smooth function $f \in C^\infty(M)$. The space $\mathrm{Vect}(M)$ is equipped with the Lie bracket $[\cdot,\cdot]$ defined as the commutator of derivations on smooth functions. It holds that $\mathcal{L}_v(w) = [v, w]$. We denote the space of (differential) k-forms as $\Omega^k(M)$ and the set of differential forms of arbitrary degree as $\Omega^\star(M)$. We use the common notation $\lrcorner$ for the interior product.

We adopt Einstein's summation convention implying a summation from 1 to n whenever an index i appears once as an upper and once as a lower index, e.g.,

$$v^i \frac{\partial}{\partial x^i} = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i} \quad \text{or} \quad \frac{\partial}{\partial u^i} \otimes \eta^i = \sum_{i=1}^{n} \frac{\partial}{\partial u^i} \otimes \eta^i\,.$$

3. **Galilean manifold and state space.** To model the generalized space-time of an n-dimensional time-dependent finite-dimensional mechanical system, we introduce an $(n{+}1)$-dimensional smooth manifold M with *time structure* ϑ, which is a non-vanishing closed one-form on M. The Poincaré lemma guarantees the existence of local *time functions* $t\colon M \supseteq U \to \mathbb{R}$ with $\mathrm{d}t = \vartheta\big|_U$ such that the temporal distance of two events $p, q \in U$ is given by $t(q) - t(p)$.

A chart (U, ϕ) of M given by

$$\phi\colon M \supseteq U \to \mathbb{R}^{n+1},\ p \mapsto \phi(p) = (t, x^1, \ldots, x^n) \tag{1}$$

is *adapted* to the time structure if its first coordinate function is a time function. The existence of adapted charts is guaranteed by the existence of time functions and the fact that ϑ is nowhere zero. Therefore, the adapted charts provide an atlas of M. In what follows, we will restrict our considerations to adapted charts. The change of coordinates

$$\psi \circ \phi^{-1}\colon \phi(U \cap V) \to \psi(U \cap V),\ (t, x^1, \ldots, x^n) \mapsto (\bar{t}, y^1, \ldots, y^n)$$

between two adapted charts (U, ϕ) and (V, ψ) of M with $U \cap V \neq \emptyset$ is given by

$$\begin{aligned} \bar{t} &= t + \text{const.}, \\ y^i &= \psi^i \circ \phi^{-1}\big(t, x^1, \ldots, x^n\big),\ i = 1, \ldots, n, \end{aligned} \tag{2}$$

where $\psi^i\colon V \to \mathbb{R}$ denotes the i-th coordinate function of the chart ψ.

The time structure ϑ is used to introduce the *spacelike bundle* A^0M of M as the vector subbundle of the tangent bundle TM resulting from the pointwise restriction of the tangent space T_pM to the *space of spacelike vectors in* p given by

$$A^0_pM := \ker \vartheta_p = \big\{ v_p \in T_pM \;\big|\; \vartheta_p(v_p) = 0 \big\} \subset T_pM \tag{3}$$

for every $p \in M$. By definition, A^0M is a distribution of rank n on M. As $\mathrm{d}\vartheta = 0$ annihilates the distribution trivially, it is completely integrable and therefore defines a foliation of M by the Frobenius theorem.[2] The leafs of this foliation are just the

[2]Theorems 19.12 and 19.21 in [18]

codimension-one submanifolds of synchronous events that can be distinguished in classical mechanics.

Each chart (U, ϕ) of an adapted atlas of M induces the smooth local sections

$$\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \colon U \to TM \tag{4}$$

that provide a basis for $A^0_q M$ at each $q \in U$.

In order to model the inertia of a mechanical system, the bundle $A^0 M$ is equipped with a bundle metric[3]

$$g = g_{ij}\, \mathrm{d}x^i \otimes \mathrm{d}x^j \tag{5}$$

called *Galilean metric*. The Galilean metric is a tensor field, for which in each $p \in M$ the tensor g_p is symmetric and positive definite. The coefficient matrix g_{ij} is the *mass matrix* of the mechanical system.

The above construction can be summarized in the following definition.

Definition 3.1. The *Galilean manifold* (M, ϑ, g) of an n-dimensional mechanical system is an $(n+1)$-dimensional smooth manifold M with a time structure ϑ and a bundle metric g that endows the subspaces $A^0_p M$ with an inner product for all $p \in M$.

The evolution of the configuration of a mechanical system is a time-parametrized curve

$$\gamma \colon \mathbb{R} \supseteq I \to M,\ \tau \mapsto \gamma(\tau) \tag{6}$$

in the Galilean manifold (M, ϑ, g). A *time-parametrized* curve is defined as a smooth sequence of events with $\vartheta(\dot\gamma) = 1$, where $\dot\gamma$ denotes the tangent field along γ. The local time coordinate t increases monotonically along a time-parametrized curve because locally

$$1 = \vartheta(\dot\gamma) = \mathrm{d}t(\dot\gamma) = \dot\gamma[t] = \frac{\mathrm{d}}{\mathrm{d}\tau}\big(t \circ \gamma(\tau)\big). \tag{7}$$

Consequently, the time coordinate is an affine function along γ, i.e.,

$$t \circ \gamma(\tau) = \tau + \tau_0,$$

where $\tau_0 \in \mathbb{R}$ is a constant. The condition $\vartheta(\dot\gamma) = 1$ motivates the following definition of the state space of a mechanical system.

Definition 3.2. Let (M, ϑ, g) be the Galilean manifold of a finite-dimensional mechanical system. The *state space* $A^1 M$ of the mechanical system is the affine subbundle of the tangent bundle TM resulting from the pointwise restriction of the tangent space $T_p M$ to the *affine space of time-normalized vectors in* p given by

$$A^1_p M \coloneqq \big\{ v_p \in T_p M \,\big|\, \vartheta_p(v_p) = 1 \big\} \subset T_p M. \tag{8}$$

for every $p \in M$.

[3] If the fibers of a vector bundle are equipped with an inner product that smoothly depends on the point in the base manifold, one speaks of a *bundle metric*. See Definition 1.8.11 in [12]. A bundle metric is the generalization of a Riemannian metric on a manifold to arbitrary vector bundles. Indeed, a Riemannian metric on a manifold is just a bundle metric on its tangent bundle. For this reason some authors, see Definition 6.42 in [17] or p. 308 in [24], designate a bundle metric as *Riemannian metric*. We abstain from doing so since it might lead to confusion.

The coordinate vector fields induced by an adapted chart $\phi\colon p \mapsto \big(t, x^1 \dots, x^n\big)$ can be used to express a time-normalized vector $v_p \in A^1_p M$ as

$$v_p = \frac{\partial}{\partial t}\bigg|_p + u^i \frac{\partial}{\partial x^i}\bigg|_p . \tag{9}$$

Accordingly, any adapted chart ϕ induces a corresponding *natural chart* of the state space A^1M as

$$\Phi\colon A^1M \supseteq \pi^{-1}(U) \to \mathbb{R}^{2n+1},\ \big(p, v_p\big) \mapsto \big(t, x^1, \dots, x^n, u^1, \dots, u^n\big)\,, \tag{10}$$

where

$$\pi\colon A^1M \to M,\ (p, v_p) \mapsto p \tag{11}$$

denotes the natural projection of the affine bundle A^1M. The state space A^1M is canonically endowed with the time structure

$$\hat{\vartheta} \coloneqq \pi^* \vartheta\,, \tag{12}$$

which is the pullback of the time structure of M by the natural projection (11). The natural chart (10) is an adapted chart with respect to the time structure (12) of A^1M because it holds that

$$\hat{\vartheta}\big|_{\pi^{-1}(U)} = \mathrm{d}t.$$

Definition 3.3. Let (M, ϑ, g) be the Galilean manifold of a finite-dimensional mechanical system and let A^1M be the corresponding state space. A *motion* of the mechanical system is a curve

$$\beta\colon \mathbb{R} \supseteq I \to A^1M,\ \tau \mapsto \beta(\tau) \tag{13}$$

in the state space A^1M that is time-parametrized with respect to $\hat{\vartheta}$ and satisfies the condition

$$\beta = \big(\pi \circ \beta\big)^{\cdot}\,. \tag{14}$$

The second-order condition (14) requires the motion β to correspond to the (time-normalized) tangent field along its (time-parametrized) projection $\gamma \coloneqq \pi \circ \beta$ onto the base manifold M. A time-parametrized curve in the state space which fulfills (14) is called a second-order curve. Thus, a motion of a mechanical system is a second-order curve. Condition (14) can be restated as

$$\beta = \dot{\gamma}\colon I \to A^1M,\ \tau \mapsto \big(\gamma(\tau), \dot{\gamma}_{\gamma(\tau)}\big), \tag{15}$$

where γ as a time-parametrized curve in the Galilean manifold M describes the evolution of the configuration of the mechanical system.

The possible motions of a finite-dimensional mechanical system can be characterized altogether by modeling a vector-field $Z \in \mathrm{Vect}(A^1M)$ such that any integral curve $\beta\colon I \to A^1M$ of Z is a motion, i.e., a motion of the mechanical system satisfies

$$\dot{\beta}(\tau) = Z\big(\beta(\tau)\big)\,. \tag{16}$$

Since the motion of a mechanical system is a second-order curve, Z cannot be arbitrary, but needs to be time-normalized such that

$$\hat{\vartheta}(Z) = 1. \tag{17}$$

Additionally, the vector field Z needs to obey the second-order condition

$$\mathrm{D}\pi\, Z = \mathrm{id}_{A^1M}\,. \tag{18}$$

Indeed, condition (14) together with (16) lead to

$$\beta = \big(\pi \circ \beta\big)^{\cdot} = \mathrm{D}\pi\, \dot{\beta} = \mathrm{D}\pi\, Z \circ \beta, \tag{19}$$

where $\mathrm{D}\pi \colon T\big(A^1M\big) \to TM$ denotes the differential of the natural projection (11). Because condition (19) has to hold for arbitrary integral curves β, the second-order condition (18) follows.

A vector field $Z \in \mathrm{Vect}(A^1M)$ that satisfies conditions (17) and (18) is called a *second-order (vector) field*. In every natural chart (10), a second-order field is locally given by

$$Z = \frac{\partial}{\partial t} + u^i \frac{\partial}{\partial x^i} + Z^i \frac{\partial}{\partial u^i}, \tag{20}$$

with n smooth (local) functions Z^i. From the local expression (20) it is apparent that second-order fields can only differ by the coefficients of their $\partial/\partial u^i$ part. Moreover, the differential equation (16) related to a second-order field Z is a system of second-order differential equations in first-order form

$$\begin{aligned} \dot{t}(\tau) &= 1, \\ \dot{\mathbf{x}}(\tau) &= \mathbf{u}(\tau), \\ \dot{\mathbf{u}}(\tau) &= \mathbf{Z}\big(t(\tau), \mathbf{x}(\tau), \mathbf{u}(\tau)\big), \end{aligned} \tag{21}$$

where we adopt the notation that quantities $a^1, \dots, a^n$ are gathered as $\mathbb{R}^n$-tuples $\mathbf{a} \coloneqq (a^1, \dots, a^n)$. The first equation of (21) can be solved to

$$t(\tau) \coloneqq t \circ \beta(\tau) = \tau + \tau_0, \tag{22}$$

where $\tau_0 \in \mathbb{R}$ denotes a constant. Because of the second-order condition (14), which expressed in the natural chart is

$$\dot{\mathbf{x}}(\tau) = \mathbf{u}(\tau)\,, \tag{23}$$

the remaining equations of (21) are equivalent to the second-order differential equations

$$\ddot{\mathbf{x}}(\tau) = \mathbf{Z}\big(t(\tau), \mathbf{x}(\tau), \dot{\mathbf{x}}(\tau)\big).$$

Since τ_0 in (22) is a constant, it is clear form the change of coordinates (2) that τ_0 can be eliminated by a change in the time coordinate. In the following we therefore use without loss of generality

$$t(\tau) = \tau\,. \tag{24}$$

4. **Action form of a second-order field.** To any second-order field Z on A^1M, a two-form $\Omega \in \Omega^2(A^1M)$ can be associated which uniquely determines Z. This allows to transfer the mechanical modeling of Z, which characterizes any motion of the mechanical system, to the modeling of Ω. In this section, we make use of the geometric structure of the state space to formulate the relation between Ω and Z.

The differential of the natural projection (11), $\mathrm{D}\pi \colon T(A^1M) \to TM$, defines the *vertical bundle* $\mathrm{Ver}(A^1M)$ as the vector subbundle of the tangent bundle $T(A^1M)$ resulting from the pointwise restriction of the tangent space $T_a(A^1M)$ to the *space of vertical vectors in* a given by

$$\mathrm{Ver}_a(A^1M) = \ker \mathrm{D}\pi_a = \big\{ w \in T_a(A^1M) \,\big|\, \mathrm{D}\pi_a(w) = 0 \big\}. \tag{25}$$

A section $V \in \Gamma\big(\mathrm{Ver}(A^1M)\big)$ of the vertical bundle is called a *vertical vector field*. Let $\big(U, \phi\big)$ be an adapted chart of M and consider the corresponding natural chart (10) on the neighbourhood $\pi^{-1}(U)$ of A^1M. A vertical vector field V can then be expressed as

$$V = V^i \frac{\partial}{\partial u^i},$$

because the vectors

$$\left.\frac{\partial}{\partial u^1}\right|_a, \dots, \left.\frac{\partial}{\partial u^n}\right|_a \tag{26}$$

provide a basis of $\mathrm{Ver}_a(A^1M)$ for points $a \in \pi^{-1}(U) \subseteq A^1M$. The vertical bundle naturally appears in the study of second-order fields because the difference of two second-order fields is always a vertical vector field. This can be seen from the local expression (20) of a second-order field.

Vertical vector fields can be used to define *semi-basic* forms on A^1M as the differential forms ω on A^1M for which $V \lrcorner\, \omega$ vanishes for any vertical vector field V. An equivalent statement is that the local representation of ω with respect to the dual basis $\mathrm{d}t, \mathrm{d}x^1, \dots, \mathrm{d}x^n, \mathrm{d}u^1, \dots, \mathrm{d}u^n$ induced by the natural chart (10) does not contain terms in $\mathrm{d}u^1, \dots, \mathrm{d}u^n$. A differential form ω on A^1M is *basic*, if there is a differential form α on M such that $\omega = \pi^*\alpha$, where π^* denotes the pullback by the natural projection. Hence, basic forms are semi-basic. However, in contrast to semi-basic forms, the chart representations of the coefficients of basic forms do not depend on $u^1, \dots, u^n$.

We introduce the operator ∂ from [20] as the anti-derivation on the exterior algebra of differential forms that increases the degree of a form by one and obeys the following rules

$$\partial f = \frac{\partial f}{\partial u^i}(\mathrm{d}x^i - u^i \mathrm{d}t), \quad \partial(\mathrm{d}x^i) = \partial(\mathrm{d}t) = 0, \quad \partial(\mathrm{d}u^i) = \mathrm{d}u^i \wedge \mathrm{d}t, \tag{27}$$

where f denotes a smooth function on A^1M. With the rules (27), one easily verifies that

$$\partial \circ \mathrm{d} = -\mathrm{d} \circ \partial\,. \tag{28}$$

However, $\partial^2 \neq 0$ but

$$\partial^2\omega = \hat{\vartheta} \wedge \partial\omega, \tag{29}$$

where $\hat{\vartheta}$ denotes the time structure on A^1M. To prove (29), it is enough to see that ∂^2 and $\omega \mapsto \hat{\vartheta} \wedge \partial\omega$ are derivations that coincide on zero- and one-forms. By induction, (29) holds for forms of arbitrary degree. From the rules (27), it becomes obvious that ∂ maps semi-basic forms to semi-basic forms.

Finally, we equip the vertical bundle $\mathrm{Ver}(A^1M)$ with a bundle metric. For this, we notice that the space $\mathrm{Ver}_a(A^1M)$ defined by (25) can be also seen as the tangent space to A^1_pM at the point $a \in A^1M$ with $p = \pi(a)$. Since A^1_pM is the affine hyperplane in T_pM defined by the equation $\vartheta_p(v) = 1$ for all $v \in T_pM$, the tangent space to A^1_pM can be identified with $\ker \vartheta_p = A^0_pM$ defined in (3). Accordingly, we have the pointwise isomorphism

$$\mathrm{Ver}_a\big(A^1M\big) \cong A^0_{\pi(a)}M \tag{30}$$

for all $a \in A^1M$, which can be locally expressed as

$$\left.\frac{\partial}{\partial u^i}\right|_a \mapsto \left.\frac{\partial}{\partial x^i}\right|_{\pi(a)} \tag{31}$$

using the basis vectors from (26) and (4). By the isomorphism (30), the Galilean metric g on the bundle A^0M of spacelike vectors induces a bundle metric $\hat{g}$ on the bundle $\mathrm{Ver}(A^1M)$ of vertical vectors that is defined as

$$\hat{g}_a\left(\frac{\partial}{\partial u^i}\bigg|_a, \frac{\partial}{\partial u^j}\bigg|_a\right) \coloneqq g_{\pi(a)}\left(\frac{\partial}{\partial x^i}\bigg|_{\pi(a)}, \frac{\partial}{\partial x^j}\bigg|_{\pi(a)}\right), \tag{32}$$

for all $a \in A^1M$. Consequently, using (5), the bundle metric $\hat{g}$ locally reads as

$$\hat{g} = g_{ij}\circ\pi \; \mathrm{d}u^i \otimes \mathrm{d}u^j. \tag{33}$$

By abuse of notation we henceforth denote the coefficients $g_{ij}\circ\pi$ in (33) by g_{ij}.

Let $Z \in \mathrm{Vect}(A^1M)$ be a second-order field with local representation (20). The *action form of* Z is defined as the two-form $\Omega \in \Omega^2(A^1M)$ which locally reads as

$$\Omega \coloneqq g_{ij}\left(\mathrm{d}u^i - Z^i\mathrm{d}t - \frac{1}{2}\frac{\partial Z^i}{\partial u^k}\left(\mathrm{d}x^k - u^k\mathrm{d}t\right)\right) \wedge \left(\mathrm{d}x^j - u^j\mathrm{d}t\right), \tag{34}$$

where g_{ij} are the coefficients of the bundle metric $\hat{g}$. A coordinate-free definition of the action form can be found in [20]. The following theorem asserts that the action form Ω of Z uniquely characterizes the second-order vector field Z. Moreover, the theorem gives necessary and sufficient conditions on a two-form for being the action form of some second-order vector field.

Theorem 4.1 ([19], p. 21 and p. 24)**.** *Let $\left(M, \vartheta\right)$ be a manifold with time structure. A two-form Ω on A^1M is the action form of a second-order field Z if and only if it satisfies the following conditions:*

(i) For any vector fields X and Y on A^1M, where each is either second-order or vertical,

$$\Omega\big(X, Y\big) = 0\,.$$

(ii) Ω defines a bundle metric g on A^0M, i.e., the matrix

$$g_{ij} = \Omega\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial x^j}\right)$$

is symmetric and positive definite for all charts.

(iii) $\partial\Omega = 0$.

The second-order field Z is the only vector field on A^1M for which

$$Z \lrcorner\, \Omega = 0, \;\; \hat{\vartheta}(Z) = 1.$$

For the proof of the theorem as well as for the coordinate-free definition of ∂ and the action form of Z (34), we also refer to [5].

Theorem 4.1 represents a cornerstone for the mechanical theory presented in this paper. In fact, postulating a two-form Ω that satisfies the conditions (i)–(iii) uniquely defines the second-order field Z characterizing the motions of the mechanical system.

It is a consequence of Theorem 4.1 that the kernel of the action form Ω of Z defined as

$$\ker\Omega \coloneqq \left\{(a, X_a) \in T(A^1M) \;\middle|\; X_a \lrcorner\, \Omega_a = 0\right\}$$

is the line bundle which is spanned by the vector field Z. This can be proven by contradiction. For that we note that by Theorem 4.1, the span of Z is a subbundle of $\ker\Omega$, because $Z \lrcorner\, \Omega = 0$. Assume that there is a section X of $\ker\Omega$ which does not lie in the span of Z. For any $f \in C^\infty(A^1M)$, fX still lies in $\ker\Omega$. Therefore, we can choose f such that $\hat{\vartheta}(fX) = 1$. Since by Theorem 4.1, Z is the only vector field lying in $\ker\Omega$ that is time-normalized, it follows that $Z = fX$ which contradicts the assumption and proves that $\ker\Omega$ is indeed the one-dimensional vector bundle over A^1M spanned by the vector field Z. Hence, in every point $a \in A^1M$

$$\dim\ker\Omega_a = 1 \tag{35}$$

implying that Ω_a, seen as a linear map of tangent vectors to covectors, has rank $2n$ in every a. For reasons of brevity, we say that Ω has rank $2n$.

5. **The kinetic energy.** We experience that whenever we perceive the motion of a mechanical system, we do this relative to some reference. For instance, we observe the motion of a car relative to the street or the motion of the sun relative to the horizon. To account for this when modeling the inertia of a mechanical system, we introduce a *reference field* as a time-normalized vector field R defined on a neighbourhood U_R of M, i.e.,

$$R\colon M \supseteq U_R \to A^1M$$

with $\pi \circ R = \mathrm{id}_M$, where again (M, ϑ, g) denotes the Galilean manifold of the mechanical system. For an adapted chart $\phi(p) = (t, x^1, \ldots, x^n)$, the reference field $R = \partial/\partial t$ is said to be the *resting field* induced by the chart.

In (13), we defined the motion of a mechanical system to be a second-order curve $\beta = \dot{\gamma}\colon I \to A^1M$, where $\gamma = \pi \circ \beta\colon I \to M$ denotes a time-parametrized curve in the Galilean manifold (M, ϑ, g). We define the *relative velocity* of the motion $\dot{\gamma}$ with respect to the reference field R as the vector field along γ which is pointwise given by

$$\dot{\gamma}_{\gamma(\tau)} - R_{\gamma(\tau)} \in A^0_{\gamma(\tau)}M.$$

As a difference of time-normalized vectors, the relative velocity is spacelike and can therefore be measured by the Galilean metric g. This brings us to the following definition.

Definition 5.1. Let (M, ϑ, g) be the Galilean manifold of a finite-dimensional mechanical system and let A^1M be the corresponding state space. The *kinetic energy* of the mechanical system with respect to a reference field $R\colon M \supseteq U_R \to A^1M$ is the function

$$T_R\colon \pi^{-1}(U_R) \to \mathbb{R},\ (p, v_p) \mapsto \tfrac{1}{2}\, g_p(v_p - R_p, v_p - R_p)\,, \tag{36}$$

with $v_p \in A^1_pM$ and $R(p) = (p, R_p)$.

The kinetic energy of the motion $\beta = \dot{\gamma}$ with respect to the reference field R is then given by

$$T_R\big(\dot{\gamma}(\tau)\big) = \tfrac{1}{2}\, g_{\gamma(\tau)}\big(\dot{\gamma}_{\gamma(\tau)} - R_{\gamma(\tau)}, \dot{\gamma}_{\gamma(\tau)} - R_{\gamma(\tau)}\big)\,.$$

Let (U, ϕ) be an adapted chart of M and let us assume for simplicity that $U \subseteq U_R$. Moreover, let $R = \partial/\partial t + R^i\partial/\partial x^i$ be an arbitrary reference field. In the natural chart induced by ϕ, the kinetic energy (36) locally reads as

$$T_R = \tfrac{1}{2} g_{ij} u^i u^j - g_{ij} u^i R^j + \tfrac{1}{2} g_{ij} R^i R^j\,, \tag{37}$$

where we have used the local expression of the metric (5) and the symmetry of the bilinear map g. In the special case where R is a resting field (i.e., $R = \partial/\partial t$), the local expression of the kinetic energy (37) reduces to

$$T_R = \tfrac{1}{2} g_{ij} u^i u^j.$$

Definition 5.2. Let (M, ϑ, g) be the Galilean manifold of a finite-dimensional mechanical system and let $\hat{\vartheta} = \pi^* \vartheta$ be the time structure of the state space A^1M. The kinetic energy T_R with respect to a reference field R of the mechanical system induces the action form

$$\Omega_R \coloneqq \mathrm{d}(T_R \hat{\vartheta} + \partial T_R)\,. \tag{38}$$

To check that (38) indeed defines an action form, we have to check the properties (i) to (iii) from Theorem 4.1. According to the rules (27), (28), and (29) of ∂, it holds that

$$\partial \Omega_R = -\mathrm{d}(\partial T_R \wedge \hat{\vartheta} + \partial^2 T_R) = -\mathrm{d}(\partial T_R \wedge \hat{\vartheta} + \hat{\vartheta} \wedge \partial T_R) = 0,$$

which shows that Ω_R enjoys property (iii). To prove properties (i) and (ii), we use the rules (27) to arrive at the local expression

$$T_R \hat{\vartheta} + \partial T_R = T_R \, \mathrm{d}t + \frac{\partial T_R}{\partial u^i}(\mathrm{d}x^i - u^i \mathrm{d}t). \tag{39}$$

Using (39) in the definition (38) leads to

$$\begin{aligned} \Omega_R &= \frac{\partial T_R}{\partial x^i} \mathrm{d}x^i \wedge \mathrm{d}t + \mathrm{d}\left(\frac{\partial T_R}{\partial u^i}\right) \wedge \big(\mathrm{d}x^i - u^i \mathrm{d}t\big) \\ &= \left(\mathrm{d}\left(\frac{\partial T_R}{\partial u^i}\right) - \frac{\partial T_R}{\partial x^i}\mathrm{d}t\right) \wedge \big(\mathrm{d}x^i - u^i \mathrm{d}t\big). \end{aligned} \tag{40}$$

It is clear from the expression (40) that Ω_R fulfills property (i). Moreover, it follows that

$$\Omega_R\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial x^j}\right) = \frac{\partial^2 T_R}{\partial u^i \partial u^j} = g_{ij}.$$

6. **Forces.** Consider two second-order fields Z_1 and Z_2 on the state space A^1M of a mechanical system. As two second-order fields can only differ by a vertical vector field, it holds that

$$Z_2 = Z_1 + V, \tag{41}$$

where V is a smooth section of the vertical bundle $\mathrm{Ver}(A^1M)$. This vertical vector field can be interpreted as (relative) acceleration between Z_1 and Z_2. The bundle metric $\hat{g}$ defined in (32) induces the bijection

$$\hat{g}\,\cdot : \Gamma\big(\mathrm{Ver}(A^1M)\big) \to \Gamma\big(\mathrm{Ver}^*(A^1M)\big), \; V \mapsto F = \hat{g} \cdot V \tag{42}$$

between smooth sections of the vertical bundle $\mathrm{Ver}(A^1M)$ and sections of the dual of the vertical bundle $\mathrm{Ver}^*(A^1M)$, where $\hat{g} \cdot V$ is the one-form $\hat{g}(\,\cdot\,, V)$.

If we consider the Galilean metric to model the mass of a finite-dimensional mechanical system and if we interpret vertical vector fields as (relative) accelerations, then with the bijection (42) we are facing Newton's second law that says "force F is equal to mass $\hat{g}$ times acceleration V". This motivates the following definition.

Definition 6.1.[4] Let (M, ϑ, g) be the Galilean manifold of a finite-dimensional mechanical system and let A^1M be the corresponding state space. A *force* is a smooth section of the dual of the vertical bundle $\mathrm{Ver}^*(A^1M)$, i.e., a force is a $C^\infty(A^1M)$-linear map

$$F \colon \Gamma(\mathrm{Ver}(A^1M)) \to C^\infty(A^1M) \tag{43}$$

on the space of vertical vector fields.

[4]This definition of force is implicitly given in Theorem 1 on page 32 of [19].

Consider the action forms Ω_1 and Ω_2 of the respective second-order fields Z_1 and Z_2. We introduce the differential two-form Φ by which the action forms Ω_1 and Ω_2 differ, i.e.,

$$\Omega_2 = \Omega_1 + \Phi. \tag{44}$$

This implies $\partial\Phi = 0$ because $\partial\Omega_1 = \partial\Omega_2 = 0$ by Theorem 4.1.

It is possible to associate the two-form Φ in (44) to the force F which by (42) is related to the vertical vector field in (41). Indeed, in terms of the coordinate fields induced by a natural chart, the coefficients of the force

$$F = F_i \, \mathrm{d}u^i \,, \tag{45}$$

which is associated to $V = Z_2 - Z_1$ by (42), are given by

$$F_i = g_{ij} V^j = g_{ij}(Z_2^j - Z_1^j) \,.$$

Using the representation (34) of the action form, the two-form $\Phi = \Omega_2 - \Omega_1$ is given as

$$\begin{aligned} \Phi &= g_{ij}(Z_2^i - Z_1^i)\, \mathrm{d}x^j \wedge \mathrm{d}t + \frac{1}{2} g_{ij} \left(\frac{\partial Z_2^i}{\partial u^k} - \frac{\partial Z_1^i}{\partial u^k} \right) (\mathrm{d}x^j - u^j \mathrm{d}t) \wedge (\mathrm{d}x^k - u^k \mathrm{d}t) \\ &= F_j \, \mathrm{d}x^j \wedge \mathrm{d}t + \frac{1}{2} \frac{\partial F_j}{\partial u^k} (\mathrm{d}x^j - u^j \mathrm{d}t) \wedge (\mathrm{d}x^k - u^k \mathrm{d}t), \end{aligned} \tag{46}$$

where the last equality uses that the coefficients $g_{ij} = g_{ij} \circ \pi$ are independent of $u^1, \ldots, u^n$ and form a symmetric matrix. The local expression (46) shows that the two-form Φ is semi-basic.

Definition 6.2. Let (M, ϑ, g) be the Galilean manifold of a finite-dimensional mechanical system and let A^1M be the corresponding state space. A *force two-form* is a differential two-form $\Phi \in \Omega^2(A^1M)$ that is semi-basic and satisfies $\partial\Phi = 0$.

A coordinate-free version of the map relating forces (45) to force two-forms (46) together with a proof that this map is a bijection can be found in [5, 19, 26]. It follows from (46) that the difference of two action forms is a force two-form. This implies that by choosing a reference field R, any action form Ω can be decomposed as

$$\Omega = \Omega_R + \Phi_R \tag{47}$$

where Ω_R is the action form given by (38) and Φ_R is a force two-form.

7. **Classification of forces.** We say that a force F^{p} is a *potential force* if the related force two-form Φ^{p} is closed, i.e., if

$$\mathrm{d}\Phi^{\mathrm{p}} = 0. \tag{48}$$

Straightforward computations in local coordinates show that a force two-form (46) is closed if and only if its coefficients F_i can be written as

$$F_i = E_i + B_{ij} u^j, \tag{49}$$

with functions E_i and B_{ij} which do not depend on $(u^1, \ldots, u^n)$. Moreover, these functions have to fulfill

$$B_{ij} = -B_{ji} \tag{50}$$

together with

$$\frac{\partial B_{ij}}{\partial x^k} + \frac{\partial B_{ki}}{\partial x^j} + \frac{\partial B_{jk}}{\partial x^i} = 0 \quad \text{and} \quad \frac{\partial B_{ij}}{\partial t} = \frac{\partial E_i}{\partial x^j} - \frac{\partial E_j}{\partial x^i} \,,$$

which thanks to the suggestive use of the letters B and E can be identified as a generalized version of Maxwell's equations. Consequently, a closed force two-form has the local form

$$\Phi^{\mathrm{p}} = E_i \,\mathrm{d}x^i \wedge \mathrm{d}t + \tfrac{1}{2} B_{ij} \,\mathrm{d}x^i \wedge \mathrm{d}x^j \,, \tag{51}$$

implying that a closed force two-form is basic.

According to the Poincaré lemma, a closed differential form is locally exact, i.e., there exists a neighbourhood $W \subseteq A^1 M$ and a one-form ϕ defined on W such that

$$\Phi^{\mathrm{p}}\big|_W = \mathrm{d}\phi. \tag{52}$$

As a one-form ϕ satisfying (52), we consider the locally defined basic one-form

$$\phi = -V \mathrm{d}t + A_i \,\mathrm{d}x^i \,, \tag{53}$$

with functions[5] V and A_i which only depend on $(t, x^1, \ldots, x^n)$. With the one-form (53) it holds that

$$E_i = -\left(\frac{\partial V}{\partial x^i} + \frac{\partial A_i}{\partial t}\right), \qquad B_{ij} = 2\frac{\partial A_j}{\partial x^i}.$$

The previous considerations allow us to split a given force two-form

$$\Phi = \Phi^{\mathrm{p}} + \Phi^{\mathrm{np}} \tag{54}$$

into a potential force two-form Φ^{p} that is locally defined by the exterior derivative of the one-form (53) and the remaining part Φ^{np} which we will refer to as *nonpotential force two-form*.

8. **Lagrangian and Cartan one-form.** Using the decomposition of forces (54) in (47), any action form Ω can be decomposed as

$$\Omega = \Omega_R + \Phi^{\mathrm{p}}_R + \Phi^{\mathrm{np}}_R \,, \tag{55}$$

where Ω_R is the action form (38) induced by the kinetic energy with respect to a reference field R of the mechanical system. At least locally, the potential force two-form Φ^{p}_R is the exterior derivative of the one-form ϕ_R given in (53). Using the rules (27), it can be seen that the chart representation (53) of ϕ_R is equivalent to

$$\phi_R = (-V_R + A^R_i u^i)\mathrm{d}t + \partial(-V_R + A^R_i u^i), \tag{56}$$

because V_R and A^R_i only depend on $(t, x^1, ..., x^n)$. By comparing (56) to the definition (38) of the action form Ω_R induced by the kinetic energy T_R of the mechanical system, it is clear that the sum $\Omega_R + \Phi^{\mathrm{p}}_R$ can be locally written as

$$\begin{aligned}\Omega_R + \Phi^{\mathrm{p}}_R = \Omega_R + \mathrm{d}\phi_R &= \mathrm{d}[(T_R - V_R + A^R_i u^i)\hat{\vartheta} + \partial(T_R - V_R + A^R_i u^i)] \\ &= \mathrm{d}(L_R \hat{\vartheta} + \partial L_R) = \mathrm{d}\omega_R \,,\end{aligned} \tag{57}$$

where we have introduced the following objects.

Definition 8.1. Let (M, ϑ, g) be the Galilean manifold of a finite-dimensional mechanical system, $\hat{\vartheta} = \pi^* \vartheta$ be the time structure of the state space $A^1 M$, and let T_R be the kinetic energy of the mechanical system with respect to a reference field R. Let Φ^{p}_R be the (locally) exact potential force two-form given in the natural chart (10) by

$$\Phi^{\mathrm{p}}_R = \mathrm{d}\phi_R \quad \text{with} \quad \phi_R = -V_R \mathrm{d}t + A^R_i \mathrm{d}x^i \,,$$

[5] In the context of a charged particle moving in an electromagnetic field the function V is known as scalar potential of the field and the $\mathbb{R}^3$-tuple (A_1, A_2, A_3) is said to be its vector potential. See p. 45 in [15].

where the component functions V_R and A^i_R of ϕ_R only depend on $(t, x^1, \dots, x^n)$. The *Lagrangian* of the mechanical system with respect to the reference field R is the function

$$L_R \coloneqq T_R - V_R + A^R_i u^i \tag{58}$$

on the state space and it defines the *Cartan one-form*

$$\omega_R = L_R \hat{\vartheta} + \partial L_R \,. \tag{59}$$

The coefficient function V_R in (58) is known as the *potential energy* with respect to the reference field R of the mechanical system. Moreover, in classical mechanics (no electromagnetism) $A^R_i = 0$ can be assumed, see [5].

In the local coordinates induced by the natural chart (10), the Cartan one-form reads as

$$\omega_R = L_R \mathrm{d}t + \frac{\partial L_R}{\partial u^i}\big(\mathrm{d}x^i - u^i \mathrm{d}t\big) = \left(L_R - u^i \frac{\partial L_R}{\partial u^i}\right)\mathrm{d}t + \frac{\partial L_R}{\partial u^i}\mathrm{d}x^i \,, \tag{60}$$

from which we can see that the Cartan one-form determines the Lagrangian by

$$L_R = Z \lrcorner\, \omega_R, \tag{61}$$

where Z is an arbitrary second-order field on the state space. As it will be needed later on, we use (60) to compute the exterior derivative of the Cartan one-form as

$$\begin{aligned} \mathrm{d}\omega_R &= \frac{\partial L_R}{\partial x^i}\mathrm{d}x^i \wedge \mathrm{d}t + \mathrm{d}\left(\frac{\partial L_R}{\partial u^i}\right) \wedge \big(\mathrm{d}x^i - u^i \mathrm{d}t\big) \\ &= \left(\mathrm{d}\left(\frac{\partial L_R}{\partial u^i}\right) - \frac{\partial L_R}{\partial x^i}\mathrm{d}t\right) \wedge \big(\mathrm{d}x^i - u^i \mathrm{d}t\big). \end{aligned} \tag{62}$$

9. **Canonical chart.** Let L_R denote the Lagrangian of the mechanical system with respect to the reference field R, then the Cartan one-form takes a particularly simple form with respect to the *canonical chart*[6]

$$\tilde{\Phi} \colon A^1 M \supseteq \pi^{-1}(U) \to \mathbb{R}^{2n+1}, (p, v_p) \mapsto (\tilde{t}, \tilde{x}^1, \dots, \tilde{x}^n, p_1, \dots, p_n)\,, \tag{63}$$

which is defined by the change of coordinates $\tilde{\Phi} \circ \Phi^{-1}$ given by

$$\tilde{t} = t, \quad \tilde{x}^i = x^i \quad \text{and} \quad p_i = \frac{\partial L_R}{\partial u^i} \circ \Phi^{-1}(t, x^1, \dots, x^n, u^1, \dots, u^n)\,. \tag{64}$$

Note that Φ denotes the natural chart (10). We call p_i *generalized momentum* coordinates, which by equations (58) and (37) together with $\partial V_R / \partial u^i = 0$ have the form

$$p_i = g_{ij}\big(u^j - R^j\big) + A^R_i. \tag{65}$$

The full rank of the Galilean metric g guarantees that the relation (65) can be resolved for $u^1, \dots, u^n$ as

$$u^i = g^{ij}\big(p_j - A^R_j\big) + R^i, \tag{66}$$

where g^{ij} are the components of the inverse matrix of g_{ij}. We refer to $\big(\tilde{t}, \tilde{x}^1, \dots, \tilde{x}^n, p_1, \dots, p_n\big)$ as *canonical coordinates.* The tildes on t and the x^i allow the distinction between the canonical coordinates and those provided by the natural chart (10).

[6] These coordinates are by no means canonically defined since they depend on the choice of a reference field R. Physically, the quantities $p_1, \dots, p_n$ are generalized momenta. In the context of time-independent mechanics playing on the cotangent bundle T^*Q of a time-independent configuration manifold Q, the position and generalized momentum coordinates provided by the Darboux theorem are *canonical.* Moreover, Hamilton's equations are also referred to as *canonical equations* (see [16], p. 132). So we use the adjective canonical because of tradition.

We rewrite the Cartan one-form (60) as

$$\omega_R = -H_R \mathrm{d}\tilde{t} + p_i \mathrm{d}\tilde{x}^i , \tag{67}$$

where we have used that $\mathrm{d}t = \mathrm{d}\tilde{t}$ and $\mathrm{d}x^i = \mathrm{d}\tilde{x}^i$ for $i = 1, \dots, n$, and defined the *Hamiltonian* as

$$H_R \colon A^1M \supseteq \pi^{-1}(U) \to \mathbb{R},\ a \mapsto H_R(a) \coloneqq -\left(L_R - u^i \frac{\partial L_R}{\partial u^i} \right)(a),$$

which is a *local* function defined on the neighbourhood $\pi^{-1}(U)$. By equations (37) and (58), the Hamiltonian takes the form

$$\begin{aligned} H_R &= \frac{1}{2} g_{ij} u^i u^j - \frac{1}{2} g_{ij} R^i R^j + V_R \\ &= \frac{1}{2} g^{ij} \big(p_i - A_i^R\big)\big(p_j - A_j^R\big) + R^j \big(p_j - A_j^R\big) + V_R \, , \end{aligned} \tag{68}$$

where we have used (66) for the second equality. In the special case where the reference field R is the resting field of the natural chart, i.e. $R^i = 0$, the Hamiltonian is the sum of the kinetic energy (36) and the potential energy V_R.

It can be seen from (68) that we can rewrite (66) as

$$u^i = \frac{\partial H_R}{\partial p_i} \, . \tag{69}$$

Expressing the basis vectors $\partial/\partial u^i$ induced by the natural chart (10) with respect to the basis vectors induced by the chart (63) gives

$$\frac{\partial}{\partial u^i} = \frac{\partial \tilde{t}}{\partial u^i} \frac{\partial}{\partial \tilde{t}} + \frac{\partial \tilde{x}^j}{\partial u^i} \frac{\partial}{\partial \tilde{x}^j} + \frac{\partial p_j}{\partial u^i} \frac{\partial}{\partial p_j} = g_{ji} \frac{\partial}{\partial p_j} \, , \tag{70}$$

where we adopt the convention that a lower index appearing in the denominator is considered to be an upper index. Using this relation and equation (69) in (46), a force two-form Φ can locally be written as

$$\Phi = F_i \mathrm{d}\tilde{x}^i \wedge \mathrm{d}\tilde{t} + \frac{1}{2} g_{rj} \frac{\partial F_i}{\partial p_r} \left(\mathrm{d}\tilde{x}^i - \frac{\partial H_R}{\partial p_i} \mathrm{d}\tilde{t} \right) \wedge \left(\mathrm{d}\tilde{x}^j - \frac{\partial H_R}{\partial p_j} \mathrm{d}\tilde{t} \right) \tag{71}$$

because $\mathrm{d}t = \mathrm{d}\tilde{t}$ and $\mathrm{d}x^i = \mathrm{d}\tilde{x}^i$ for $i = 1, \dots, n$.

Finally, it follows from (69) and (70) that a second-order field (20) has the local representation

$$Z = \frac{\partial}{\partial t} + u^i \frac{\partial}{\partial x^i} + Z^i \frac{\partial}{\partial u^i} = \frac{\partial}{\partial \tilde{t}} + \frac{\partial H_R}{\partial p_i} \frac{\partial}{\partial \tilde{x}^i} + Z_i \frac{\partial}{\partial p_i} \tag{72}$$

with respect to the natural and the canonical chart. Note that we have introduced the coefficients $Z_i = g_{ij} Z^j$.

10. **Dynamics of finite-dimensional mechanical systems.** It is possible to characterize any action form Ω by means of a Lagrangian L_R and a nonpotential force two-form Φ_R^{np}, which can be seen from (55) and (57). Moreover, the action form Ω uniquely determines a second-order field by Theorem 4.1. This empowers us to model the second-order field which gives the motions of a mechanical system by modeling a Lagrangian together with a nonpotential force two-form. In other words, for a specific mechanical system, e.g., a rigid-body model of an industrial robot, the modeling process consists in finding an appropriate Lagrangian together with a nonpotential force two-form. Hence, we formulate the following fundamental

postulate for the description of the dynamics of time-dependent finite-dimensional mechanical systems.

Postulate 1. *Let (M, ϑ, g) be the Galilean manifold of a finite-dimensional mechanical system and $\hat{\vartheta} = \pi^*\vartheta$ be the time structure of the state space A^1M. Moreover, let ω_R be the Cartan one-form induced by a Lagrangian L_R of the mechanical system and let Φ_R^{np} be the nonpotential force two-form, each one with respect to the reference field R.*

A motion β of the mechanical system is an integral curve of the vector field $X \in \mathrm{Vect}(A^1M)$ characterized by

$$\hat{\vartheta}(X) = 1 \quad \textit{and} \quad X \lrcorner\, \Omega = 0, \tag{73}$$

where

$$\Omega := \mathrm{d}\omega_R + \Phi_R^{\mathrm{np}} \tag{74}$$

is the action form Ω of the mechanical system. Consequently, a motion β is a solution of the equations of motion

$$\dot{\beta}(\tau) = X(\beta(\tau))\,. \tag{75}$$

Using (57) in (74) allows to write the action form Ω of the mechanical system as

$$\Omega = \Omega_R + \Phi_R^{\mathrm{p}} + \Phi_R^{\mathrm{np}}\,,$$

where Ω_R is the action form (38) induced by the kinetic energy T_R of the system and Φ_R^{p} is the potential force two-form of the system. Consequently, the action form Ω_R describes the motion of a mechanical system which with respect to the reference field R is not subjected to forces (i.e., $\Phi_R^{\mathrm{p}} = \Phi_R^{\mathrm{np}} = 0$). It is important to note, that the same system might be subjected to forces if a different reference field $\tilde{R}$ is chosen. Indeed, by (47) we have

$$\Omega = \Omega_R = \Omega_{\tilde{R}} + \Phi_{\tilde{R}}$$

for the two different reference fields. We call the force two-form $\Phi_{\tilde{R}}$ *inertia force* as it appears due to the change of reference field, see Sect. 39 in [16] or Sects. IV.4–5 in [14]. Two examples thereof are the Coriolis forces and the centrifugal forces.

We call a mechanical system *exact (closed)* if its action form Ω is exact (closed). In view of (74), a mechanical system is closed if it is only subjected to potential forces. According to the Poincaré lemma, a closed differential form is locally exact. Hence, a closed mechanical system is locally exact and its action form can be locally written as $\Omega = \mathrm{d}\omega_R$. It can be shown that inertia forces stemming from a change of reference field are potential forces [5]. Thus, the statement that a mechanical system is only subjected to potential forces is independent of the reference field.

11. **Virtual displacements.** In its development, classical mechanics has been intimately related to the calculus of variations, which led to the definition of virtual displacements as the derivative with respect to the parameter ε of a one-parameter family of curves, where the motion of the mechanical system is the curve given by $\varepsilon = 0$. Thus, a virtual displacement is induced by a one-parameter family of curves and can be seen as a vector field along the motion. As our Postulate 1 is not based on the calculus of variations, we consider virtual displacements as vector fields that (locally) induce one-parameter families of curves. Indeed, a vector field $Y \in \mathrm{Vect}(A^1M)$ on the state space A^1M induces the one-parameter family

$$\kappa_\varepsilon(\tau) = \varphi_\varepsilon^Y \circ \beta(\tau) \tag{76}$$

by displacing every point $\beta(\tau)$ of the motion with the flow φ_ε^Y of the vector field Y, where $\varphi_0^Y = \mathrm{id}_{A^1M}$.

It turns out, that not every vector field Y can be seen as a virtual displacement. In fact, in classical mechanics a virtual displacement occurs at fixed time,[7] which in our context means that the temporal distance between the point $\beta(\tau)$ of the motion and the associated displaced point $\kappa_\varepsilon(\tau)$ is zero, i.e.

$$t\big(\kappa_\varepsilon(\tau)\big) - t\big(\beta(\tau)\big) = t \circ \varphi_\varepsilon^Y\big(\beta(\tau)\big) - t\big(\beta(\tau)\big) = 0\,, \tag{77}$$

where t is a time function on the state space A^1M and where we have used (76) for the second equality. Condition (77) is fulfilled, if the flow of Y preserves the time function t such that

$$t \circ \varphi_\varepsilon^Y - t = (\varphi_\varepsilon^Y)^* t - t = 0\,, \tag{78}$$

which using $\varphi_0^Y = \mathrm{id}_{A^1M}$ implies

$$\mathcal{L}_Y t = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\big[(\varphi_\varepsilon^Y)^* t - t\big] = 0\,. \tag{79}$$

Hence, the vector field Y is required to be spacelike by $\mathcal{L}_Y t = \mathrm{d}t(Y) = 0$. Consequently, we define a *virtual displacement field* on the state space A^1M as a vector field $Y \in \mathrm{Vect}(A^1M)$ which is spacelike, i.e., a virtual displacement field must satisfy the condition

$$\hat{\vartheta}(Y) = 0 \tag{80}$$

and is therefore a section of the spacelike bundle $A^0(A^1M)$ of A^1M. Hereby, $A^0(A^1M)$ is defined by replacing M and ϑ with A^1M and $\hat{\vartheta}$ in (3). We introduce

$$\mathrm{Virt}(A^1M) = \Gamma\big(A^0(A^1M)\big) \tag{81}$$

denoting the set of virtual displacement fields on A^1M. By (80), a virtual displacement field is locally given by

$$Y = \delta x^i \frac{\partial}{\partial x^i} + \delta u^i \frac{\partial}{\partial u^i} = \delta \tilde{x}^i \frac{\partial}{\partial \tilde{x}^i} + \delta p_i \frac{\partial}{\partial p_i}\,, \tag{82}$$

where the basis vector fields are induced by the natural chart (10) and the canonical chart (63), respectively. To agree with the notation used in classical literature, the coefficients of the virtual displacement fields are decorated with deltas.

As the motion β of a mechanical system is a second-order curve in the state space, we consider next a virtual displacement field Y, such that its induced one-parameter family of motions κ_ε defined by (76) is a second-order curve for every ε. Let Z be the second-order vector field of which the motion β is an integral curve, then

$$\dot{\kappa}_\varepsilon = (\varphi_\varepsilon^Y \circ \beta)^{\cdot} = \mathrm{D}\varphi_\varepsilon^Y \dot{\beta} = \mathrm{D}\varphi_\varepsilon^Y Z \circ \beta\,.$$

Since the flow of Y in (76) is a (local) diffeomorphism, we obtain

$$\dot{\kappa}_\varepsilon = \mathrm{D}\varphi_\varepsilon^Y Z \circ (\varphi_\varepsilon^Y)^{-1} \circ \kappa_\varepsilon = (\varphi_\varepsilon^Y)_* Z \circ \kappa_\varepsilon\,.$$

Therefore, the curves κ_ε of the one-parameter family induced by Y are integral curves of the vector field $(\varphi_\varepsilon^Y)_* Z$ given by the push-forward of Z with the flow of Y. As κ_ε is a family of second-order curves, $(\varphi_\varepsilon^Y)_* Z$ is a second-order field. Moreover, since the difference of two second-order fields is vertical,

$$\mathcal{L}_Y Z = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\big[(\varphi_\varepsilon^Y)_* Z - Z\big] \in \Gamma\big(\mathrm{Ver}(A^1M)\big)\,, \tag{83}$$

[7]Pars [21] calls these virtual displacements ‘contemporaneous variations’.

where we have used the definition of the Lie derivative and $\varphi_0^Y = \mathrm{id}_{A^1M}$. Hence, the set of virtual displacement fields that for any integral curve of Z induce one-parameter families of second-order curves is

$$\mathrm{Virt}_Z(A^1M) \coloneqq \big\{ Y \in \mathrm{Virt}(A^1M) \;\big|\; \mathcal{L}_Y Z = [Y,Z] \in \Gamma\big(\mathrm{Ver}(A^1M)\big) \big\} . \tag{84}$$

With respect to a natural chart (10), $Y \in \mathrm{Virt}_Z(A^1M)$ has the local form

$$Y = \delta x^i \frac{\partial}{\partial x^i} + \mathcal{L}_Z \delta x^i \frac{\partial}{\partial u^i} . \tag{85}$$

Indeed, let x^i and u^i denote the coordinate functions of the natural chart then the $\partial/\partial x^i$ component of $[Y,Z]$ is

$$\big[Y,Z\big][x^i] = Y\big[Z[x^i]\big] - Z\big[Y[x^i]\big] = Y[u^i] - Z[\delta x^i] \tag{86}$$

where we have used the definition of the Lie bracket as well as the local representation (20) of the second-order field Z. It can be seen from (86) that the $\partial/\partial x^i$ components of $[Y,Z]$ vanish if and only if Y has the form (85). This is equivalent to $[Y,Z]$ being vertical, since the vertical bundle is spanned by $\partial/\partial u^1, \dots, \partial/\partial u^n$.

In classical mechanics yet another class of one-parameter family of motions is often encountered. Its construction uses that a second-order curve β on A^1M is the tangent field $\dot\gamma$ to the time-parametrized curve $\gamma = \pi \circ \beta$ on M. By the same argument as in (78), a *virtual displacement field on* M is a spacelike vector field

$$\bar Y \in \mathrm{Virt}(M) \coloneqq \Gamma(A^0M) , \tag{87}$$

which with respect to the adapted chart (1) has the form

$$\bar Y = \delta x^i \frac{\partial}{\partial x^i} . \tag{88}$$

The virtual displacement field $\bar Y$ on M induces the one-parameter family $\varphi_\varepsilon^{\bar Y} \circ \gamma$ of time-parametrized curves on M. By construction, $(\varphi_\varepsilon^{\bar Y} \circ \gamma)^{\dot{}}$ is a one-parameter family of second-order curves on A^1M. Let $\hat Y$ be the virtual displacement field on A^1M, which induces the one-parameter family of second-order curves $\kappa_\varepsilon = \varphi_\varepsilon^{\hat Y} \circ \dot\gamma$, chosen such that

$$\kappa_\varepsilon = \varphi_\varepsilon^{\hat Y} \circ \dot\gamma = (\varphi_\varepsilon^{\bar Y} \circ \gamma)^{\dot{}} . \tag{89}$$

This vector field $\hat Y$ is called the *complete lift of* $\bar Y$. It can be shown that with respect to a natural chart, the virtual displacement field $\hat Y$ on A^1M has the local form

$$\hat Y = \delta x^i \circ \pi \frac{\partial}{\partial x^i} + \mathcal{L}_Z(\delta x^i \circ \pi) \frac{\partial}{\partial u^i} , \tag{90}$$

where δx^i are the component functions of (88) and π denotes the natural projection (11) of A^1M. A detailed treatment of lifts is given in [27].

12. **Principle of virtual work.** By Postulate 1, a motion β of the mechanical system is an integral curve of the time-normalized vector field X on A^1M characterized by

$$X \lrcorner\, \Omega = 0 , \tag{91}$$

where Ω is the action form of the mechanical system. The virtual displacement fields can be used to restate (91) in the equivalent variational form

$$(X \lrcorner\, \Omega)(Y) = \Omega(X,Y) = 0 \quad \forall Y \in \mathrm{Virt}(A^1M) . \tag{92}$$

In fact, we know from (35) that the map $\Omega(X,\cdot)$ has constant rank $2n$ and that it is blind on the line bundle over A^1M spanned by the time-normalized vector field

X that satisfies (91). As $\hat{\vartheta}\big(X\big) = 1$ and $\hat{\vartheta}\big(Y\big) = 0$, virtual displacement fields Y are used to test the $2n$ complementary directions to $\operatorname{span}\big\{X\big\}$.

Since Ω is an action form, we know that the vector field X defined by (92) is a second-order field. Using this information a priori it follows from (92) that a motion β of the mechanical system is an integral curve of the second-order field Z on A^1M characterized by

$$\Omega(Z, Y) = 0 \quad \forall Y \in \mathrm{Virt}(A^1M)\,. \tag{93}$$

Definition 12.1. Let Ω be the action form of a finite-dimensional mechanical system with state space A^1M. Moreover, let $Z \in \mathrm{Vect}(A^1M)$ be a second-order field and $Y \in \mathrm{Virt}(A^1M)$ be a virtual displacement field on the state space. The *virtual work* of the mechanical system is the smooth function $\Omega(Z, Y)$.

In order to derive the coordinate representation of the virtual work of the mechanical system, we use the definition of the action form (74) and compute the summands of $Z \lrcorner\, \Omega = Z \lrcorner\, \mathrm{d}\omega_R + Z \lrcorner\, \Phi_R^{\mathrm{np}}$ separately. Using (62), (72) and $Z \lrcorner\, \mathrm{d}f = \mathcal{L}_Z f$ for real-valued functions f, the representation with respect to the natural chart of the first summand is

$$Z \lrcorner\, \mathrm{d}\omega_R = \left(\mathcal{L}_Z\left(\frac{\partial L_R}{\partial u^i}\right) - \frac{\partial L_R}{\partial x^i}\right)\big(\mathrm{d}x^i - u^i \mathrm{d}t\big)\,. \tag{94}$$

Similarly, straightforward computations with (67) and (72) lead to the representation

$$\begin{aligned} Z \lrcorner\, \mathrm{d}\omega_R &= -\mathcal{L}_Z(H_R)\mathrm{d}\tilde{t} + \mathrm{d}H_R + Z_i\, \mathrm{d}\tilde{x}^i - \frac{\partial H_R}{\partial p_i}\mathrm{d}p_i \\ &= -\left(\mathcal{L}_Z H_R - \frac{\partial H_R}{\partial \tilde{t}}\right)\mathrm{d}\tilde{t} + \left(\frac{\partial H_R}{\partial \tilde{x}^i} + Z_i\right)\mathrm{d}\tilde{x}^i \end{aligned} \tag{95}$$

with respect to the canonical chart. Finally, using the chart representations (72) in (46) and (71) respectively, the contribution of the nonpotential force is

$$Z \lrcorner\, \Phi_R^{\mathrm{np}} = F_i u^i \mathrm{d}t - F_i\, \mathrm{d}x^i = F_i \frac{\partial H_R}{\partial p_i}\mathrm{d}\tilde{t} - F_i\, \mathrm{d}\tilde{x}^i\,. \tag{96}$$

Hence, in the local coordinates used for the virtual displacement field Y in (82) the virtual work reads as

$$\Omega(Z, Y) = \left[\mathcal{L}_Z\left(\frac{\partial L_R}{\partial u^i}\right) - \frac{\partial L_R}{\partial x^i} - F_i\right]\delta x^i = \left(Z_i + \frac{\partial H_R}{\partial \tilde{x}^i} - F_i\right)\delta\tilde{x}^i\,. \tag{97}$$

It can be seen from the absence of the coefficients δu^i and δp_i of Y in (97) that the virtual work is blind on $\mathrm{Ver}(A^1M)$. Therefore, only virtual displacement fields in the n complementary directions to $\mathrm{Ver}(A^1M)$ need to be tested in (93). A look at (85) and (90) reveals that both the virtual displacements in $\mathrm{Virt}_Z(A^1M)$ and the complete lifts of virtual displacements in $\mathrm{Virt}(M)$ can be used to test n complementary directions to $\mathrm{Ver}(A^1M)$ in (93). Thus, we have established the following theorem.

Theorem 12.2 (Principle of virtual work)**.** *Let (M, ϑ, g) be the Galilean manifold of a finite-dimensional mechanical system and $\hat{\vartheta} = \pi^*\vartheta$ be the time structure of the state space A^1M. Moreover let Ω be the action form (74) of the mechanical system.*

A motion β of the mechanical system is an integral curve of the second-order field Z on A^1M equivalently characterized by one of the following conditions

$$(i) \qquad \Omega(Z,Y)=0 \quad \forall Y\in \mathrm{Virt}(A^1M) \tag{98}$$
$$(ii) \qquad \Omega(Z,Y)=0 \quad \forall Y\in \mathrm{Virt}_Z(A^1M) \tag{99}$$
$$(iii) \qquad \Omega(Z,\hat{Y})=0 \quad \forall \bar{Y}\in \mathrm{Virt}(M)\,, \tag{100}$$

where $\hat{Y}$ is the complete lift of the virtual displacement field $\bar{Y}$.

As the characterizations of the motion in Theorem 12.2 by (97) coincide in local coordinates, we refer to either of the characterizations as principle of virtual work. In [21] on p. 74, the local expression of the virtual work (97) involving the Lagrangian is referred to as the 'fourth form of the fundamental equation'. Note that one could postulate the principle of virtual work as the starting point of a theory for the description of finite-dimensional mechanical systems. In our presentation, which is based on Postulate 1, the observation that the virtual work vanishes is a theorem rather than a principle. Nevertheless, we speak of the principle of virtual work in order to relate our presentation to classical approaches. The observation that Postulate 1 generalizes the principle of virtual work can be found in the book by Souriau, where it is written that the virtual work is a truncated version of the action form Ω.

13. **Equations of motion.** Let us consider a motion $\beta\colon I\to A^1M$, $\tau\mapsto\beta(\tau)$ of the mechanical system, which by Theorem 12.2 is an integral curve of the second-order field Z characterized by the principle of virtual work, i.e., the motion satisfies the equations of motion

$$\dot{\beta}(\tau)=Z\big(\beta(\tau)\big)\,. \tag{101}$$

With (97), it follows from Theorem 12.2 that Z is characterized by *Lagrange's equations* of the second kind

$$\mathcal{L}_Z\left(\frac{\partial L_R}{\partial u^i}\right)-\frac{\partial L_R}{\partial x^i}=F_i. \tag{102}$$

With (24), the motion, being a second-order curve, has the chart representation $\Phi\circ\beta(\tau)=(\tau,\mathbf{x}(\tau),\dot{\mathbf{x}}(\tau))$ with respect to the natural chart Φ. Equation (102) needs to be satisfied along the motion, such that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L_R}{\partial u^i}\circ\beta(\tau)\right)-\frac{\partial L_R}{\partial x^i}\circ\beta(\tau)=F_i\circ\beta(\tau) \tag{103}$$

by (101) and the definition of the Lie derivative. We recognize (103) as Lagrange's equations of the second kind in their classical form.[8]

Similarly, with (97) it follows from Theorem 12.2 that Z is characterized by

$$Z_i=-\frac{\partial H_R}{\partial\tilde{x}^i}+F_i. \tag{104}$$

Hence, by (72), the second-order field Z locally reads as

$$Z=\frac{\partial}{\partial\tilde{t}}+\frac{\partial H_R}{\partial p_i}\frac{\partial}{\partial\tilde{x}^i}+\left(-\frac{\partial H_R}{\partial\tilde{x}^i}+F_i\right)\frac{\partial}{\partial p_i}$$

[8] See [13], p. 24, [25], p. 63, [21], p. 75 or [16], p. 3.

and the equations of motion (101) take the form

$$\dot{\tilde{x}}^i(\tau) = \frac{\partial H_R}{\partial p_i} \circ \beta(\tau), \tag{105}$$

$$\dot{p}_i(\tau) = -\frac{\partial H_R}{\partial \tilde{x}^i} \circ \beta(\tau) + F_i \circ \beta(\tau), \tag{106}$$

where $\tilde{\Phi} \circ \beta(\tau) = (\tau, \tilde{\mathbf{x}}(\tau), \mathbf{p}(\tau))$ is the representation of the motion in canonical coordinates. We recognize (105) and (106) as *Hamilton's equations.*[9] It is clear by construction, that (105) is the second-order condition (23) expressed in the canonical chart.

14. **Central equations of Lagrange and Hamel.** We pursue our endeavor to establish a firm link to the classical results by deriving Lagrange's central equation and Hamel's generalized version of it as coordinate representations of two different versions of the principle of virtual work given in Theorem 12.2. We use the definition of the action form of a mechanical system Ω from Postulate 1 to write the virtual work of the system as

$$\Omega\big(Z, Y\big) = \mathrm{d}\omega_R\big(Z, Y\big) + \Phi_R^{\mathrm{np}}\big(Z, Y\big), \tag{107}$$

where

$$\mathrm{d}\omega_R = \mathrm{d}L_R \wedge \mathrm{d}t + \mathrm{d}\bigg(\frac{\partial L_R}{\partial u^i}\bigg) \wedge \big(\mathrm{d}x^i - u^i \mathrm{d}t\big) - \frac{\partial L_R}{\partial u^i}\mathrm{d}u^i \wedge \mathrm{d}t$$

and

$$\Phi_R^{\mathrm{np}} = F_i \mathrm{d}x^i \wedge \mathrm{d}t + \frac{1}{2}\frac{\partial F_i}{\partial u^j}\big(\mathrm{d}x^i - u^i \mathrm{d}t\big) \wedge \big(\mathrm{d}x^j - u^j \mathrm{d}t\big),$$

as can be seen from from (60) and (46). Considering that $Z \lrcorner \big(\mathrm{d}x^i - u^i \mathrm{d}t\big) = 0$ because Z is a second-order field and that $Y \lrcorner\, \mathrm{d}t = 0$ since Y is a virtual displacement field, it follows that

$$\Omega\big(Z, Y\big) = -\mathcal{L}_Y L_R + \mathcal{L}_Z\bigg(\frac{\partial L_R}{\partial u^i}\bigg)\delta x^i + \frac{\partial L_R}{\partial u^i}\delta u^i - F_i \delta x^i\,, \tag{108}$$

where we used the local representation (82) of the virtual displacement field Y.

Using (108), the principle of virtual work given by Theorem 12.2 (i) reads as

$$\mathcal{L}_Z\bigg(\frac{\partial L_R}{\partial u^i}\delta x^i\bigg) - \mathcal{L}_Y L_R - F_i \delta x^i + \frac{\partial L_R}{\partial u^i}\Big(\delta u^i - \mathcal{L}_Z \delta x^i\Big) = 0 \tag{109}$$

for all $\delta x^i, \delta u^i \in C^\infty(\pi^{-1}(U))$, where we have used the product rule for the Lie derivative and $\pi^{-1}(U)$ denotes the domain of the natural chart. We refer to equation (109) as *Hamel's generalized central equation.*[10]

On the other hand, using (108) in the principle of virtual work given by Theorem 12.2 (ii), the principle of virtual work takes the form of *Lagrange's central equation*[11]

$$\mathcal{L}_Z\bigg(\frac{\partial L_R}{\partial u^i}\delta x^i\bigg) - \mathcal{L}_Y L_R - F_i \delta x^i = 0 \tag{110}$$

for all $\delta x^i \in C^\infty(\pi^{-1}(U))$.

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[9]See p. 132 of [16] or p. 63 of [25].

[10]See [9], p. 480 and [8], p. 424. Note that the generalized central equation in [8] is formulated in more general coordinates that comprise the coordinates used in (109).

[11]See [7], p. 15, [2], p. 47, or [3], p. 21.

15. **Hamilton's principle.** In this section, we derive three versions of Hamilton's principle, which characterize the motion of an exact mechanical system as the solution of a variational problem.

By the definition of the action form Ω of a mechanical system in Postulate 1, it is clear that Ω is exact if the mechanical system is only subjected to potential forces. By Postulate 1, the action form Ω of an exact mechanical system is given by the exterior derivative of the Cartan one-form $\omega_R = L_R\hat{\vartheta} + \partial L_R$, where L_R denotes the Lagrangian of the mechanical system with respect to a reference field R.

Let $\mathcal{C}(\gamma_0,\gamma_1)$ denote the set of time-parametrized curves $\beta\colon I = [\tau_0,\tau_1] \to A^1M$ with fixed position endpoints $\gamma_0,\gamma_1 \in M$, i.e.

$$\mathcal{C}(\gamma_0,\gamma_1) \coloneqq \big\{\beta \text{ time-param.} \;\big|\; \pi\circ\beta(\tau_0) = \gamma_0,\ \pi\circ\beta(\tau_1) = \gamma_1\big\}\,. \tag{111}$$

We define the *action* of a mechanical system as the functional

$$A\colon \mathcal{C}(\gamma_0,\gamma_1) \to \mathbb{R},\ \beta \mapsto A\big[\beta\big] = \int_{\beta(I)} \iota^*\omega_R, \tag{112}$$

where

$$\iota\colon \beta(I) \hookrightarrow A^1M \tag{113}$$

denotes the inclusion map of the subset $\beta(I)$ into A^1M. The set $\beta(I) \subset A^1M$ is an immersed submanifold of A^1M. Indeed, the map $\beta\colon I \to A^1M$ is an injective immersion because of equation (22) and since the tangent vector of a time-parametrized curve never vanishes.

We know from definition (80) that a virtual displacement field $Y \in \mathrm{Virt}(A^1M)$ on the state space induces a one-parameter family of time-parametrized curves

$$\kappa_\varepsilon(\tau) = \varphi_\varepsilon^Y \circ \beta(\tau) \tag{114}$$

by displacing the curve $\beta(\tau)$ with the flow φ_ε^Y of Y. To assure that the family κ_ε lies in $\mathcal{C}(\gamma_0,\gamma_1)$ we demand

$$\pi\circ\kappa_\varepsilon(\tau_0) = \pi\circ\beta(\tau_0) = \gamma_0 \quad \text{and} \quad \pi\circ\kappa_\varepsilon(\tau_1) = \pi\circ\beta(\tau_1) = \gamma_1$$

for every ε, which implies that

$$Y(\beta(\tau_0)) \in \mathrm{Ver}(A^1M) \quad \text{and} \quad Y(\beta(\tau_1)) \in \mathrm{Ver}(A^1M)\,. \tag{115}$$

Following [11], we use the one-parameter family induced by the virtual displacement field Y to define the *first variation* of the action (112) in β as

$$\delta A\big[Y\big] \coloneqq \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} A\big[\varphi_\varepsilon^Y\circ\beta\big]\,, \tag{116}$$

where $\varphi_0^Y = \mathrm{id}_{A^1M}$. It is important to point out, that the one-parameter family defining the first variation has to lie in the set of curves for which the functional is defined, i.e., the set $\mathcal{C}(\gamma_0,\gamma_1)$ in this case. Using Prop. 16.6 of [18] in (116), the first variation of the action (112) can be recast as

$$\begin{aligned} \delta A\big[Y\big] &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \int_{\varphi_\varepsilon^Y\circ\beta(I)} \iota_\varepsilon^*\,\omega_R = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \int_{\beta(I)} \iota^*\Big(\big(\varphi_\varepsilon^Y\big)^*\omega_R\Big) \\ &= \int_{\beta(I)} \iota^*\bigg(\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \big(\varphi_\varepsilon^Y\big)^*\omega_R\bigg) = \int_{\beta(I)} \iota^*(\mathcal{L}_Y\omega_R) \end{aligned} \tag{117}$$

where $\iota_\varepsilon\colon \kappa_\varepsilon(I) \hookrightarrow A^1M$ denotes the inclusion map of the subset $\kappa_\varepsilon(I) \subset A^1M$ and $\iota = \iota_0$ denotes the inclusion map (113). Finally, by Cartan's magic formula and Stoke's theorem, the first variation (117) becomes

$$\delta A\big[Y\big] = \int_{\beta(I)} \iota^*\big(Y \lrcorner\, \mathrm{d}\omega_R\big) + \int_{\partial\beta(I)} \hat{\iota}^*\big(Y \lrcorner\, \omega_R\big)\,, \tag{118}$$

where $\partial\beta(I)$ denotes the boundary of $\beta(I)$ and $\hat{\iota}\colon \partial\beta(I) \hookrightarrow A^1M$ its inclusion map. The integral over the boundary in (118) vanishes as the virtual displacement fields are vertical on the boundary by (115) and ω_R is semi-basic, see (60). Consequently, the first variation (118) of the action (112) can be written as

$$\delta A\big[Y\big] = \int_I \big(Y \lrcorner\, \mathrm{d}\omega_R\big)\big(\dot{\beta}(\tau)\big)\mathrm{d}\tau = \int_I \mathrm{d}\omega_R(Y, \tilde{X}) \circ \beta(\tau)\mathrm{d}\tau, \tag{119}$$

where the vector field $\tilde{X}$ satisfies $\dot{\beta}(\tau) = \tilde{X}(\beta(\tau))$. We say that a curve β with given fixed position endpoints *stationarizes* the action (112) if its first variation (119) in β vanishes for all virtual displacement fields satisfying (115).

Theorem 15.1 (Hamilton's principle I)**.** *Let (M, ϑ, g) be the Galilean manifold of an exact finite-dimensional mechanical system and let ω_R be its Cartan one-form with respect to the reference field R, such that the action form of the system is given by $\Omega = \mathrm{d}\omega_R$. A time-parametrized curve β with fixed position endpoints $\gamma_0, \gamma_1 \in M$ stationarizes the action*

$$A\colon \mathcal{C}(\gamma_0, \gamma_1) \to \mathbb{R},\ \ \beta \mapsto A\big[\beta\big] = \int_{\beta(I)} \iota^*\omega_R \tag{120}$$

if and only if the curve β is a motion of the mechanical system.

Proof. By Postulate 1, a motion β of the mechanical system is an integral curve of the unique time-normalized vector field X on A^1M characterized by

$$\mathrm{d}\omega_R(X, Y) = 0 \quad \forall Y \in \mathrm{Virt}(A^1M)\,, \tag{121}$$

see (92). Consequently, the first variation in β of the action (120) vanishes for all virtual displacement fields satisfying (115) as X plays the role of $\tilde{X}$ in (119). Hence, a motion of the mechanical system stationarizes the action.

To show the converse, we assume that the time-normalized curve β stationarizes the action, i.e., the first variation (119) in β vanishes for all virtual displacement fields satisfying (115). We first show by reductio ad absurdum that it follows from the stationarity condition that

$$\mathrm{d}\omega_R(Y, \tilde{X}) \circ \beta(\tau) = 0 \quad \forall Y \in \mathrm{Virt}(A^1M) \text{ and } \forall \tau \in I\,, \tag{122}$$

where the virtual displacement fields Y must satisfy (115) and the tangent field of β is given by the vector field $\tilde{X}$ defined only along β. Without loss of generality, we assume that there exists a virtual displacement Y^* and a τ^* such that $\mathrm{d}\omega_R(Y^*, \tilde{X}) \circ \beta(\tau^*) > 0$. Since $\mathrm{d}\omega_R(Y^*, \tilde{X}) \circ \beta$ seen as a function of τ is a smooth real-valued function defined on $I \subseteq \mathbb{R}$, it follows by continuity that there exists an open subset $\tilde{I} \subseteq I$ containing τ^* such that

$$\mathrm{d}\omega_R(Y^*, \tilde{X}) \circ \beta(\tau) > 0 \quad \forall \tau \in \tilde{I}\,. \tag{123}$$

We choose an open neighborhood $W \subseteq A^1M$ of $\beta(\tau^*)$ such that $W \cap \beta(I) \subseteq \beta(\tilde{I})$. Moreover, we select a closed neighborhood $A \subset W$ of $\beta(\tau^*)$ and denote by $\psi \in C^\infty(A^1M)$ a smooth bump function for A supported in W defined as a real-valued function with the properties $0 \leq \psi \leq 1$ on A^1M and $\psi = 1$ on A.[12] Using the bump function to define the virtual displacement field $\tilde{Y} = \psi Y^*$, it follows from (123) and the C^∞-linearity of $\mathrm{d}\omega_R$ that

$$\mathrm{d}\omega_R(\tilde{Y}, \tilde{X}) \circ \beta(\tau) \geq 0 \quad \forall \tau \in \mathcal{I} := \beta^{-1}\big(W \cap \beta(I)\big) \subseteq \tilde{I}\,, \tag{124}$$

because $\psi \geq 0$. Since $\psi = 1$ on A, the inequality in (124) is strict for $\tau \in \beta^{-1}\big(A \cap \beta(I)\big)$. This, together with the fact that by construction $\tilde{Y} = 0$ on the complement of W, implies that

$$\delta A\big[\tilde{Y}\big] = \int_I \mathrm{d}\omega_R(\tilde{Y}, \tilde{X}) \circ \beta(\tau)\mathrm{d}\tau = \int_{\mathcal{I}} \mathrm{d}\omega_R(\tilde{Y}, \tilde{X}) \circ \beta(\tau)\mathrm{d}\tau > 0\,, \tag{125}$$

which is a contradiction to the stationarity condition and therefore proves (122) for virtual displacement fields Y that satisfy (115).

Since the vector field X characterized by (121) is unique, it follows from (122) that $\tilde{X} \circ \beta = X \circ \beta$. Hence, β is a motion of the mechanical system as it is an integral curve of the vector field X of Postulate 1.

□

In order to relate this version of Hamilton's principle to classical mechanics, we rewrite the action (120) as

$$A\big[\beta\big] = \int_I \beta^* \omega_R = \int_I \omega_{\beta(\tau)}\big(\dot{\beta}(\tau)\big)\mathrm{d}\tau\,, \tag{126}$$

where we have dropped the R for notational reasons, and derive its chart representation with respect to the natural and the canonical chart. For the natural chart (10), it holds by (24) that $\Phi \circ \beta(\tau) = \big(\tau, \mathbf{x}(\tau), \mathbf{u}(\tau)\big)$, such that

$$\dot{\beta}(\tau) = \frac{\partial}{\partial t}\bigg|_{\beta(\tau)} + \dot{x}^i(\tau)\frac{\partial}{\partial x^i}\bigg|_{\beta(\tau)} + \dot{u}^i(\tau)\frac{\partial}{\partial u^i}\bigg|_{\beta(\tau)}\,.$$

Therefore, the action (126) locally reads as

$$A\big[\beta\big] = \int_I \left[\mathrm{L}_R\big(\tau, \mathbf{x}(\tau), \mathbf{u}(\tau)\big) + \frac{\partial \mathrm{L}_R}{\partial u^j}\big(\tau, \mathbf{x}(\tau), \mathbf{u}(\tau)\big)\Big(\dot{x}^j(\tau) - u^j(\tau)\Big)\right]\mathrm{d}\tau\,, \tag{127}$$

where we have used (60) and introduced $\mathrm{L}_R := L_R \circ \Phi^{-1}$ to denote the chart representation of the Lagrangian defined in (58). The action functional (127) can be found for instance in [21, p. 531]. In the canonical chart $\tilde{\Phi}$, the motion has the representation $\tilde{\Phi} \circ \beta(\tau) = \big(\tau, \tilde{\mathbf{x}}(\tau), \mathbf{p}(\tau)\big)$, such that

$$\dot{\beta}(\tau) = \frac{\partial}{\partial \tilde{t}}\bigg|_{\beta(\tau)} + \dot{\tilde{x}}^i(\tau)\frac{\partial}{\partial \tilde{x}^i}\bigg|_{\beta(\tau)} + \dot{p}_i(\tau)\frac{\partial}{\partial p_i}\bigg|_{\beta(\tau)}\,.$$

Using this together with the representation (67) of the Cartan one-form, it follows from (126) that the action locally reads as

$$A\big[\beta\big] = \int_I \Big[p_i(\tau)\dot{\tilde{x}}^i(\tau) - \mathrm{H}_R\big(\tau, \tilde{\mathbf{x}}(\tau), \mathbf{p}(\tau)\big)\Big]\mathrm{d}\tau\,, \tag{128}$$

where $\mathrm{H}_R := H_R \circ \tilde{\Phi}^{-1}$ denotes the chart representation of the Hamiltonian introduced in (67). The action functional (128) is found in [4], p. 12; [14], p. 169; [21], p. 531; [25], p. 110 and [6], p. 354.

[12]The existence of bump functions is guaranteed by Proposition 2.25 in [18].

In order to derive a version of Hamilton's Principle which is often found in classical mechanics, we restrict the action (112) to second-order curves, i.e., to curves in the set

$$\mathcal{C}_{\mathrm{so}}(\gamma_0,\gamma_1) \coloneqq \left\{\beta \in \mathcal{C}(\gamma_0,\gamma_1) \;\middle|\; \beta = (\pi\circ\beta)^{\cdot}\right\}. \tag{129}$$

For such curves, the action locally reads as

$$\begin{aligned} A\big[\beta\big] &= \int_I \mathrm{L}_R\big(\tau,\mathbf{x}(\tau),\dot{\mathbf{x}}(\tau)\big)\mathrm{d}\tau \\ &= \int_I \Big[p_i(\tau)\frac{\partial \mathrm{H}_R}{\partial p_i}\big(\tau,\tilde{\mathbf{x}}(\tau),\mathbf{p}(\tau)\big) - \mathrm{H}_R\big(\tau,\tilde{\mathbf{x}}(\tau),\mathbf{p}(\tau)\big)\Big]\mathrm{d}\tau\,, \end{aligned} \tag{130}$$

where we have used that the second-order curve β fulfills the second-order conditions (23) and (105) in (127) and (128), respectively. As the action (130) is defined on second-order curves, the variational families used to compute the first variation of the action consist of second-order curves for any ε. Such variational families are induced by the virtual displacement fields $\mathrm{Virt}_{\tilde{Z}}(A^1M)$ defined in (84), where $\tilde{Z}$ denotes the second-order field of which β is an integral curve. Using this in the first variation (119) of the action leads to the stationarity condition

$$\delta A\big[Y\big] = \int_I \mathrm{d}\omega_R(Y,\tilde{Z})\circ\beta(\tau)\mathrm{d}\tau = 0 \quad \forall Y \in \mathrm{Virt}_{\tilde{Z}}(A^1M)\,, \tag{131}$$

of the restricted action (130), where Y fulfills condition (115).

Theorem 15.2 (Hamilton's Principle II)**.** *Let (M,ϑ,g) be the Galilean manifold of an exact finite-dimensional mechanical system and let ω_R be its Cartan one-form with respect to the reference field R, such that the action form of the system is given by $\Omega = \mathrm{d}\omega_R$. A second-order curve β with fixed position endpoints $\gamma_0,\gamma_1 \in M$ stationarizes the action*

$$A\colon \mathcal{C}_{so}(\gamma_0,\gamma_1)\to\mathbb{R},\;\; \beta\mapsto A\big[\beta\big] = \int_{\beta(I)} \iota^*\omega_R \tag{132}$$

if and only if the curve β is a motion of the mechanical system.

The proof is verbatim the proof of Theorem 15.1 except that the principle of virtual work (Theorem 12.2 (ii)) plays the role of Postulate 1 for the characterization of the motion of the mechanical system.

Looking at the local representation (130) of the action (132), we find that this version of Hamilton's principle corresponds to the one found in classical mechanics. The representation (130) in both charts is found in [10] on pp. 98–99. Moreover the natural chart representation is for example found in [16], p. 2 or [6], p. 35 f. However, here the correspondence lies in the eye of the beholder, as the action found in these texts is defined without specifying the set of functions it takes as arguments.

The arguably most prevalent version of Hamilton's principle in classical mechanics uses that the action of a mechanical system can also be defined on the set of time-parametrized curves $\gamma\colon I = [\tau_0,\tau_1] \to M$ with fixed position endpoints $\gamma_0,\gamma_1 \in M$ given by

$$\mathcal{D}(\gamma_0,\gamma_1) \coloneqq \left\{\gamma \text{ time-param.} \;\middle|\; \gamma(\tau_0)=\gamma_0,\ \gamma(\tau_1)=\gamma_1\right\}. \tag{133}$$

By definition, a second-order curve β on A^1M is the tangent field to the time-parametrized curve $\gamma = \pi\circ\beta$ on M. We use this in (132) to define the action of a mechanical system

$$A\big[\gamma\big] \coloneqq \int_{\dot\gamma(I)} \iota^*\omega_R = \int_I \mathrm{L}_R\big(\tau,\mathbf{x}(\tau),\dot{\mathbf{x}}(\tau)\big)\mathrm{d}\tau\,, \tag{134}$$

on the set $\mathcal{D}(\gamma_0,\gamma_1)$. Such curves have the representation $\phi\circ\gamma = \big(\tau,\mathbf{x}(\tau)\big)$ with respect to an adapted chart.

Since the one-parameter family $\varphi_\varepsilon^{\bar Y}\circ\gamma$ induced by a virtual displacement field $\bar Y\in \mathrm{Virt}(M)$ on M is a one-parameter family of time-parametrized curves on M, the first variation of (134) at γ is

$$\delta A\big[\bar Y\big] = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0}\int_{(\varphi_\varepsilon^{\bar Y}\circ\gamma)\dot{}(I)} \iota^*\omega_R\,, \tag{135}$$

where we essentially used (116). Similar arguments as used for (115) lead to the conditions

$$\bar Y(\gamma_0) = 0 \quad\text{and}\quad \bar Y(\gamma_1)=0 \tag{136}$$

on the virtual displacement field $\bar Y\in\mathrm{Virt}(M)$ such that its induced one-parameter family lies in $\mathcal{D}(\gamma_0,\gamma_1)$. Denoting the complete lift of $\bar Y$ with $\hat Y$ and using condition (89), we obtain from (135) the stationarity condition

$$\delta A\big[\bar Y\big] = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0}\int_{\varphi_\varepsilon^{\hat Y}\circ\dot\gamma(I)} i^*\omega_R = \int_I \mathrm{d}\omega_R(\hat Y,\tilde Z)\circ\dot\gamma(\tau)\mathrm{d}\tau = 0 \tag{137}$$

for all $\bar Y\in\mathrm{Virt}(M)$ which fulfill condition (136). The derivation of the last representation of the first variation in (137) follows the lines of (117)–(119) and uses the fact that by construction we have $\dot\gamma\in\mathcal{C}(\gamma_0,\gamma_1)$ if $\gamma\in\mathcal{D}(\gamma_0,\gamma_1)$.

Theorem 15.3 (Hamilton's Principle III)**.** *Let (M,ϑ,g) be the Galilean manifold of an exact finite-dimensional mechanical system and let ω_R be its Cartan one-form with respect to the reference field R, such that the action form of the system is given by $\Omega=\mathrm{d}\omega_R$. The time-parametrized curve γ with fixed position endpoints $\gamma_0,\gamma_1\in M$ stationarizes the action*

$$A\colon \mathcal{D}(\gamma_0,\gamma_1)\to\mathbb{R},\ \gamma\mapsto A\big[\gamma\big] = \int_{\dot\gamma(I)} \iota^*\omega_R, \tag{138}$$

if and only if the curve $\beta=\dot\gamma$ is a motion of the mechanical system.

The proof follows the lines of the proof of Theorem 15.1. Here, the principle of virtual work (Theorem 12.2 (iii)) plays the role of Postulate 1 for the characterization of the motion of the mechanical system.

This version of Hamilton's principle with the local representation (134) of the action is usually found in classical texts. See for example [16], p. 2; [6], p. 35; [1], p. 59; [21], p. 35; [25], p. 107; [9], p. 235; or [4], p. 10.

16. **Conclusion.** We presented a geometric theory for time-dependent finite-dimensional mechanical systems which may be subjected to nonpotential forces. The theory is formulated on a Galilean manifold (M,ϑ,g) modeling the generalized space-time. The state space containing the system's information on time, position and velocity is defined as the affine bundle A^1M. The fundamental idea behind the theory is that a differential two-form with the properties from Theorem 4.1, i.e., an action form, uniquely characterizes a second-order vector field whose integral curves are the motions of the system. We have seen that any action form Ω can always be attributed to a Lagrangian and a nonpotential force two-form. This led

us eventually to the formulation of Postulate 1, which is considered here as the fundamental law for time-dependent finite-dimensional mechanical systems. Starting from Postulate 1, we derived alternative coordinate-free principles that traditionally are used as fundamental laws in dynamics. This geometric framework allowed us to recognize if two classical principles indeed differ by the mathematical objects they involve or if they are just two chart representations of the same principle.

With Theorem 12.2, we proved the equivalence between the principle of virtual work and Postulate 1. In this theorem, the principle of virtual work is given for three different types of virtual displacement fields, all of which lead to the same indistinguishable coordinate representation of the virtual work. Striving to present other classical equations, we could show that these three types lead to different classical principles when represented in coordinates. Indeed, two types of virtual displacement fields can respectively be distinguished in the central equation of Lagrange and in Hamel's generalized version thereof.

The study of the variational families related to three different types of virtual displacement fields led to the formulation of three versions of Hamilton's principle. Their equivalence to Postulate 1 is established by Theorems 15.1–15.3. These three principles are traced back to six versions of the principle of Hamilton found in classical mechanics.

If one is willing to pay the price of a more involved mathematical framework, the proposed geometric theory leads definitively to a deeper understanding of the theory of time-dependent finite-dimensional mechanical systems and notably allows a definition of forces. While the coordinate-free theory comprises the Lagrangian and the Hamiltonian approaches as mere chart representations, it enables to precisely distinguish concepts which appear as similar in the classical formalisms.

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