Geometric description of time-dependent finite-dimensional mechanical systems

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Abstract: Using the non-standard geometric structure proposed by Loos [29], we present a coordinate-free formulation of the theory for time-dependent finite-dimensional mechanical systems with n degrees of freedom. The state space containing the system's information on time, position and velocity is defined as a $(2n+1)$-dimensional affine bundle over an $(n+1)$-dimensional generalized space-time. The main goal is to present a geometric postulate that characterizes a second-order vector field whose integral curves describe the motions of a time-dependent finite-dimensional mechanical system. The core objects of the postulate are differential two-forms on the state space, called action forms, which are in a bijective relation with second-order vector fields. The requirements for a differential two-form to be an action form allow for a coordinate-free definition of non-potential forces, which may depend on time, position and velocity. Finally, we show that not only Lagrange's equations but also Hamilton's equations follow directly as mere coordinate representations of the same coordinate-free postulate.

1 Introduction

In a coordinate-free description of time-independent finite-dimensional mechanical systems the n-dimensional configuration manifold Q and its $2n$-dimensional tangent or cotangent bundles TQ and T^*Q, respectively, play a central role. For time-dependent mechanical systems, however, time needs to be included in the space on which the related physical theory is formulated. A straightforward approach to incorporate explicit time-dependence is to consider the extended state space $\mathbb{R}\times TQ$, or to extend the phase space as $\mathbb{R}\times T^*Q$. Both are spaces on which time-dependent Lagrangian and Hamiltonian formalisms have been established, see [9, 33, 36] and [1, 2], respectively. However, the physical interpretation of these spaces is problematic because their structure as Cartesian product assumes the existence of an absolute space Q, which is independent of time. This assumption can be dropped when using the concept of a Galilean manifold as introduced by Dombrowski and Horneffer [10, 11]. Therein the Galilean manifold M is defined as an $(n+1)$-dimensional smooth manifold equipped with a time-structure and a Galilean metric. While the time-structure allows to measure the temporal distance between two points of the space-time manifold, the Galilean metric captures the inertia of the mechanical system.

It was Loos in [29, 30], who seized the idea of the Galilean manifold and defined the state space of a mechanical system as a $(2n+1)$-dimensional affine subbundle of the tangent bundle of M. Following Souriau [40], Loos characterizes the motion of a finite-dimensional mechanical systems using a differential two-form defined on this state space. The general approach of using two-forms in this context can be traced back to Élie Cartan's lectures on integral invariants

[7]. Including the study of bilaterally and unilaterally constrained mechanical systems, Gallissot [14] demonstrates that the use of differential two-forms leads to a far-reaching approach in the description of finite-dimensional mechanical systems. Souriau's book [40] clearly pursues the way taken by Élie Cartan and François Gallissot. Indeed, the link can be formally made because the work of Gallissot is one of the few references given by Souriau.

By his "Maxwell's principle", Souriau [40] focuses on the study of mechanical systems that are subjected exclusively to potential forces. In [29, 30], Loos recognizes this principle as too restrictive and proposes a geometric theory for time-dependent finite-dimensional systems that can also deal with nonpotential forces, i.e., a rigorous differential geometric formulation that can cope with the requirements from the classical formulations of analytical mechanics, cf. pp. 79 of [35]. For the time-independent case, already Godbillon [15] incorporated nonpotential forces in his theory by studying the geometry of the double tangent bundle of the configuration manifold. Much of the mathematical structures exposed by Godbillon reappear in the description of time-dependent systems. The works of Lichnerowicz [28] and of his student Klein [18] deal with the description of mechanical systems involving nonpotential forces within the calculus of variations.

An alternative branch of research that applies and explores the geometric structures of the Galilean space-time has emerged under the name Newton–Cartan theory [44, 16, 11, 20, 12, 31, 38]. Newton–Cartan theory formulates Newton gravity within an intrinsic geometric framework in an analogue way to general relativity. More recently, Bekaert [3] has come up with a generalization of the Newton-Cartan theory in which an extensive discussion on different space-time models is presented.

Let us come back to Loos' theory, whose transmission has been quite an odyssey. In fact, his contributions have almost fallen into oblivion. The main source [29], which is written in German, is a typescript related to a seminar held in the winter semester 1981/1982 by Ottmar Loos and Josef Rothleitner at the University of Innsbruck in Austria. Since the script has never been officially published, it can only be found at an antiquarian bookseller or one can get it accidentally from one's PhD advisor; this is what happened to the first author of this publication. Ironically, the declared objective of the script [29] was to make the results known in the French school of mechanics available to the German-speaking scientific community. With the rigorous definition of time-dependent nonpotential forces, the typescript written by Loos goes definitively beyond the French school in those times and also beyond recent findings. Among other results, only parts of the script are contained in the paper [30] that was published in English. Nevertheless, until the submission of this work in 2019, the latter paper has been ignored anyway.[1] To the authors' knowledge, the discussed results have only partly made their way into the English literature centered around the standard textbook [1] and the English references therein. Many of the mathematical results are available in English in [27] and [33]. More recent publications on the subject such as [32], [34], [6], [39], [4] or [8] ignore the contributions of Loos. It is not astonishing that one can find recent publications such as [5], which claim to extend the theory while ignoring the existing results.

One aim of this paper is therefore to communicate and popularize the con-

[1] According to Google Scholar it has been cited only three times until December 2019.

tributions of Loos; an objective that has started with the PhD thesis [45]. The theory of Loos provides a geometric description of finite-dimensional mechanical systems that does not involve restrictive assumptions such as the limitation to time-independent systems or the exclusion of nonpotential forces. Another goal is to establish a single postulate from which various classical results about finite-dimensional mechanical systems can be deduced. In particular, we show in this paper how Lagrange's and Hamilton's equations follow directly as different coordinate representations of the same coordinate-free postulate. In the time-dependent case, we can thus unite the Lagrangian and the Hamiltonian side which often appear as "separate" worlds in time-independent formulations.

The present paper uses the language of differential geometry as presented for example in [26, 25, 42]. In Sect. 2, we introduce frequently-used notation to facilitate reading and to avoid ambiguities. In Sect. 3, we define the generalized space-time as Galilean manifold. The state space and the motion of a mechanical system are discussed in Sect. 4. Thereafter, the main achievements of Loos are presented in the Sects. 5 – 9. In Sect. 10, we study the appearance of inertia forces when changing the frame of reference in a generalized sense. Moreover, we give a distinct condition for which a force two-form is a potential force. Subsequently, the Lagrangian and the Cartan one-form are discussed in Sect. 11. Finally, in Sect. 12, we give a coordinate-free postulate that describes the dynamics of a time-dependent finite-dimensional mechanical system subjected to nonpotential forces.

2 Notation

Let M be a smooth manifold, then T_pM denotes the tangent space of M in p. The tangent bundle of M is denoted as TM. The set of all smooth sections of a vector bundle E over M is $\Gamma(E)$, such that the set of all vector fields on M is given by $\mathrm{Vect}(M) \coloneqq \Gamma(TM)$. The evaluation of a vector field $v \in \mathrm{Vect}(M)$ in a point $p \in M$ gives

$$v(p) = (p, v_p), \quad \text{where } v_p \in T_pM\,.$$

We will use an analog notation for tensor fields. To distinguish the evaluation $v(p)$ of a vector field v in a point $p \in M$ from its action as a derivation on a smooth function $f \in C^\infty(M)$, we write the latter as $v[f]$. We denote by $\mathcal{L}_v$ the Lie derivative with respect to the vector field $v \in \mathrm{Vect}(M)$. Using d for the exterior derivative, we have the identity

$$\mathcal{L}_v f = v[f] = \mathrm{d}f(v)$$

for any smooth function $f \in C^\infty(M)$. The space $\mathrm{Vect}(M)$ is equipped with the Lie bracket $[\cdot,\cdot]$ defined as the commutator of derivations on smooth functions. The space of (differential) k-forms is denoted as $\Omega^k(M)$ and the set of differential forms of arbitrary degree as $\Omega^\star(M)$. By $v \lrcorner \omega$ we denote the interior product between a vector field $v \in \mathrm{Vect}(M)$ and a differential form $\omega \in \Omega^\star(M)$.

We use Einstein's summation convention implying a summation from 1 to n whenever an index i appears once as an upper and once as a lower index, e.g.

$$v^i \frac{\partial}{\partial x^i} = \sum_{i=1}^{n} v^i \frac{\partial}{\partial x^i} \quad \text{or} \quad \frac{\partial}{\partial u^i} \otimes \eta^i = \sum_{i=1}^{n} \frac{\partial}{\partial u^i} \otimes \eta^i\,.$$

3 Space-time

To model the generalized space-time of an n-dimensional time-dependent finite-dimensional mechanical system, we introduce an $(n+1)$-dimensional smooth manifold M. We assume M to be endowed with a non-vanishing closed one-form ϑ, which we refer to as *time structure* on M. By the Poincaré lemma, the time structure defines local *time functions* $t\colon M \supseteq U \to \mathbb{R}$ such that $\mathrm{d}t = \vartheta\big|_U$. The temporal distance of two events $p, q \in U$ is then $t(q) - t(p)$.

A chart (U, ϕ) of M given by

$$\phi\colon M \supseteq U \to \mathbb{R}^{n+1},\ p \mapsto \phi(p) = (x^0, \dots, x^n) \tag{1}$$

is *adapted* to the time structure if $\vartheta\big|_U = \mathrm{d}x^0$, i.e., the first coordinate function x^0 is a time function. In this case, the coordinate x^0 is a local *time coordinate* and we will often use t instead of x^0 to denote it. In what follows, we will restrict our considerations to adapted charts. The existence of adapted charts is guaranteed by the existence of time functions and the fact that ϑ is nowhere zero. Therefore, the adapted charts provide an atlas of M. The change of coordinates

$$\psi \circ \phi^{-1}\colon \phi(U \cap V) \to \psi(U \cap V),\ (x^0, \dots, x^n) \mapsto (y^0, \dots, y^n)$$

between two adapted charts (U, ϕ) and (V, ψ) of M with $U \cap V \neq \emptyset$ is given by

$$\begin{aligned} y^0 &= x^0 + \text{const.},\\ y^i &= \psi^i \circ \phi^{-1}\big(x^0, \dots, x^n\big),\ i = 1, \dots, n, \end{aligned}$$

where $\psi^i\colon V \to \mathbb{R}$ denotes the i-th coordinate function of the chart ψ.

The $(n+1)$-dimensional manifold M is foliated by the time structure ϑ. To see this, we introduce the *space of spacelike vectors in* $p \in M$ as

$$A^0_p M \coloneqq \ker \vartheta_p = \big\{v_p \in T_pM \,\big|\, \vartheta_p(v_p) = 0\big\} \subset T_pM \tag{2}$$

and the corresponding subbundle of the tangent bundle TM to the generalized spacetime M as

$$A^0M \coloneqq \bigcup_{p \in M} \Big(\{p\} \times A^0_pM\Big) \subset TM\,, \tag{3}$$

which we call the *spacelike bundle* of M. Indeed, A^0M is a subbundle of the vector bundle TM because each chart $\big(U, \phi\big)$ of an adapted atlas of M induces the smooth local sections

$$\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\colon U \to TM \tag{4}$$

that provide a basis for A^0_pM at each $p \in U$. Consequently, A^0M is a distribution of rank n on M defined by the time structure ϑ. This distribution is involutive by Proposition 19.8 in [26] as $\mathrm{d}\vartheta = 0$ annihilates the distribution trivially. By the Frobenius theorem (see Theorem 19.12 in [26]), the distribution (3) is completely integrable. Moreover, A^0M defines a foliation according to the global Frobenius theorem (Theorem 19.21 in [26]). The leafs of this foliation are just the codimension-one submanifolds of synchronous events that can be distinguished in classical mechanics.

In order to model the inertia of the system, the vector bundle A^0M is equipped with a bundle metric[2] g called *Galilean metric.* The Galilean metric is a tensor field, for which in each $p \in M$ the tensor g_p is symmetric and positive definite.

The above construction can be summarized in the following definition.

Definition 1 ([29], pp. 5–6)**.** An $(n+1)$-dimensional smooth manifold M with a time structure ϑ and a bundle metric g that endows the subspaces A^0_pM with an inner product for all $p \in M$ is called a *Galilean manifold* denoted as (M, ϑ, g).

4 State space and motion

In the previous section, we defined the spacelike bundle A^0M of a manifold M equipped with a time structure $\vartheta \in \Omega^1(M)$. Similarly, we define the *time-normalized bundle* as the affine subbundle of TM given by

$$A^1M := \bigcup_{p \in M} \Big(\{p\} \times A^1_pM\Big) \subset TM\,, \tag{5}$$

where in each point $p \in M$, the *affine space of time-normalized vectors in* p is defined as

$$A^1_pM := \big\{v_p \in T_pM \;\big|\; \vartheta_p(v_p) = 1\big\} \subset T_pM. \tag{6}$$

The evolution of the configuration of a mechanical system is a time-parametrized curve

$$\gamma\colon \mathbb{R} \supseteq I \to M,\;\; \tau \mapsto \gamma(\tau) \tag{7}$$

in the Galilean manifold (M, ϑ, g). A *time-parametrized* curve is defined as a smooth sequence of events with $\vartheta(\dot\gamma) = 1$, where $\dot\gamma$ denotes the tangent field along γ. The local time coordinate t increases monotonically along a time-parametrized curve because locally

$$1 = \vartheta(\dot\gamma) = \mathrm{d}t(\dot\gamma) = \dot\gamma[t] = \frac{\mathrm{d}}{\mathrm{d}\tau}\big(t \circ \gamma(\tau)\big). \tag{8}$$

Consequently, the time coordinate is an affine function along γ, i.e.,

$$t \circ \gamma(\tau) = \tau + \tau_0,$$

where $\tau_0 \in \mathbb{R}$ is a constant. The condition $\vartheta(\dot\gamma) = 1$ motivates the following definition of the state space of a mechanical system.

Definition 2. Let (M, ϑ, g) be the Galilean manifold of a finite-dimensional mechanical system. The *state space* of the mechanical system is the time-normalized bundle A^1M.

[2] If the fibers of a vector bundle are equipped with an inner product that smoothly depends on the point in the base manifold, one speaks of a *bundle metric.* See Definition 1.8.11 in [17]. A bundle metric is the generalization of a Riemannian metric on a manifold to arbitrary vector bundles. Indeed, a Riemannian metric on a manifold is just a bundle metric on its tangent bundle. For this reason some authors, see Definition 6.42 in [25] or p. 308 in [42], designate a bundle metric as *Riemannian metric.* We abstain from doing so since it might lead to confusion.

The coordinate vector fields induced by an adapted chart $\phi\colon p \mapsto \big(t, x^1 \ldots, x^n\big)$ can be used to express a time-normalized vector $v_p \in A^1_p M$ as

$$v_p = \frac{\partial}{\partial t}\bigg|_p + u^i \frac{\partial}{\partial x^i}\bigg|_p . \tag{9}$$

Accordingly, any adapted chart ϕ induces a corresponding *natural chart* of the state space $A^1 M$ as

$$\Phi\colon A^1 M \supseteq \pi^{-1}(U) \to \mathbb{R}^{2n+1},\ \big(p, v_p\big) \mapsto \big(t, x^1, \ldots, x^n, u^1, \ldots, u^n\big)\,, \tag{10}$$

where

$$\pi\colon A^1 M \to M,\ (p, v_p) \mapsto p \tag{11}$$

is the natural projection of the affine bundle $A^1 M$. The state space $A^1 M$ is canonically endowed with the time structure

$$\hat{\vartheta} \coloneqq \pi^* \vartheta\,, \tag{12}$$

which is the pullback of the time structure of M by the natural projection (11). The natural chart (10) is an adapted chart with respect to the time structure (12) of $A^1 M$ because it holds that

$$\hat{\vartheta}\big|_{\pi^{-1}(U)} = \mathrm{d}t.$$

A *second-order curve* is a curve

$$\beta\colon \mathbb{R} \supseteq I \to A^1 M,\ \tau \mapsto \beta(\tau) \tag{13}$$

in the state space $A^1 M$ that is time-parametrized with respect to $\hat{\vartheta}$, i.e. $\hat{\vartheta}(\dot{\beta}) = 1$, and satisfies the condition

$$\beta = \big(\pi \circ \beta\big)^{\dot{}}\,. \tag{14}$$

Hence, the curve β corresponds to the (time-normalized) tangent field along its (time-parametrized) projection $\gamma \coloneqq \pi \circ \beta$ onto the base manifold M, i.e., by condition (14) it follows that

$$\beta = \dot{\gamma}\colon I \to A^1 M,\ \tau \mapsto \big(\gamma(\tau), \dot{\gamma}_{\gamma(\tau)}\big). \tag{15}$$

Definition 3. Let (M, ϑ, g) be the Galilean manifold of a finite-dimensional mechanical system and let $A^1 M$ be the corresponding state space. A *motion* of the mechanical system is a second-order curve in the state space.

Let a second-order curve $\beta\colon I \to A^1 M$ be the integral curve of a local vector field $Z \in \mathrm{Vect}\big(A^1 M\big)$, i.e.,

$$\dot{\beta}(\tau) = Z\big(\beta(\tau)\big)\,, \tag{16}$$

then the vector field Z cannot be arbitrary. First, the latter needs to be time-normalized such that

$$\hat{\vartheta}(Z) = 1. \tag{17}$$

Second, the vector field Z needs to obey the second-order condition

$$\mathrm{D}\pi\, Z = \mathrm{id}_{A^1 M}. \tag{18}$$

Indeed, condition (14) together with (16) lead to

$$\beta = \big(\pi \circ \beta\big)^{\dot{}} = \mathrm{D}\pi\, \dot{\beta} = \mathrm{D}\pi\, Z \circ \beta, \tag{19}$$

where $\mathrm{D}\pi\colon T\big(A^1M\big) \to TM$ denotes the differential of the natural projection (11). Because condition (19) has to hold for arbitrary integral curves $\beta\colon I \to A^1M$, the second-order condition (18) follows.

A vector field $Z \in \mathrm{Vect}(A^1M)$ that satisfies conditions (17) and (18) is called a *second-order (vector) field*. Second-order fields can be equivalently characterized using local coordinates by requiring a vector field $Z \in \mathrm{Vect}(A^1M)$ to be locally expressible in every natural chart (10) as

$$Z = \frac{\partial}{\partial t} + u^i \frac{\partial}{\partial x^i} + Z^i \frac{\partial}{\partial u^i}, \tag{20}$$

with n smooth functions Z^i. From the local expression (20) it is apparent that second-order fields can only differ by the coefficients of their $\partial/\partial u^i$ part. Moreover, the differential equation (16) related to a second-order field Z is a system of second-order differential equations in first-order form

$$\begin{aligned} \dot{t}(\tau) &= 1, \\ \dot{\mathbf{x}}(\tau) &= \mathbf{u}(\tau), \\ \dot{\mathbf{u}}(\tau) &= \mathbf{Z}\big(t(\tau), \mathbf{x}(\tau), \mathbf{u}(\tau)\big), \end{aligned} \tag{21}$$

where we adopt the notation that quantities $a^1, \ldots, a^n$ are gathered as $\mathbb{R}^n$-tuples $\mathbf{a} := (a^1, \ldots, a^n)$. The first equation of (21) can be solved to

$$t(\tau) := t \circ \beta(\tau) = \tau + \tau_0, \tag{22}$$

where $\tau_0 \in \mathbb{R}$ denotes a constant. The remaining equations of (21) are equivalent to the second-order differential equations

$$\ddot{\mathbf{x}}(\tau) = \mathbf{Z}\big(t(\tau), \mathbf{x}(\tau), \dot{\mathbf{x}}(\tau)\big).$$

In the study of finite-dimensional mechanical systems, we are interested in modeling the second-order field Z, which characterizes all possible motions of the system by (16). Differential forms are particularly useful for the characterization of vector fields. With the time structure ϑ, we have already used a differential one-form to define the sets of spacelike (2) and of time-normalized vectors (6) on M, respectively. Furthermore, we have used the pullback $\hat{\vartheta}$ of the time structure ϑ on M to characterize time-normalized vector fields on A^1M (see equation (17)). From the local expression (20), we deduce a characterization of second-order fields using differential forms. We define the local one-forms $\theta^1, \ldots, \theta^n \in \Omega^1(\pi^{-1}(U))$ as

$$\theta^i := \mathrm{d}x^i - u^i \mathrm{d}t, \qquad \text{with} \quad i = 1, \ldots, n \tag{23}$$

and formulate the second-order condition as

$$Z \in \ker(\theta^1) \cap \cdots \cap \ker(\theta^n) \quad \text{and} \quad \hat{\vartheta}(Z) = 1,$$

i.e., on $\pi^{-1}(U) \subseteq A^1M$ the vector field Z on A^1M needs to be time-normalized and it has to lie in the distribution defined by the differential one-forms (23).

The remaining n free coefficients in the local representation of Z can be prescribed by requiring Z to lie in the distribution on A^1M defined by the n one-forms

$$\lambda^i \coloneqq \mathrm{d}u^i - Z^i \mathrm{d}t, \qquad \text{with} \quad i = 1, \dots, n. \tag{24}$$

Vector and covector fields (or one-forms) on the state space A^1M are sections of the bundles $T(A^1M)$ and $T^*(A^1M)$, respectively. Therefore, we start by studying the geometric structure of these two vector bundles following [29].

5 Galilean manifolds and their related bundles

The differential of the natural projection (11), $\mathrm{D}\pi\colon T(A^1M) \to TM$, defines the *vertical bundle*

$$\mathrm{Ver}(A^1M) \coloneqq \ker \mathrm{D}\pi = \bigcup_{a \in A^1M} \Big(\{a\} \times \ker \mathrm{D}\pi_a \Big), \tag{25}$$

which is a subbundle of the tangent bundle $T(A^1M)$. For any point $a \in A^1M$ the *space of vertical vectors in* a is denoted by

$$\mathrm{Ver}_a(A^1M) = \ker \mathrm{D}\pi_a = \big\{ w \in T_a(A^1M) \,\big|\, \mathrm{D}\pi_a(w) = 0 \big\}. \tag{26}$$

A section $V \in \Gamma\big(\mathrm{Ver}(A^1M)\big)$ of the vertical bundle is called a *vertical vector field*. Let $\big(U, \phi\big)$ be an adapted chart of M and consider the corresponding natural chart (10) on the neighbourhood $\pi^{-1}(U)$ of A^1M. A vertical vector field V can then be expressed as

$$V = V^i \frac{\partial}{\partial u^i},$$

because the vectors

$$\left.\frac{\partial}{\partial u^1}\right|_a, \dots, \left.\frac{\partial}{\partial u^n}\right|_a \tag{27}$$

provide a basis of $\mathrm{Ver}_a(A^1M)$ for points $a \in \pi^{-1}(U) \subseteq A^1M$. The vertical subbundle (25) naturally appears in the study of second-order fields because the difference of two second-order fields is always a vertical vector field. This can be seen from the local expression (20) of a second-order field.

The space $\mathrm{Ver}_a(A^1M)$ defined by (26) can be also seen as the tangent space to A^1_pM at the point $a \in A^1M$ with $p = \pi(a)$. Since A^1_pM is the affine hyperplane in T_pM defined by the equation $\vartheta_p(v) = 1$ for all $v \in T_pM$, the tangent space to A^1_pM can be identified with $\ker \vartheta_p = A^0_pM$ defined in (2). Accordingly, we have the pointwise isomorphism

$$\mathrm{Ver}_a\big(A^1M\big) \cong A^0_{\pi(a)}M \tag{28}$$

for all $a \in A^1M$. This isomorphism can be locally expressed as

$$\left.\frac{\partial}{\partial u^i}\right|_a \mapsto \left.\frac{\partial}{\partial x^i}\right|_{\pi(a)} \tag{29}$$

using the basis vectors from (27) and (4).

By the isomorphism (28), the bundle metric g (see Definition 1) on the bundle A^0M of spacelike vectors induces a bundle metric $\hat{g}$ on the bundle $\mathrm{Ver}(A^1M)$ of vertical vectors that is defined as

$$\hat{g}_a\left(\frac{\partial}{\partial u^i}\bigg|_a, \frac{\partial}{\partial u^j}\bigg|_a\right) := g_{\pi(a)}\left(\frac{\partial}{\partial x^i}\bigg|_{\pi(a)}, \frac{\partial}{\partial x^j}\bigg|_{\pi(a)}\right), \tag{30}$$

for all $a \in A^1M$. While the bundle metric g on A^0M can be locally written as

$$g = g_{ij}\,\mathrm{d}x^i \otimes \mathrm{d}x^j \tag{31}$$

it follows by (30) that

$$\hat{g} = g_{ij} \circ \pi \ \mathrm{d}u^i \otimes \mathrm{d}u^j. \tag{32}$$

For the sake of brevity, we will often write g_{ij} instead of $g_{ij} \circ \pi$ for the coefficients in (32).

An arbitrary tangent vector $w \in T_a(A^1M)$ is mapped to a spacelike vector at the point $\pi(a)$ by the map

$$\xi\colon T_a(A^1M) \to A^0_{\pi(a)}M,\ w \mapsto D\pi_a(w) - \vartheta_{\pi(a)}\big(\mathrm{D}\pi_a(w)\big)a\,. \tag{33}$$

To see this, we need to check if $\xi(w)$ lies in the kernel of the time structure ϑ. Using $p = \pi(a)$, we calculate that

$$\vartheta_p\big(\xi(w)\big) = \vartheta_p\big(D\pi_a(w)\big) - \vartheta_p\big(\mathrm{D}\pi_a(w)\big)\vartheta_p(a) = 0$$

because $\vartheta_p(a) = 1$. Using the concatenation of the isomorphism (28) and the map (33), we can define a vector bundle homomorphism over A^1M

$$\mu\colon T(A^1M) \to \mathrm{Ver}(A^1M), \tag{34}$$

which we call the *vertical homomorphism* of the state space A^1M. Its local expression with respect to the natural chart (10) is given by

$$\mu\big|_{\pi^{-1}(U)} = \frac{\partial}{\partial u^i} \otimes \theta^i = \frac{\partial}{\partial u^i} \otimes \big(\mathrm{d}x^i - u^i \mathrm{d}t\big),$$

where the θ^i are the one-forms of (23). Apparently, the map (34) is surjective and it holds that $\mu(V) = 0$ for all local sections V of $\mathrm{Ver}(A^1M)$ and that $\mu(Z) = 0$ for all second-order fields Z.[3] The map (34) defines an endomorphism of the bundle $T(A^1M)$ when considered as map

$$\mu\colon T(A^1M) \to T(A^1M).$$

There is no canonically defined 'horizontal' subbundle

$$\mathrm{Hor}(A^1M) := \bigcup_{a \in A^1M} \Big(\{a\} \times \mathrm{Hor}_a(A^1M)\Big) \subset T(A^1M) \tag{35}$$

[3]A theory for time-independent mechanical systems can be formulated on the tangent bundle of a time-independent configuration manifold. We refer to the work of Godbillon [15] for such a presentation. Godbillon uses a similar homomorphism as (34) that canonically exists on the double tangent bundle of any differentiable manifold. It is known as the vertical endomorphism of the double tangent bundle (see [15, Chapter X] or [33, Section 2]).

that would complement the vertical bundle $\mathrm{Ver}\big(A^1M\big)$ such that the tangent bundle $T\big(A^1M\big)$ would split as

$$T(A^1M) = \mathrm{Hor}(A^1M) \oplus \mathrm{Ver}(A^1M)\,. \tag{36}$$

In the study of tangent bundles (and double tangent bundles), it is well-known that the choice of a *particular* second-order field induces such a splitting, see for example [46] or [33]. There are several ways to define the horizontal bundle that results from the selection of a second-order field $Z \in \mathrm{Vect}(A^1M)$.

Following [30, p. 280], we consider the vector bundle homomorphism

$$\eta\colon T(A^1M) \to T(A^1M) \tag{37}$$

defined by

$$\eta(X) = \frac{1}{2}\big([Z, \mu(X)] - \mu([Z, X]) + X - \hat{\vartheta}(X)Z\big) \tag{38}$$

for all $X \in \mathrm{Vect}\big(A^1M\big)$. Indeed, one easily verifies that

$$\eta\big(fX + gY\big) = f\eta\big(X\big) + g\eta\big(Y\big)$$

for all $f, g \in C^\infty(A^1M)$ and all $X, Y \in \mathrm{Vect}\big(A^1M\big)$ such that (38) defines a vector bundle homomorphism over A^1M. The local coordinate expression of η reads

$$\eta = \frac{\partial}{\partial u^i} \otimes \eta^i = \frac{\partial}{\partial u^i} \otimes \left(\mathrm{d}u^i - Z^i \mathrm{d}t - \frac{1}{2}\frac{\partial Z^i}{\partial u^j}\big(\mathrm{d}x^j - u^j \mathrm{d}t\big)\right),$$

where we have introduced the one-forms

$$\eta^i \coloneqq \mathrm{d}u^i - Z^i \mathrm{d}t - \frac{1}{2}\frac{\partial Z^i}{\partial u^j}\big(\mathrm{d}x^j - u^j \mathrm{d}t\big).$$

Note that the coefficients Z^i denote the defining coefficients of the second-order field Z from (20). The vector bundle homomorphism η has the property that

$$\eta\big|_{\mathrm{Ver}(A^1M)} = \mathrm{id}_{\mathrm{Ver}(A^1M)}.$$

Hence, it holds that $\eta \circ \eta = \eta$ such that η is a projection onto $\mathrm{Ver}\big(A^1M\big)$ and consequently

$$T\big(A^1M\big) = \ker\eta \oplus \mathrm{Ver}\big(A^1M\big)\ .$$

In view of (36), the kernel of η defines a horizontal subbundle

$$\mathrm{Hor}(A^1M) \coloneqq \ker\eta\ .$$

One can easily convince oneself, that $\ker\mu \cap \ker\eta$ is a line bundle that is spanned by the second-order field Z.

6 Basic and semi-basic differential forms

We have already seen that differential forms can be used to characterize vector fields. There are two types of differential forms that will reveal useful in the definition of forces, the so-called basic and semi-basic forms. The natural projection $\pi\colon A^1M \to M$ is a surjective submersion. Hence, the pullback by the natural projection

$$\pi^*\colon \Omega^\star(M) \to \Omega^\star(A^1M)$$

is an injection of the differential forms on M to those on A^1M. These forms on A^1M that are given by $\operatorname{im}\pi^* \subset \Omega^\star(A^1M)$ are called *basic* differential forms, because they result from pulling differential forms on the base manifold M back to A^1M.

A differential l-form ω on A^1M is called *semi-basic* if $\omega\big(X_1,\dots,X_l\big) = 0$ as soon as one of the vector fields X_i is vertical, or differently said, if the interior product $V \lrcorner \omega$ vanishes for any vertical vector field V. An equivalent statement is that the local representation of ω with respect to the dual basis induced by the natural chart (10) does not contain terms in $\mathrm{d}u^1,\dots,\mathrm{d}u^n$. The chart representations of the coefficients however may depend on $u^1,\dots,u^n$, contrary to those of basic forms. Note that basic forms are semi-basic.

The vertical homomorphism (34) allows us to define a differentiation operation[4] on differential forms on A^1M with the property that the subalgebra of semi-basic forms is closed under its operation. Using the vertical homomorphism, we first define the derivation

$$\mathcal{D}_\mu\colon \Omega^\star(A^1M) \to \Omega^\star(A^1M)$$

by the condition

$$(\mathcal{D}_\mu\omega)(X_1,\dots,X_l) \coloneqq \sum_{i=1}^{l} \omega\big(X_1,\dots,\mu(X_i),\dots,X_l\big)\,, \tag{39}$$

where $X_1,\dots,X_l$ are arbitrary vector fields on A^1M. The operator $\mathcal{D}_\mu$ is linear, does not alter the degree of the form, is distributive over the wedge product and satisfies

$$\mathcal{D}_\mu f = 0, \quad \mathcal{D}_\mu(\mathrm{d}x^i) = \mathcal{D}_\mu(\mathrm{d}t) = 0, \quad \mathcal{D}_\mu(\mathrm{d}u^i) = \mathrm{d}x^i - u^i\mathrm{d}t,$$

where f is a smooth function on A^1M. Using (39) and the exterior derivative d, we define the linear operator

$$\partial \coloneqq \mathcal{D}_\mu \circ \mathrm{d} - \mathrm{d} \circ \mathcal{D}_\mu.$$

The operator ∂ is an anti-derivation on the exterior algebra of differential forms, increases the degree of a form by one and obeys the following rules

$$\partial f = \frac{\partial f}{\partial u^i}(\mathrm{d}x^i - u^i\mathrm{d}t), \quad \partial(\mathrm{d}x^i) = \partial(\mathrm{d}t) = 0, \quad \partial(\mathrm{d}u^i) = \mathrm{d}u^i \wedge \mathrm{d}t, \tag{40}$$

where again f denotes a smooth function on A^1M. Moreover, it holds that

$$\partial \circ \mathrm{d} = -\mathrm{d} \circ \partial \tag{41}$$

[4]The vertical homomorphism is an example of a vector-valued differential form. It holds in general that vector-valued differential forms come along with certain derivations. We refer to [13] for the general theory. We will make use of the differential concomitant of the vertical homomorphism (34) while in *time-independent* mechanics, the differential associate of the vertical endomorphism of the double tangent bundle is of interest (see [15, Chapters X and XI] as well as [33, Section 2]). See also [19].

because of $\mathrm{d}^2 = 0$. However, $\partial^2 \neq 0$ but

$$\partial^2\omega = \hat{\vartheta} \wedge \partial\omega, \tag{42}$$

where $\hat{\vartheta}$ denotes the time structure on A^1M. To prove (42), it is enough to see that ∂^2 and $\omega \mapsto \hat{\vartheta} \wedge \partial\omega$ are derivations that coincide on zero- and one-forms. By induction (42) holds for forms of arbitrary degree. From the rules (40), it becomes obvious that ∂ maps semi-basic forms to semi-basic forms. Let Z be a second-order field and let ω be a semi-basic l-form. Then $Z \lrcorner \omega$ is a semi-basic $(l-1)$-form that is independent of the specific choice of Z. Indeed, if Z' denotes another second-order field, then it holds that $Z' = Z + V$ where V is a vertical vector field. The following formula holds

$$\partial\big(Z \lrcorner \omega\big) + Z \lrcorner \partial\omega + \hat{\vartheta} \wedge \big(Z \lrcorner \omega\big) = l\omega. \tag{43}$$

To prove (43), one considers that the left-hand and the right-hand side represent derivations of the algebra of semi-basic differential forms that agree on the semi-basic zero-forms (smooth functions) and on the semi-basic one-forms and, therefore, are equal by induction.

7 Action form of a second-order field

In the study of finite-dimensional mechanical systems, we are interested in modeling the second-order field Z, whose integral curves define the motions of the mechanical system. As mentioned at the end of Section 4, differential forms are particularly useful for the characterization of vector fields. For this purpose, as it was suggested by [29, p. 20],[5] the surjective vector bundle homomorphisms $\eta\colon T(A^1M) \to \mathrm{Ver}(A^1M)$ and $\mu\colon T(A^1M) \to \mathrm{Ver}(A^1M)$ together with the bundle metric (30) can be used to define a differential two-form Ω on A^1M as

$$\Omega(X, Y) := \hat{g}\big(\eta(X), \mu(Y)\big) - \hat{g}\big(\eta(Y), \mu(X)\big), \tag{44}$$

for all $X, Y \in \mathrm{Vect}\big(A^1M\big)$. Because η depends on the choice of a second-order field Z, we call Ω the *action form of Z*. The local expression of the action form (44) reads

$$\begin{aligned} \Omega\big|_{\pi^{-1}(U)} &= g_{ij}\, \eta^i \wedge \theta^j \\ &= g_{ij}\left(\mathrm{d}u^i - Z^i\mathrm{d}t - \frac{1}{2}\frac{\partial Z^i}{\partial u^k}\big(\mathrm{d}x^k - u^k\mathrm{d}t\big)\right) \wedge \big(\mathrm{d}x^j - u^j\mathrm{d}t\big). \end{aligned} \tag{45}$$

The following theorem gives necessary and sufficient conditions for a two-form to be the action form of a second-order field.

Theorem 1 ([29], p. 21 and p. 24)**.** *Let $\big(M, \vartheta\big)$ be a manifold with time structure. A two-form Ω on A^1M is the action form of a second-order field Z if and only if it satisfies the following conditions:*

(i) Ω vanishes on $\ker\mu$, i.e.,

$$\Omega\big(X, Y\big) = 0$$

for all X, Y with $\mu(X) = \mu(Y) = 0$.

[5] See also p. 280 in [30].

(ii) Ω defines a bundle metric g on A^0M, i.e., the matrix

$$g_{ij} = \Omega\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial x^j}\right)$$

is symmetric and positive definite for all charts.

(iii) $\partial\Omega = 0$.

The second-order field Z is the only vector field on A^1M for which

$$Z \lrcorner \Omega = 0,\ \hat{\vartheta}(Z) = 1.$$

Proof. The necessity of (i) is clear by definition (44) and the necessity of (ii) follows directly from the local expression (45). Direct calculation with (45) and the rules (40) shows that $\partial\Omega = 0$. We have just proven, that the action form of Z defined by (44) has the properties (i)–(iii).

To prove sufficiency of (i)–(iii), we assume the two-form Ω to satisfy these conditions. Consider the vector bundle homomorphism

$$\hat{f} \colon T(A^1M) \to T^*(A^1M) \tag{46}$$

defined by

$$\hat{f}(X) = X \lrcorner \Omega. \tag{47}$$

for all $X \in \mathrm{Vect}(A^1M)$. By property (i), the homomorphism (47) cannot have full rank $2n{+}1$. Because the $2n$ one-forms $\partial/\partial u^i \lrcorner \Omega$, $\partial/\partial x^i \lrcorner \Omega$ are linearly independent, the homomorphism has constant rank $2n$. Indeed, by properties (i) and (ii), it follows from $a^i \partial/\partial u^i \lrcorner \Omega + b^i \partial/\partial x^i \lrcorner \Omega = 0$ that

$$\begin{aligned}
0 &= \left(a^i \frac{\partial}{\partial u^i} \lrcorner \Omega\right)\left(\frac{\partial}{\partial u^j}\right) + b^i \left(\frac{\partial}{\partial x^i} \lrcorner \Omega\right)\left(\frac{\partial}{\partial u^j}\right) \\
&= a^i\, \Omega\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right) + b^i\, \Omega\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial u^j}\right) = 0 - b^i g_{ij},
\end{aligned}$$

i.e., that $b^i = 0$, and thereby that

$$0 = \left(a^i \frac{\partial}{\partial u^i} \lrcorner \Omega\right)\left(\frac{\partial}{\partial x^j}\right) = a^i\, \Omega\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial x^j}\right) = a^i g_{ij},$$

i.e., that $a^i = 0$. For reasons of brevity, we also say that Ω has rank $2n$. Consequently,

$$\ker\Omega \coloneqq \ker\hat{f} = \left\{(a, X_a) \in T(A^1M) \,\middle|\, X_a \lrcorner \Omega_a = 0\right\}$$

is a line bundle.

For a subspace $W \subseteq T_a(A^1M)$, the orthogonal complement with respect to the bilinear form Ω_a is defined as

$$W^\perp = \left\{X_a \in T_a(A^1M) \,\middle|\, \Omega_a(X_a, Y_a) = 0 \ \ \forall Y_a \in W\right\}\ .$$

By condition (i) $\ker\mu_a \subseteq (\ker\mu_a)^\perp$, which implies

$$n + 1 = \dim\ker\mu_a \le \dim(\ker\mu_a)^\perp\ .$$

By definition, the orthogonal complement $\big(T_a(A^1M)\big)^\perp$ coincides with the one-dimensional subspace $\ker\Omega_a$. This leads to an upper bound of

$$\begin{aligned}\dim(\ker\mu_a)^\perp &= \dim T_a(A^1M) - \dim\ker\mu_a + \dim\ker\mu_a \cap \big(T_a(A^1M)\big)^\perp\\ &\le (2n+1) - (n+1) + 1 = n+1\,,\end{aligned}$$

where we have used the dimension formula of Proposition 2.13 in [45]. By dimension, we conclude that

$$\ker\mu_a = (\ker\mu_a)^\perp \ .$$

It is easy to see, that any $Z_a \in \ker\Omega_a$ is an element of $(\ker\mu_a)^\perp$ and therefore

$$\ker\Omega_a \subseteq (\ker\mu_a)^\perp = \ker\mu_a\,.$$

By (ii), Z_a cannot be a vertical vector. Moreover, since $\ker\mu_a$ is spanned by $\mathrm{Ver}_a(A^1M)$ and $\partial/\partial t|_a + u^i(a)\partial/\partial x^i|_a$, there exists an element Z_a in $\ker\Omega_a$ of the form

$$Z_a = \frac{\partial}{\partial t}\bigg|_a + u^i(a)\frac{\partial}{\partial x^i}\bigg|_a + Z^i(a)\frac{\partial}{\partial u^i}\bigg|_a\,. \tag{48}$$

which is the sole vector in $\ker\Omega_a$ satisfying $\hat{\vartheta}_a(Z_a) = 1$. Thus, the conditions $Z\lrcorner\Omega = 0$ and $\hat{\vartheta}(Z) = 1$ characterize the unique second-order field Z pointwise given by (48).

It remains to be shown that Ω is the action form of the second-order field Z defined by (44). For this, we consider the basis Z, $\partial/\partial x^i$, $\partial/\partial u^i$ and its dual one-forms $\mathrm{d}t$, $\theta^i = \mathrm{d}x^i - u^i\mathrm{d}t$, $\lambda^i = \mathrm{d}u^i - Z^i\mathrm{d}t$. By (i) and (ii), Ω has the form

$$\begin{aligned}\Omega &= \Omega\bigg(\frac{\partial}{\partial u^i},\frac{\partial}{\partial x^j}\bigg)\big(\mathrm{d}u^i - Z^i\mathrm{d}t\big)\wedge\big(\mathrm{d}x^j - u^j\mathrm{d}t\big)\\ &\quad+\frac{1}{2}\Omega\bigg(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\bigg)\big(\mathrm{d}x^i - u^i\mathrm{d}t\big)\wedge\big(\mathrm{d}x^j - u^j\mathrm{d}t\big)\\ &= g_{ij}\big(\mathrm{d}u^i - Z^i\mathrm{d}t\big)\wedge\big(\mathrm{d}x^j - u^j\mathrm{d}t\big)\\ &\quad+\frac{1}{2}\Omega\bigg(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\bigg)\big(\mathrm{d}x^i - u^i\mathrm{d}t\big)\wedge\big(\mathrm{d}x^j - u^j\mathrm{d}t\big).\end{aligned}$$

Direct calculation of $\partial\Omega = 0$ shows that

$$\Omega\bigg(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\bigg) = \frac{1}{2}\bigg(g_{ik}\frac{\partial Z^k}{\partial u^j} - g_{jk}\frac{\partial Z^k}{\partial u^i}\bigg)$$

and, by (45), proves the assertion. □

In view of (23) and (24), one might be tempted to use the locally defined two-form

$$\Omega' = g_{ij}\,\lambda^i\wedge\theta^j = g_{ij}\big(\mathrm{d}u^i - Z^i\mathrm{d}t\big)\wedge\big(\mathrm{d}x^j - u^j\mathrm{d}t\big), \tag{49}$$

to characterize the second-order field Z. Computations in local coordinates show that Ω' has the properties (i) and (ii) of Theorem 1 and defines a second-order field Z by $Z\lrcorner\Omega' = 0$ and $\hat{\vartheta}(Z) = 1$. As discussed in [30] however, this two-form is not well-defined as it is not independent of the choice of the adapted chart.

Note that the two-form (49) can essentially be found in [14], p. 153, and [40], p. 132 (respectively on p. 129 of [41]). As the authors of the references do not work on Galilean manifolds, some caution is advised here. Nevertheless, both authors define a two-form which looks like (49).

8 Forces

Consider two second-order fields Z_1 and Z_2 on the state space A^1M of a mechanical system. As two second-order fields can only differ by a vertical vector field, it holds that

$$Z_2 = Z_1 + V, \tag{50}$$

where V is a smooth section of the vertical bundle $\mathrm{Ver}(A^1M)$. This vertical vector field can be interpreted as (relative) acceleration between Z_1 and Z_2. The bundle metric $\hat{g}$ defined in (30) induces the bijection

$$\hat{g}\cdot\, : \Gamma\big(\mathrm{Ver}(A^1M)\big) \to \Gamma\big(\mathrm{Ver}^*(A^1M)\big),\ V \mapsto F = \hat{g}\cdot V \tag{51}$$

between smooth sections of the vertical bundle $\mathrm{Ver}(A^1M)$ and sections of the dual of the vertical bundle $\mathrm{Ver}^*(A^1M)$, where $\hat{g}\cdot V$ is the one-form $\hat{g}(\,\cdot\,, V)$.

If we consider the Galilean metric to model the mass of a finite-dimensional mechanical system and if we interpret vertical vector fields as (relative) accelerations, then with the bijection (51) we are facing Newton's second law that says "force F is equal to mass $\hat{g}$ times acceleration V". This motivates the following definition.

Definition 4. Let (M, ϑ, g) be the Galilean manifold of a finite-dimensional mechanical system and let A^1M be the corresponding state space. A *force* is a smooth section of the dual of the vertical bundle $\mathrm{Ver}^*(A^1M)$, i.e., a force is a $C^\infty(A^1M)$-linear map

$$F\colon \Gamma(\mathrm{Ver}(A^1M)) \to C^\infty(A^1M) \tag{52}$$

on the space of vertical vector fields.

Consider the action forms Ω_1 and Ω_2 of the respective second-order fields Z_1 and Z_2. We introduce the differential two-form Φ by which the action forms Ω_1 and Ω_2 differ, i.e.,

$$\Omega_2 = \Omega_1 + \Phi. \tag{53}$$

It is clear that $\partial\Phi = 0$ because $\partial\Omega_1 = \partial\Omega_2 = 0$ by Theorem 1. In terms of the coordinate fields induced by a natural chart, the coefficients of the force $F = F_i \mathrm{d}u^i$, which is associated to $V = Z_2 - Z_1$ by (51), are given by

$$F_i = g_{ij}V^j = g_{ij}(Z_2^j - Z_1^j)\ .$$

Using the representation of the action form in the natural chart (45), the two-form $\Phi = \Omega_2 - \Omega_1$ is given as

$$\begin{aligned}\Phi &= g_{ij}(Z_2^i - Z_1^i)\,\mathrm{d}x^j{\wedge}\mathrm{d}t + \frac{1}{2}g_{ij}\left(\frac{\partial Z_2^i}{\partial u^k} - \frac{\partial Z_1^i}{\partial u^k}\right)(\mathrm{d}x^j{-}u^j\mathrm{d}t){\wedge}(\mathrm{d}x^k{-}u^k\mathrm{d}t)\\ &= F_j\,\mathrm{d}x^j{\wedge}\mathrm{d}t + \frac{1}{2}\frac{\partial F_j}{\partial u^k}(\mathrm{d}x^j{-}u^j\mathrm{d}t){\wedge}(\mathrm{d}x^k{-}u^k\mathrm{d}t),\end{aligned} \tag{54}$$

where the last equality uses that the coefficients $g_{ij} = g_{ij}\circ\pi$ are independent of $u^1, \ldots, u^n$ and form a symmetric matrix. The local expression shows that the two-form Φ is semi-basic and that it can be associated with the force F.

Definition 5. Let (M, ϑ, g) be the Galilean manifold of a finite-dimensional mechanical system and let A^1M be the corresponding state space. A *force two-form* is a differential two-form $\Phi \in \Omega^2(A^1M)$ that is semi-basic and satisfies $\partial\Phi = 0$.

The following theorem establishes a bijective relation between forces (52) and force two-forms (54).

Theorem 2 ([29], p. 32)**.** *Let $Z \in \mathrm{Vect}(A^1M)$ be any second-order field. The formulae*

$$\varphi = F \circ \mu, \quad \varphi = -Z \lrcorner \Phi, \quad \Phi = -\tfrac{1}{2}\big(\partial\varphi + \hat{\vartheta} \wedge \varphi\big)$$

define bijections between

(i) the forces, i.e., smooth sections $F \in \Gamma(\mathrm{Ver}^*(A^1M))$,

(ii) the semi-basic one-forms φ with $Z \lrcorner \varphi = 0$,

(iii) the force two-forms, i.e., the semi-basic two-forms Φ with $\partial\Phi = 0$.

In local coordinates given by a natural chart (10)*, it holds that*

$$F = F_i \, \mathrm{d}u^i, \tag{55}$$

$$\varphi = F_i(\mathrm{d}x^i - u^i \mathrm{d}t), \tag{56}$$

$$\Phi = F_i \, \mathrm{d}x^i \wedge \mathrm{d}t + \frac{1}{2}\frac{\partial F_i}{\partial u^j}(\mathrm{d}x^i - u^i \mathrm{d}t) \wedge (\mathrm{d}x^j - u^j \mathrm{d}t). \tag{57}$$

Proof. Because for any vertical vector field V it holds that $\mu\big(V\big) = \mu(Z) = 0$, $F \circ \mu$ is a semi-basic one-form with $Z \lrcorner (F \circ \mu) = 0$. Conversely, a semi-basic one-form φ with $Z \lrcorner \varphi = 0$ vanishes on $\ker \mu$ and, therefore, defines a linear form on

$$T(A^1M)/\ker \mu \cong \mathrm{im}\, \mu = \mathrm{Ver}(A^1M). \tag{58}$$

The isomorphism (58) follows by the first isomorphism theorem [37, Theorem 3.5]. This proves the bijection between (i) and (ii). According to the properties (40) and (42), it holds that

$$\begin{aligned} \partial(\partial\varphi + \hat{\vartheta} \wedge \varphi) &= \partial^2\varphi + \partial\hat{\vartheta} \wedge \varphi - \hat{\vartheta} \wedge \partial\varphi \\ &= \hat{\vartheta} \wedge \partial\varphi - \hat{\vartheta} \wedge \partial\varphi = 0 \end{aligned}$$

and $Z \lrcorner (-Z \lrcorner \Phi) = -\Phi(Z, Z) = 0$. Finally, since $Z \lrcorner \varphi = 0$, it holds that

$$\begin{aligned} -Z \lrcorner \big(- \tfrac{1}{2}(\partial\varphi + \hat{\vartheta} \wedge \varphi)\big) &= \tfrac{1}{2}\big(Z \lrcorner \partial\varphi + (Z \lrcorner \hat{\vartheta}) \wedge \varphi - \hat{\vartheta} \wedge (Z \lrcorner \varphi)\big) \\ &= \tfrac{1}{2}(Z \lrcorner \partial\varphi + \varphi) = \varphi, \end{aligned}$$

where in the last equality we have used $Z \lrcorner \partial\varphi = \varphi$ induced by the rule (43). Using the same rule we have

$$-\tfrac{1}{2}\big(\partial(-Z \lrcorner \Phi) + \hat{\vartheta} \wedge (-Z \lrcorner \Phi)\big) = \tfrac{1}{2}\big(\partial(Z \lrcorner \Phi) + \hat{\vartheta} \wedge (Z \lrcorner \Phi)\big) = \Phi \, .$$

This proves the assertion. The coordinate expressions (55) to (57) follow by straightforward computation. □

Theorem 3 ([29], p. 33)**.** *Let* Ω *denote the action form of a mechanical system, let* Z *be its related second-order field and let* F *be a force. By the one-to-one correspondence* (51)*,* F *is associated with a vertical vector field* V*. Moreover,* F *can be uniquely related to a force two-form* Φ *by Theorem 2. It then holds that the vector field* $Z' = Z + V$ *is the second-order field related to the action form* $\Omega' = \Omega + \Phi$*.*

Proof. One easily verifies that $\Omega' = \Omega + \Phi$ is an action form, i.e., that it respects the properties from Theorem 1. Furthermore, one observes that Ω' and Ω induce the same Galilean metric g. It remains to be shown that $Z' \lrcorner \Omega' = (Z+V) \lrcorner (\Omega + \Phi) = 0$. Because $Z \lrcorner \Omega = 0$ and $V \lrcorner \Phi = 0$, it holds that

$$Z' \lrcorner \Omega' = (Z + V) \lrcorner (\Omega + \Phi) = V \lrcorner \Omega + Z \lrcorner \Phi.$$

By definition (44), it holds for $V \lrcorner \Omega$ that

$$\big(V \lrcorner \Omega\big)(Y) = \hat{g}\big(\eta(V), \mu(Y)\big) - \hat{g}\big(\eta(Y), \mu(V)\big) = \hat{g}\big(V, \mu(Y)\big) = F \circ \mu(Y),$$

where we have used the properties $\mu(V) = 0$ and $\eta(V) = V$ of the vector bundle homomorphisms μ and η. The last equality follows by (51) and the symmetry of $\hat{g}$. By Theorem 2, it holds that $Z \lrcorner \Phi = -F \circ \mu$ and consequently

$$Z' \lrcorner \Omega' = V \lrcorner \Omega + Z \lrcorner \Phi = 0.$$

□

9 Modeling inertia – the kinetic energy

We experience that whenever we perceive the motion of a mechanical system, we do this relative to some reference. For instance, we observe the motion of a car relative to the street or the motion of the sun relative to the horizon. To account for this when modeling the inertia of a mechanical system, we introduce a *reference field* as a time-normalized vector field R defined on a neighborhood U_R of M, i.e.,

$$R\colon M \supseteq U_R \to A^1 M$$

with $\pi \circ R = \mathrm{id}_M$, where again (M, ϑ, g) denotes the Galilean manifold of the mechanical system. For an adapted chart $\phi(p) = (t, x^1, \ldots, x^n)$ of M, the reference field $R = \partial/\partial t$ is said to be the *resting field* induced by the chart.

In (13), we have defined the motion of a mechanical system to be a second-order curve $\beta = \dot{\gamma}\colon I \to A^1 M$, where $\gamma = \pi \circ \beta\colon I \to M$ denotes a time-parametrized curve in the Galilean manifold (M, ϑ, g). We define the *relative velocity* of the motion $\dot{\gamma}$ with respect to the reference field R as the vector field along γ which is pointwise given by

$$\dot{\gamma}_{\gamma(\tau)} - R_{\gamma(\tau)} \in A^0_{\gamma(\tau)} M.$$

As a difference of time-normalized vectors, the relative velocity is spacelike and can therefore be measured by the Galilean metric g, which models the mass of a mechanical system. This is accounted for in the following definition.

Definition 6. Let (M, ϑ, g) be the Galilean manifold of a finite-dimensional mechanical system and let A^1M be the corresponding state space. The *kinetic energy* of the mechanical system with respect to a reference field $R\colon M \supseteq U_R \to A^1M$ is the function

$$T_R\colon \pi^{-1}(U_R) \to \mathbb{R},\ (p, v_p) \mapsto \tfrac{1}{2}\, g_p(v_p - R_p, v_p - R_p)\,, \tag{59}$$

with $v_p \in A^1_pM$ and $R(p) = (p, R_p)$.

The kinetic energy of the motion (15) with respect to the reference field R is then given by

$$T_R\big(\dot\gamma(\tau)\big) = \tfrac{1}{2}\, g_{\gamma(\tau)}\big(\dot\gamma_{\gamma(\tau)} - R_{\gamma(\tau)}, \dot\gamma_{\gamma(\tau)} - R_{\gamma(\tau)}\big)\,.$$

Let (U, ϕ) be an adapted chart of M and let us assume for simplicity that $U \subseteq U_R$. Moreover, let $R = \partial/\partial t + R^i\partial/\partial x^i$ be an arbitrary reference field. In the natural chart induced by ϕ, the kinetic energy (59) locally reads

$$T_R = \tfrac{1}{2} g_{ij}u^iu^j - g_{ij}u^iR^j + \tfrac{1}{2} g_{ij}R^iR^j\,, \tag{60}$$

where we have used the local expression of the metric (31) and the symmetry of the bilinear map g. By equation (60), the kinetic energy is the sum of

$$T_{R,2} \coloneqq \tfrac{1}{2} g_{ij}u^iu^j, \quad T_{R,1} \coloneqq -g_{ij}u^iR^j, \quad \text{and } T_{R,0} \coloneqq \tfrac{1}{2} g_{ij}R^iR^j.$$

The number in the subscript describes the respective degree of positive homogeneity[6] of each term with respect to $(u^1, \ldots, u^n)$. In the special case where R is a resting field, i.e. $R = \partial/\partial t$, the local expression of the kinetic energy (60) reduces to

$$T_R = \tfrac{1}{2} g_{ij}u^iu^j.$$

Definition 7. Let (M, ϑ, g) be the Galilean manifold of a finite-dimensional mechanical system and let $\hat\vartheta = \pi^*\vartheta$ be the time structure of the state space A^1M. The kinetic energy T_R with respect to a reference field R of the mechanical system induces the action form

$$\Omega_R \coloneqq \mathrm{d}(T_R\hat\vartheta + \partial T_R)\,. \tag{61}$$

The following proposition justifies Definition 7.

Proposition 1 ([29], p. 35)**.** *The differential two-form Ω_R defined by* (61) *is indeed an action form that induces a bundle metric g on A^0M. The difference between an (arbitrary) action form Ω and Ω_R is a force two-form $\Phi_R \coloneqq \Omega - \Omega_R$.*

Proof. To check that (61) defines an action form, we have to check the properties (i) to (iii) from Theorem 1. According to the rules (40), (41), and (42) of ∂, it holds that

$$\partial\Omega_R = -\mathrm{d}(\partial T_R \wedge \hat\vartheta + \partial^2 T_R) = -\mathrm{d}(\partial T_R \wedge \hat\vartheta + \hat\vartheta \wedge \partial T_R) = 0,$$

[6]Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be differentiable. The function f is called *positively homogeneous* of degree k if $f(\alpha\mathbf{x}) = \alpha^k f(\mathbf{x})$ for all $\alpha \in \mathbb{R}_0^+$ and all $\mathbf{x} \in \mathbb{R}^n$.

which shows that Ω_R enjoys property (iii). To prove properties (i) and (ii), we use the rules (40) to arrive at the local expression

$$T_R\hat{\vartheta} + \partial T_R = T_R\,\mathrm{d}t + \frac{\partial T_R}{\partial u^i}(\mathrm{d}x^i - u^i\mathrm{d}t). \tag{62}$$

With (62), definition (61) leads to

$$\begin{aligned}\Omega_R &= \frac{\partial T_R}{\partial x^i}\mathrm{d}x^i \wedge \mathrm{d}t + \mathrm{d}\left(\frac{\partial T_R}{\partial u^i}\right) \wedge \left(\mathrm{d}x^i - u^i\mathrm{d}t\right) \\ &= \left(\mathrm{d}\left(\frac{\partial T_R}{\partial u^i}\right) - \frac{\partial T_R}{\partial x^i}\mathrm{d}t\right) \wedge \left(\mathrm{d}x^i - u^i\mathrm{d}t\right),\end{aligned} \tag{63}$$

It is clear from the expression (63) that Ω_R vanishes on $\ker\mu$. Moreover, it follows that

$$\Omega_R\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial x^j}\right) = \frac{\partial^2 T_R}{\partial u^i \partial u^j} = g_{ij}.$$

Therefore, Ω_R is an action form by Theorem 1. As seen in (54), the difference of two action forms $\Omega - \Omega_R$ is semi-basic. The assertion that $\Phi_R = \Omega - \Omega_R$ is a force two-form follows because $\partial(\Omega - \Omega_R) = \partial\Omega - \partial\Omega_R = 0$. □

10 Classification of forces

10.1 Inertia forces

If in classical mechanics the motion of a particle is studied with respect to a non-inertial frame of reference,[7] additional force effects appear in the equations of motions. These forces that result from the use of a non-inertial frame of reference instead of an inertial one are referred to as fictitious, apparent or as inertia forces. Two examples are the Coriolis force and the centrifugal force. In our presentation, these forces are provided by the inertia force two-form stemming from a change in reference field.

Let R and $\tilde{R}$ be two reference fields and Ω be a given action form. By Proposition 1, we know that Ω decomposes as

$$\Omega = \Omega_R + \Phi_R = \Omega_{\tilde{R}} + \Phi_{\tilde{R}}$$

and, therefore,

$$\Omega_{\tilde{R}} - \Omega_R = \Phi_R - \Phi_{\tilde{R}}.$$

Being the difference of Φ_R and $\Phi_{\tilde{R}}$, the two-form $\Omega_{\tilde{R}} - \Omega_R$ is a force two-form

$$\Psi_{R,\tilde{R}} \coloneqq \Omega_{\tilde{R}} - \Omega_R\,, \tag{64}$$

which we call the *inertia force two-form* between the reference fields R and $\tilde{R}$. The force Φ_R with respect to R is composed of the force $\Phi_{\tilde{R}}$ with respect to $\tilde{R}$ and of the inertia force two-form (64). As force two-form, the latter is a kinematic quantity because it does not depend on the motion Z. It depends only on the Galilean manifold (M, ϑ, g) and on the reference fields R and $\tilde{R}$, as can be seen from the following considerations.

[7] See Section 39 in [24] or Sections IV.4–5 in [22].

Because the inertia force two-form (64) is given by the difference of two exact two-forms, it is exact. This means that there exists a one-form $\hat{\alpha}_{R,\tilde{R}}$ such that

$$\Psi_{R,\tilde{R}} = \Omega_{\tilde{R}} - \Omega_R = \mathrm{d}\hat{\alpha}_{R,\tilde{R}}\,.$$

By definition (61) and the linearity of the differential operators d and ∂, we know that

$$\hat{\alpha}_{R,\tilde{R}} = (T_{\tilde{R}} - T_R)\hat{\vartheta} + \partial(T_{\tilde{R}} - T_R)\,.$$

Let the reference field R be defined on U_R such that $R = \partial/\partial t + R^i\partial/\partial x^i$ and let the field $\tilde{R}$ be given by $\tilde{R} = \partial/\partial t + \tilde{R}^i\partial/\partial x^i$ on $U_{\tilde{R}}$ such that $U_R \cap U_{\tilde{R}} \neq \emptyset$. Equations (60) and (62) lead to

$$\begin{aligned}\hat{\alpha}_{R,\tilde{R}} &= (T_{\tilde{R}} - T_R)\hat{\vartheta} + \partial(T_{\tilde{R}} - T_R)\\ &= \tfrac{1}{2}g_{ij}(\tilde{R}^i\tilde{R}^j - R^iR^j)\mathrm{d}t + g_{ij}(R^j - \tilde{R}^j)\mathrm{d}x^i\,.\end{aligned} \tag{65}$$

on $\pi^{-1}\big(U_R\big) \cap \pi^{-1}\big(U_{\tilde{R}}\big)$. The local expression (65) reveals that $\hat{\alpha}_{R,\tilde{R}}$ is a basic form. This motivates the following alternative definition of the one-form $\hat{\alpha}_{R,\tilde{R}}$. Indeed, let $\alpha_{R,\tilde{R}}$ be the one-form on $U_R \cap U_{\tilde{R}}$ defined by requiring

$$\begin{aligned}\alpha_{R,\tilde{R}}\big(R\big) &\overset{!}{=} \tfrac{1}{2}g\big(R-\tilde{R}, R-\tilde{R}\big)\,,\\ \alpha_{R,\tilde{R}}\big(v\big) &\overset{!}{=} g\big(v, R-\tilde{R}\big)\end{aligned}$$

for all spacelike vector fields v, i.e., for all local sections of the spacelike bundle A^0M. Then the one-form $\hat{\alpha}_{R,\tilde{R}}$ is the pullback of $\alpha_{R,\tilde{R}}$ with the natural projection, i.e.,

$$\hat{\alpha}_{R,\tilde{R}} = \pi^*\alpha_{R,\tilde{R}}\,.$$

This shows that the inertia force two-form (64) for the reference fields R and $\tilde{R}$ does indeed depend only on the Galilean manifold (M, ϑ, g) and, therefore, is a kinematic quantity.

10.2 Potential and nonpotential forces

We say that a force F_R is a *potential force* if the related force two-form Φ_R^{p} is closed, i.e. if

$$\mathrm{d}\Phi_R^{\mathrm{p}} = 0. \tag{66}$$

According to the Poincaré lemma there exists a neighborhood $W \subseteq A^1M$ and a one-form ϕ_R defined on W such that

$$\Phi_R^{\mathrm{p}}\big|_W = \mathrm{d}\phi_R\,. \tag{67}$$

The closedness (exactness) of the force two-form implies the closedness (exactness) of the action form. Indeed, with Proposition 1, we have seen that an action form is the sum of an exact form (61) and the force two-form Φ_R. Therefore, it makes sense to speak of a *closed (exact) mechanical system* if the force two-form is closed (exact).

Using the local coordinates of the force two-form (57), condition (66) gives

$$
\begin{aligned}
0 = \mathrm{d}\bigg[\Big(-F_i + \frac{1}{2}u^j\Big(\frac{\partial F_i}{\partial u^j} - \frac{\partial F_j}{\partial u^i}\Big)\Big)\mathrm{d}t\wedge\mathrm{d}x^i + \frac{1}{2}\frac{\partial F_i}{\partial u^j}\mathrm{d}x^i\wedge\mathrm{d}x^j\bigg] \\
= \Big(-\frac{\partial F_i}{\partial x^k} + \frac{1}{2}\frac{\partial^2 F_i}{\partial t\partial u^k} + \frac{1}{2}u^j\Big(\frac{\partial^2 F_i}{\partial x^k\partial u^j} - \frac{\partial^2 F_j}{\partial x^k\partial u^i}\Big)\Big)\mathrm{d}t\wedge\mathrm{d}x^i\wedge\mathrm{d}x^k \\
+ \Big(-\frac{\partial F_i}{\partial u^k} + \frac{1}{2}\Big(\frac{\partial F_i}{\partial u^k} - \frac{\partial F_k}{\partial u^i}\Big) + \frac{1}{2}u^j\Big(\frac{\partial^2 F_i}{\partial u^k\partial u^j} - \frac{\partial^2 F_j}{\partial u^k\partial u^i}\Big)\Big)\mathrm{d}t\wedge\mathrm{d}x^i\wedge\mathrm{d}u^k \\
+ \frac{1}{2}\frac{\partial^2 F_i}{\partial x^k\partial u^j}\mathrm{d}x^i\wedge\mathrm{d}x^j\wedge\mathrm{d}x^k + \frac{1}{2}\frac{\partial^2 F_i}{\partial u^k\partial u^j}\mathrm{d}x^i\wedge\mathrm{d}x^j\wedge\mathrm{d}u^k,
\end{aligned}
$$

where we have dropped the R by writing F instead of F_R for notational convenience. The above condition leads to restrictions on the coefficient functions F_i of F. The last term disappears if and only if

$$F_i = E_i + B_{ij}u^j, \tag{68}$$

with functions E_i and B_{ij} which do not depend on $(u^1, \ldots, u^n)$. The vanishing of the $\mathrm{d}t\wedge\mathrm{d}x^i\wedge\mathrm{d}u^k$-term requires that

$$B_{ij} = -B_{ji}.$$

The annihilation of the $\mathrm{d}x^i\wedge\mathrm{d}x^j\wedge\mathrm{d}x^k$-term leads to

$$\frac{\partial B_{ij}}{\partial x^k} + \frac{\partial B_{ki}}{\partial x^j} + \frac{\partial B_{jk}}{\partial x^i} = 0. \tag{69}$$

Finally, the annihilation of the first term imposes that

$$\frac{\partial B_{ij}}{\partial t} = \frac{\partial E_i}{\partial x^j} - \frac{\partial E_j}{\partial x^i}\,. \tag{70}$$

Consequently, a closed two-form Φ_R^{p} has the local form

$$\Phi_R^{\mathrm{p}} = E_i\,\mathrm{d}x^i\wedge\mathrm{d}t + \tfrac{1}{2}B_{ij}\,\mathrm{d}x^i\wedge\mathrm{d}x^j\,, \tag{71}$$

where the component functions only depend on $(t, x^1, \ldots, x^n)$. The suggestive use of the letters B and E lets us identify (69) and (70) as a generalized version of Maxwell's equations. We see from the local expression (71) that a closed force two-form is basic. Therefore, as a one-form ϕ_R satisfying (67), we consider the locally defined basic one-form

$$\phi_R = -V_R\mathrm{d}t + A_i^R\mathrm{d}x^i\,, \tag{72}$$

with functions[8] V_R and A_i^R which only depend on $(t, x^1, \ldots, x^n)$. With the one-form (72) it holds that

$$E_i = -\Big(\frac{\partial V_R}{\partial x^i} + \frac{\partial A_i^R}{\partial t}\Big), \qquad B_{ij} = 2\frac{\partial A_j^R}{\partial x^i}.$$

[8]In the context of a charged particle moving in an electromagnetic field the function V_R is known as scalar potential of the field and the $\mathbb{R}^3$-tuple (A_1^R, A_2^R, A_3^R) is said to be its vector potential. See p. 45 in [23].

It is important to notice that the Poincaré lemma guarantees the existence of a one-form ϕ_R and *not* its uniqueness. Indeed, two one-forms ϕ_R and

$$\phi'_R = \phi_R + \mathrm{d}f\,,$$

differing by the differential $\mathrm{d}f$ of a function $f = f(t, x^1, \dots, x^n)$, lead to the same force two-form (67) because $\mathrm{d} \circ \mathrm{d} = 0$. This implies that the coefficient functions of ϕ_R from (72) are related to those of

$$\phi'_R = -V'\mathrm{d}t + A'_i\mathrm{d}x^i$$

by

$$V' = V - \frac{\partial f}{\partial t}, \quad \text{and} \quad A'_i = A_i + \frac{\partial f}{\partial x^i}, \tag{73}$$

without changing the resulting force two-form Φ_R. Note that we dropped the letter R in equation (73) for notational convenience. The invariance property (73) of the coefficient functions of the one-form ϕ_R is known as *gauge invariance*.[9]

In classical mechanics (no electromagnetism), one assumes $B_{ij} = 0$ such that the coefficient functions (68) are independent of $(u^1, \dots, u^n)$. In this case, the closed force two-form (71) reduces to

$$\Phi^{\mathrm{p}}_R = E_i\,\mathrm{d}x^i \wedge \mathrm{d}t.$$

Accordingly, the one-form ϕ_R from (72) reduces to

$$\phi_R = -V_R(t, x^1, \dots, x^n)\mathrm{d}t + A^R_i(t)\mathrm{d}x^i.$$

Because of the gauge invariance (73), we can add a differential $\mathrm{d}f$ without changing the resulting force two-form. We choose $f(t, x^1, \dots, x^n) = -A^R_i(t)x^i$ such that

$$\phi'_R = \phi_R + \mathrm{d}f = \Big(- V_R - \frac{\mathrm{d}A^R_i}{\mathrm{d}t}x^i\Big)\mathrm{d}t =: -V'_R\mathrm{d}t.$$

This proves that in classical mechanics the force two-form of a potential force can be locally derived from a one-form

$$\phi'_R = -V'_R(t, x^1, \dots, x^n)\,\mathrm{d}t \tag{74}$$

without loss of generality. The coefficient function V'_R in (74) is known as *potential energy* with respect to the reference field R. In what follows, we will consider one-forms of the form (72) because they comprise the form (74) used in classical mechanics.

The previous considerations allow us to split a given force two-form

$$\Phi_R = \Phi^{\mathrm{p}}_R + \Phi^{\mathrm{np}}_R$$

into a part $\Phi^{\mathrm{p}}_R = \mathrm{d}\phi_R$ that is defined by a one-form (72) and the remaining part Φ^{np}_R which we will refer to as *nonpotential force two-form*.

[9] See Section 18 in [23].

11 Lagrangian and Cartan one-form

Definition 8. Let (M, ϑ, g) be the Galilean manifold of a finite-dimensional mechanical system, $\hat{\vartheta} = \pi^*\vartheta$ be the time structure of the state space A^1M, and let T_R be the kinetic energy of the mechanical system with respect to a reference field R. Let Φ_R^{p} be the (locally) exact potential force two-form given in the natural chart (10) by

$$\Phi_R^{\mathrm{p}} = \mathrm{d}\phi_R \quad \text{with} \quad \phi_R = -V_R \mathrm{d}t + A_i^R \mathrm{d}x^i \,,$$

where the component functions V_R and A_R^i of ϕ_R only depend on $(t, x^1, \dots, x^n)$, see (72). The *Lagrangian* of the mechanical system with respect to the reference field R is the function

$$L_R \coloneqq T_R - V_R + A_i^R u^i \tag{75}$$

on the state space and induces the *Cartan one-form*

$$\omega_R = L_R \hat{\vartheta} + \partial L_R \,. \tag{76}$$

In the local coordinates induced on the neighborhood $\pi^{-1}(U) \subseteq A^1M$ by the natural chart (10), the Cartan one-form reads

$$\omega_R = L_R \mathrm{d}t + \frac{\partial L_R}{\partial u^i}\big(\mathrm{d}x^i - u^i \mathrm{d}t\big) = \left(L_R - u^i \frac{\partial L_R}{\partial u^i}\right)\mathrm{d}t + \frac{\partial L_R}{\partial u^i}\mathrm{d}x^i \,, \tag{77}$$

from which we can see that the Cartan one-form determines the Lagrangian by

$$L_R = Z \lrcorner \omega_R, \tag{78}$$

where Z is an arbitrary second-order field on the state space.

Using the rules (40), it can be seen that the chart representation of ϕ_R used in Definition 8 is equivalent to

$$\phi_R = (-V_R + A_i^R u^i)\mathrm{d}t + \partial(-V_R + A_i^R u^i). \tag{79}$$

By comparing (79) to the definition (61) of the action form Ω_R induced by the kinetic energy T_R of the mechanical system, it is clear that the sum $\Omega_R + \Phi_R^{\mathrm{p}}$ can be written as

$$\begin{aligned} \Omega_R + \Phi_R^{\mathrm{p}} = \Omega_R + \mathrm{d}\phi_R &= \mathrm{d}[(T_R - V_R + A_i^R u^i)\hat{\vartheta} + \partial(T_R - V_R + A_i^R u^i)] \\ &= \mathrm{d}(L_R \hat{\vartheta} + \partial L_R) = \mathrm{d}\omega_R \,. \end{aligned} \tag{80}$$

By Proposition 1, the sum of the action form Ω_R and a force two-form is again an action form, which in view of equation (80) implies that $\mathrm{d}\omega_R$ is an action form.

If two Cartan one-forms ω_R and ω_R' define the same action form

$$\Omega = \mathrm{d}\omega_R = \mathrm{d}\omega_R',$$

they may still differ by the differential of a function $f \in C^\infty(A^1M)$, i.e.

$$\omega_R' - \omega_R = \mathrm{d}f \tag{81}$$

because $\mathrm{d}\circ\mathrm{d}=0$. Since the difference $\omega_R'-\omega_R$ is semi-basic, the function f in (81) needs to satisfy

$$f=\pi^*h=h\circ\pi$$

for some function $h\in C^\infty(M)$. The fact that the same action form can be determined by different Cartan one-forms transfers to the Lagrangians by equation (78). The Lagrangians L_R and L_R' defining the Cartan one-forms ω_R and ω_R' can locally differ by

$$L_R'-L_R=Z\lrcorner(\omega_R'-\omega_R)=Z\lrcorner\mathrm{d}f=\frac{\partial f}{\partial t}+u^i\frac{\partial f}{\partial x^i} \tag{82}$$

and still define the same action form Ω. If the difference (82) is evaluated along a second-order curve $\dot\gamma$, then we retrieve the classical statement[10] that (the chart representations of) the Lagrangians L_R and L_R' may differ by the total derivative with respect to t of a function f, which depends on time t and the positions $(x^1,\dots,x^n)$. Note that the statement (82) is directly related to the gauge invariance (73). This can be seen by using (73) in the definition of the Lagrangian (75).

The following considerations allow us to pin down the form of Lagrangians. First, a Lagrangian needs to define the bundle metric by

$$g_{ij}=\Omega\left(\frac{\partial}{\partial u^i},\frac{\partial}{\partial x^j}\right)=\frac{\partial^2L_R}{\partial u^i\partial u^j},$$

according to Theorem 1(ii), where the second equality follows from straight forward computations in a natural chart. Because it holds for the bundle metric that $\partial g_{ij}/\partial u^k=0$, the Lagrangian needs to satisfy

$$\frac{\partial^3L_R}{\partial u^k\partial u^i\partial u^j}=0,$$

and it therefore has the local form

$$L_R=\tfrac12g_{ij}u^iu^j+a_iu^i+a_0 \tag{83}$$

with coefficients $a_0,\dots,a_n$ that do not depend on $u^1,\dots,u^n$. This means that there are functions $\bar a_0,\dots,\bar a_n\colon U\to\mathbb{R}$ defined on the neighborhood $U\subseteq M$ such that

$$a_\alpha\coloneqq\pi^*\bar a_\alpha=\bar a_\alpha\circ\pi$$

with $\alpha=0,\dots,n$. With the local expression (60) of the kinetic energy the Lagrangian (75) can be written as

$$L_R=T_R-V_R+A_i^Ru^i=\tfrac12g_{ij}u^iu^j+(A_i^R-g_{ij}R^j)u^i+\tfrac12g_{ij}R^iR^j-V_R.$$

The comparison with (83) leads to the equalities

$$\begin{aligned}a_0&=\tfrac12g_{ij}R^iR^j-V_R,\\ a_i&=A_i-g_{ij}R^j,\end{aligned} \tag{84}$$

with $i=1,\dots,n$.

[10] See [24] p. 4.

The case of classical mechanics where the one-form ϕ_R reduces to (74) can be studied by setting $A_i = 0$. It follows from equation (84) that the reference field R and the potential V_R can be determined from the coefficients $a_0, \ldots, a_n$ of a given Lagrangian (83) as

$$R^i = -g^{ij} a_j \quad \text{and} \quad V_R = \tfrac{1}{2} g_{ij} R^i R^j - a_0. \tag{85}$$

The coefficients g^{ij} in equation (85) are given by the inverse matrix to the coefficient matrix of the Galilean metric g.

12 Dynamics of finite-dimensional mechanical systems

Postulate 1. *Let (M, ϑ, g) be the Galilean manifold of a finite-dimensional mechanical system and let $\hat{\vartheta} = \pi^*\vartheta$ be the time structure of the state space A^1M. Moreover, let ω_R be the Cartan one-form induced by a Lagrangian L_R of the mechanical system and let Φ_R^{np} be the nonpotential force two-form, each one with respect to the reference field R.*

A motion β of the mechanical system is an integral curve of the vector field $X \in \mathrm{Vect}(A^1M)$ characterized by

$$\hat{\vartheta}(X) = 1 \quad \textit{and} \quad X \lrcorner \Omega = 0, \tag{86}$$

where

$$\Omega := \mathrm{d}\omega_R + \Phi_R^{\mathrm{np}} \tag{87}$$

is the action form of the mechanical system. Consequently, the motion β is a solution of the equations of motion

$$\dot{\beta}(\tau) = X(\beta(\tau))\,. \tag{88}$$

This postulate is fundamental to the description of the dynamics of finite-dimensional mechanical systems, as it links the motion of the mechanical system to a Lagrangian and the nonpotential force two-form of the system. This means that for a specific finite-dimensional mechanical system, e.g., an industrial robot, the modeling process consists in finding an appropriate Lagrangian together with a nonpotential force two-form. Postulate 1 then links these two quantities to the motion of the system.

In Section 11 we showed that $\mathrm{d}\omega_R$ is an action form, which by Proposition 1 implies that the action form Ω defined in (87) is indeed an action form. Moreover, Theorem 1 guarantees that the vector field X is uniquely characterized by (86) and that it is a second-order field, such that its integral curves are motions of the mechanical system.

Using (80), equation (87) allows to write the action form Ω of the mechanical system as

$$\Omega = \Omega_R + \Phi_R^{\mathrm{p}} + \Phi_R^{\mathrm{np}}\,, \tag{89}$$

where Ω_R is the action form (61) induced by the kinetic energy T_R of the system and Φ_R^{p} is the potential force two-form of the system. Equation (89) shows, that the action form Ω_R describes the motion of a mechanical system which with respect to the reference field R is not subjected to forces (i.e., $\Phi_R^{\mathrm{p}} = \Phi_R^{\mathrm{np}} = 0$).

12.1 Lagrange's equations of the second kind

Using the coordinate representation of Postulate 1 with respect to the local coordinates provided by the natural chart (10), we show that the equations of motion (88) take the form of Lagrange's equations of the second kind.

Consider that by equation (77), the action form $\mathrm{d}\omega_R$ is locally given by

$$\mathrm{d}\omega_R = \mathrm{d}\Big(L - u^i \frac{\partial L}{\partial u^i}\Big) \wedge \mathrm{d}t + \mathrm{d}\Big(\frac{\partial L}{\partial u^i}\Big) \wedge \mathrm{d}x^i \tag{90}$$

and, by equation (57), the nonpotential force two-form Φ_R^{np} in (87) reads

$$\Phi_R^{\mathrm{np}} = F_i \,\mathrm{d}x^i \wedge \mathrm{d}t + \frac{1}{2}\frac{\partial F_i}{\partial u^j}\big(\mathrm{d}x^i - u^i \mathrm{d}t\big) \wedge \big(\mathrm{d}x^j - u^j \mathrm{d}t\big). \tag{91}$$

Note that we lightened the notation by suppressing the reference field R when writing the Lagrangian in (90).

By Postulate 1, we know that for the action form $\Omega = \mathrm{d}\omega_R + \Phi_R^{\mathrm{np}}$, the conditions

$$\hat{\vartheta}\big(X\big) = 1, \quad \text{and} \quad X \lrcorner \Omega = 0 \tag{92}$$

determine the vector field $X \in \mathrm{Vect}\big(A^1M\big)$ that describes the motion of the mechanical system. Condition (92) requires the vector field X to be time-normalized such that it can be locally written as

$$X = \frac{\partial}{\partial t} + A^i \frac{\partial}{\partial x^i} + B^i \frac{\partial}{\partial u^i}, \tag{93}$$

where the coefficients A^i and B^i with $i = 1, \dots, n$ are smooth functions. Moreover, condition (92) can be rewritten as

$$0 = X \lrcorner \Omega = X \lrcorner \mathrm{d}\omega_R + X \lrcorner \Phi_R^{\mathrm{np}}. \tag{94}$$

Using equations (90), (91) and (93), together with the relation $\mathcal{L}_X f = \mathrm{d}f(X)$ for $f \in C^\infty(A^1M)$, we compute both terms separately. The first term of equation (94) can be written as

$$\begin{aligned} X \lrcorner \mathrm{d}\omega_R &= \mathcal{L}_X\Big(L - u^i \frac{\partial L}{\partial u^i}\Big)\mathrm{d}t - \mathrm{d}\Big(L - u^i \frac{\partial L}{\partial u^i}\Big) + \mathcal{L}_X\Big(\frac{\partial L}{\partial u^i}\Big)\mathrm{d}x^i - A^i \mathrm{d}\Big(\frac{\partial L}{\partial u^i}\Big) \\ &= \Big[\mathcal{L}_X\Big(L - u^i \frac{\partial L}{\partial u^i}\Big) - \frac{\partial L}{\partial t} - \frac{\partial^2 L}{\partial t \partial u^i}\big(A^i - u^i\big)\Big]\mathrm{d}t \\ &\quad + \Big[\mathcal{L}_X\Big(\frac{\partial L}{\partial u^i}\Big) - \frac{\partial L}{\partial x^i} - \frac{\partial^2 L}{\partial x^i \partial u^j}\big(A^j - u^j\big)\Big]\mathrm{d}x^i \\ &\quad + \frac{\partial^2 L}{\partial u^i \partial u^j}\big(u^j - A^j\big)\mathrm{d}u^i \,. \end{aligned} \tag{95}$$

The second term of equation (94) reads as

$$\begin{aligned} X \lrcorner \Phi_R^{\mathrm{np}} = &\Big[A^i F_i - \frac{1}{2}u^i\big(A^j - u^j\big)\Big(\frac{\partial F_j}{\partial u^i} - \frac{\partial F_i}{\partial u^j}\Big)\Big]\mathrm{d}t \\ &+ \Big[- F_i + \frac{1}{2}\big(A^j - u^j\big)\Big(\frac{\partial F_j}{\partial u^i} - \frac{\partial F_i}{\partial u^j}\Big)\Big]\mathrm{d}x^i \,. \end{aligned} \tag{96}$$

By equation (94), the sum of (95) and (96) has to vanish. In particular, the $\mathrm{d}u^i$-component of (95) must be zero. This implies that

$$A^j = u^j \quad \text{for } j = 1, \ldots, n \tag{97}$$

because the matrix

$$\frac{\partial^2 L}{\partial u^i \partial u^j} = g_{ij}$$

is positive definite and thus has full rank. Equation (97) requires the vector field X to be a second-order field, which in consideration of Theorem 1 is no surprise. The annihilation of the $\mathrm{d}x^i$-part of the sum (94) together with (97) leads to *Lagrange's equations* of the second kind

$$\mathcal{L}_X\left(\frac{\partial L}{\partial u^i}\right) - \frac{\partial L}{\partial x^i} = F_i. \tag{98}$$

Let us consider a motion $\beta\colon I \to A^1M,\ \tau \mapsto \beta(\tau)$ of the mechanical system, which by Postulate 1 is an integral curve of the vector field X defined by equation (98). Since X is a second-order field by (93) and (97), we know that the integral curve β of X is a second-order curve. Hence, the motion has the chart representation $\Phi \circ \beta(\tau) = (t(\tau), \mathbf{x}(\tau), \dot{\mathbf{x}}(\tau))$ and needs to satisfy equation (98) such that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L}{\partial u^i} \circ \beta(\tau)\right) - \frac{\partial L}{\partial x^i} \circ \beta(\tau) = F_i \circ \beta(\tau) \tag{99}$$

by the definition of the Lie derivative. We recognize (99) as Lagrange's equations of the second kind in their classical form.[11] Note, $t(\tau) = \tau + \tau_0$ by (22).

Since the action form Ω has rank $2n$, the $\mathrm{d}t$-component depends on the $2n$ equations (97) and (98), which fully determine the vector field X. Accordingly, the $\mathrm{d}t$-component must vanish for the vector field X and provides the *equation of energy*

$$\mathcal{L}_X\left(u^i \frac{\partial L}{\partial u^i} - L\right) = -\frac{\partial L}{\partial t} + u^i F_i\,. \tag{100}$$

Evaluated along the motion β this relation takes the form

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\dot{x}^i(\tau)\frac{\partial L}{\partial u^i} \circ \beta(\tau) - L \circ \beta(\tau)\right) = -\frac{\partial L}{\partial t} \circ \beta(\tau) + \dot{x}^i(\tau) F_i \circ \beta(\tau)\,. \tag{101}$$

We immediately recognize that the *Hamiltonian*

$$H_R\colon A^1M \supseteq \pi^{-1}(U) \to \mathbb{R},\ a \mapsto H_R(a) := \left(u^i \frac{\partial L_R}{\partial u^i} - L_R\right)(a) \tag{102}$$

is conserved along the motion β, if the mechanical system is not subjected to nonpotential forces and its Lagrangian has no explicit time dependence. Note that $\pi^{-1}(U)$ denotes as usual the domain of the natural chart (10), for which, by equations (60) and (75), the Hamiltonian takes the form

$$H_R = u^i \frac{\partial L_R}{\partial u^i} - L_R = \frac{1}{2} g_{ij} u^i u^j - \frac{1}{2} g_{ij} R^i R^j + V_R\,. \tag{103}$$

In the special case where the reference field R is the resting field of the natural chart, i.e. $R^i = 0$, the Hamiltonian is the sum of the kinetic energy (59) and the potential energy V_R.

[11] See [21], p. 24, [43], p. 63, [35], p. 75 or [24], p. 3.

12.2 Hamilton's equations

Using a different chart as in the previous section, the equations of motion take the form of Hamilton's equations. To show this, we represent Postulate 1 with respect to the local coordinates provided by the chart

$$\tilde{\Phi}\colon A^1M \supseteq \pi^{-1}(U) \to \mathbb{R}^{2n+1}, (p, v_p) \mapsto (\tilde{t}, \tilde{x}^1, \ldots, \tilde{x}^n, p_1, \ldots, p_n)\,. \tag{104}$$

Denoting the natural chart (10) with Φ, the chart (104) is given by the change of coordinates $\tilde{\Phi} \circ \Phi^{-1}$ defined as

$$\tilde{t} = t, \quad \tilde{x}^i = x^i \quad \text{and} \quad p_i = \frac{\partial L_R}{\partial u^i} \circ \Phi^{-1}(t, x^1, \ldots, x^n, u^1, \ldots, u^n)\,. \tag{105}$$

We call p_i the *generalized momentum* coordinates, which by equations (75) and (60) together with $\partial V_R/\partial u^i = 0$ have the form

$$p_i = g_{ij}\big(u^j - R^j\big) + A_i^R. \tag{106}$$

The full rank of the Galilean metric g guarantees that the relation (106) can be resolved for $u^1, \ldots, u^n$ as

$$u^i = g^{ij}\big(p_j - A_j^R\big) + R^i, \tag{107}$$

where g^{ij} are the components of the inverse matrix g_{ij}. We refer to $(\tilde{t}, \tilde{x}^1, \ldots, \tilde{x}^n, p_1, \ldots, p_n)$ as *canonical coordinates*.[12] The tildes on t and the x^i allow the distinction between the canonical coordinates and those provided by the natural chart (10).

With the Hamiltonian (102), we rewrite the Cartan one-form (77) as

$$\omega_R = \left(L_R - u^i \frac{\partial L_R}{\partial u^i}\right)\mathrm{d}t + \frac{\partial L_R}{\partial u^i}\mathrm{d}x^i = -H_R \mathrm{d}\tilde{t} + p_i \mathrm{d}\tilde{x}^i, \tag{108}$$

where we have used that $\mathrm{d}t = \mathrm{d}\tilde{t}$ and $\mathrm{d}x^i = \mathrm{d}\tilde{x}^i$ for $i = 1, \ldots, n$. Inserting (107) in (103), the Hamiltonian in canonical coordinates reads as

$$H_R = \frac{1}{2} g^{ij}\big(p_i - A_i^R\big)\big(p_j - A_j^R\big) + R^j\big(p_j - A_j^R\big) + V_R\,.$$

This implies that we can rewrite (107) as

$$u^i = \frac{\partial H_R}{\partial p_i}\,. \tag{109}$$

We compute the exterior derivative of the Cartan one-form (108) as

$$\mathrm{d}\omega_R = -\mathrm{d}H \wedge \mathrm{d}\tilde{t} + \mathrm{d}p_i \wedge \mathrm{d}\tilde{x}^i\,, \tag{110}$$

[12] These coordinates are by no means canonically defined since they depend on the choice of a reference field R. Physically, the quantities $p_1, \ldots, p_n$ are generalized momenta. In the context of time-independent mechanics playing on the cotangent bundle T^*Q of a time-independent configuration manifold Q, the position and generalized momentum coordinates provided by the Darboux theorem are *canonical*. Moreover, Hamilton's equations are also referred to as *canonical equations* (see [24], p. 132). So we use the adjective canonical because of tradition.

where we stick to our policy of dropping the letter R whenever it is hindering. Expressing the basis vectors $\partial/\partial u^i$ induced by the natural chart (10) with respect to the basis vectors induced by the chart (104) gives

$$\frac{\partial}{\partial u^i} = \frac{\partial \tilde{t}}{\partial u^i}\frac{\partial}{\partial \tilde{t}} + \frac{\partial \tilde{x}^j}{\partial u^i}\frac{\partial}{\partial \tilde{x}^j} + \frac{\partial p_j}{\partial u^i}\frac{\partial}{\partial p_j} = g_{ji}\frac{\partial}{\partial p_j}\,,$$

where we adopt the convention that a lower index appearing in the denominator is considered to be an upper index. With this relation and equation (109), the nonpotential force Φ_R^{np} of (57) adopts the local form

$$\Phi_R^{\mathrm{np}} = F_i \mathrm{d}\tilde{x}^i \wedge \mathrm{d}\tilde{t} + \frac{1}{2} g_{rj}\frac{\partial F_i}{\partial p_r}\left[\mathrm{d}\tilde{x}^i - \frac{\partial H}{\partial p_i}\mathrm{d}\tilde{t}\right] \wedge \left[\mathrm{d}\tilde{x}^j - \frac{\partial H}{\partial p_j}\mathrm{d}\tilde{t}\right] \tag{111}$$

because $\mathrm{d}t = \mathrm{d}\tilde{t}$ and $\mathrm{d}x^i = \mathrm{d}\tilde{x}^i$ for $i = 1, \ldots, n$.

The time-normalized field X from (93) that describes the motion can be locally expressed as

$$X = \frac{\partial}{\partial \tilde{t}} + A^i \frac{\partial}{\partial \tilde{x}^i} + C_i \frac{\partial}{\partial p_i}\,. \tag{112}$$

By Postulate 1, the time-normalized vector field X is determined by

$$0 = X \lrcorner \Omega = X \lrcorner \mathrm{d}\omega_R + X \lrcorner \Phi_R^{\mathrm{np}}\,.$$

As before, we compute $X \lrcorner \mathrm{d}\omega_R$ and $X \lrcorner \Phi_R^{\mathrm{np}}$ separately. With the local expressions (110) and (112), we get

$$\begin{aligned} X \lrcorner \mathrm{d}\omega_R &= -\mathcal{L}_X(H)\mathrm{d}\tilde{t} + \mathrm{d}H + C_i\,\mathrm{d}\tilde{x}^i - A^i \mathrm{d}p_i \\ &= -\left(\mathcal{L}_X H - \frac{\partial H}{\partial \tilde{t}}\right)\mathrm{d}\tilde{t} + \left(\frac{\partial H}{\partial \tilde{x}^i} + C_i\right)\mathrm{d}\tilde{x}^i + \left(\frac{\partial H}{\partial p_i} - A^i\right)\mathrm{d}p_i\,. \end{aligned} \tag{113}$$

Using (111) and (112), we obtain

$$\begin{aligned} X \lrcorner \Phi_R^{\mathrm{np}} = &\left[A^i F_i - \frac{1}{2}\frac{\partial H}{\partial p_i}\left(A^j - \frac{\partial H}{\partial p_j}\right)\left(g_{ri}\frac{\partial F_j}{\partial p_r} - g_{rj}\frac{\partial F_i}{\partial p_r}\right)\right]\mathrm{d}\tilde{t} \\ &+ \left[-F_i + \frac{1}{2}\left(A^j - \frac{\partial H}{\partial p_j}\right)\left(g_{ri}\frac{\partial F_j}{\partial p_r} - g_{rj}\frac{\partial F_i}{\partial p_r}\right)\right]\mathrm{d}\tilde{x}^i. \end{aligned} \tag{114}$$

The one-form $X \lrcorner \Omega = X \lrcorner \mathrm{d}\omega_R + X \lrcorner \Phi_R^{\mathrm{np}}$ is zero if each component vanishes. Since the coefficient of the $\mathrm{d}p_i$-component in (113) has to vanish, it follows that

$$A^i = \frac{\partial H}{\partial p_i} = g^{ij}\big(p_j - A_j^R\big) + R^i\,. \tag{115}$$

With equation (115), the annihilation of the $\mathrm{d}\tilde{x}^i$-component of $X \lrcorner \Omega$ implies

$$C_i = -\frac{\partial H}{\partial \tilde{x}^i} + F_i. \tag{116}$$

Finally, consider a motion β of the mechanical system, which by Postulate 1 is an integral curve of the vector field X. The tangent field $\dot{\beta}$ is time-parametrized because of (112). Consequently, using equations (115) and (116) the equations of motion (88) locally have the form

$$\begin{aligned}\dot{\tilde{x}}^i(\tau) &= \frac{\partial H}{\partial p_i} \circ \beta(\tau), \\ \dot{p}_i(\tau) &= -\frac{\partial H}{\partial \tilde{x}^i} \circ \beta(\tau) + F_i \circ \beta(\tau),\end{aligned} \tag{117}$$

where $\tilde{\Phi} \circ \beta(\tau) = (\tilde{t}(\tau), \tilde{\mathbf{x}}(\tau), \mathbf{p}(\tau))$ with $\tilde{t}(\tau) = \tau + \tau_0$ by (22). We recognize (117) as *Hamilton's equations.*[13]

Similarly to the previous section, the $\mathrm{d}\tilde{t}$-component depends on the $2n$ equations (115) and (116), which fully determine the vector field X. Accordingly, the $\mathrm{d}\tilde{t}$-component must vanish for the vector field X and provides the *equation of energy*

$$\mathcal{L}_X H = \frac{\partial H}{\partial \tilde{t}} + \frac{\partial H}{\partial p_i} F_i \,. \tag{118}$$

Evaluated along the the motion β this relation takes the form

$$\frac{\mathrm{d}}{\mathrm{d}\tau} H \circ \beta(\tau) = \frac{\partial H}{\partial \tilde{t}} \circ \beta(\tau) + \left(\frac{\partial H}{\partial p_i} F_i \right) \circ \beta(\tau) \,. \tag{119}$$

Also here, we recognize that the *Hamiltonian* is conserved along the motion β, if the mechanical system is not subjected to nonpotential forces and its Hamiltonian has no explicit time-dependence.

13 Conclusion

We have shown that the Galilean manifold (M, ϑ, g) together with the affine subbundle A^1M are appropriate spaces on which the dynamics of time-dependent finite-dimensional mechanical systems subjected to nonpotential forces can be formulated. The motion as a curve in the state space is given as an integral curve of a second-order field on the state space. Theorem 1 is a major result that was obtained by Loos in [29] and which uniquely connects a differential two-form Ω with the second-order field Z whose interior product with Ω vanishes. This theorem justifies to model the dynamics of a mechanical system by stating a particular action form. Maybe the most original and outstanding contribution of Loos is the treatment of forces within the time-dependent theory. With the idea that any force leads to a change of acceleration, forces are introduced as smooth sections of the dual of the vertical bundle $\mathrm{Ver}^*(A^1M)$. Theorem 2 then provides a bijective relation between forces and force two-forms, which are semi-basic ∂-closed differential two-forms on the state space. Eventually, for the dynamics described by an action form, Theorem 3 guarantees that changing the dynamics by adding an additional force corresponds with summing up action form and force two-form. The essential gain within the geometric framework provided by Loos lies then in the coordinate-free formulation of Postulate 1, which comprises the governing equations of a theory for time-dependent finite-dimensional mechanical systems subjected to nonpotential forces. Most remarkably, this postulate unifies the Lagrangian and the Hamiltonian approaches, which in this intrinsic theory just result from different coordinate representations.

[13]See p. 132 of [24] or p. 63 of [43].

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