Exegesis of the Introduction and Sect. I from "Fundamentals of the Mechanics of Continua"* by E. Hellinger

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The fundamental review article DIE ALLGEMEINEN ANSÄTZE DER MECHANIK DER KONTINUA in the Encyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, Bd. IV-4, Hft. 5 (1913) by Ernst Hellinger has not been translated into English so far. We believe that such a circumstance is really deprecative, as the insight reached by Hellinger in the mathematical structure of continuum mechanics seems, in some aspects, unsurpassed even nowadays. Hellinger's scientific manuscripts do not fill more than one and half boxes in library storage [2], but their impact on mathematics and mechanical sciences is profound. Indeed, the Hellinger-Reissner variational principle is still a fundamental tool in theoretical and numerical mechanics. The intent of this paper is threefold: i) to allow to those who cannot understand German to enjoy the reading of a crystal-clear and still topical article whose content has some enlightening parts, ii) to show that only one century ago the principle of virtual work (or virtual velocities) was regarded as the central principle in continuum mechanics and that Hellinger did forecast already then the main lines of its development, iii) to discuss some technical and conceptual aspects of the variational principles in continuum mechanics which some authors consider still controversial.

1 Introduction

In this paper the presented English translation of the original text written in German by Hellinger is typed in *italic* and is indented on both sides. The comments and explanatory remarks are in standard text style. Naturally, the main burden of the translation from German into English has been on the stronger shoulders of the younger author, whose mother language is a Germanic one. However, the revisions and the analysis of logical coherence of the translated text are responsibility of both authors, who must, therefore, share for this work criticism and, hopefully, credit. We tried to produce a word-by-word translation totally refraining from allowing us to include any comment or interpretation of the original text because of translation.[1] When we were obliged to introduce a word which had no correspondence in the German text we included it in square brackets as follows: [xxxx].

It seems to the authors that, for some reasons to be understood and studied, a well-known phenomenon of degeneration of scientific knowledge can be observed also in modern times and in some contemporary scientific ambients and groups: exactly this phenomenon was described in the enlightening monograph by Lucio Russo [82] referring to the development and decadence of science during the Hellenistic and subsequent cultures and civilizations. One fundamental cause[2] of the erasure of some parts of science during its transmission from one generation to the following ones consists in the presence of changes in the universal communication language used by scientists. In other words, every time the *lingua franca* of science is changed, some relevant losses of important bodies of knowledge may occur.

Russo describes in detail what happened when Greek was abandoned as a universal scientific language, and when Arabic and Latin, among the others, competed to replace it. Similarly, when Latin was abandoned as universal scientific

* DIE ALLGEMEINEN ANSÄTZE DER MECHANIK DER KONTINUA. Encyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. Bd. IV-4, Hft. 5 (1913).

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[1] Soon a side-by-side translation of the whole article will be published by the same authors and the reader who can understand German hopefully will agree with the previous statement.

[2] However, it seems to us that other concauses can be recognized: further investigations will concern their discussion.

language, only after a long competition among many national languages, English has become the new *lingua franca*. When Hellinger wrote his article, Latin language (notwithstanding the efforts of Giuseppe Peano [72] to establish the "latino sine flexione") had been abandoned definitively.[3] As a consequence the well-educated scientist needed to know several languages, including at least German, French and English. Until well into the 20th century, Russians tried to impose also their language, and even Italians struggled for continuing to use their one (see the discussion about the destiny of the works of Gabrio Piola, strongly influenced by this effort [29], [25]). For several reasons, in the end English has become dominant: consequently, many important contributions written in "lost" languages were erased or misunderstood or were rediscovered, i.e. published in English as if they were original.

We share completely the idea of history of science presented by Lucio Russo [82]: it seems to us that science, human knowledge and technology are not developing by simple additions, improvements and progress, so that one can state that surely more modern textbooks, theories and civilizations are bound to be more advanced than the preceding ones. Actually, science is moving some steps forward and some steps backward (see [82]), periods with higher scientific culture are followed by more primitive ones, and, for making the described phenomena more complex, in one stage of history the scientific knowledge may be declining while the technological byproducts of the previous more advanced scientific stages may be causing a great development of economy and life-style. This is, in Russo's opinion, the situation which occurred during the rise of Roman Empire, whose decadence started exactly when Hellenistic Science was being forgotten while Hellenistic technology was still in use and, in some aspect, even continuing to be improved (see again [82]).

Russo's views seem to us to be similar to the ideas of Giambattista Vico. Vico was a Neapolitan philosopher whose contributions to epistemology and theory of history (see e.g. [24][4]) were, for a long period, known only to the restricted circle of Neapolitan intellectuals: it is ironic that the reasons for which his ideas were for so long time ignored by the mainstreams of philosophical thought can be investigated by the same methods which Vico himself envisioned. Vico discusses the evolution of societies and nations, but his ideas are without any doubt applicable also to the social groups constituted by scientists, which may be organized into universities or, more loosely, into schools of intellectuals. Vico introduces the concept of "corso"[5]: i.e. an organizational stage of human societies. He also conceives the concept of "ricorso" (see again [24][6]): i.e. a regressive stage of the societies experience after a progressive one.

The historical stage in mechanical sciences which is very well described by Hellinger's article, suffered a "ricorso" where the variational ansatz was rejected as logically inconsistent and not physically well-grounded (see [26] for a detailed discussion of this point). Our intention, also with this paper, is to contribute to the return to the ancient "corso", which although it is anterior seems to us to be more advanced.

The destiny of the nice piece of work by Hellinger, which we translate here, is undoubtedly related to the many tragedies occurring in Europe in the 20th century: Hellinger escaped from Dachau (see [63]) and could proceed his academic career with many difficulties in the USA; however this destiny is also related to the change in the dominant scientific language. Some English speaking authors managed to express their (in our opinion wrong) point of view with greater momentum, compared with the style of an old-fashioned German Jewish Professor, who was not aware of (or wanted to ignore) the fact that also scientific ideas need to be advertised.[7]

We want to "rediscover" Hellinger's point of view about continuum mechanics by offering to the wider audience of English readers the commented translation of his DIE ALLGEMEINEN ANSÄTZE DER MECHANIK DER KONTINUA, as a tribute to a great mind and a continuator of Archimedean and more generally Hellenistic scientific spirit, which joins physical understanding with rigorous mathematical practice. We are happy to have conceived this challenge: Hellinger's pedagogical style, his clear understanding of mechanics (obtained by his deep understanding of mathematics) improved our own understanding of the subject, and we believe that the same will happen to the interested reader.

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[3] For instance, the first professor in the Università di Napoli who gave his lectures in Italian seems to have been Antonio Genovesi starting from 1754.

[4] In [24] one reads: «"Thus our Science", Vico says near the beginning of his ("Poetic Wisdom," §368, p.112) "comes to be at once a history of the ideas, the customs, the deeds of mankind. From these three we shall derive the principles of the history of human nature, which we shall show to be the principles of universal history, which principles it seems hitherto to have lacked".»

[5] In [24] one reads that Giambattista Vico is a «philosopher who, by deciphering and thus recovering its content, can discover an "ideal eternal history traversed in time by the histories of all nations" (Proposition XLII, §114, p.57).» and that «The result of this [discovery], in Vico's view, is to appreciate history as at once "ideal" – since it is never perfectly actualized – and "eternal," because it reflects the presence of a divine order or Providence guiding the development of human institutions. Nations need not develop at the same pace – less developed ones can and do coexist with those in a more advanced phase – but they all pass through the same distinct stages (corsi): the ages of gods, heroes, and men. Nations "develop in conformity to this division," Vico says, "by a constant and uninterrupted order of causes and effects present in every nation" ("The Course the Nations Run," §915, p.335).

[6] Again in [24] one reads that «Although from a general point of view history reveals a progress of civilization through actualizing the potential of human nature, Vico also emphasizes the cyclical feature of historical development. Society progresses towards perfection, but without reaching it (thus history is "ideal"), interrupted as it is by a break or return (ricorso) to a relatively more primitive condition.»

[7] It is lucky that his sister Hanna managed to save him from Dachau and did what she could to have her brother's contributions to science to be recognized.

2 Annotated translation of No. 1 and No. 2 (pp. 602–611)

In a nutshell, Hellinger's style can be classified as Tacitean: sharp, concise and meaningful.[8]

2.1 Hellinger's introductory remarks

At the very beginning, Hellinger emphasizes that he will only consider mechanical systems whose spaces of configurations are infinite dimensional. As a mathematician he stresses this feature of continua, which in his eyes appears (rightly) as the most important structural property to be remarked.

> ***1. Introduction.*** *With respect to a [mathematically] consistent point of view, the paper at hand shall give a recapitulatory overview on various* ***forms*** *of the axiomatic foundations, which in the particular fields of the "mechanics of continua" in the broadest sense, i. e. the mechanics and physics of continuously extended media, enable the determination of the time behavior or also the state of equilibrium of the analyzed processes; thereby these continua are kept in mind, which, due to any restricting conditions, in particular are not reduced to continua with finitely many degrees of freedom.*

The reader will notice that in Hellinger's mind there is a clear place for the theory of extended media: the need and even the logical possibility of introducing such media has been considered controversial for many years after Hellinger's article (the interested reader is referred to [26] for a discussion of this point).

> *The possibility, to bring the fundamental equations of various theories into similar forms, has been noticed soon: the "mechanical" theories of physics, which try to explain the physical phenomena only by the motion of matter, contain from a formal-mathematical point of view the idea that the equations of physics appear as special cases of equations of a general system with moving masses or mass points; thus such [mechanical] theories must generate those analogies.*
>
> *Besides the intrinsic mechanical theories, which put more or less detailed images of the constitution of matter at the basis [of their formulation], one has, partially in the beginning, but in a wider extent in the mid-19th-century, developed a new path following J. L. Lagrange's analytical mechanics; indeed as in there all analyzed problems are based on a few very general principles, one has tried to bring the foundations of more and more physical disciplines into the form of those principles, by identifying the appearing quantities — energy, forces, and so on — with certain physical quantities [previously introduced] from a purely phenomenological point of view. For systems with finitely many degrees of freedom this development is presented in particular in the analysis inaugurated by W. Thomson (Lord Kelvin), J. J. Thomson and H. v. Helmholtz on cyclic systems with its applications and on the reciprocity theorems of mechanics.*

In the previous sentences, Hellinger clearly distinguishes reductionist modeling from direct (phenomenological) modeling: he is aware of the fact that both approaches are possible and consistent. Reading them one understands also that some interesting (and advanced) results concerning finite dimensional micro-models for mechanical systems are already available in 1913.

> *Already Lagrange applied his principles directly to certain continuous systems (fluids, flexible wires and plates and similar ones)[9]; in connection with the further developments of these fundamentals, particularly with that one concerning the development of the theory of elasticity by following A. L. Cauchy[10], as well as under the influence of the extension of other physical, especially optical theories, one got increasingly accustomed to consider a continuous system as an independent object of study in mechanics (with infinitely many degrees of freedom), which has to be in formal analogy to the well-known point mechanics, but which can be treated independently. This "mechanics of the deformable continuum" developed as an independent discipline, contains, due to the formal approaches [used], in addition to the common theory of elasticity and hydrodynamics all physical phenomena which are accounted for in [the theory of] continuously extended media.*

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[8] We find that to Hellinger's style some of the comments about Tacitus' can be applied which can be found in [70] being available also online. «While it is important to bear in mind F. R. D. Goodyear's point that Tacitean style is protean (both across his oeuvre and within a single work) and his writings constitute an 'endless experiment with his medium, the discontent with and reshaping of what had been achieved before, the obsessive restlessness of a stylist never satisfied that he had reached perfection', it is nevertheless possible to identify some pervasive features that are also amply on display in the set text: (a) [. . .] the name of Tacitus' game is brevity (brevitas) [. . .] (b) [. . .] Tacitus delights in the unusual lexical choice (c) [. . .] Tacitus goes for disjunctive varietas. His 'studied avoidance of syntactical balance and the pursuit of asymmetry' is in evidence [...]»

[9] *cf. especially 1. part., sect. IV, § II of "Mécanique analytique".*

[10] *Crucial were his analyses on the notion of stress from 1822 (Bull. de la Soc. philom. 1823, p. 9). For further details see IV 23, No.* ***3****a, Müller-Timpe.*

Hellinger recognizes here that the theory of continuous media includes and generalizes the theory of elasticity and hydrodynamics: the battle for having this logical hierarchy recognized in engineering and mathematical curricula is still in progress. Actually (and unfortunately!), Hellinger's point of view is in the present historical "ricorso" loosing some positions. It is remarkable that many scholars many years after Hellinger rediscovered this point of view, claimed to have conceived it. One has also to remark that Hellinger attributes to Cauchy the merit of having established the notion of stress: Hellinger, however, spends no word to describe Cauchy's foundational ansatz for mechanics.

> *The advancement of this theory has been influenced significantly by **thermodynamics**, which aims in principle to cover the entire field of physics, and by considering the energy function or rather the potential as the most fundamental concept, naturally yields the fundamental equations of various specific fields in similar forms.*

Therefore, it is clear that the concept of Continuum Thermodynamics was already formulated in Hellinger's scientific milieu.

> *All these equations have been treated in the mechanics and physics literature in many cases; much that has been stated explicitly in point mechanics or for systems with finitely many degrees of freedom, can immediately be extended to continuous systems. At a preeminent place, just the names of a few authors are mentioned, which have especially considered the relations treated here which often show to be useful in the following: W. Voigt*[11]*, P. Duhem*[12] *and E. and F. Cosserat.*[13,14]
>
> *The objective of this paper requires that the purely **formal**-mathematical aspects must have priority in what follows: The formulation of the **ansatz** of various problems as well as their collection to a unified and at most simple and convenient formula. Both the analysis of **mechanical** and **physical** interpretation of the quantities and equations as well as the essential **analytic**-mathematical theory of the particular disciplines are covered by various papers in the volumes IV and V.*

Here the Archimedean point of view accepted by Hellinger is formulated shortly but exactly. He reduces the complete process of founding a physical theory to Duns Scotus' statement: "Pluralitas non est ponenda sine necessitate" ("Plurality is not to be posited without necessity") which is actually also known as the law of parsimony or Occam's razor. Also in this idea, we trace some Greek antecedents, as such a prescription is attributed by Proclus to Pythagoreans in his "Hypotyposis astronomicarum positionum" Chapter 1, section 34.

> *As unifying **mathematical** form, in which all individual [methodological] fundamentals are included in the easiest, the **variational principle** is applied.*

With this statement, paraphrasing Gabrio Piola's words (see [16, 25, 26, 29]), Hellinger "accepts as our schoolmaster Lagrange".

> *Although, the form commonly considered in the calculus of variations, in which the unknown function has to to be determined such that a certain definite integral, containing the function, has an extremum, is not adequate. On the contrary, it concerns here particularly the form, which the calculus of variations yields as the necessary criterion of the extremum, and the form in which **the principle of virtual work** is expressed of old: "Given a series of quantities $X, \ldots, X_a, \ldots$ depending on the unknown functions $x, \ldots$ on $a, \ldots, c$ and on the derivatives thereof; these functions shall satisfy the condition, that a definite integral of a linear form on the arbitrary functions $\delta x, \ldots$ of $a, \ldots c$ and their derivatives composed with those [functions] $X, \ldots, X_a, \ldots$ as coefficients*
>
> $$\int \cdots \int \left\{ X\delta x + \cdots + X_a \frac{\partial \delta x}{\partial a} + \cdots \right\} da \cdots dc$$
>
> *— or the sum of such integrals — vanishes identically for all (or however for all constraint satisfying) $\delta x, \ldots$.*

This point seems still not to be accepted by some contemporary scholars. Even if it is clearly stated by Hellinger, we want to reinforce his statement in the following lines. The calculus of variations is a mathematical theory whose aim is to find extrema for functionals, usually expressed by means of integrals. To find these extrema one can calculate the first variation of the integral operators involved, by obtaining some linear functionals of the variations of the unknown fields.

[11] *Besides many individual works especially in his compendium of theoretical physics, 2 Bde., Leipzig 1895/96.*

[12] *In numerous works to cite later on; cf. also his Traité d'énergétique ou de thermodynamique générale, t. I. II, Paris 1911.*

[13] *Cf. "théorie des corps déformables" (Paris 1909), appearing as appendix to the french edition of O. D. Chwolson, Traité de physique, and which is added partially as a note to the 2. Edn. of the 3. vol. of P. Appell Traité de mécanique rationelle (Paris 1909).*

[14] *The following is influenced in many ways also from similar developments treated in some of the Göttinger lectures of D. Hilbert.*

To base continuum mechanics on an extremum principle may be regarded as a too hazardous choice. Therefore, following Lagrange, we prefer to base the postulation of mechanics by formulating a principle HAVING THE FORM of the necessary criterion for being an extremum. This point is rather abstract, but its implications have a marvelous impact, allowing for a very general postulation of physical theories. This postulation is based on the principle of virtual work or equivalently on the principle of virtual velocities. Remark, that this principle has been attributed again to a Pythagorean philosopher: Archytas of Tarentum (see [94]).

Therefore, as done by Lagrange and Piola, Hellinger calls "Variational Principle" also a principle formulated in a formal way similarly as the necessary condition for being an extremum, even if in the postulation process there is no explicit consideration of any functional to be minimized.

> *The advantage, which the application of such a variational principle as foundation allows versus other possible formulations or also the direct consideration of the fundamental equations, consists especially therein that the variational principle is capable to determine the behavior of the considered media in all points and to* ***every*** *instant of time in a* ***single*** *formula, in particular to contain besides the equations for the interior also the* ***boundary conditions*** *and the* ***initial conditions****.*

In these words the echoes from the works of Gabrio Piola are striking: Piola talks about "di quella formula una dalla quale discendono innumerevoli verità" i.e. that unique formula from which one can deduce innumerable truths. Remark that Hellinger accepts in a paper dated 1913 a statement as obvious which is still nowadays denied by some authors. This statement is the following: when formulating a variational principle (in the wider sense given to this expression by Lagrange and Piola, but also when accepting the more restrictive sense considered by Hamilton) one gets "for free" the required boundary conditions. Note that those who refuse variational principles sometimes claim that boundary conditions need to be determined on "physical grounds": in this way they seem to contravene the law of parsimony.

> *Furthermore from a certain point of view, it is in its concise brevity clearer than the equations and consequently it is [more suitable] for the treatment of new fields, for the formulation of further generalizations and it is therefore of essential* ***heuristic*** *relevance;*

Here clearly the Pythagorean spirit of Hellinger emerges: the heuristic power of a method strongly based on firmly based mathematical methods is obvious in his eyes. The economy of thought allowed by the application of Occam's razor is to him the only tool which allows for the discovery of new theories and formulation of new models.

Hellinger's point of view was not isolated in German speaking epistemology: his appeal to "concise brevity" is completely coherent with the concept of "economy of science" by Ernst Mach.[15] Mach's positivistic views are greatly influenced by the Vienna Circle and by Ludwig Wittgenstein, whose rigorous style (including the resuming synoptic side notes in their monographs) is adopted also by Mach, when dealing with the history of mechanics. On page 481 of [60], having as a synoptic side note "The basis of science, economy of thought." one can read:

> «It is the object of science to replace, or save, experiences, by the reproduction and anticipation of facts in thought. Memory is handier than experience, and often answers the same purpose. This economical office of science, which fills its whole life, is apparent at first glance; and with its full recognition all mysticism in science disappears. Science is communicated by instruction, in order that one man may profit by the experience of another and be spared the trouble of accumulating it for himself; and thus, to spare posterity, the experiences of whole generations are stored up in libraries.»

Maybe Mach's most important statement starts at the bottom of page 489 of [60]:

> «But, as a matter of fact, within the short span of a human life and with man's limited powers of memory, any stock of knowledge worthy of the name is unattainable except by the greatest mental economy.»

It seems to us that this greatest mental economy has been reached in the presentation by Hellinger and can be reached in mechanics only when working with variational formulations.[16]

[15] See e.g. Section 4 of Chapter IV of the book "The science of mechanics; a critical and historical account of its development", [60].

[16] Unfortunately, nowadays variational principles do not seem to be trendy, and the mystic appearance from nowhere of unknown and unexplained theories seems to be the preferred approach. Deep considerations about this point can be found in [82] pages 391-397. Russo starts them by stating: «The closer we get to the deepest aspects of Hellenistic science, which are the methodological ones, the longer they took to reappear. One important methodological step in the evolution of modern mechanics was the introduction of variational principles, which correspond to ways to formulate a dynamical problem not as a search for solutions of ordinary differential equations with chosen initial conditions (Cauchy problems) but as a search for minimum points of appropriate functionals. Instead of deducing the future from the past (a process regarded as causal, if only unconsciously), the variational formulation in principle allows the whole motion to be obtained simultaneously. This "radically new" way of setting problems was derived from its first attested example, transmitted by Heron of Alexandria and having to do with optics. It was natural to draw ideas from Hellenistic science in trying to formulate the advances of modern dynamics within the lucid geometric framework that Archimedes used for the creation of mechanics.»

this [circumstance] is emphasized especially through the profound relation of the variational principle to thermodynamics, as through requirement of generality it gains evident value for the foundations of physical theories. Also for the evaluation of ***coordinate transformations*** *the variational principle is in advantage versus the explicit equations; in many cases it is possible to identify the* ***invariance*** *of the considered problem, i. e. the question of the transformation group letting the problem unchanged, easier and without the requirement to introduce a specific symbolism. —*

Note that many have argued against the variational principles as they seem to be in contradiction with Thermodynamics. In contrast, Hellinger recognizes here that the logical unity of all physical theory is obtained by means of variational principles (this is also the point of view that Landau presents in his celebrated textbook on Quantum Mechanics [55]).

Subsequent to some introductory discussions about the notion of a continuum and the kinematics thereof, in the ***first*** *section of the paper the foundations of* ***statics****, in the* ***second*** *the foundations of* ***kinetics*** *are treated, each time without any consideration of what classes force effects influence the continuum in particular. The nature of these force effects, especially their dependence on the position and the motion of the continuum (****dynamics****) is discussed in the* ***third*** *section, whereat the individual disciplines are classified; in this connection, on the one hand the relation to the methods of thermodynamics, on the other hand the behavior of the individual constitutive laws with respect to transformations of space-time-coordinates and thereby the concept of the* ***theory of relativity of electrodynamics*** *eventually are profitably presented together in a short outline.*

Hellinger concludes this section adopting the Lagrangian scheme: kinematics, statics and kinetics (as he calls dynamics). He underlines that he is able to develop his presentation for every kind of possible applied forces (no assumption about them being conservative!) and that he can establish the due links with the concept of invariance and the basic ideas of thermodynamics.

In a few lines Hellinger clarifies his philosophical and epistemological point of view. He answers clearly to essentially all objections always repeated against the postulation based on variational principles and gives a wonderful example of an elegant and clear pedagogical text. In a few pages he expresses much more than what is written by many in hundreds of pages. Tacitean indeed.

2.2 How Hellinger introduces the concept of a continuum.

We discuss next the notion of a continuous body as described by Hellinger. It has to be remarked that there are some indications that the ultimate source of Hellinger, also for what concerns this part of his work, is Gabrio Piola. For instance, the coordinates in the reference configuration for the generic particle are indicated by Hellinger with the triple (a, b, c) which is exactly the notation chosen by Piola. From the philological point of view this is not an evidence: however it can be for sure regarded as a meaningful clue.[17]

Piola considers a continuum model for a material as an approximation needed to deduce results with tractable mathematics. Piola's idea is simple: the "true" (or most accurate) mathematical model for matter is given by a discrete molecular theory. However, the problems to be solved in using this theory directly are "formidabili" (i.e. formidable). Therefore Piola suggests to homogenize the discrete micro-theory and to deduce the most suitable macro theory according to the following steps: i) by formulating the principle of virtual work at micro and macro levels, ii) by specifying the "most likely" micro-motion once a macro-motion is chosen and iii) by identifying micro with macro expended (virtual!) works. It is then the macro theory which he hopes to use for formulating and solving deformation problems of interest in applications. A concept that is being rediscovered and applied with great difficulty in recent engineering literature dealing with granular media such as [64,66,67]. To aim this, Piola intends to use the methods of mathematical analysis (as the method of solution by separation of variables or by series of PDEs). It is ironic that in modern times, instead, the equations deduced by Piola (those of higher gradient continua) are rendered discrete by introducing suitable finite elements or even so-called molecular dynamics.

It has to be remarked that in the "corso" of history of science, during which Piola was a protagonist, the molecular theory of matter was regarded as "fundamental" while in contrast the theory of continua was regarded as a "computational tool". In the subsequent "ricorso", few years after Piola's flourishing period, Boltzmann's more rigorous homogenization methods were bitterly criticized and opposed (see [23] for the most careful monograph about this subject). Indeed during the "ricorso", where Boltzmann was the main character on the scene, the Hertzian view about the intrinsic "continuum" nature of matter was (temporary) dominant. It is remarkable how Hellinger manages to describe rigorously and precisely continuum mechanics without being involved in such controversies.

[17] We do not know how well Hellinger could read Italian. However, we know that Piola's Italian style is difficult and very elaborate (as imposed by the habits of the academic rhetorics at his times, see [29]) and we do not believe that it could have been easy for Hellinger to master Piola's works.

2. The notion of a continuum.

2a. The continuum and its deformation. *The general* ***three-dimensional extended continuous medium****, on which the following presentation relates to, stands — under abstraction of all more specific properties of matter — for an aggregate of material particles, which first of all are* ***distinguishable*** *from each other and second which* ***continuously occupy*** *the space or rather a continuously bounded part of the space. The first property finds its expression [by assuming] that every particle is identified by the specification of three variable values a, b, c such that different particles always have different positions in every state in which the continuum can be considered in any case; [and that] the domain of variability V_0 of these a, b, c, bounded by the continuous and closed surfaces S_0, characterizes the portion of matter, which is taken into consideration. The second requirement states, that the positions of all particles occupy a part of the space V bounded by a continuous and closed surface S. Determining the position of a particle by cartesian coordinates, analytically such a state is given by three functions of a, b, c,*

$$x = x(a, b, c), \quad y = y(a, b, c), \quad z = z(a, b, c), \tag{1}$$

mapping V_0 to V, and by their Jacobian

$$\Delta = \frac{\partial(x, y, z)}{\partial(a, b, c)} \tag{2}$$

being different from zero within V_0, for instance positive. For a, b, c one can take the coordinates of a fixed chosen initial position; then $x - a$, $y - b$, $z - c$ are the components of the displacement, which every particle undergoes by shifting them to the position (1), and the functions (1) become ***continuous*** *functions of a, b, c, as long as the common assumptions are taken, that initially neighboring particles always remain neighboring. Moreover, we will always assume the functions (1) to have sufficiently many continuous difference quotients with respect to their arguments; only at individual points, lines or surfaces, discontinuities may occur (cf. IV 1, No.* ***9****, Voss). In general, we will have to make the same assumptions tacitly for the upcoming functions which describe physical processes.*

In a few lines Hellinger resumes in an elegant and rigorous way the kinematical assumptions which characterize the space of configurations used for continuous bodies. Indeed:

Every system of functions (1) describes entirely a certain ***state of deformation*** *of the continuum; in general* ***every*** *state of deformation, i. e. every triple of functions (1), which satisfies the just characterized continuity assumptions, is admissible; Restrictions on the class of possible functions express particular properties of special media. In any case, the partial derivatives of the functions (1) assign in the well-known manner the displacements, the rotations and the shape change, which each very little portion (volume element) undergoes during its deformation (cf. IV 14, No.* ***16****, Abraham).*

Next Hellinger introduces the tangent space of the previously introduced space of configuration:

We obtain the basis for the analysis of the equilibrium conditions of an arbitrary state of deformation (1), by superimposing it with a so-called ***infinitesimal virtual displacement****, called* ***virtual****, since it is added arbitrarily to the actually occurring state of deformation.*[18]

For the English "displacement", in German there are the two synonyms "Verschiebung" and "Verrückung". While "Verrückung" is a rather old-fashioned word, nowadays it is more common to use "Verschiebung". Nevertheless, throughout his paper, Hellinger distinguishes between actual and virtual displacements by attributing to them the words "Verschiebung" and "Verrückung", respectively. This underlines once more the Tacitean style of Hellinger in being extremely precise, even in the choice of his words. In modern literature mainly the word "Verschiebung" is in use.

Footnote 18 makes apparent that even Gauss has considered the principle of virtual work as fundamental tool in mechanics. In addition, Hellinger touches in there the subject of admissible (or compatible) and non-admissible (or incompatible) virtual displacements in the context of constraints. We refer to [49, Sect. 3.6] for a detailed discussion on this topic.

Some reader may consider the introduction of the modern concept of "space of configuration" and "tangent space" exaggerated which we have evoked in the previous comments. However, the following sentences prove that Hellinger is perfectly aware of what is happening in functional analysis during the historical "corso" which he witnesses. Indeed what follows is exactly the technical definition of the Gâteaux derivative adapted to the present context.

[18] *Thus in coincidence with the terminology of Voss (IV 1, No.* ***30****), which is also often common in textbooks. Others (e. g. Voigt, Kompendium I, p. 27) speak of "virtual" displacements only when the otherwise arbitrary displacements are admissible with respect to any constraints of the system; C. Neumann (Ber. Ges. Wiss. Leipzig 31 (1879), p. 53 ff.) occasionally has adopted the suggestion of Gauss, to speak then of* ***optional*** *displacements.*

To obtain this notion in a mathematical rigorous way, without dropping the convenient and common expression and application of "infinitesimal" quantities, one considers at first a deformation, depending on a parameter σ, superimposed on the deformation (1) which vanishes for $\sigma = 0$ and shifts the particle being at the initial position (x, y, z) to the position

$$\bar{x} = x + \xi(x, y, z; \sigma) \qquad \begin{pmatrix} x, y, z \\ \xi, \eta, \zeta \end{pmatrix};[19]$$

thereby ξ, η, ζ are known functions of x, y, z and of the parameter σ, which varies in an (arbitrary small) surrounding of $\sigma = 0$. Using (1) for the elimination of x, y, z, one can also write these new arising deformations in the old form:

(3) $$\bar{x} = \bar{x}(a, b, c; \sigma), \quad \textit{where} \quad \bar{x}(a, b, c; 0) = x \qquad (x, y, z).$$

Before proceeding in the reading of Hellinger's lucid exposition, one has to remark that the true structure of tensorial notation is already fully present at this time. Ricci and Levi-Civita have published their fundamental paper in 1900 (see [79]). Hellinger is aware of their results. Nevertheless, he prefers an intermediate notation between the fully tensorial one and the most familiar one based on components. The reasons of such a choice can be understood and maybe, with the wisdom of hindsight, also criticized.

Let f be any expression depending on the deformation functions (1) and their derivatives, then we generally denote its "variation" by the expression

$$\delta f(x, \ldots, x_a, \ldots) = \left\{ \frac{\partial}{\partial \sigma} f(\bar{x}, \ldots, \bar{x}_a, \ldots) \right\}_{\sigma=0}, \ \textit{with}\ x_a = \frac{\partial x}{\partial a}, \ldots;$$

thereby a, b, c remain constant during the differentiation; thus, the operation δ commutes with the differentiation with respect to a, b, c:

$$\delta \frac{\partial f}{\partial a} = \frac{\partial \delta f}{\partial a}.$$

When the 3 functions

$$\left(\frac{\partial \bar{x}}{\partial \sigma} \right)_{\sigma=0} = \left(\frac{\partial \xi}{\partial \sigma} \right)_{\sigma=0} = \delta x(x, y, z) \qquad (x, y, z),$$

which, due to (1), can be seen as functions of x, y, z, do not vanish identically in x, y, z, then one can set according to the common continuity postulates

(3′) $$\bar{x} = x + \sigma \delta x(x, y, z) \qquad (x, y, z),$$

provided σ is so small, that σ^2 becomes sufficiently small with respect to σ; up to the factor σ, the infinitesimal virtual displacement of the continuum as given is determined by the 3 functions $\delta x, \delta y, \delta z$.

The last formulas are at the basis of the deduction of the Euler-Lagrange equation being the necessary condition for the search of extrema of a Lagrangian functional and are the basis of the development of modern calculus of variations. The reader will also remark how, in the presented rigorous formalism, the controversial commutation rule between variation and differentiation is an obvious statement. Here Hellinger shows how he is a first rank mathematician (consider that the present paper was published in 1913).

One can immediately bring these displacements into line with the notion of "infinitesimal deformations" considered in the kinematics of elastic media (cf. IV 14, No. ***18****, Abraham) and finds particularly, that the "****virtual shape change****" of each volume element is determined by the 6 quantities*

(4) $$\frac{\partial \delta x}{\partial x}, \ \frac{\partial \delta y}{\partial y}, \ \frac{\partial \delta z}{\partial z}, \ \frac{\partial \delta y}{\partial z} + \frac{\partial \delta z}{\partial y}, \ \frac{\partial \delta z}{\partial x} + \frac{\partial \delta x}{\partial z}, \ \frac{\partial \delta x}{\partial y} + \frac{\partial \delta y}{\partial x},$$

*and its "****virtual rotation****" is determined by*

(4′) $$\frac{1}{2}\left(\frac{\partial \delta z}{\partial y} - \frac{\partial \delta y}{\partial z} \right), \ \frac{1}{2}\left(\frac{\partial \delta x}{\partial z} - \frac{\partial \delta z}{\partial x} \right), \ \frac{1}{2}\left(\frac{\partial \delta y}{\partial x} - \frac{\partial \delta x}{\partial y} \right)$$

caused [by the virtual displacement] — in each case apart from the factor σ.

[19] *This notation and the analogous ones which follow, denote that besides the equation being written out also those [equations] are valid, which arise by simultaneous cyclic permutation of x, y, z and ξ, η, ζ.*

The notation adopted here is exactly the one used by Piola (see [29]): the migration towards a more tensorial notation seems to be gradual. In the following the notion of motion is mathematically made rigorous in terms of time-depending "deformations" (remark that in a more modern nomenclature for the set of fields (1) it is used the name "placement").

A motion of the continuum is considered as a sequence of states of deformations depending on a time parameter t and is hence represented by the three [following] deformation functions which are now also depending on t

$$x = x(a, b, c; t), \quad y = y(a, b, c; t), \quad z = z(a, b, c; t), \tag{5}$$

which as functions of all four variables are sufficiently continuous and differentiable; for fixed a, b, c, (5) represents the trajectory of a certain particle.

As above by taking the variable t into the formulas, one considers then besides the motion (5) also the family of motions

$$\bar{x} = \bar{x}(a, b, c; t; \sigma) = x + \sigma\delta x(x, y, z; t) \qquad (x, y, z) \tag{6}$$

holding for small values of the parameter σ and implying (5) for $\sigma = 0$ and [one] denotes $\delta x, \delta y, \delta z$ as the characteristic quantities of the ***virtual displacements*** *being superimposed on the motion (5).*

The following subsection deals with the next question: is the deformation (placement) field enough for describing the kinematics of a generic continuum model?

2b. Introduction of physical parameters, in particular density and orientation. *Every physical property of a medium is described by one or more functions of a, b, c, t, which [may need to be] added to the deformation functions.*

Subsequently, it will be generally made use of the following property: the existence of a ***fixed mass*** *m for any portion V_0' of the medium, expressed by the integral over the domain V_0' with the integrand being a* ***density function*** *$\varrho_0 = \varrho_0(a, b, c)$ which is characteristic for the medium. By transition to the deformed position (1) the* ***actual mass density*** *ϱ of the distribution of the medium appears as*

$$\varrho = \frac{\varrho_0}{\Delta}, \tag{7}$$

and the mass within the part V' of V is

$$m = \iiint\limits_{(V')} \varrho\, dx\, dy\, dz = \iiint\limits_{(V_0')} \varrho_0\, da\, db\, dc.$$

Changes in the position of the continuum determine nothing concerning the behavior of such an introduced physical parameter; the mass of any portion, i. e. the function $\varrho_0(a, b, c)$ is in the meanwhile left unchanged for virtual displacements and one exchanges therefore the density ϱ by

$$\bar{\varrho} = \bar{\varrho}(x, y, z; \sigma) = \varrho + \sigma\delta\varrho(x, y, z), \tag{8}$$

such that (analogous to the continuity conditions, cf. IV 15, No. ***7****, p. 59f. A. E. H. Love):*

$$\delta\varrho_0 = \delta(\varrho\Delta) = 0 \;\; or \;\; \delta\varrho + \varrho\frac{\partial(\delta x)}{\partial x} + \varrho\frac{\partial(\delta y)}{\partial y} + \varrho\frac{\partial(\delta z)}{\partial z} = 0. \tag{8'}$$

The same shall hold for a motion, i. e. $\varrho_0(a, b, c)$ shall be independent of t and ϱ is consequently determined by (7).

Hellinger has not left any non-explicit assumptions. In the previous lines he rigorously formulates the hypothesis that a material particle of a continuum has, in all configurations and therefore during its motion, always the same mass. This assumption, although it may seem so natural not to be questioned, indeed needs to be removed when some particular problems are studied, e.g. when dealing with "growth" phenomena occurring in living tissues (see e.g. [13, 14, 45–47, 58] and references cited therein).

Additionally, it will be frequently made use of a conceptualization which must be presented here, namely the assumption, ***that for any particle of the continuum different directions radiating from these particles may present characteristically different behavior, and that therefore the specification of its orientation is essential in the description of the state of the continuum****. Such perceptions have been developed in the molecular theories, in which the bodies of crystalline structure are thought of as molecules, and already S. D. Poisson*[20] *has tried to use [such perceptions] to arrive at a better molecular theory of elasticity. Recently, without referring to molecular perceptions, E. and F. Cosserat*[21] *have extensively treated continua in which any particle is given a certain orientation.*

The concept of generalized continuum is introduced precisely by means of the specification of the concept of directors. Hellinger has perceived the importance of the pioneering works by the Cosserat brothers. It has to be remarked that for many years their results have been nearly ignored: we conjecture (see [62]) that this circumstance is related to the fact that their presentation is systematically based on Hamilton's Principle.

In the most general way such a notion of oriented particles of the continuum can be formulated analytically[22]*, by thinking that every particle a, b, c of the continuum is endowed with an attached* ***orthonormal triad****, whose 3 axes have the directional cosines $\alpha_i, \beta_i, \gamma_i$ $(i = 1, 2, 3)$; besides the functions (1), three independent parameters λ, μ, ν (e. g. Euler angles) must be given as functions of a, b, c to determine the orientation of such a triad with respect to the x-y-z-coordinate system:*

$$\lambda = \lambda(a, b, c), \quad \mu = \mu(a, b, c), \quad \nu = \nu(a, b, c), \tag{9}$$

in order to describe the state of such a medium completely.

With every virtual displacement of the continuum a ***virtual rotation*** *comes along by taking a family of rotations with respect to the orientation of the triads (9) depending on a parameter σ which vanishes for $\sigma = 0$ and by exchanging λ, μ, ν for sufficiently small values of σ by*

$$\bar{\lambda} = \bar{\lambda}(a, b, c; \sigma) = \lambda + \sigma\delta\lambda(a, b, c) \qquad (\lambda, \mu, \nu). \tag{10}$$

Thereby, incidentally one can consider both λ, μ, ν and $\delta\lambda, \delta\mu, \delta\nu$ always either as functions of a, b, c, or with the help of (1) as functions of x, y, z. The variations $\delta\alpha_1, \ldots, \delta\gamma_3$ of the directional cosines of the three axes are themselves linear homogeneous functions of $\delta\lambda, \delta\mu, \delta\nu$ being obtained by differentiation of the explicit expressions of $\alpha_1, \ldots, \gamma_3$ with respect to σ; the components $\delta\pi, \delta\kappa, \delta\varrho$ of the angular velocity of the virtual rotation with respect to the 3 axes, which are connected to $\delta\alpha_1, \ldots, \delta\gamma_3$ by the formulas

$$\delta\pi = \beta_1\delta\gamma_1 + \beta_2\delta\gamma_2 + \beta_3\delta\gamma_3 = -(\gamma_1\delta\beta_1 + \gamma_2\delta\beta_2 + \gamma_3\delta\beta_3) \quad \binom{\pi, \kappa, \varrho}{\alpha, \beta, \gamma}, \tag{11}$$

$$\delta\alpha_i = \gamma_i\delta\kappa - \beta_i\delta\varrho \qquad \left(i = 1, 2, 3;\ \begin{smallmatrix}\alpha, \beta, \gamma\\ \pi, \kappa, \varrho\end{smallmatrix}\right) \tag{11'}$$

and which by the way, in contrast to the former application of the symbol δ, are not variations of a particular function of a, b, c, are in the same way linear homogeneous functions of $\delta\lambda, \delta\mu, \delta\nu$, and we set

$$\delta\lambda = l_1\delta\pi + m_1\delta\kappa + n_1\delta\varrho \qquad \binom{\lambda, \mu, \nu}{1, 2, 3}. \tag{12}$$

Thus $\delta\pi, \delta\kappa, \delta\varrho$ (given as functions of a, b, c or x, y, z) determine the virtual rotation of the continuum.[23]

By adding the time parameter t, all these formulas can immediately be extended to the case of a ***motion****.*

The previous sentences and formulas prove that in Hellinger's vision generalized continua are a natural concept which can be introduced in continuum mechanics and that the most convenient formulation of the corresponding theory is based on the principle of virtual work.

[20] *Paris, Mém. de l'Acad. 18 (1842), p. 3, as well as some preceding works; cf. the detailed citations in IV 23, No.* **4***c, p. 39 (Müller-Timpe).*

[21] *Paris C. R. 145 (1907), p. 1409; 146 (1908), p. 68. They have given a summarizing version in their "théorie des corps déform."*[13] *Cf. also IV 11, II. Teil, K. Heun.*

[22] *Cf. a note of P. Duhem, Ann. Éc. Norm. (3) 10 (1893), p. 206.*

[23] *There are the known kinematic methods of the geometry of surfaces (cf. for instance III D 3, No.* **10***, R. v. Lilienthal and G. Darboux, Leçons sur la théorie générale des surfaces), which are applied here by E. and F. Cosserat (see for the detailed version in „corps déform.").*

2c. Two- and one-dimensional continua. *By suppressing one or two of the three parameters a, b, c, one obtains immediately the basis for the treatment of* ***two- and one-dimensional continua****, which are embedded* ***in the three-dimensional space****.*[24] *The position in every state is given by*

(13) $$x = x(a, b) \quad or \quad x = x(a) \qquad (x, y, z);$$

the parameters vary in the domains S_0 and C_0 of the a-b-plane and the a-axis, respectively, which are mapped by (13) onto a surface S and a curve C, respectively. Here, too, one can think of every particle with an attached triad consisting of ***three*** *orthogonal directions*[25]*,which is determined by the functions*

(14) $$\lambda = \lambda(a, b) \quad or \quad \lambda = \lambda(a) \qquad (\lambda, \mu, \nu).$$

This subsection proves that the subdivision of the matter already present in Piola works, and also used by Cosserat brothers, is also exploited by Hellinger.

A final comment has to be made about the presentation order of the subject chosen by Hellinger. He starts with kinematics, postponing the discussion about the concept of work and of force. This is reminiscent of D'Alembert's views (see his "Traité de dynamique", page xvj. As, to our knowledge, there is not any available translation of D'Alembert's masterpiece into English, the translation in the footnote of the following excerpt into English is ours):

> «A l'égard des démonstrations de ces Principes en eux-même, le plan que j'ai suivi pour leur donner toute la clarté & la simplicité dont elles m'ont paru susceptibles, a été de les déduire toujours de la considération seule du Mouvement, envisagé de la maniére la plus simple & la plus claire. Tout ce que nous voyons bien distinctement dans le Mouvement d'un Corps, c'est qu'il parcourt un certain espace, & qu'il employe un certain tems à le parcourir.»[26]

Kinematics relates to experimental evidence. Nobody has ever experienced directly anything concerning a force. Also the most abstract mathematical model must try to relate to experimental evidence: it is therefore clear that mechanics must be based on a postulation where kinematics plays a fundamental role. On page xvj (loc. cit.) D'Alembert continues by stating that:

> «C'est donc de cette seule idée qu'on doit tirer tous les Principes de la Méchanique, quand on veut les démontrer d'une maniére nette & précise; ainsi on ne sera point surpris qu'en conséquence de cette réfléxion, j'ai, pour ainsi dire, détourné la vûe de dessus les causes motrices, pour n'envisager uniquement que le Mouvement qu'elles produisent; que j'aie entiérement proscrit les forces inhérentes au Corps en Mouvement, êtres obscurs & Métaphysiques, qui ne sont capables que de répandre les ténèbres sur une Science claire par elle-même»[27]

A stylistic comment is required here: the reader will note the difference between the style of presentation adopted by Hellinger compared with that used by D'Alembert. One could state that while Hellinger was Tacitean, D'Alembert is Ciceronian. Somebody could find D'Alembertian style baroque, manneristic or even bombastic. The controversies arisen by D'Alembert were attributed sometimes to his style: which could be somehow true. However, while D'Alembertian overstatements attracted the attention of many scientists the more concise and less polemical style used by Hellinger was not more successful. Nevertheless it is clear that the highly abstract mathematical concepts needed for mastering variational methods and principles are the only true obstacle to their diffusion.

3 Annotated translation of No. 3 (pp. 611–622)

The study of statics, i.e. equilibrium, of mechanical systems is presented by Hellinger after the study of kinematics. Hellinger follows the nomenclature established already in the works of Lagrange and maintained by Piola. The fundamental principle of mechanics is still called the "Principle of Virtual Displacements": this proves that the change in nomenclature into "Principle of Virtual Work" became universal later.

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24 *In a certain manner these problems are easier than the ones for three-dimensional media; In fact, a few of them belong to the earliest problems which have been treated thoroughly in the mechanics of continua; (cf. IV, 6, No.* ***22—24****, P. Stäckel and IV 11, No.* ***19, 20****, K. Heun).*

25 *Cf. the notes of Paris referred to in 21) of E. and F. Cosserat and Cap. II, III of their "corps déform.".*

26 Concerning the demonstrations of those Principles in them-selves, the plan, which I have followed to give them all the clarity and the simplicity which they could have, has been of deducing them always by considering only the Motion, always considered in the clearest and simplest possible way. All that which can see really distinctly in the Motion of a Body, is that it is moving in space and that its need some time to move along its trajectory.

27 It is therefore from this only idea that one must deduce all the Principles of Mechanics when one wants to show them in a precise and neat way; therefore one will not be surprised that, as a consequence of this reflexion I have, as to say, turned away my eyes from the moving (efficient) causes, for uniquely considering the Movement which they are producing; [and] that I had entirely proscribed the forces relative to the bodies in motion, [as] entities obscure and metaphysical which are only able to throw obscurity on a Science which is instead clear by her-self.

I. The foundations of statics.

> ***3. The principle of virtual displacements.***
>
> ***3a. Forces and stresses.*** *To build the dynamic properties of the continuum on this kinematic scheme, we take up the* ***notion of work****.*

This sentence is crucial and shows again clearly that Hellinger is a follower of D'Alembertian or Lagrangian general idea of mechanics. First one considers the (admissible) kinematics and then one considers the laws of dynamics governing the motion of the considered systems. The basic notion in this construction is the notion of work. In fact, Hellinger starts formulating the fundamental problem of mechanics exactly following the conceptual frame set up by D'Alembert (see again "Traité de dynamique", page viij,ix (end,beginning)):

> «Mais comment arrive-t'il que le Mouvement d'un corps suive telle ou telle loi particuliére? C'est sur quoi la Geométrie seule ne peut rien nous apprendre, & c'est aussi ce qu'on peut regarder comme le premier Problême qui appartienne immédiatement à la Méchanique.
>
> On voit d'abort fort clairement, qu'un Corps ne peut se donner le Mouvement lui-même. Il ne peut donc être tiré du repos, que par l'action de quelque cause étrangére.»[28]

Hellinger calls the "cause étrangére" (the external cause) evoked by D'Alembert force (or stress).

> *All the forces and stresses of any kind which act on the continuum due to the current state of deformation, due the position in space or due to any external circumstances — for the moment [considering this set of forces and stresses] in its entirety without considering its cause —, have in common, that for any virtual displacement they expend a* ***"virtual work"*** *δA;*

Hellinger uses the word force exactly in the same spirit and with the same intentions as D'Alembert. The forces and the stresses respectively "applied on" or "applied in" a continuous body have in common, or, are characterized by the fact that they expend a virtual work on virtual displacements. Indeed on page xxv (loc. cit.) D'Alembert warns the reader:[29]

> «Au reste, comme cette seconde Partie est destinée principalement à ceux, qui déja instruits du calcul différentiel & intégral, se seront rendus familiers les principes établis dans la premiére, ou seront déja exercés à la solution des Problêmes connus & ordinaires de la Méchanique ; je dois avertir que pour éviter les circonlocutions, je me suis souvent servi du terme obscur de force, & de quelques autres qu'on employe communément quand on traite du Mouvement des Corps ; mais je n'ai jamais prétendu attacher à ces termes d'autres idées que celles qui résultent des principes que j'ai établis, soit dans cette Préface, soit dans la premiére Partie de ce Traité.»[30]

> *we see this [virtual work] as primitive and determine it as follows:* ***Let δA be a linear homogeneous function of the entirety of values of the displacement components δx, δy, δz within the continuum and let it be a scalar quantity, independent of the choice of the coordinate system.*** *The coefficients, which enter δA together with the values of $\delta x, \ldots$, are the characteristic quantities of the individual acting force systems; the independence of these components of the virtual displacements (i. e. the linearity of δA), expresses the assumption that these displacements, due to their smallness, do not modify the force effects exerted on every particle.*

This statement in its brief clarity should not need any comment if it were accepted without controversies. Unfortunately, many debates were started about its content even in relatively more modern works and conference discussions. Hence, we remark here that:

i) Some mechanicians doubted about the generality of the postulation presented by Hellinger objecting that the linear dependence of work functional limits the range of applicability to variational principles (the reader should remark how elegantly this objection is answered by Hellinger, without formulating the question, though).

[28] «But how it happens that the motion of a body follows this or this other particular law? This is where the Geometry, alone, cannot teach us anything and this is what one can regard as the first Problem which belongs immediately to Mechanics. One can see immediately [and] really clearly that a body cannot give to him-self a motion. It therefore can be subtracted from a state of rest only by the action of some external cause.»

[29] It is unfortunate that this warning has been ignored or removed from their mind by many mechanicians.

[30] On the other hand, as this second part is addressed mainly to those who being already learned in differential and integral calculus managed to become familiar with the principles established in the first one, I must warn [these readers] that for avoiding the circumlocutions I have often used the obscure term "force", and some other terms which one commonly employs when he treats the motion of bodies; but I never wanted to attribute to these terms any other ideas [different] from those which result from the principles which I have established, either in this Preface, or in the first Part of this Treatise.

ii) The forces and stresses appear naturally in variational postulations as the dual quantities with respect to virtual displacements and virtual deformations, respectively (for more details see e.g. [31, 33, 41, 42, 44]): they are univocally characterized by the introduced work functionals, so that they do not need to be introduced as independent concepts.

iii) Again, Hellinger's mathematical knowledge becomes apparent in the elegant way in which he treats this point. He is a contemporary of Fréchet and Gâteaux and therefore, being a mathematician at the boundary of knowledge, it is most likely that he knew and mastered their ideas and methods. However, in the few lines we have translated before, he shows a vision on the concept of distributions which is anticipating the revolutionary results by Laurent Schwartz and their applications to continuum mechanics (concerning this point the reader is referred to [33,44] and also to the enlightening textbook [85]).

> *In order to include all fundamental equations of the mechanics of continua, it is not necessary to begin with the most general expression of the described form of δA, which would consist of a sum of linear functions of values of $\delta x, \delta y, \delta z$ and their derivatives at certain locations of the continuum as well as line, surface and volume integrals of such expressions.*

Here, Hellinger anticipates the structure theorems later proven by Laurent Schwartz (see [88]) for distributions which has been later explicitly considered in [33] and [28,30,32]. He also assumes the correct attitude towards generality: the correct procedure for presenting a theory and for advancing science is to start with meaningful particular cases before trying to push towards generality.

> *Instead, we consider at first an expression — which we will extend later on — , which consists of a volume integral over the whole domain V of the continuum as well as a surface integral of its surface S and thereby the former includes in addition a linear form of the 9 derivatives of $\delta x, \delta y, \delta z$ with respect to x, y, z:*[31]

$$
\begin{aligned}
\delta A = & \iiint_{(V)} \varrho(X\delta x + Y\delta y + Z\delta z)dV && = \delta A_1 \\
(1) \qquad & - \iiint_{(V)} \Big(X_x \frac{\partial \delta x}{\partial x} + X_y \frac{\partial \delta x}{\partial y} + \cdots + Z_z \frac{\partial \delta z}{\partial z}\Big)dV && + \delta A_2 \\
& + \iint_{(S)} (\overline{X}\delta x + \overline{Y}\delta y + \overline{Z}\delta z)dS && + \delta A_3
\end{aligned}
$$

> *The 15 coefficients of the displacement quantities, which appear in here and which are going to be discussed immediately in more detail, shall be for any deformation of the considered medium definite **functions** of x, y, z or a, b, c **being along with their derivatives everywhere bounded and, possibly with exceptions at individual surfaces, continuous**; in that case, the concrete meaning of the ansatz (1) is that we merely consider **forces**, which are in general continuously distributed on spatial domains as well as on individual surfaces, and [that we only take] continuously distributed **stresses** [into account].*

Hellinger shows here to know about the existence of actions being more general than forces (in the sense of usually attributed to Newton). The reader should remark that in the subsequent literature the possibility of introducing more general actions has been questioned both from the physical and the mathematical point of view. An explicit introduction of the concept of double-force and multipolar stresses can be found e.g. in the later works [41] and [50].

> *To begin with, the first and the last summand of δA are similar to the familiar work expressions of point mechanics, besides the appearance of the mass of a volume element ϱdV and the surface element dS as factor, respectively; thus X, Y, Z and $\overline{X}, \overline{Y}, \overline{Z}$ are to be interpreted as components of forces per unit mass of the medium and per unit area, respectively, acting at their corresponding position. Since $\delta x, \delta y, \delta z$ are the components of a polar vector, and since δA remains as scalar invariant under coordinate transformations, for a change of the orthogonal coordinate system these force components transform like $\delta x, \delta y, \delta z$:*[32] ***these forces are polar vectors.***

[31] *Such fundamental equations for the virtual work have early been developed as obvious generalization to the formulas of point mechanics for many special problems. Almost naturally was the form of the summands δA_1, δA_3, which replaces merely the sigma sign from point mechanics with the integral (cf. for instance Lagrange, Méc. an., 1. part, IV, 11); but also terms of the form δA_2 only in very special form have been used by Lagrange e. g. for the treatment of the extensible wire and the compressible fluid, namely terms, which are proportional to the variation of the length or density, respectively (see Méc. an., 1. part, V, 42; VIII,1). Moreover, the development of the generalized approach (5) has been initiated by the opinion to consider the virtual work as variation of a "potential" (see No. **7**), as introduced by C. L. Navier in the theory of elasticity (see IV 23, No. **5**, Müller-Timpe).*

[32] *Cf. IV 14, No. **2**, Abraham.*

Hellinger deduces here the vector nature of forces by imposing the invariance of work functional and the vector nature of virtual displacements: also this is a very modern concept, whose appearance in Hellinger's paper is somehow surprising.

> *Rather characteristic for the mechanics of continua is the summand δA_2. The 9 coefficients $X_x, X_y, \ldots, Z_z$— in the familiar notation of Kirchhoff*[33] *— , which measure the influence of the individual characteristic quantities of the virtual deformation on the expended work, can be interpreted as* ***components of the stress state*** *at the corresponding position, computed by their action per unit volume. Their behavior under coordinate transformation follows from the remark, that the 9 derivatives $\frac{\partial \delta x}{\partial x}, \ldots, \frac{\partial \delta z}{\partial z}$ of vector components transform under an orthogonal coordinate transformation in the same way as the 9 products of the components of two vectors (a so called* ***dyad***[34]*)*
>
> $$X_1 \cdot X_2, \quad X_1 \cdot Y_2, \quad \ldots, \quad Z_1 \cdot Z_2$$
>
> *while the bilinear aggregate $X_x \cdot \frac{\partial \delta x}{\partial x} + \cdots$ remains invariant; hence the stress components must also transform like the components of a dyad, with the result that one speaks of a* ***stress dyadic****. One can compose it, as any dyad, into a (symmetric) part with 6 components (a* ***tensor triple***[35]*)*
>
> $$X_x, Y_y, Z_z, \tfrac{1}{2}(Y_z + Z_y), \tfrac{1}{2}(Z_x + X_z), \tfrac{1}{2}(X_y + Y_x) \tag{2}$$
>
> *and a (skew-symmetric) part with 3 components*
>
> $$Z_y - Y_z,\ X_z - Z_x,\ Y_x - X_y, \tag{2'}$$
>
> *representing an* ***axial vector****. This decomposition corresponds to the two separate parts (4), (4′) of the virtual deformation of the continuum considered in No.* ***2****, and is obtained directly by decomposing the integrand of δA_2 as follows:*
>
> $$\sum_{\binom{x\,y\,z}{X\,Y\,Z}} \left\{ X_x \frac{\partial \delta x}{\partial x} + \tfrac{1}{2}(Y_z + Z_y) \left(\frac{\partial \delta y}{\partial z} + \frac{\partial \delta z}{\partial y} \right) + (Z_y - Y_z) \tfrac{1}{2} \left(\frac{\partial \delta z}{\partial y} - \frac{\partial \delta y}{\partial z} \right) \right\}.^{36}$$
>
> *(Cf. the derivation in IV 14, No.* ***19****, Abraham.)*

Two remarks about the used concepts of tensor calculus have to be added:

i) Again, the contributions by Ricci and Levi-Civita to tensor algebra and calculus are not used in this part of the paper: Nevertheless, the presented concepts are more advanced than those used in many contemporary textbooks. The dyadic approach to tensor algebra is here the preferred approach.

ii) Hellinger uses the notation introduced by Kirchhoff for representing stress in the actual configuration: this notation is nearly fully tensorial.

> *In particular it follows, that the 6 quantities (2) determine that part of the stress state, which expends work for an infinitesimally pure shape change of the continuum, i. e. the* ***actual elastic action****, the vector (2′) on the other hand determines that part, which can be considered for a virtual rotation of the volume element, also without shape change, i. e.* ***torques*** *induced by the stress state. From the negative sign of (1) it follows furthermore, that for positive X_x and negative $\frac{\partial \delta x}{\partial x}$ positive work is expended, so that* ***pressure*** *is consequently measured* ***positive****.*
>
> *To obtain from the ansatz (1) finally the interpretation of the stress components as* ***surface forces***[37]*, one considers the virtual work contribution expended by a subdomain V_1 bounded by a closed surface S_1, which is*

[33] *J. f. Math. 56 (1858) = G. Kirchhoff Ges. Abhandl. (Leipzig 1882), p. 287.*

[34] *The herewith indicated definition of the dyad as complex of quantities with a particular behavior with respect to an orthogonal coordinate transformation ("basic group" of spatial transformations), which lies definitively within the notion of F. Klein's geometry, vector analysis and more (cf. in particular Zeitschr. f. Math. Phys. 47 (1902), p. 237 and Math. Ann. 62 (1906), p. 419, the presentation in IV 14, Abraham as well as F. Klein, Elementarmath. v. höh. Standp. aus, Bd. 2, 2. Aufl., Leipzig 1913, p. 90ff., p. 534) seems hitherto not to be the basis of an independent presentation. The name "dyadics" originates from J. W. Gibbs (see Gibbs and Wilson, Vektor Analysis, New York 1901, p. 260ff.), who gives rise to them starting with so called linear vector functions; from here they have been transmitted to the German literature (cf. IV 11, No.* ***1****c, K. Heun). If one considers a dyad as a matrix of $3 \cdot 3$ elements, then the dyadic calculus is included within Cayley's matrix calculus (cf. for this I A 4, No.* ***10*** [19]*), Study).*

[35] *In the notation of W. Voigt; cf. in addition IV 14, No.* ***17****, M. Abraham.*

[36] *The indices of the sigma sign and similar ones in the following denote that the expressions to be summed arise by simultaneous cyclic permutation of x, y, z and X, Y, Z*

[37] *The following includes the ideas, which are set since C. L. Navier and G. Green, to obtain from the ansatz concerning the elastic potential the fundamental equations in addition with its intuitive explanation; one should compare the historical presentation in IV 23, No.* ***5*** *(Müller-Timpe) as well as e. g. the presentation in H. v. Helmholtz, Vorles. über theoret. Phys. II (Leipzig 1902), § 23.*

the integral over the part V_1 of δA_2; for continuous stress components within V_1, [this virtual work contribution] is transformed further by integration by parts (using the "Theorem of Gauss", s. IV 14, p. 12), to

$$\iiint\limits_{(V_1)} \sum_{\binom{x\,y\,z}{X\,Y\,Z}} \Big(\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z}\Big)\delta x \cdot dV + \iint\limits_{(S_1)} \sum_{\binom{x\,y\,z}{X\,Y\,Z}} (X_x \cos nx + X_y \cos ny + X_z \cos nz)\delta x \cdot dS_1,$$

where n denotes the normal of the surface S_1 at the position of the element dS_1 pointing in direction of V_1. By comparison with (1) it follows consequently, that the stress state in V_1 expends the same virtual work, i. e. acts equally as if besides volume forces in V_1 the force per area

$$X_n = X_x \cos nx + X_y \cos ny + X_z \cos nz, \qquad (X, Y, Z) \tag{3}$$

were acting on the surface element dS_1 of S_1. This "pressure theorem" of Cauchy provides then, by specializing the directions of n, immediately the interpretation of the 9 components (cf. IV 23, No. ***3****a, Müller-Timpe).*

A remark is needed here to establish how much of the work of Gabrio Piola was known to Hellinger: if the transmission of knowledge was direct or indirect is a question which we will try to address in further investigations.

It is in our opinion clear that (see [25, 29]) the integration by parts presented in the previous sentences by Hellinger is already performed by Piola because of exactly the same reason: i.e. to transform the volume expression of internal work into the expression of work expended by surface forces. The delicate question concerning the priority between Piola and Cauchy in the introduction of surface contact forces (we mean the concept generalizing the concept of pressure to solids) needs a very detailed scrutiny, if ever one will be able to solve it. However, the priority of the introduction of the aforementioned integration by parts process for "deducing" the existence of contact forces inside a deformable continuous body seems to us indisputable: this is to be attributed to Piola, in both the reference and the actual configuration.

A further remark has to be added: it is not clear how in the literature Piola's change of variable from the reference to the actual configuration and the related vector and tensor formulas (including the one concerning the transformation of the normal to a surface, see again [29]) could be attributed to Edward J. Nanson (see [3]). The works of Piola, although written in Italian, remained always available in all English speaking Universities.

3b. Formulation of the principle of virtual displacements. *Due to these conceptualizations the* ***principle of virtual displacements****, dominating the statics of discrete mechanical systems*[38]*, can be adopted immediately for the mechanics of continua:* ***A continuous medium in a particular state of deformation, for certain volume and surface forces $X, \ldots$ and $\overline{X}, \ldots$, respectively, and for a certain stress state $X_x, \ldots$, is in equilibrium if and only if the total virtual work of these forces and stresses vanish for every virtual displacement, which is admissible with respect to the possibly imposed constraints of the continuum:***

$$\iiint\limits_{(V)} \Big\{\varrho \sum_{\binom{x\,y\,z}{X\,Y\,Z}} X\delta x - \sum_{\binom{x\,y\,z}{X\,Y\,Z}} \Big(X_x \frac{\partial \delta x}{\partial x} + X_y \frac{\partial \delta x}{\partial y} + X_z \frac{\partial \delta x}{\partial z}\Big)\Big\} dV + \iint\limits_{(S)} \sum_{\binom{x\,y\,z}{X\,Y\,Z}} \overline{X}\delta x \cdot dS = 0. \tag{4}$$

This ansatz has been implemented in fact already by J. L. Lagrange[39]*, after postulating Bernoulli's principle of virtual displacements as foundation of his analytical mechanics; for him the natural consequences of the validity of this principle of point mechanics is its applicability to problems of the mechanics of continua accessible for himself, whenever he is able to obtain the work expression by a limit process of discrete systems or by direct intuition. Since then one has also applied the principle of virtual displacements to further fields of the mechanics of continua, and has, like Lagrange, often based oneself on the perception, to be able to approximate the continuum by a system of finitely many mass points and that at the same time all physical processes in the continuum can be approximated by corresponding processes in these approximated systems; however it does not seem that such an axiomatic clarification of this connection have already been given, [a clarification] concerning the transformation of those intuitions into rigorous deductions would particularly have to postulate the necessary continuity requirements.*

The reduction problem to which Hellinger refers here is nowadays called the problem of homogenization of discrete systems: it consists, exactly as stated by Hellinger in a very clear way, to prove that suitable ε-families of solutions of discrete problems converge, when suitable continuation processes are introduced, to the solution of a continuous problem. In this

[38] *Cf. IV 1, No.* ***30****, Voss.*
[39] *Mécan. anal., 1. part., sect. IV. § II, as well as for a series of particular problems in sect. V—VIII.*

kind of rigorous problems the concept of Gamma convergence is now playing a crucial role. The literature in the subject is becoming immense: we quote here [20, 78] among the most interesting papers obtaining first gradient continua as continuous limit while, for what concerns the papers where a higher gradient continuum limit is obtained, we cite [4] and [5, 21, 89] (see also the effort at continuum modeling of granular systems [64, 65, 95]).

> *Thus one may prefer in the meantime for the mechanics of continua, to choose the principle formulated at the beginning as the* ***highest axiom*** *(cf. IV I, p. 72, Voss); one prefers to take up this position anyway, when one considers the concept of continuously distributed media as more natural than the abstract "mass points" of point mechanics.*[40]

In the last sentences Hellinger shortly refers to the controversy and/or duality between discrete and continuous systems concerning their "logical" or "physical" relative prevalence and priority. This controversy is, indeed, still disputed between those who believe in the ultimate discrete nature of matter (as did Boltzmann) and those who, instead, believe that matter has, at the most fundamental level, a continuous nature. We can only note that Hellinger is very well aware of the existence of this duality (as it was obvious, considered his scientific standing) which is so old to have been discussed already by Galen. The interested reader can find in [18] an interesting description of the so called Galen divide between Democritus (together with all Epicureans School) versus Empedocles, Parmenides and the theorists of continuum (among whom Galen seems to have placed himself).

> *The certainty of the validity of this axiom is justified that such an ansatz corresponds with our general physical intuition and habitual ways of thinking, but in particular therein, that it is adaptive to represent the empirical facts sufficiently enough.*

Hellinger is a follower of D'Alembert in accepting and formulating as shown before the principle of virtual work and in basing on it continuum mechanics. It seems to us that he also has been influenced by the positivistic views and anticipates here the idea of the Vienna Circle and Karl Popper about the role of science in human activities. Hellinger refuses the postulation of Mechanics based on the balance of force (and eventually other quantities) exactly as D'Alembert did in (loc. cit.) where on page xj,xij (end,beginning) one indeed reads:

> «Pourquoi donc aurions-nous recours à ce Principe dont tout le monde fait usage aujourd'hui, que la force accélératrice ou retardatrice est proportionelle à l'Elément de la vitesse; principe appuyé sur cet unique axiome vague & obscur, que l'effet est proportionnel à sa cause. Nous n'examinerons point si ce Principe est de vérité nécessaire; nous avouerons seulement que les preuves qu'on en a données jusqu'ici, ne nous paroissent pas fort convaincantes: nous ne l'adopterons pas non plus, avec quelque Geométres, comme de verité purement contingent, ce qui ruineroit la certitude de la Méchanique, & la réduiroit à n'être plus qu'une Science expérimentale: nous nous contenterons d'observer, que vrai ou douteux, clair ou obscur, il est inutile à la Méchanique, & que par conséquent il doit en être banni.»[41]

We conclude our comments on this subsection by stating that there is not a "petitio principii"[42] hidden in Hellinger's statement of the principle of virtual displacements, how unfortunately too often sustained by the opposers of D'Alembertian-Lagrangian postulation of mechanics and in particular in [92] page 595 where one can read in the first footnote:

> «The derivation given by HELLINGER [...] fails through *petitio principi*, since the stress components appear in the original variational principle. [...] Existence of the stress tensor can be proved from variational principles which assume the existence of an internal energy having a special functional form.»

[40] *This perception has recently been represented in particular by G. Hamel (Math. Ann. 66 (1908), p. 350 and Jahresb. d. Math.-Ver. 18 (1909), p. 357; cf. also his textbook "Elementare Mechanik", Leipzig 1912); therein he introduces a complete axiomatic system of the mechanics of continua, in which the fundamental principle, used here, follows from a sequence of independent theorems.*

[41] Why ever should we resort to such a Principle which is nowadays used by everybody, that the accelerating or retarding force is proportional to the Element of velocity; principle based only on that vague and obscure axiom [stating] that the effect is proportional to the cause. We will not examine at all if this is a Principle of necessary truth; we will only confess that the evidence given up to now to this aim does not seem to us at all persuasive: we will not adopt it anymore, together with some Geometers, as a purely contingent truth because in this way we would ruin the certitude of Mechanics and would reduce it to be anything more than an experimental Science; we will limit ourselves to observe that, be it true or doubtful, clear or obscure, it is useless to Mechanics and that, for this reason, it has to be banished out of it.

[42] We resist to use in this context the most common English expression "begging the question", as it is usually phrased, as unfortunately it originated in the 16th century as a wrong translation of the Latin correct expression "petitio principii". A correct English translation could be: "assuming the initial point" or even better "a fallacy in which a conclusion is taken for granted in the premises".

Remark that obviously very often the conclusion may be accepted in an indirect way such that its presence within the premise is hidden or at least not easily apparent.

The footnote is a comment on the following passage:

> «[...] no variational principle has ever been shown to yield Cauchy's fundamental theorem in its basic sense as asserting that existence of the stress vector implies the existence of the stress tensor.»

Simply the authors of the aforementioned statements do not want to follow the reasonings presented in the works by D'Alembert, Lagrange, Piola and finally Hellinger: the fundamental, primitive concept in mechanics is the work while contact force is a derived concept. One postulates that work is a linear and continuous functional on a set of test functions (i.e. virtual displacements) and then, via the celebrated theory of distributions by L. Schwartz (if one wants to be mathematically sophisticated) or via a suitable series of regularity ansatz, one gets a representation of work in terms of n-th order stresses which are defined as the dual in work of n-th gradient of virtual displacements. There is no logical reason for which contact actions (in the case of first gradient continua they reduce to contact surface forces) must be the most fundamental concept. Actually Piola, Hellinger and many others (see [25, 33, 42, 50] and references cited therein) prefer to consider as fundamental, primitive concepts the stresses and to deduce the contact actions as concepts derived in terms of stresses.

To be more precise, D'Alembertian postulation of Mechanics is based on the principle of virtual work which is formulated following the subsequent steps:

i) to introduce an admissible set of configurations and an admissible kinematics, specifying the set of all possible motions,

ii) to introduce the required work functionals in order to model ALL interactions of the system, including the inertial work, which was considered explicitly by D'Alembert and is given in terms of kinematic quantities including accelerations.[43]

iii) to postulate that the sum of internal work plus external work plus inertial work, i.e. the total virtual work, is vanishing.

In this context the word force is used (as it should always be) simply to describe the structure of postulated work functionals and is not primitive. In particular we do not need to postulate any balance of forces: actually Hellinger treats the concept and the word "force" exactly in the same spirit as D'Alembert (see supra our comments concerning D'Alembert's usage of the word force).

It seems to us that the elegant "Tetrahedron Theorem" by Cauchy, which has been considered the only possible way for founding continuum mechanics in [92] (see there page 595), although very interesting and meaningful, cannot be regarded as the "unavoidable" basis of continuum mechanics (see e.g. [30]). We could not find in the works of Cauchy such a strong statement[44]: Cauchy followers (like Aristotelians) seem much more extreme than Cauchy himself.

> ***3c. Application to continuously deformable bodies.*** *The established formal operations of the calculus of variations enable easily, to transform the principle of virtual displacements into a number of equations between forces and stresses.*[45] *Consider at first only the* ***arbitrarily continuously deformable medium*** *which is typically not at all constrained, then the condition (4) must be satisfied for every system of continuous functions $\delta x, \delta y, \delta z$. For forces, stresses and partial derivatives being continuous everywhere in V, the transformation of (4) yields by integration by parts the equations:*
>
> *1) for every point of the domain V*
>
> (5a) $$\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} + \varrho X = 0 \qquad (X, Y, Z),$$
>
> *2) for every point on the surface S with* ***outward pointing*** *normal direction n*
>
> (5b) $$X_x \cos nx + X_y \cos ny + X_z \cos nz = \overline{X} \qquad (X, Y, Z).$$

[43] This treatment of the inertia forces is a little unsatisfactory as it already relates force quantities with kinematic quantities, i.e. it includes the relation that the inertia forces are proportional to the accelerations. In No. 5d of Hellinger's work (this number will follow in the next annotated translation) a general principle for dynamics is introduced in which momentum is considered as dual quantity to the time derivative of the virtual displacement field. Thus, in such an ansatz the relation between momentum and kinematical quantities remains unspecified.

[44] While Cauchy's lemma and the symmetry of the stress tensor are formulated in [22] as "Théorème I" and "Théorème II", respectively. The celebrated stress theorem of Cauchy has to be extracted out of the text and the formulas on pp. 68-69.

[45] *Already Lagrange proceeded in the Méc. an. in this way to treat the problems therein; see remark 23.*

> *Thereby the so-called "equations of stress" together with the corresponding surface conditions are obtained, which give* ***the necessary and sufficient conditions that a particular force and stress system, acting on a freely deformable continuum in a certain position, is in equilibrium.***[46]

While we do not try to dispute the priority of Cauchy in the formulation of these equations we indeed state that in the works of Piola one can find the just mentioned deduction from a variational principle (and we even do not claim that Piola was the first in presenting such a deduction).

> *Certainly these conditions are not enough, to* ***determine*** *the stress and force components: To this we must add the relations which will be treated later on and which express the dependence of forces and stresses on the actual deformation of the continuum or on any external causes (cf. IV 6, No.* ***26****, Stäckel and IV 23,* ***3****b, Müller-Timpe).*
>
> *In (4), (5) the independent variables are the coordinates of the* ***deformed*** *state of the continuum, and also force and stress components have their descriptive meaning as action per unit mass or surface of the medium in the deformed state. On the other hand, since S. D. Poisson*[47]*one often refers to a, b, c, being the coordinates of the initial position of the medium as independent variables;*

As in Poisson (loc. cit.) one cannot find explicitly the triple (a, b, c), Hellinger attributes here the concept of using coordinates of the initial configuration as independent variables to Poisson. Consequently and unfortunately, the source of the triple (a, b, c) used by Hellinger remains an issue to speculate about.

> *indeed, this leads to force components of physical interpretation less immediate, but it is for many cases analytically more convenient. Setting*
>
> $$k \cdot dS_0 = dS \tag{6}$$
>
> *and considering No.* ***2****, (7), we obtain namely:*
>
> $$\delta A = \iiint\limits_{(V_0)} \Big[\varrho_0 \sum_{(xyz)} X\delta x - \sum_{\left(\substack{x\,y\,z \\ X\,Y\,Z}\right)} \Big(X_a \frac{\partial \delta x}{\partial a} + X_b \frac{\partial \delta x}{\partial b} + X_c \frac{\partial \delta x}{\partial c} \Big) \Big] dV_0 + \iint\limits_{(S_0)} \sum_{\left(\substack{x\,y\,z \\ X\,Y\,Z}\right)} \overline{X} k \delta x \cdot dS_0, \tag{7}$$
>
> *whereas*
>
> $$\Delta \cdot X_x = X_a \frac{\partial x}{\partial a} + X_b \frac{\partial x}{\partial b} + X_c \frac{\partial x}{\partial c} \qquad (X, Y, Z; x, y, z). \tag{8}$$
>
> *Therefore, by solving and comparing with (3) it follows, that X_a, Y_a, Z_a are the components of the surface force, computed per unit area of the initial position in the a-b-c-space, which due to the stress state acts via an element of the surface $a = \mathrm{const.}$ on the matter lying on this side for which a is increasing.*[48] *From (7) a* ***new form of the equilibrium conditions***[48] *arises, in the same manner as (5a), (5b) arise from (4):*
>
> $$\frac{\partial X_a}{\partial a} + \frac{\partial X_b}{\partial b} + \frac{\partial X_c}{\partial c} + \varrho_0 X = 0 \quad \text{in } V_0 \qquad (X, Y, Z), \tag{9a}$$
>
> $$X_a \cos n_0 a + X_b \cdot \cos n_0 b + X_c \cos n_0 c = k\overline{X} \quad \text{on } S_0 \quad (X, Y, Z); \tag{9b}$$
>
> *hereby n_0 denotes the outward pointing normal direction of the surface element dS_0 in the a-b-c-space.*

Once more, the contributions by Piola are overlooked here: the tensor appearing in the just written equations should be, in our opinion, called Piola stress and not Piola-Kirchhoff stress as the subsequent contribution by Kirchhoff in this particular part of continuum theory does not seem particularly relevant or original.

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[46] *These equations can be traced back to A. L. Cauchy, Exerc. de math. 2 (1827) = Oeuvres 7, sér. II, p. 141. Cf. the further references about this in IV 23, No.* ***3****b, Müller-Timpe.*

[47] *Paris Mém. de l'Acad. 8 (1829), p. 387; J. éc. polyt. 20 (1831), p. 54. This difference has been frequently overlooked, since it vanishes in fact for the consideration of infinitesimal deformations from a stress free state of equilibrium; so it have shown to be useful only when the development of the theory of elasticity of finite deformations (cf. below No.* ***7*** *and* ***9****) [was achieved].*

[48] *Cf. IV 23, No.* ***6*** *(Müller-Timpe) and for instance the detailed presentation (which presumes certainly the symmetry of the stress dyad) of E. and F. Cosserat; Ann. de Toulouse, X (1896), p. 146; the notation $X_a, X_b, \ldots$ seems to be more consistent than $A_x, B_x, \ldots$, used there, since it remains the capital letters for the denotation of the components, but the indices for the characterization of the considered surface element.*

3d. Relation to the mechanics of rigid bodies. *It is also possible to derive the equilibrium conditions (5) in a slightly different way starting with the principle (4) and thereby one obtains the connection to the "rigidifying principle", frequently used for the direct derivation of the equilibrium conditions according to the approach of A. L. Cauchy*[49]*, stating that every part cut out of the deformable continuum exposed to the volume forces applied within the part and the forces applied on the surface (3) must be in equilibrium like a rigid body. For this, one only has to consider certain* ***discontinuous*** *displacements, which certainly violate the connection of the continuously deformable continuum and for which δA does not have to vanish at first: though one succeeds by approximating it with a family of* ***continuous*** *virtual displacements.*

Hence, a displacement, which on a subset V_1 of V with boundary S_1 has constant values $\delta x = \alpha$, $\delta y = \beta$, $\delta z = \gamma$, but is 0 outside of V_1 (which is a ***translation*** *of the domain V_1), is approximated by continuous virtual displacements, by surrounding V_1 with an arbitrary small region V_2, in which $\delta x, \delta y, \delta z$ decrease continuously from α, β, γ to 0. For such a virtual displacement it follows from (4):*

$$\iiint\limits_{(V_1)} \varrho(X\alpha + Y\beta + Z\gamma)dV_1 + \iint\limits_{(S_1)} (X_n\alpha + Y_n\beta + Z_n\gamma)dS_1 + \iiint\limits_{(V_2)} \sum_{\left(\substack{x\,y\,z\\ X\,Y\,Z}\right)} \big(\varrho X + \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z}\big)\delta x \cdot dV_2 = 0,$$

where n is the normal of dS_1 being outward pointing with respect to V_1. Letting V_2 become smaller and smaller, the second integral gets arbitrary small, since $X, X_x, \ldots$ and the derivatives thereof remain finite, and since α, β, γ are arbitrary, one yields the three equations

$$\text{(10)} \qquad \iiint\limits_{(V_1)} \varrho X dV_1 + \iint\limits_{(S_1)} X_n dS = 0 \qquad (X, Y, Z).$$ [50]

Hellinger has proven here the balance of force for every subbody of the considered continuous body starting from his (conventional!) formulation of the principle of virtual displacements. Moreover, in the previous sentences he simply explains how fallacious the following statement in [37] is:

> «To do so we use a nonstandard form of the principle of virtual power (Gurtin [12][51]). Conventional versions of this principle are formulated for the body B as a whole rather than for control volumes and as such generally involve particular boundary conditions applied to the boundary ∂B of B. Such formulations allow for a weak statement of the basic force balances and when combined with constitutive equations result in weak statements of the resulting boundary-value problems. *Here the principle of virtual power is used instead as a basic tool to determine the structure of the tractions and of the local force balances.*»

Indeed:

- The version in which the principle is formulated in [37] is not at all nonstandard: Hellinger formulated it in 1913!
- Piola and Germain [29, 41] and here Hellinger use the principle of virtual work "as a basic tool to determine the structure of the tractions and of the local force balances".
- Conventional versions of the principle of virtual work are formulated for all subbodies of a considered body (see the textbook used by Salençon at the École Polytechnique in Paris [85] and the English translation thereof [84]).
- Using the preceding limit argument presented here, it seems possible to give an answer to the question: In formulating the principle of virtual work do we need to assume that the virtual work vanishes for all (regular) virtual displacements of all (suitably regular) subbodies of the considered body? Or is it sufficient to assume that it vanishes for all regular displacements of the whole body only? Indeed Hellinger masters the concept of mollifiers in three-dimensional Euclidean space whose existence Urysohn will prove in a more general setting few years later.[52]

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[49] *Bull. soc. philomath. 1823, p. 9 and Exerc. de math. 2 (1827) = Oeuvres, sér. II, t. 7, p. 141; cf. the references in IV 6, No.* ***26****, Stäckel and IV 23, No.* ***3****b, Müller-Timpe.*

[50] Due to a typo in the original source, the subscript 1 in the surface element dS_1 is missing.

[51] This corresponds to reference [51] in the paper at hand.

[52] Urysohn lemma: For any two disjoint closed sets A and B of a normal space X there exists a real-valued function f, continuous at all points, taking the value 0 at all points of A, the value 1 at all points of B and for all $x \in X$ satisfying the inequality $0 \leq f(x) \leq 1$. See e.g. [15, 53].

These are precisely the equations obtained by the application of the so-called ***center-of-mass theorem*** *on the part V_1 being cut out of the continuum and being regarded as rigid in the above mentioned manner. Due to the arbitrariness of the domain V_1, as is generally known, one can gain from (10) the equations (5a). (cf. IV 23, Müller-Timpe, p. 23).*

He next imposes gradual rotations in a volume of a little bit larger than the considered subbody and gets the balance of moment of force (which he continues to call as done by Piola, the law of equal area, reminiscent of Kepler's law in the motion of planets).

Proceeding on the assumption of a rigid rotation of the subset V_1 with components $qz - ry$, $rx - pz$, $py - qx$, consequently three equations follow:

$$(11) \qquad \iiint\limits_{(V_1)} \{\varrho(Zy - Yz) + Y_z - Z_y\} dV_1 + \iint\limits_{(S_1)} (Z_n y - Y_n z) dS_1 = 0 \qquad (X, Y, Z).$$

This coincides exactly with the ***law of equal area*** *applied to V_1 if one adds to the moments of the spatially distribute forces X, Y, Z and the surface forces X_n, Y_n, Z_n an opposed torque exerted directly at the volume element corresponding to the vector components (2′) of the stress dyad. By postulating the law of equal areas in the usual form, i. e. the sum of the moments of the volume and surface forces vanishes, thereof the symmetry of the stress dyad follows immediately.*[53] *In close relationship to these facts is another notion of the principle of virtual displacements considering at first only the actual* ***forces****, i. e. the forces per unit mass X, Y, Z and the surface forces $\overline{X}, \overline{Y}, \overline{Z}$, as given; it is the following slightly advanced formulation by G. Piola*[54]*:* ***For the equilibrium it is necessary that the virtual work of the applied forces***

$$\iiint\limits_{(V)} (X\delta x + Y\delta y + Z\delta z) dV + \iint\limits_{(S)} (\overline{X}\delta x + \overline{Y}\delta y + \overline{Z}\delta z) dS$$

vanishes for all purely translational virtual displacements of the whole domain *V. Expressing the [corresponding] constraints of the displacements identical to the 9 partial differential equations:*

$$\frac{\partial \delta x}{\partial x} = 0, \ \frac{\partial \delta x}{\partial y} = 0, \ \ldots, \ \frac{\partial \delta z}{\partial z} = 0,$$

then, due to the well-known calculus of variations one can introduce 9 corresponding Lagrange multipliers $-X_x, -X_y, \ldots, -Z_z$ and [one] obtains precisely equation (4) of the old principle, whereas the ***components of the stress dyad*** *appear as* ***Lagrange multiplier of certain rigidity constraints****. Certainly, they are not determined by this variational principle and play in fact exactly the same role as the internal stresses of the statically indeterminate problems of rigid body mechanics.*[55]

Assuming the same requirement for ***all*** *rigid motions of V at all (instead of mere translations), one obtains precisely Piola's ansatz given in IV 23, p. 23, which provides due to the 6 constraints only 6 Lagrange multiplier and consequently a symmetric stress dyad.*

In this part of Hellinger paper it is not clear if he is aware of the true content of Piola's works. Indeed what is referred to above are statements which can be found in Piola's works. However, Piola develops the Lagrangian theory of deformable bodies and, by considering the subset of rigid virtual displacements, he proves (Piola's Theorem) that balance of force and moment of force are necessary conditions for the equilibrium. Moreover, Piola proves that introducing the constraint of rigidity makes the stress undetermined and therefore he assesses the logical necessity of the introduction of the theory of deformable bodies. It is not clear how the linguistic barrier prevented Hellinger to appreciate completely the value of Piola's works (see [25, 29]).

3e. Two- and one-dimensional continua in the three-dimensional space. *All these fundamentals can immediately be formulated also for two- and one-dimensional continua being embedded in the three-dimensional space, which have been mentioned at the end of No.* ***2****.*[56] *The only modification is the change in the dimension of*

[53] *This requirement denoted as "Boltzmann axiom" has been included by G. Hamel*[40] *into his axioms of the mechanics of volume elements.*

[54] *Modena Mem. 24, parte 1 (1848), p. 1; vgl. IV 23, No.* ***3****b, Müller-Timpe.*

[55] *Cf. also IV 6, No.* ***26*** *(Stäckel), p. 550 and IV 23, No.* ***3****b (Müller-Timpe), p. 24.*

[56] *For a series of particular problems these fundamentals can already be found in Lagrange, Mécan. anal.; s. 1. part, sect IV, No. 25 ff.; sect. V, chap. III.*

the domain of integration and that instead of the derivatives of the virtual displacements with respect to the three spatial directions, the derivatives with respect to the two or one coordinate within the deformed medium enter.

At first, we consider in particular a ***two-dimensional continuum****, which consists in the deformed state of a simply connected surface S with boundary curve C; on S let — for the sake of simplicity — u, v be an* ***orthogonal*** *system of parameters and let the line and surface elements be*

$$ds^2 = Edu^2 + Gdv^2,\ dS = hdudv,\ h = \sqrt{EG}$$

and ϱ denotes the surface density of the mass distribution. Then we consider the virtual work [expression]:

$$\delta A = \iint\limits_{(S)} \sum_{\binom{x\,y\,z}{X\,Y\,Z}} \Big\{ \varrho X \delta x - \Big(\frac{X_u}{\sqrt{E}} \frac{\partial \delta x}{\partial u} + \frac{X_v}{\sqrt{G}} \frac{\partial \delta x}{\partial v} \Big) \Big\} dS + \int\limits_{(C)} \sum_{\binom{x\,y\,z}{X\,Y\,Z}} \overline{X} \delta x \, ds. \tag{12}$$

Herein $X, Y, Z, \overline{X}, \overline{Y}, \overline{Z}$ denote the components of the applied forces in S per unit mass and on C per unit length, for the quantities $X_u, \ldots$ similar conclusions can be drawn as above for $X_x, \ldots$; on the one hand they result in certain forces exerted on the masses of S, on the other hand [they cause] a stress state within S such that due to this stress state on one side of every line element lying on S the force per unit length

$$X_\nu = X_u \cos(\nu, u) + X_v \cos(\nu, v) \tag{13}$$

acts; herein n denotes the normal direction lying within S and pointing in direction of the considered side of the element.

For a medium, which allows for all continuous displacements, one can solve the condition $\delta A = 0$ of the principle of virtual displacements with respect to 6 equilibrium conditions[57]*, by transforming δA using the familiar methods of integration by parts:*

$$\frac{1}{h} \left(\frac{\partial \sqrt{G} X_u}{\partial u} + \frac{\partial \sqrt{E} X_v}{\partial v} \right) + \varrho X = 0 \quad \textit{on } S \quad (X, Y, Z), \tag{14a}$$

$$X_u \cos \nu u + X_v \cos \nu v = \overline{X} \quad \textit{on } C \quad (X, Y, Z), \tag{14b}$$

here ν denotes the direction which is within the surface S and is normal to the curve C pointing away from the considered surface. — Also these equations can easily be transformed with respect to the initial parameters a, b, when starting with the transformed expression of the virtual work

$$\delta A = \iint\limits_{(S_0)} \sum_{\binom{x\,y\,z}{X\,Y\,Z}} \Big\{ \varrho_0 X - \Big(X_a \frac{\partial \delta x}{\partial a} + X_b \frac{\partial \delta x}{\partial b} \Big) \Big\} dadb + \int\limits_{(C_0)} \sum_{\binom{x\,y\,z}{X\,Y\,Z}} \overline{X} \delta x \frac{ds}{ds_0} ds_0 \tag{15}$$

where

$$h \frac{\partial(u, v)}{\partial(a, b)} X_u = X_a \frac{\partial u}{\partial a} + X_b \frac{\partial u}{\partial b} \qquad (X, Y, Z; u, v); \tag{16}$$

a comparison with (13) results in the interpretation, that $X_a, \ldots$ denote the forces due to the stress state acting at the line elements $a = \text{const.}$, $b = \text{const.}$, computed with respect to the unit of length in the a-b-plane.

Everything is entirely analogous for ***one-dimensional continua****.*[58] *Let s $(0 \leqq s \leqq l)$ be the arc length of the curve representing the deformed state, then one has*

$$\delta A = \int\limits_0^l \sum_{\binom{x\,y\,z}{X\,Y\,Z}} \Big\{ \varrho X \delta x - X_s \frac{\partial \delta x}{\partial s} \Big\} ds + \Big[\sum_{\binom{x\,y\,z}{X\,Y\,Z}} \overline{X} \delta x \Big]_{s=0}^{s=l}, \tag{17}$$

[57] *The general form of these equations from different viewpoints are given by E. and F. Cosserat, Corps déform., chap. III, by the way directly for the case of oriented particles (see No. 4b; cf. also IV 11, No. 20, K. Heun). For the particular problems treated since Lagrange's fundamental studies*[56]*) cf. also IV 6, No. 24, Stäckel.*

[58] *Cf. E. and F. Cosserat, Corps déformables, chap. II as well as IV 11, No. 19 (K. Heun) and IV 6, No. 23 (P. Stäckel).*

in which the interpretation of each quantity is obtained as above, and for arbitrary continuous variations the equilibrium conditions are

(18a) $$\frac{dX_s}{ds} + \varrho X = 0 \quad \textit{for} \quad 0 < s < l \quad (X, Y, Z)$$

(18b) $$X_s = \overline{X} \quad \textit{for} \quad s = 0,\ s = l \quad (X, Y, Z).$$

By using the formula

(19) $$\delta A = \int_0^{l_0} \sum_{\left(\substack{x\,y\,z \\ X\,Y\,Z}\right)} \Big\{ \varrho_0 X \delta x - X_a \frac{\partial \delta x}{\partial a} \Big\} da + \Big[\sum_{\left(\substack{x\,y\,z \\ X\,Y\,Z}\right)} \overline{X} \delta x \Big]_{a=0}^{a=l_0}, \quad X_s \frac{ds}{da} = X_a$$

it is also here convenient to introduce the initial parameter a as independent quantity.

In the present subsection simply first gradient two-dimensional and one-dimensional continua are considered. The application range of this kind of continua is rather limited: remark, for instance that in (14a) the equations of a plate or a shell are not included. However some interesting applications of the continuum models introduced here have been found (see e.g. [71]). Some interesting generalizations of the class of one-dimensional continua considered by Hellinger have been proposed in the subsequent literature. One-dimensional continua can be indeed used as "reduced-order" approximate models for three-dimensional bodies having one dimension preponderant with respect to the other two, even if some relevant deformation energy is stored in the changes of shape "along" neglected dimensions. he generalizations consists in introducing some extra kinematical descriptors to include in the picture the effects of such changes. The principle of virtual work plays a relevant role also in the deduction of evolution equations for reduced-order one-dimensional continuum models: the reader is referred for instance to the papers [9, 11, 12, 38, 40, 80, 87] and references cited therein for some examples of the application of such deduction processes.

4 Annotated translation of No. 4 (pp. 622–628)

In this section we comment on the No. 4 of Hellinger's article. It is rather astonishing that a huge part of the successive literature in mechanical sciences greatly underestimated its content. The prestige of the author, the prestige of the Encyklopädie der Mathematischen Wissenschaften, the lucid and crystal-clear exposition were not enough to save such a masterpiece in mathematical physics from a long oblivion. The text has been continuously available in many libraries, the language in which it is written is not dead, and the reputation of the author, although not universally recognized, has always been recognized, although not universally by a large part of the experts in the field (He is rather famous for the Hellinger-Reissner variational principle). However, the content of this number has been ignored practically for many decades by the greatest majority of the investigators in the field. In the preliminary conclusions we will try to find a justification for this circumstance.

4. Enhancement of the principle of virtual displacements.
4a. Appearance of higher order derivatives of displacements. *One can apply a number of enhancements to the ansatz of the principle of virtual displacements presented in No. **3**, which enable it to cover in the broadest sense all laws appearing in the mechanics of continua.*

Here Hellinger has formulated what can be regarded as an explicit conjecture about the structure theorem for distributions which as been proved later by Laurent Schwartz.

*The most obvious is to add to the virtual work per unit volume a linear form of the 18 **second derivatives** of the virtual displacements $\frac{\partial^2 \delta x}{\partial x^2}, \ldots$. In fact there have been problems, in which it was necessary to let the energy function depend on the second derivatives, which have led to expressions belonging to here; primarily this comes into consideration for one- and two-dimensional continua (wires and plates).*[59]

In these lines one can read one of the first formulation of the second gradient continua theory which after many decades will be developed in detail by Toupin and Germain (see [91], [41]).

[59] *Cf. the discussions about the potential-based approaches in No. **7a**, p. 645 as well as No. **8a**, p. 660.*

*A thorough treatment of this [new] ansatz from a more general point of view seems not to be available [in the literature] and becomes unnecessary by remarking, that one can transform using integration by parts the new additional terms in the volume integral to terms which include merely the **first** derivatives of $\delta x, \delta y, \delta z$; hence, the new actions within the body are classified in the sense of the old notion of the stress dyad. Certainly, a new **surface integral** of the form*

$$(1) \qquad \iint\limits_{(S)} \sum_{\left(\substack{x\,y\,z\\ X\,Y\,Z}\right)} \Big(\overline{X}_x \frac{\partial \delta x}{\partial x} + \overline{X}_y \frac{\partial \delta x}{\partial y} + \overline{X}_z \frac{\partial \delta x}{\partial z}\Big) dS$$

*appears, which at one point proves the existence of a **surface tension** as it has been considered in No. **3**e for an independently existing two-dimensional continuum, it may contain in addition expressions which are not included in (12) of No. **3**e [and] which depend on the derivatives of the $\delta x, \dots$ **normal** to the surface. These new effects of stresses applied at the surface, seem not to have found any application so far, while in contrast the other [remaining] terms simply contribute to the old boundary conditions (5b) of No. **3** in the same form as the terms appearing in (14a), and possibly lead to line distributed forces in the sense of (14b) at interfaces or lines of discontinuities.*[60]

The contact actions described here by Hellinger were intensively studied by Germain [41] and rediscovered in [37].

4b. Media with oriented particles. *When we enhance our consideration furthermore to the media with oriented particles defined in No. **2**b, then a new assumption must become valid, **that, also for every virtual rotation of the continuum, virtual work is expended being a linear homogeneous function of the totality of values of the rotational components** $\delta\pi, \delta\kappa, \delta\varrho$ for which we make the ansatz analogous to No. **3**, (1):*

$$(2) \qquad \iiint\limits_{(V)} \varrho(L\delta\pi + M\delta\kappa + N\delta\varrho)dV + \iint\limits_{(S)} (\overline{L}\delta\pi + \overline{M}\delta\kappa + \overline{N}\delta\varrho)dS - \iiint\limits_{(V)} \Big(L_x \frac{\partial \delta\pi}{\partial x} + L_y \frac{\partial \delta\pi}{\partial y} + \dots + N_z \frac{\partial \delta\varrho}{\partial z}\Big) dV.$$

*Here one can discuss similar arguments as in No. **3**a, where one naturally assumes again the requirements of the finiteness and the continuity of the 15 appearing coefficients. At first L, M, N and $\overline{L}, \overline{M}, \overline{N}$ represent the components of an **axial vector**, which has to be understood as a **torque** at the point within the body (per unit of mass) or at the point on the surface (per unit of area), respectively; then in fact we have here an force effect of exactly the same kind as in rigid body mechanics. Under coordinate transformations, the quantities $L_x, \dots, N_z$ still transform like the components of a dyad with the modification that the sign changes for reflections*[61]*; their interpretation can be found therein, that*

$$(3) \qquad L_n = L_x \cos nx + L_y \cos ny + L_z \cos nz \qquad (L, M, Z)^{62}$$

represents the components of the torque per unit area, which is exerted via a surface element on the matter being on the side of the positive normal direction n.

Again the ideas presented here by Hellinger are bound to be developed later for instance by Toupin and Germain (see [91], [41]). The present "corso" of mechanical science has seen a flourishing of the ideas presented here by Hellinger: without trying to write a complete list of the papers developing more recently the subject, we refer here to the following papers and textbooks: [16, 25, 35, 36, 43, 45, 46, 48, 50, 56–58, 62, 64–67, 77, 91, 95] and the references cited therein. All of the cited papers accept the point of view of Hellinger and base their treatment on the solid ground of suitable variational principles. The reader should remark that the extended kinematics considered here includes micro-rotations, but does not consider micro-deformations: a clear variational treatment of continua with micro-stretch is presented in [42], where the ideas presented by Hellinger are fully developed.

[60] *Cf. below No. **12**.*

[61] *For tensor components (i. e. for a symmetric dyad) W. Voigt (cf. Lehrbuch der Kristallphysik, Leipzig 1910, p. 132 ff.) has denoted the corresponding behavior using the adjective axial, in contrast to polar tensors, whose components do not change sign under inversion. About this classification one compares also the literature cited in 34).*

[62] There is a typo in the original source. Eq. (3) should hold for (L, M, N).

We now assume the principle of virtual displacements for the new continuum in enhanced form, ***that in the equilibrium position described by the 6 functions*** *No.* ***2****, (1)* ***and*** *(9),* ***the virtual work augmented by*** *(2)* ***must vanish for every admissible set of virtual displacements.*** *Being assured that the continuously deformable continuum is completely free, for which the triads can be each other relatively rotated independently also of the magnitude of the displacements, then $\delta x, \ldots, \delta\pi, \ldots$ are 6 completely arbitrary continuous functions, and by repeating the considerations of No.* ***3****c one finds, that the conditions (5) formulated therein remain unchanged and that they have to be completed only by following two sets of three equations formulated first by W. Voigt*[63] *and recently discussed in detail in the work of Cosserat*[64,65]*:*

(4a) $$\frac{\partial L_x}{\partial x} + \frac{\partial L_y}{\partial y} + \frac{\partial L_z}{\partial z} + \varrho L = 0 \quad \textit{in } V \qquad (L, M, N),$$

(4b) $$L_x \cos nx + L_y \cos ny + L_z \cos nz = \bar{L} \quad \textit{on } S \qquad (L, M, N).$$

Also these equations can be transformed to be formulated with respect to the initial parameters a, b, c, by transforming the virtual work of the internal surface torques to the form

(2′) $$-\iiint\limits_{(V_0)} \Big(\sum_{\binom{\pi\ \kappa\ \varrho}{L\,M\,N}} L_a \frac{\partial \delta\pi}{\partial a} + L_b \frac{\partial \delta\pi}{\partial b} + L_c \frac{\partial \delta\pi}{\partial c} \Big) dV_0,$$

where

(5) $$\Delta \cdot L_x = L_a \frac{\partial x}{\partial a} + L_b \frac{\partial x}{\partial b} + L_c \frac{\partial x}{\partial c} \qquad (L, M, N; x, y, z),$$

and where L_a, M_a, N_a denote the torque acting on an element of the surface $a = \text{const}$, computed with respect to the unit of area in the undeformed state. The equations of (4) are then substituted besides No. ***3****, (9) by the triple of equations*[66]*):*

(6a) $$\frac{\partial L_a}{\partial a} + \frac{\partial L_b}{\partial b} + \frac{\partial L_c}{\partial c} + \varrho_0 L = 0 \quad \textit{in } V \qquad (L, M, N),$$

(6b) $$L_a \cos n_0 a + L_b \cos n_0 b + L_c \cos n_0 c = k\bar{L} \quad \textit{on } S \qquad (L, M, N).$$

Also here one can obtain a connection to the equilibrium conditions of the rigid body, by starting in one case with a translation, then with a rotation, of a subset V_1 cut out of V thought of as being rigid, within which the triads are rigidly fixed with the continuum, considering them consequently to be carried along in parallel and rigidly, respectively; approximating these discontinuous displacements exactly as in No. ***3****d by continuous displacements, one finds on the one hand the unchanged equations No.* ***3*** *(10) of the center-of-mass theorem, but then one finds instead of the formulas (11) three equations*

(7) $$\iiint\limits_{(V_1)} \{\varrho(Zy - Yz + L) + Y_z - Z_y\}\, dV_1 + \iint\limits_{(S_1)} \{Z_n y - Y_n z + L_n\}\, dS_1 = 0 \qquad \binom{L, M, N}{X, Y, Z},$$

which express the law of equal areas in the current context. From these 6 integral conditions, which have to be satisfied for ***every*** *subset V_1, one can again derive the equilibrium conditions (4).*[67]

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[63] *Gött. Abhandl. 34 (1887), p. 11, where Voigt builds on the notions of Poisson.*[20] *Cf. also the discussion in Voigts presentation at the international congress of physicists in Paris 1900 (Rapp. prés. au congr. T. I, p. 277 = Gött. Nachr., math.-phys. Kl. 1900, p. 117) and the exposition in Voigts Kompendium I, p. 219ff, being free of any direct reference to molecular perceptions.*

[64] *E. and F. Cosserat, Corps déform., chap. IV, in particular p. 137. Cf. also IV 11, No.* ***21****, K. Heun.*

[65] *Except for the assignment of the signs, these equations are insofar different from the ones of Voigt and Cosserat, because there the entire torque $\varrho L + Y_z - Z_y, \ldots$ acting on a particle is denoted by a single letter.*

[66] *In a slightly different notation in E. and F. Cosserat, Corps déformables, p. 132.*

[67] *In this way Voigt, Kompendium I, p. 219 proceeds.*

When the triads are not any more free to move, then the equilibrium conditions (4) and No. ***3****, (5) are modified, since the summands (2) and No.* ***3****, (1) of the virtual work cannot be treated separately anymore. We just want to mention the case when the axes of the triad are fixed to the medium; then for every virtual displacement this will imply a rotation of the triad with the magnitude No.* ***2****, (4′) as components, and thus in particular new terms will be added to the components of the stress dyad. One have used this to interpret the appearance of torques even when using a symmetric stress dyad* $(X_y = Y_x, \dots)$.[68]

A remark is needed here: More recently, Richard Toupin has continued the study of the continuum models just described by Hellinger in [91]. The reasons for which these developments were waited for so long need a close inquiry.

For ***two- and one-dimensional media*** *with oriented particles (see No.* ***2****c) it yields similarly, by the application of the earlier used notation, that to the virtual work of the surface (No.* ***3****e, (12)) the summand*

$$(8)\qquad \iint\limits_{(S)} \sum_{\binom{\pi\,\kappa\,\varrho}{L N M}} \left\{ \varrho L \delta\pi - \Big(\frac{L_u}{\sqrt{E}} \frac{\partial \delta\pi}{\partial u} + \frac{L_v}{\sqrt{G}} \frac{\partial \delta\pi}{\partial v} \Big) \right\} dS + \int\limits_{(C)} \sum_{\binom{\pi\,\kappa\,\varrho}{L N M}} \bar{L} \delta\pi ds,$$

is added and that to the virtual work of the curve (No. ***3****e, (17)) a corresponding [term]*

$$(9)\qquad \int_0^l \sum_{\binom{\pi\,\kappa\,\varrho}{L N M}} \left\{ \varrho L \delta\pi - L_s \frac{\partial \delta\pi}{\partial s} \right\} ds + \Big[\sum_{\binom{\pi\,\kappa\,\varrho}{L N M}} \bar{L} \delta\pi \Big]_0^l$$

is added; accordingly, one obtains in the first case in addition to No. ***3****e, (14), the equilibrium equations*[69]

$$(10)\qquad \begin{array}{rl} \frac{1}{h}\Big(\frac{\partial \sqrt{G} L_u}{\partial u} + \frac{\partial \sqrt{E} L_v}{\partial v} \Big) + \varrho L = 0 & \text{on } S \\ L_u \cos \nu u + L_v \cos \nu v = \bar{L} & \text{on } C \end{array} \qquad (L, M, N),$$

in the second case one obtains in addition to No. ***3****e, (18)) the following*[70]

$$(11)\qquad \begin{array}{rl} \frac{dL_s}{ds} + \varrho L = 0 & \text{for} \quad 0 < s < l \\ L_s = \bar{L} & \text{for} \quad s = 0, s = l \end{array} \qquad (L, M, N).$$

Also the interpretation of $L_u, \dots$ *as specific torques formulated with respect to the deformed state is obtained similarly to No.* ***3****e; they are connected to the corresponding quantities formulated with respect to the undeformed configuration with the equations of the kind*

$$(12)\qquad h \frac{\partial(u,v)}{\partial(a,b)} L_u = L_a \frac{\partial u}{\partial a} + L_b \frac{\partial u}{\partial b} \quad \text{or} \quad L_s \frac{ds}{da} = L_a.$$

How much of the so called “modern” theory of continua with directors was already available to Hellinger is again surprising. Except for, maybe, the notation which often became more compact using tensorial algebra, the equations of the present subsection have been rediscovered several times, since 1913. Again the ideas presented by Hellinger have been continued after many years of apparent neglect: among those papers which seems to us have been animated by Hellinger’s spirit we found particularly interesting: [5,7,10,27,39,59,73–75,83,90] for the generalized theory of beams and [6,8,13,17,19,34, 35,39,75,76,80,81,86] for the generalized theory of plates and shells.

4c. Appearance of constraints. *Hitherto the principle of virtual displacements has been applied particularly for cases in which the continuum was continuously deformable in all possible ways. In the formulation of the principle also such continua are immediately included whose* ***movability is constrained by restrictions of any kind****, and in fact just some of the first problems in the mechanics of continua, treated by Lagrange*[71]*, are cases of this kind. Primarily, these constraints are expressed by* ***equations*** *for the functions (1), (9) of No.* ***2*** *describing*

[68] *See for instance J. Larmor, London math. Soc. Proc. 23 (1892), p. 127, Combébiac, Bull. soc. de math. 30 (1902), p. 108, 242.*
[69] *Cf. F. and E. Cosserat, Corps déform., chap. III as well as IV 11, No.* ***20****, (K. Heun).*
[70] *Cf. F. and E. Cosserat, Corps déform., chap. II, as well as IV 11, No.* ***19****, (K. Heun).*
[71] *Mécan. anal., 1. Part., Sect. V, Chap. III (inextensible wire and similar ones), Sect. VIII (incompressible fluid).*

the deformation, in which besides the functions also their derivatives with respect to a, b, c can enter; typically is an equation

$$(13) \qquad \omega(a,b,c;x,y,z;x_a,\ldots,z_a;\lambda,\mu,\nu;\lambda_a,\ldots,\nu_c)=0, \text{ where } x_a=\frac{\partial x}{\partial a},\ldots$$

*for **every** point of the domain V_0, but it is also possible to formulate similar equations for subsets, interfaces or similar ones. In any case thereby possible deformations or possible rotations of the adjoint triads are restricted, or particular relations between rotation of the triad and the deformation are demanded (e. g. a particular orientation of the triad with respect to the space or the medium; cf. above p. 626); The appearance of a, b, c in (13) indicates, that the type of the condition can vary from particle to particle. Inserting the variation of the deformation No. **2**, (3) or (10) in (13), then differentiation with respect to σ yields*

$$(14) \quad \delta\omega \equiv \sum_{(x\,y\,z)} \Big(\frac{\partial\omega}{\partial x}\delta x + \frac{\partial\omega}{\partial x_a}\delta x_a + \frac{\partial\omega}{\partial x_b}\delta x_b + \frac{\partial\omega}{\partial x_c}\delta x_c\Big) + \sum_{(\lambda\,\mu\,\nu)} \Big(\frac{\partial\omega}{\partial \lambda}\delta \lambda + \frac{\partial\omega}{\partial \lambda_a}\delta \lambda_a + \frac{\partial\omega}{\partial \lambda_b}\delta \lambda_b + \frac{\partial\omega}{\partial \lambda_c}\delta \lambda_c\Big) = 0,$$

*and since due to No. **2**, p. 608 the $\delta x_a, \ldots$ coincide with the derivatives of $\delta x, \ldots$, there is a **linear homogeneous condition for the virtual displacements**.*

***The principle of virtual displacements** then claims, **that δA vanishes for all functions $\delta x, \ldots$ which satisfy (14)**, and this can be realized, when equation (14) does not allow accidentally the elimination of a displacement component, by the introduction of a Lagrange multiplier[72] λ in the form*

$$(15) \qquad \delta A + \iiint_{(V)} \lambda\delta\omega dV = 0 \quad \text{for all} \quad \delta x, \ldots$$

*which corresponds exactly with the original principle; instead of volume integrals possibly there appear surface or curve integrals, when (13) exists only along individual surfaces or curves, or when the continuum is in fact merely a surface or a curve. The denotation of the factor λ as "pressure" will be addressed later on (No. **8b**, p. 662).*

*Finally, one should think of the possibility, likewise well-known from the mechanics of discrete systems, that **"unilateral" constraints** appear, which have the form of **inequalities** — let it be e. g., that the boundary of the continuum is restricted in its movability in one direction, let it be that the deformation quantities in the inside are subjected to certain inequalities (one can think of bodies, which do not allow any compression beyond a certain threshold, or similar conditions). Then also here, the equilibrium will be determined by Fourier's formulation[73] of the principle of virtual displacements, that for any system of virtual displacements, satisfying the constraints, the virtual work is negative or zero:*

$$\delta A \leqq 0.$$

It is remarkable that in the paper of Hellinger one can find already the conceptual frame which has been used, more than one century later, to study, with numerical methods, the problem of three point bending of second gradient three-dimensional bodies. Indeed in [61] the conceptual frame presented in the previous lines is applied to forecast the behavior of composite reinforcements using a micro-structured continuum model where suitable Lagrange multipliers are introduced to impose both bilateral and unilateral constraints. The bilateral constraints impose that the kinematical descriptors of micro-deformation coincide with macro-deformation, in such a way that the numerical scheme developed for first gradient micro-structured continua can be applied to the study of second gradient continua. The unilateral constraints are imposed to impose the impenetrability of the body on the rigid support on which it is supported. Again a very abstract mathematical treatment has proven to be an effective "practical" tool (see again [82] for a historical view on this point).

[72] *The treatment of multidimensional variational problems has been developed for the first time by Lagrange for the problems referred to in 71); cf. II A8, p. 622, Kneser.*

[73] *Cf. IV 1, No. **34**, Voss; The formulation in Gauss (Principia generalia theoriae figurae fluidorum in statu aequilibrii, Gott. Comment. rec. 7 (1830) = Werke 5, p. 35, german of R. H. Weber in Ostwald's Klassiker der exakten Wiss. No. 135, Leipzig 1903) a priori considers the enhancement to continua.*

5 Some preliminary conclusions

The Latin language has been used in all European universities between the 11th and 18th century. The elimination of (Neo-)Latin language as the universal language of science and higher education (see e.g. [93]) caused the establishment of a new Babel. Many languages competed for replacing Latin: French language nearly succeeded becoming for a while a nearly dominant one. However, Russian, English and German languages were always extensively used, and even Italian language tried to get a place in the internationally recognized panorama of scientific languages.

The competition ended with the establishment of English as the modern *lingua franca*. This success is not only related to the economical and political force of British Empire and United States, but also to two features of this language: i) even if it was syntactically and grammatically formalized by scholars whose education had been strongly influenced by Latin culture, its structure is relatively simple when compared with all other spoken languages, ii) it includes a large volume of words whose etymology is Latin, Anglo-Saxon or Greek.

The establishment of English as a dominant language caused a natural attitude in those scientists whose mother language was English: they developed the tendency to ignore everything which was not written in English, so that they preferred to "rediscover" what was not written in English instead of getting the translation of the original sources into English. The case of peridynamics [25] seems to us exemplary: the results of Gabrio Piola have been rediscovered some 150 years after their publication by a group of scientists whose linguistic competences did not allow them to read the many copies of Piola's works available in their libraries, simply because they are written in Italian, a nearly "forgotten" scientific language.

We do not want to claim that the linguistic barrier is the only cause of the loss of the content of a paper or a monograph. Many other concauses are active and need to be considered. In the case of the article by Hellinger which we are discussing, the fact that during the career of Hellinger himself English became the preeminent scientific language plays a relevant role.

More technical remarks are needed, concerning the intrinsic mathematical difficulties which a reader must confront in order to understand the ideas presented by Hellinger. The technical knowledge of the mathematical tools is an essential prerequisite to understand any advanced text in mathematical physics, i.e. a text placing itself in the tradition represented in an exemplary way by its most ancient highly and unanimously regarded champion: i.e. Archimedes of Syracuse.

Some texts written by Archimedes are being rediscovered and understood in their full scientific value and impact only since recently (see [1, 69]). Then only after Dedekind, the mathematical understanding of the structure of the real numbers coincides again with the understanding reached during the peaks of the Hellenistic times (see [82]). Indeed without a precise definition of the "continuum" of real numbers, as placed along a line (in Euclidean space) and incorporating rational numbers (as represented by fractions) one cannot understand the reasonings of Archimedes, which include a heuristic conjecture of a theorem of integration (we use here the modern nomenclature: but we are not afraid to do so, as in the literature such a "modernistic" interpretation has been nearly universally accepted at the end, see e.g. [52, 68]) and also the rigorous (in the modern sense) demonstration of such a theorem. Archimedes was the main character of the scientific and technological revolution which occurred during Hellenistic times: no ancient author did dispute his standing both as a scientist and as an engineer. Notwithstanding this reputation, many of his books were not reproduced enough to survive the Middle Age centuries and some of his most important contributions reached us via a sequence of unbelievable fortuitous circumstances (see again [1, 69]). This means that when a body of knowledge is based on a particularly abstract mathematical technique, it is more likely that its rather complex content can be lost. Actually, the more a theory is based on abstract mathematics the more difficult its preservation and transmission can be. Reputation of the author plays a strange role: Archimedes (as Einstein) has been placed in an Empyrean Heaven of great geniuses which cannot be understood because of their exceptionality: this sanctification has as final result that the ideas of these presumed semi-gods are lost because nobody tries to understand them. In [82] it is, in our opinion, proven without doubts that the same concept of "genius out of his times" or "outstanding exceptional mind" who "cannot be understood by his contemporaries" is false. Archimedes was maybe *primus* but for sure he was *inter pares*. There was at least a *beta* (i.e. Eratosthenes of Cyrene) after Archimedes and a beloved friend whom he treated as a peer, i.e. Conon of Samos. The true situation is that every great advancement of science is a choral challenge attained by a group of scientists which constructs a "cultural paradigm", in the sense of Kuhn [54]. Using the tools constructed and developed by this group, finally, remarkable results are obtained, maybe thanks to the last effort of one of its most talented representative. Archimedes was without any doubt an outstanding genius: however he was surrounded by peers who understood and appreciated his work, he had been a pupil belonging to a strong school (most likely he studied in Alexandria) and he contributed to the formation of successors (in his *On the Method* he explicitly states that he writes to show to his successors the proof strategy he had used to get his results). In several aspects Archimedes believed to be a solitary genius who was predestined not to be understood: however, he was member of a community where mathematical knowledge was highly regarded and cultivated.

Similarly, Einstein is not a single mind surrounded by the empty. Hellinger is one of the most prominent scientists who formed the mathematical-physics community which appreciated Einstein's works, which worked together with him for the development of science and which prepared new generations to understand his works. The translated and commented article by Hellinger is exactly one cornerstone in such an activity: the interested reader will see in the subsequent forthcoming translations that the Relativity is framed exactly in the same formal scheme outfitted for continuum mechanics. Hellinger is perfectly aware of the perfect continuity between Continuum Mechanics and Relativity and teaches his point of view showing a rarely surpassed vision.

The main message by Hellinger can be resumed rather simply. To master and produce original research in continuum theories in mathematical physics one needs to be familiar with:

- Tensor Algebra and Calculus (be this theory formulated *à la Levi-Civita* or *à la Klein-Gibbs-Kronecker* is not important),
- Calculus of Variations (a pillar of ancient and modern mathematics whose modern form was shaped by Euler and Lagrange),
- Functional Analysis and Theory of Distributions (where Hellinger contributed personally, following the mainstream of the French school headed by Fréchet, Gâteaux and Laurent Schwartz).

Hellinger does not try to present a simplified version of a mathematically complex theory: instead his presentation is self-consistent and comprehensive. For this reason his work has been overlooked and definitively underestimated. However, we are sure that he shared the persuasion that[74]

μή εἶναι βασιλικήν ἀτραπόν ἐπί γεωμετρίαν
Non est regia [inquit Euclides] ad Geometriam via
There is no royal road to geometry

The reader will not have any difficulty in being convinced by Hellinger that this statement is true also when replacing geometry with mathematical-physics and the First Book of Euclid's Elements with a textbook of Lagrangian school.

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