Variational methods in the theory of beams and lattices

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Synonyms

minimum principles, calculus of variations, rod theory

Definitions

With the minimum total potential energy principle, the question concerning equilibrium configurations of beam systems can be formulated as minimization problems. After postulating a particular form of the potential energy for a conservatively loaded system, the calculus of variations leads to necessary and sometimes even sufficient conditions for energy-minimizing equilibrium configurations.

Introduction

Intrinsic beam theory makes use of one-dimensional generalized continua to model the mechanical behavior of three-dimensional beam-like objects. While a one-dimensional continuum corresponds to a deformable curve in space, parameterized by a single parameter, say $s \in I \subset \mathbb{R}$, a generalized continuum is augmented by further kinematical quantities whose state depends merely on the very same param-

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eter. Accordingly, a configuration of a beam is fully described by a set of functions $\{y_i(s)\}$ where $i = 1,\ldots,m$. In this article the Euler–Bernoulli beam theory is addressed as a representative of the simplest nonlinear spatial beam theories. In this theory, the spatial curve, called the *centerline*, is augmented by an orthonormal director triad modeling the cross-sections of a thin three-dimensional elastic body as sections which remain plane and rigid for all configurations. Furthermore, the directors are restricted such that the centerline's tangents remain always orthogonal to these sections. The special theory considered here requires relatively modest empirical input with regard to constitutive equations, and incorporates a reasonably broad range of applications, ranging from flexible cables to beams with significant flexural and torsional rigidity. A system of n beams is called lattice, when the beams are interconnected such that they span a two-dimensional surface in space.

Let $E[y_i]$ be the postulated total potential energy functional of a conservatively loaded beam. The minimum total potential energy principle defines a stable equilibrium as *minimizers* of E in some suitable class of competing configurations $\{y_i^*\}$. In particular, a configuration $\{y_i\}$ with energy $E[y_i]$ is stable if and only if

$$E[y_i] \leq E[y_i^*] \,, \quad \text{for all admissible } y_i^* \,. \tag{1}$$

Admissibility does not only account for conditions such as e.g. fixed boundary conditions, but also for the specification of the function class from which the configurations are chosen. Minimizing configurations $\{y_i\}$ for which competing configurations $\{y_i^*\}$ satisfy $\sum_i |y_i^* - y_i| < \delta$ for some $\delta > 0$ and for all $s \in I$ are called *strong relative minimizers*. Alternatively, configurations are called *weak relative minimizers* if $\sum_i |y_i^* - y_i| + \sum_i |y_i^{*\prime} - y_i'| < \delta$, where prime denotes here and henceforth derivatives with respect to s. Among others, the calculus of variations aims to give necessary and sufficient conditions for (1). The most widespread necessary condition for a minimizer $\{y_i\}$ of (1) is that the first variation of the total potential energy evaluated at $\{y_i\}$ vanishes for all admissible variations. In beam theory this corresponds to the static equilibrium equations of a beam. However, there are also alternative conditions which give rise to requirements for constitutive parameters.

The minimum total potential energy principle is not only a method to derive conditions for energy-minimizing equilibrium configurations but also a heuristic method to develop beam theories. Especially, the additional requirement of the *invariance under superimposed rigid body motions* of the internal strain energy provides a suitable guideline for modeling the internal force effects within a beam.

Kinematics and constitutive hypothesis for beams

Configurations of spatial beams within the Euclidean vector space $\mathbb{E}^3$ are defined by mappings $\mathbf{r}, \mathbf{e}_i : I \to \mathbb{E}^3$, $i = 1,2,3$, where the closed interval $I \subset \mathbb{R}$ parametrizes the set of beam points, the centerline $\mathbf{r}$ typically represents the line of centroids of the body, and the $\mathbf{e}_i$ are vector-valued functions that specify the orientations of the cross-sections for each beam point, cf. Fig. 1. The reference placement is a configuration in which the functions $\mathbf{r}$ and $\mathbf{e}_i$ take the values $\mathbf{x}$ and $\mathbf{E}_i$, respectively. A convenient choice to parametrize the set of beam points is the arclength parameter $s \in I = [0, L]$ of the reference centerline curve $\mathbf{x}$ with total arclength L. In order that the set $\{\mathbf{E}_i\}$ specifies the orientations of the cross-sections, they are assumed to form an orthonormal triad for every $s \in I$. Furthermore, $\mathbf{E}_1$ is identified with the unit tangent to the centerline $\mathbf{x}$, i.e. $\mathbf{E}_1 = \mathbf{x}'$. The vectors $\mathbf{E}_2(s)$ and $\mathbf{E}_3(s)$ span the plane normal to the centerline at $\mathbf{x}(s)$. Without loss of generality, $\mathbf{E}_1 \cdot (\mathbf{E}_2 \times \mathbf{E}_3) = 1$ and $\mathbf{E}_2(s)$ and $\mathbf{E}_3(s)$ are identified with the geometric principle axes of the cross-sections.

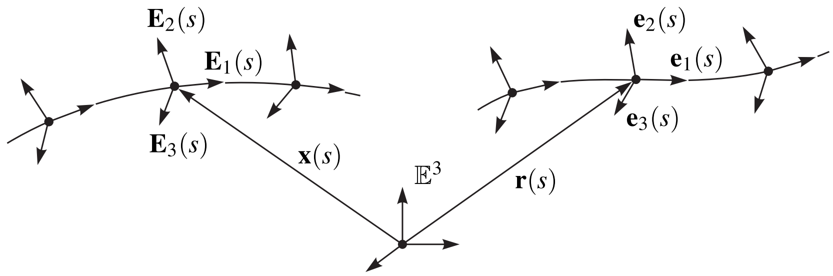

Fig. 1 Reference placement $\{\mathbf{x}, \mathbf{E}_i\}$ and configuration $\{\mathbf{r}, \mathbf{e}_i\}$ of a spatial Euler-Bernoulli beam.

A spatial Euler-Bernoulli beam is characterized by its constrained kinematics which demands the set $\{\mathbf{e}_i\}$ to be orthonormal for every s, with $\mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3) = 1$, and further requires $\mathbf{e}_1$ to coincide with the unit tangent, $\mathbf{t}$, to the centerline $\mathbf{r}$. Thus

$$\mathbf{r}' = \lambda \mathbf{t} \,, \tag{2}$$

where the local *stretch* λ is recognized, which in turn is defined as

$$\lambda := \|\mathbf{r}'\| = \sqrt{\mathbf{r}' \cdot \mathbf{r}'} \,. \tag{3}$$

Sometimes it is convenient to parametrize the beam configurations with respect to a non-arclength parameter $\nu \in \bar{I}$. The choice of a strictly increasing function $\phi : \bar{I} \to I, \nu \mapsto s = \phi(\nu)$ then induces the reparametrizations $\bar{\mathbf{x}} = \mathbf{x} \circ \phi, \bar{\mathbf{r}} = \mathbf{r} \circ \phi$ and $\bar{\lambda} = \lambda \circ \phi$ of the mappings $\mathbf{x}$, $\mathbf{r}$ and λ, respectively. Using the properties of the arclength parametrization, the local stretch $\bar{\lambda}$ then takes the form

$$\bar{\lambda} = \frac{\|\mathrm{d}\bar{\mathbf{r}}/\mathrm{d}\nu\|}{\|\mathrm{d}\bar{\mathbf{x}}/\mathrm{d}\nu\|} \,. \tag{4}$$

According to (4), the local stretch of a configuration can be interpreted as the ratio between the lengths of the centerline's tangent vectors in the actual configuration and the reference placement.

The constraints on $\{\mathbf{E}_i\}$ and $\{\mathbf{e}_i\}$ allow for the relation

$$\mathbf{e}_i(s) = \mathbf{R}(s)\mathbf{E}_i(s) \;, \tag{5}$$

where $\mathbf{R}(s)$ is a *rotation*, thus satisfying $\mathbf{R}(s)^{\mathrm{T}}\mathbf{R}(s) = \mathbf{R}(s)\mathbf{R}(s)^{\mathrm{T}} = \mathbf{1}$ ($\mathbf{1}$ denotes the identity tensor for $\mathbb{E}^3$) and $\det \mathbf{R} = +1$. Making here and in what follows use of Einstein's summation convention, the rotation can explicitly be written as

$$\mathbf{R} = \mathbf{R}\mathbf{1} = \mathbf{R}(\mathbf{E}_i \otimes \mathbf{E}_i) = (\mathbf{R}\mathbf{E}_i) \otimes \mathbf{E}_i = \mathbf{e}_i \otimes \mathbf{E}_i \;. \tag{6}$$

The kinematical description is completed by introducing a tensor $\mathbf{W}$ defined by

$$\mathbf{W} = \mathbf{e}_i' \otimes \mathbf{e}_i = W_{ij}\mathbf{e}_i \otimes \mathbf{e}_j \;, \qquad W_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j' \;. \tag{7}$$

This furnishes the rate of change of $\{\mathbf{e}_i\}$ with respect to s:

$$\mathbf{e}_i' = \mathbf{W}\mathbf{e}_i \;. \tag{8}$$

The differentiated orthonormality condition $(\mathbf{e}_i \cdot \mathbf{e}_j)' = \mathbf{e}_i' \cdot \mathbf{e}_j + \mathbf{e}_i \cdot \mathbf{e}_j' = 0$ together with (7b) implies that $W_{ij} = -W_{ji}$ and that $\mathbf{W}$ is consequently skew, i.e. $\mathbf{W}^{\mathrm{T}} = -\mathbf{W}$. Thus $\mathbf{W}(s)$ has an associated vector-valued function $\mathbf{w}(s)$, in the sense that $\mathbf{W}(s)\mathbf{u} = \mathbf{w}(s) \times \mathbf{u}$ for any $\mathbf{u} \in \mathbb{E}^3$. The use of $\mathbf{w}$ allows (8) to be written in the form

$$\mathbf{e}_i' = \mathbf{w} \times \mathbf{e}_i \tag{9}$$

The relation between the components $w_i = \mathbf{w} \cdot \mathbf{e}_i$ and W_{ij} is well known:

$$w_i = \tfrac{1}{2} e_{ijk} W_{kj} \;, \qquad W_{kj} = w_i e_{ijk} \;, \tag{10}$$

where e_{ijk} is the Levi-Civita permutation symbol, i.e. $e_{123} = e_{231} = e_{312} = +1$, $e_{213} = e_{132} = e_{321} = -1$, and zero else. It follows from (9) that the components w_α ($\alpha \in \{2,3\}$) account for the rate of change of the unit tangent $\mathbf{t} (= \mathbf{e}_1)$ with respect to s, while w_1 measures the projection onto the cross-section of the rate of change of the cross-sectional axes $\mathbf{e}_2$ and $\mathbf{e}_3$.

Let $\boldsymbol{\Omega}$ be the skew tensor defined by

$$\boldsymbol{\Omega} = \mathbf{R}^{\mathrm{T}}\mathbf{W}\mathbf{R} = W_{ij}\mathbf{E}_i \otimes \mathbf{E}_j \tag{11}$$

with its associated vector-equivalent

$$\boldsymbol{\kappa} = \kappa_i \mathbf{E}_i \;, \qquad \text{with} \quad \kappa_i = \tfrac{1}{2} e_{ijk} W_{kj} = \tfrac{1}{2} e_{ijk} \mathbf{e}_k \cdot \mathbf{e}_j' \;. \tag{12}$$

Evidently $\kappa_i = w_i$ and $\mathbf{w} = \mathbf{R}\boldsymbol{\kappa}$. A reparametrization of the beam configuration with respect to a non-arclength parameter $\nu \in \bar{I}$ induces additionally to $\bar{\mathbf{x}}, \bar{\mathbf{r}}, \bar{\lambda}$ also $\bar{\mathbf{e}}_i = \mathbf{e}_i \circ \phi$ and $\bar{\kappa}_i = \kappa_i \circ \phi$ for which

$$\bar{\kappa}_i = \frac{1}{2} e_{ijk} \bar{\mathbf{e}}_k \cdot \frac{\mathrm{d}\bar{\mathbf{e}}_j}{\mathrm{d}\nu} \frac{1}{\|\mathrm{d}\bar{\mathbf{x}}/\mathrm{d}\nu\|} \;. \tag{13}$$

If the beam has a precurved reference configuration this results in the presence of non-zero values of the functions

$$\kappa_i^0 = \tfrac{1}{2} e_{ijk} \mathbf{E}_k \cdot \mathbf{E}_j' \,, \tag{14}$$

which are the values of κ_i in the configuration $\{\mathbf{x}, \mathbf{E}_i\}$.

Strain measures must guarantee two invariance properties. Their values should neither change under reparametrization of the kinematic functions nor under superimposed rigid deformations. Invariance with respect to reparametrization is obtained for both strain measures λ and $\boldsymbol{\kappa}$, since they have been introduced with the aid of the reference arclength s being an invariant property of the reference centerline $\mathbf{x}$. When deciding for a non-arclength parametrizations ν, the strain measures have to be introduced in the sense of (4) and (13). Invariance of λ and $\boldsymbol{\kappa}$ under superimposed rigid deformations

$$\mathbf{r}(s) \mapsto \mathbf{Q}\mathbf{r}(s) + \mathbf{c} \,, \qquad \mathbf{e}_i(s) \mapsto \mathbf{Q}\mathbf{e}_i(s) \,, \qquad \mathbf{e}_i'(s) \mapsto \mathbf{Q}\mathbf{e}_i'(s) \,, \tag{15}$$

where $\mathbf{Q}$ is an arbitrary fixed rotation and $\mathbf{c}$ is an arbitrary fixed vector, follow straightforward from inserting (15) into (3) and (12).

Due to the latter invariance property, which is also required for the internal strain energy function, it is natural to formulate a theory for elastic beams by introducing a strain energy w per unit length of the reference placement, that depends on λ and $\boldsymbol{\kappa}$:

$$w = w(\lambda, \boldsymbol{\kappa}) \,. \tag{16}$$

Note that possible dependence on arclength s are suppressed which may arise due to nonuniformity of the material properties, or to the presence of non-zero values of the functions $\kappa_i^0(s)$. This dependence is left tacit, as it does not affect the considerations in the upcoming statements.

A model of elastic cables may be obtained by eliminating κ_i (and κ_i^0) from the list of arguments of the strain energy function. Alternatively, if dependence on λ is eliminated by constraining the local stretch such that $\lambda \equiv 1$, the theory of inextensible beams is obtained.

Total strain energy and its first variation

The strain energy of the beam is the functional of the configuration $\{\mathbf{r}, \mathbf{e}_i\}$ defined by

$$S[\mathbf{r}, \mathbf{e}_i] = \int_0^L w(\lambda(\mathbf{r}'(s)), \boldsymbol{\kappa}(\mathbf{e}_i(s), \mathbf{e}_i'(s)))\, \mathrm{d}s \,. \tag{17}$$

Let $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ for some positive number ε_0, and consider a smooth one-parameter family of kinematically admissible configurations $\{\mathbf{r}^*(s,\varepsilon), \mathbf{e}_i^*(s,\varepsilon)\}$, with $\{\mathbf{r}^*(s,0), \mathbf{e}_i^*(s,0)\} = \{\mathbf{r}(s), \mathbf{e}_i(s)\}$. Here kinematic admissibility means that, for each fixed ε, $\mathbf{r}^*(\cdot,\varepsilon)$ and $\mathbf{e}_i^*(\cdot,\varepsilon)$ are at least piecewise C^2 on $[0,L]$ and satisfy (2) and (3), i.e.

$$\mathbf{r}^{*\prime} = \lambda^* \mathbf{t}^* \,, \qquad \lambda^* = \|\mathbf{r}^{*\prime}\| \,, \qquad \mathbf{t}^* = \mathbf{e}_1^* \,, \tag{18}$$

as well as the orthogonality condition $\mathbf{e}_i^* \cdot \mathbf{e}_j^* = \delta_{ij}$. It is possible to relax the continuity hypothesis if equilibria with discontinuities in λ or $\boldsymbol{\kappa}$ are of interest, see Weierstrass–Erdmann corner conditions and Weierstrass inequality below. Considering (12), the variation of the configuration $\{\mathbf{r}^*, \mathbf{e}_i^*\}$ induces a variation of the change of orientation along s:

$$\boldsymbol{\kappa}^* = \kappa_i^* \mathbf{E}_i \,, \qquad \text{with} \quad \kappa_i^* = \tfrac{1}{2} e_{ijk} \mathbf{e}_k^* \cdot \mathbf{e}_j^{*\prime} \,. \tag{19}$$

Let superimposed dots denote the derivatives of functions with respect to ε, evaluated at $\varepsilon = 0$. For $h(\varepsilon) = S[\mathbf{r}^*(\cdot,\varepsilon), \mathbf{e}_i^*(\cdot,\varepsilon)]$, the first variation of S at the configuration $\{\mathbf{r}, \mathbf{e}_i\}$ is

$$\frac{\mathrm{d}h}{\mathrm{d}\varepsilon}(0) = \dot{h} = \int_0^L [(\partial w/\partial \lambda)\dot{\lambda}^* + (\partial w/\partial \kappa_i)\dot{\kappa}_i^*]\mathrm{d}s \,. \tag{20}$$

To analyze the structure of the first variation near $\varepsilon = 0$, the one-parameter families $\mathbf{r}^*, \mathbf{e}_i^*$ can be written as

$$\mathbf{r}^*(s,\varepsilon) = \mathbf{r}(s) + \varepsilon \mathbf{u}(s) + o(\varepsilon) \,, \qquad \text{where} \quad \mathbf{u} = \dot{\mathbf{r}}^* \,, \tag{21}$$

and

$$\mathbf{e}_i^*(s,\varepsilon) = \mathbf{e}_i(s) + \varepsilon \dot{\mathbf{e}}_i^* + o(\varepsilon) \,. \tag{22}$$

Similar to $\mathbf{W}$ of (7), the virtual rotation is defined as the skew tensor

$$\boldsymbol{\alpha}^* = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathbf{e}_i^* \otimes \mathbf{e}_i^* = \alpha_{ij}^* \mathbf{e}_i^* \otimes \mathbf{e}_j^* \,, \qquad \alpha_{ij}^* = \mathbf{e}_i^* \cdot \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathbf{e}_j^* \,, \tag{23}$$

which has its vector-equivalent $\mathbf{a}^*(s,\varepsilon)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathbf{e}_i^* = \boldsymbol{\alpha}^* \mathbf{e}_i^* = \mathbf{a}^* \times \mathbf{e}_i^* \,. \tag{24}$$

Consequently, the admissible variation of $\mathbf{e}_i$ in the sense of (22) is

$$\dot{\mathbf{e}}_i^* = \mathbf{a} \times \mathbf{e}_i \,, \qquad \text{where} \quad \mathbf{a} = \mathbf{a}^*(\cdot, 0) \,. \tag{25}$$

The smooth one-parameter family of the local stretch introduced by

$$\lambda^*(s,\varepsilon) = \lambda(s) + \varepsilon a(s) + o(\varepsilon) \,, \qquad \text{where} \quad a = \dot{\lambda}^* \,, \tag{26}$$

is not independent and must satisfy the compatibility condition

$$\mathbf{u}' = a\mathbf{t} + \lambda \mathbf{a} \times \mathbf{t} = a\mathbf{t} + \mathbf{a} \times \mathbf{r}' \,. \tag{27}$$

This compatibility follows from inserting (21), (22), (25) and (26) into the condition (18a). By multiplying (27) with $\mathbf{t} = \mathbf{e}_1$ and $\mathbf{e}_\alpha$ $(\alpha = 2,3)$, the condition can also be written as

$$\dot{\lambda}^* = a = \mathbf{t}\cdot\mathbf{u}' , \qquad \mathbf{u}'\cdot\mathbf{e}_\alpha + \mathbf{r}'\cdot\mathbf{a}\times\mathbf{e}_\alpha = 0 \text{ , for } \alpha = 2,3 \ . \tag{28}$$

The variation of $\kappa_i^*(s,\varepsilon)$ follows directly from (19) together with (22) and (25):

$$\dot{\kappa}_i^* = \tfrac{1}{2}e_{ijk}(\dot{\mathbf{e}}_k^*\cdot\mathbf{e}_j' + \mathbf{e}_k\cdot\dot{\mathbf{e}}_j^{*\prime}) = \tfrac{1}{2}e_{ijk}[\mathbf{a}\times\mathbf{e}_k\cdot\mathbf{e}_j' + \mathbf{e}_k\cdot(\mathbf{a}'\times\mathbf{e}_j + \mathbf{a}\times\mathbf{e}_j')] \ . \tag{29}$$

The terms involving **a** cancel. Making use of the relation $e_{ijk}e_{jkm} = 2\delta_{im}$ the result can be simplified further to

$$\dot{\kappa}_i^* = \tfrac{1}{2}e_{ijk}\mathbf{a}'\cdot\mathbf{e}_j\times\mathbf{e}_k = \mathbf{a}'\cdot(\tfrac{1}{2}e_{ijk}e_{jkm}\mathbf{e}_m) = \mathbf{e}_i\cdot\mathbf{a}' \ . \tag{30}$$

Introducing $\mathbf{M} = M_i\mathbf{e}_i$ with $M_i(s) = \partial w/\partial\kappa_i(\lambda(s),\boldsymbol{\kappa}(s))$, the relation (30) allows to rewrite the first variation (20) as

$$\dot{h} = \int_0^L [(\partial w/\partial\lambda)a + \mathbf{M}\cdot\mathbf{a}']\mathrm{d}s \tag{31}$$

In particular, when the constraints of (28) are satisfied, $\dot{h}$ coincides with the augmented functional

$$\begin{aligned} I &= \int_0^L [(\partial w/\partial\lambda)\mathbf{t}\cdot\mathbf{u}' + \mathbf{M}\cdot\mathbf{a}' + F_\alpha(\mathbf{u}'\cdot\mathbf{e}_\alpha + \mathbf{r}'\cdot\mathbf{a}\times\mathbf{e}_\alpha)]\mathrm{d}s \\ &= \int_0^L [\mathbf{F}\cdot\mathbf{u}' + \mathbf{M}\cdot\mathbf{a}' + \mathbf{r}'\cdot\mathbf{a}\times(F_\alpha\mathbf{e}_\alpha)]\mathrm{d}s \ , \end{aligned} \tag{32}$$

where $F_\alpha(s), \alpha\in\{2,3\}$, are Lagrange multipliers taking explicitly into account that the variations $\mathbf{u}(s)$ and $\mathbf{a}(s)$ are not independent, and

$$\mathbf{F} = (\partial w/\partial\lambda)\mathbf{t} + F_\alpha\mathbf{e}_\alpha \ . \tag{33}$$

For an inextensible beam the impressed force $\partial w/\partial\lambda$ also becomes a Lagrange multiplier and the force **F** is then interpreted as the constraint forces guaranteeing $\mathbf{r}' = \mathbf{t}$ whose variation is obtained from (27) for $\lambda\equiv 1$ and $a = 0$, i.e. $\mathbf{u}' = \mathbf{a}\times\mathbf{r}'$.

Equilibrium conditions for beams

For the sake of illustration, a beam is considered being clamped at $s = 0$ such that $\mathbf{r}(0)$ and $\mathbf{e}_i(0)$ are the only assigned kinematical data. Furthermore, the beam is subjected to dead forces **f**, applied at the end $s = L$, together with a dead distributed force $\mathbf{b}(s)$ per unit reference length of the beam. The distributed force **b** is at least piecewise continuous. The associated negative potential is

$$-P[\mathbf{r},\mathbf{e}_i] = -P[\mathbf{r}] \equiv \int_0^L \mathbf{b}\cdot\mathbf{r}\,\mathrm{d}s + \mathbf{f}\cdot\mathbf{r}(L) \ . \tag{34}$$

The one-parameter families $\mathbf{r}^*$ and $\mathbf{e}_i^*$ are in accordance with (21) and (22) together with the requirements $\mathbf{u}(0)=\mathbf{a}(0)=0$. A necessary condition for (1) and a sufficient for static equilibrium is that the associated first variation of the total potential energy functional $E[\mathbf{r},\mathbf{e}_i]=S[\mathbf{r},\mathbf{e}_i]-P[\mathbf{r}]$ vanishes for all admissible variations of the kinematical variables. Introducing $f(\varepsilon)=E[\mathbf{r}^*(\cdot,\varepsilon),\mathbf{e}_i^*(\cdot,\varepsilon)]$, this yields the condition $\dot{f}=0$. This variation can be computed explicitly using the result of (32) together with (34) and (21) under consideration that $\mathbf{u}(0)=\mathbf{a}(0)=0$:

$$0=\int_0^L[\mathbf{F}\cdot\mathbf{u}'+\mathbf{M}\cdot\mathbf{a}'+\mathbf{r}'\cdot\mathbf{a}\times\mathbf{F}]\mathrm{d}s-\int_0^L\mathbf{b}\cdot\mathbf{u}\,\mathrm{d}s-\mathbf{f}\cdot\mathbf{u}(L) \tag{35}$$

Since $\mathbf{r}'$ is parallel to $\mathbf{t}$, the last term in (32) has been replaced with $\mathbf{r}'\cdot\mathbf{a}\times\mathbf{F}$. Integration by parts of (35) directly leads to

$$0=-\int_0^L[\mathbf{u}\cdot(\mathbf{F}'+\mathbf{b})+\mathbf{a}\cdot(\mathbf{M}'-\mathbf{F}\times\mathbf{r}')]\mathrm{d}s+\mathbf{u}(L)\cdot(\mathbf{F}(L)-\mathbf{f})+\mathbf{a}(L)\cdot\mathbf{M}(L)\,. \tag{36}$$

According to the fundamental lemma of calculus of variations, (36) is only fulfilled if $\mathbf{M}(L)$ and $\mathbf{F}(L)$ satisfy the natural end conditions

$$\mathbf{M}(L)=0\,,\qquad \mathbf{F}(L)=\mathbf{f} \tag{37}$$

and only if the Euler-Lagrange equations

$$\mathbf{F}'+\mathbf{b}=0 \tag{38}$$

$$\mathbf{M}'-\mathbf{F}\times\mathbf{r}'=0 \tag{39}$$

are satisfied for $0<s<L$.

Equation (37b) identifies $\mathbf{F}(L)$ as the force supplied at the end $s=L$ by an external agency. Integration of (38) yields

$$\mathbf{F}(s)=\mathbf{F}(L)-\int_s^L\mathbf{F}'(\bar{s})\mathrm{d}\bar{s}=\mathbf{f}+\int_s^L\mathbf{b}(\bar{s})\mathrm{d}\bar{s}\,. \tag{40}$$

Integration of (39), together with (37a) and subsequent integration by parts leads to

$$\begin{aligned}\mathbf{M}(s)&=\mathbf{M}(L)-\int_s^L\mathbf{M}'(\bar{s})\mathrm{d}\bar{s}=\int_s^L\mathbf{r}'(\bar{s})\times\mathbf{F}(\bar{s})\mathrm{d}\bar{s}\\&=-\int_s^L\mathbf{r}(\bar{s})\times\mathbf{F}'(\bar{s})\mathrm{d}\bar{s}+\mathbf{r}(L)\times\mathbf{F}(L)-\mathbf{r}(s)\times\mathbf{F}(s)\,.\end{aligned} \tag{41}$$

For a fixed $s\in[0,L]$, let $\boldsymbol{\rho}(\bar{s})=\mathbf{r}(\bar{s})-\mathbf{r}(s)$ be the vector connecting the point $\mathbf{r}(s)$ with $\mathbf{r}(\bar{s})$ for an arbitrary $\bar{s}\in[0,L]$. Inserting $\mathbf{r}(\bar{s})=\mathbf{r}(s)+\boldsymbol{\rho}(\bar{s})$, (38) and (40) in (41), $\mathbf{M}(s)$ can be manipulated further to

$$\mathbf{M}(s) = \int_s^L (\mathbf{r}(s) + \boldsymbol{\rho}(\bar{s})) \times \mathbf{b}(\bar{s})\mathrm{d}\bar{s} + (\mathbf{r}(s) + \boldsymbol{\rho}(L)) \times \mathbf{f} - \mathbf{r}(s) \times \mathbf{F}(s)$$
$$= \int_s^L \boldsymbol{\rho}(\bar{s}) \times \mathbf{b}(\bar{s})\mathrm{d}\bar{s} + \boldsymbol{\rho}(L) \times \mathbf{f}\,. \tag{42}$$

From (40) and (42) it becomes apparent that $\mathbf{F}(s)$ and $\mathbf{M}(s)$ are, respectively, the force and moment exerted by the segment $(s,L]$ on the part $[0,s]$. Equation (37a) then requires that the moment vanishes at the unrestrained end. With these interpretations of $\mathbf{F}$ and $\mathbf{M}$, (38) and (39) are the classical equilibrium equations of beam theory, cf. (Antman 2005; Love 1944). In view of (33), for the Euler–Bernoulli beam, it is only the tangential component of the force that is determined by a constitutive equation. The transverse components F_α are shear reactions that do no virtual work for any variation of the configuration compatible with (28). They are determined by equilibrium considerations alone. For inextensible beams, even the tangential component is determined only by equilibrium considerations.

Further necessary conditions for beams (Steigmann and Faulkner 1993)

For the special case of an inextensible Euler–Bernoulli beam with the strain energy density $w = w(\boldsymbol{\kappa})$ the continuity assumptions on $\mathbf{e}_i$ can be relaxed such that the functions may have discontinuous derivatives at a finite number of points in the interval $(0,L)$ i.e. $\mathbf{e}_i \in C^1$, piecewise. The terminology used in the calculus of variations denotes the points of discontinuity of the derivatives $\mathbf{e}_i'$ as *corners*. This terminology is a bit misleading in the present context as the configurations considered have continuously turning tangents. The corners are points of discontinuity of the curvatures and twist. For such relaxed assumptions, minimizing configurations must also satisfy, besides the stationarity condition (35), further conditions.

Weierstrass–Erdmann corner conditions: Denoting the left and the right limit of a function with superscript $-$ and $+$, the *Weierstrass–Erdmann corner conditions*

$$\mathbf{M}(s^+) - \mathbf{M}(s^-) = 0\,, \quad w(\boldsymbol{\kappa}(s^+)) - w(\boldsymbol{\kappa}(s^-)) = \mathbf{M} \cdot (\kappa_j(s^+) - \kappa_j(s^-))\mathbf{e}_j\,, \tag{43}$$

must hold at the corners.

Weierstrass inequality: Let $\hat{\boldsymbol{\kappa}} = \hat{\kappa}_i \mathbf{E}_i$ with $\hat{\kappa}_i = \kappa_i + \frac{1}{2} e_{ijk} \mathbf{e}_k \cdot \boldsymbol{\alpha} \mathbf{e}_j$ for a skew $\boldsymbol{\alpha}$. A necessary condition for configurations with corners that minimize the energy with respect to strong variations is the *Weierstrass inequality*

$$w(\hat{\boldsymbol{\kappa}}) - w(\boldsymbol{\kappa}) - (\hat{\kappa}_j - \kappa_j)\partial w / \partial \kappa_j \geq 0, \forall s \in (0,L)\,. \tag{44}$$

Legendre condition: For the *stiffness* $C_{ij} = \partial^2 w / \partial \kappa_i \partial \kappa_j$, which is evaluated at the configuration of the weak relative minimizer, the Weierstrass inequality (44) implies the Legendre condition

$$C_{ij} a_i a_j \geq 0, \forall a_i, a_j\,. \tag{45}$$

Equilibrium conditions for lattices

A system of n beams is called lattice, when the beams are interconnected such that they span a two-dimensional surface in space. Let the jth beam have arclength L_j in its reference configuration. Suppose these are joined together at l nodes located at the positions $\mathbf{x}_k$ with $k = 1,\dots,l$. After deformation, the nodes displace to the (unknown) positions $\mathbf{y}_k$. Let the collection of index labels of these nodes be the set K. At each node $k \in K$, a dead load $\mathbf{q}_k$ is prescribed. In addition, we suppose that m nodes are fixed at the locations $\mathbf{z}_h$ with $h = 1,\dots,m$. Their labels belong to the set H. The collection of all node labels is $K \cup H$.

Let $s_j \in [0, L_j]$ measure the reference arclength along the jth beam. The position function in a typical configuration is $\mathbf{r}_j(s_j)$ and the director triad is $\{\mathbf{e}_i(s_j)\}_j$. It is convenient to introduce the sets

$$\begin{aligned} I_k &= \{j : s_j = 0 \text{ at node } k \in K\}, \qquad & E_k &= \{j : s_j = L_j \text{ at node } k \in K\}\ , \\ I_h &= \{j : s_j = 0 \text{ at node } h \in H\}, \qquad & E_k &= \{j : s_j = L_j \text{ at node } h \in H\}\ . \end{aligned} \tag{46}$$

Henceforth superscripts 0 and L denote the values of functions at $s_j = 0$ and $s_j = L_j$, respectively, such that $\mathbf{r}_j^0 = \mathbf{r}_j(0)$ and $\mathbf{r}_j^L = \mathbf{r}_j(L)$. These are subject to the conditions

$$\mathbf{r}_j^0 = \mathbf{y}_k,\ j \in I_k\ ; \quad \mathbf{r}_j^L = \mathbf{y}_k,\ j \in E_k\ ; \quad \mathbf{r}_j^0 = \mathbf{z}_h,\ j \in I_h\ ; \quad \mathbf{r}_j^L = \mathbf{z}_h,\ j \in E_h\ , \tag{47}$$

which ensure the continuity of the lattice at the nodes.

The total potential energy, E, of a configuration of the entire lattice is

$$E[\mathbf{r}_j, \{\mathbf{e}_i\}_j, \mathbf{y}_k] = \sum_{j=1}^{n} S[\mathbf{r}_j, \{\mathbf{e}_i\}_j] - \sum_{k=1}^{l} \mathbf{q}_k \cdot \mathbf{y}_k\ , \tag{48}$$

where S is the total strain energy functional (17). Let $f(\varepsilon) = E[\mathbf{r}_j^*(\cdot,\varepsilon), \{\mathbf{e}_i^*(\cdot,\varepsilon)\}_j, \mathbf{y}_k^*(\cdot,\varepsilon)]$. A configuration of the lattice is equilibrated if and only if, for all admissible variations of the kinematical variables,

$$0 = \dot{f} = \sum_{j=1}^{n} \int_0^{L_j} [\mathbf{F}\cdot\mathbf{u}' + \mathbf{M}\cdot\mathbf{a}' + \mathbf{r}'\cdot\mathbf{a}\times\mathbf{F}]\mathrm{d}s - \sum_{k=1}^{l} \mathbf{q}_k\cdot\mathbf{u}_k\ , \qquad \mathbf{u}_k = \dot{\mathbf{y}}_k^*\ , \tag{49}$$

where (32) has been invoked, in which the last term has been replaced with $\mathbf{r}' \cdot \mathbf{a} \times \mathbf{F}$, and the index j has been suppressed in the integrand for the sake of clarity. Integration by parts of the first two terms in (49) leads to

$$\begin{aligned} 0 = &-\sum_{j=1}^{n} \int_0^{L_j} [\mathbf{F}'\cdot\mathbf{u} + \mathbf{a}\cdot(\mathbf{M}' - \mathbf{F}\times\mathbf{r}')]\mathrm{d}s - \sum_{k=1}^{l} \mathbf{u}_k \cdot \Big[\mathbf{q}_k - \Big(\sum_{j\in E_k} \mathbf{F}_j^L - \sum_{j\in I_k} \mathbf{F}_j^0\Big)\Big] \\ &+ \sum_{k=1}^{l} \Big(\sum_{j\in E_k} \mathbf{M}_j^L\cdot\mathbf{a}_j^L - \sum_{j\in I_k} \mathbf{M}_j^0\cdot\mathbf{a}_j^0\Big) + \sum_{h=1}^{m} \Big(\sum_{j\in E_h} \mathbf{M}_j^L\cdot\mathbf{a}_j^L - \sum_{j\in I_h} \mathbf{M}_j^0\cdot\mathbf{a}_j^0\Big) \end{aligned} \tag{50}$$

wherein the following constraints have been imposed

$$\mathbf{u}_j^0 = \mathbf{u}_k,\ j \in I_k\ ; \quad \mathbf{u}_j^L = \mathbf{u}_k,\ j \in E_k\ ; \quad \mathbf{u}_j^0 = 0,\ j \in I_h\ ; \quad \mathbf{u}_j^L = 0,\ j \in E_h\ , \tag{51}$$

wherein restrictions on the virtual rotations $\mathbf{a}_j^0$ and $\mathbf{a}_j^L$ must also be imposed in accordance with the particular type of nodal connection under consideration.

Now (50) must be satisfied for all admissible $\mathbf{u}_k$, $\mathbf{a}_j^0$ and $\mathbf{a}_j^L$. Null values of these variations are admissible in all lattice types, and for this choice, the first sum of (50) must vanish. By choosing $\mathbf{u}(s)$ and $\mathbf{a}(s)$ to be non-zero in each of the n beams in succession, the fundamental lemma of calculus of variations then immediately leads in each beam to the equilibrium equations (38) and (39) in the absence of distributed load

$$\mathbf{F}' = 0\ , \qquad \mathbf{M}' = \mathbf{F} \times \mathbf{r}'\ . \tag{52}$$

With (52) satisfied in each beam all integral expressions in (50) vanish, and the remaining expression must be satisfied for all $\mathbf{u}_k$ and for all admissible $\mathbf{a}_j^0$ and $\mathbf{a}_j^L$. On setting $\mathbf{a}_j^0 = 0$ and $\mathbf{a}_j^L = 0$ and taking all but one of the $\mathbf{u}_k$ to be zero in succession, one obtains the nodal force balance equations

$$\sum_{j \in E_k} \mathbf{F}_j^L - \sum_{j \in I_k} \mathbf{F}_j^0 = \mathbf{q}_k\ , \qquad k \in K\ . \tag{53}$$

If the rotations are unrestricted in a particular node $k' \in K$, the $\mathbf{a}_j^0$ and $\mathbf{a}_j^L$ may be specified independently for each $j \in E_{k'} \cup I_{k'}$. Consequently, the node is equilibrated only if it transmits no moment to any of the attached beams:

$$\mathbf{M}_j^0 = 0\ , \quad j \in I_{k'}\ , \qquad \mathbf{M}_j^L = 0\ , \quad j \in E_{k'}\ . \tag{54}$$

Alternative restrictions, such as constrained rotations or concurrent axes of rotation, can be found in (Steigmann 1996).

Further necessary conditions for lattices (Steigmann 1996)

The second variation of (48) is defined as $\ddot{f} = \mathrm{d}^2 f/\mathrm{d}\varepsilon^2(0)$. Considering weak relative minimizers, for an equilibrium to be stable, it is necessary that the second variation of the energy, evaluated at that configuration, is nonnegative. Introducing for the jth beam $\mathbf{M} = \partial w/\partial \kappa_i \mathbf{e}_i$, $\mathbf{D} = \partial^2 w/\partial \lambda \partial \kappa_i \mathbf{e}_i$, $\mathbf{C} = \partial^2 w/\partial \kappa_i \kappa_j \mathbf{e}_i \otimes \mathbf{e}_j$ together with the integral expressions

$$\begin{aligned} F_j &= \int_0^{L_j} (\partial^2 w/\partial \lambda^2) a^2 + \mathbf{a}' \cdot \mathbf{C}\mathbf{a}' + 2a\mathbf{D} \cdot \mathbf{a}' \mathrm{d}s \\ G_j &= \int_0^{L_j} \mathbf{M} \cdot \mathbf{a}' \times \mathbf{a} - 2a\mathbf{F} \cdot \mathbf{a} \times \mathbf{t} - \mathbf{F} \cdot \mathbf{a} \times (\mathbf{a} \times \mathbf{r}'), \end{aligned} \tag{55}$$

the non-negativity of the second variation can be written as

$$\sum_j (F_j + G_j) \geq 0\,. \tag{56}$$

Legendre inequality: Considering local variations $a(s)$ and $\mathbf{a}(s)$ in (56) that vanish identically in all but one of the beams, together with an estimation of the orders of magnitudes, the *Legendre necessary condition* is obtained:

$$(\partial^2 w/\partial\lambda^2)e^2 + \mathbf{e}\cdot\mathbf{C}\mathbf{e} + 2e\mathbf{D}\cdot\mathbf{e} \geq 0 \qquad \forall e, \mathbf{e}\,. \tag{57}$$

This holds at every point $s \in [0, L]$ in a minimizing configuration. Obvious necessary conditions for this are

$$\partial^2 w/\partial\lambda^2 \geq 0 \quad \text{and} \quad \mathbf{e}\cdot\mathbf{C}\mathbf{e} \geq 0 \quad \forall \mathbf{e}\,. \tag{58}$$

The first requires that the extensional modulus be nonnegative, while the second states that the tensor of moduli associated with torsion and flexure is nonnegative definite. The latter result is equivalent to the Legendre necessary condition for inextensible rods for strong relative minimizers (45).

Cable network: For a system of elastic cables, which are modeled by suppressing the dependence of the strain energy (16) on the variable $\boldsymbol{\kappa}$, the non-negativity of the second variation induces the conditions

$$\mathrm{d}w/\mathrm{d}\lambda \geq 0 \quad \text{and} \quad \mathrm{d}^2 w/\mathrm{d}\lambda^2 \geq 0\,. \tag{59}$$

Hence, the cable cannot support a compressive force in stable equilibrium. This condition is not only necessary but also sufficient.

Cross-References

- Geometrically exact equations for beams
- Principle of virtual work and Lagrange multipliers
- Postulations of continuum mechanics, history of development

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