Dynamic simulation of the Wilberforce pendulum using constrained spatial nonlinear beam finite elements

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More than 100 years ago, Lionel Robert Wilberforce did investigations \textit{On the Vibrations of a Loaded Spiral Spring} \cite{1}. The spring was clamped at its upper side and on the other side, perpendicular to the spring axis, a steel cylinder was attached. Four screws with adjustable nuts were symmetrically attached around the cylinder in order to change its moment of inertia (Fig. 1). In this paper the Wilberforce pendulum is modeled by a rigid body attached to a constrained spatial nonlinear Timoshenko beam, discretized with B-spline shape functions. As shown by a numerical experiment, the presented model is capable of reproducing the characteristic pendulum motion.

The pendulum’s spring is made of spring steel EN 10270-1 with density \( \rho_0 = 7850 \text{ kg/m}^3 \), Young’s modulus \( E = 206 \cdot 10^9 \text{ N/m}^2 \) and shear modulus \( G = 81.5 \cdot 10^9 \text{ N/m}^2 \). Has the undeformed shape of a perfect helix with \( n = 20 \) coils, coil radius \( R = 16 \text{ mm} \), wire diameter \( d = 1 \text{ mm} \) and unloaded pitch \( c = 1 \text{ mm} \). It is discretized using 128 cubic uniformly distributed B-spline finite elements of the Timoshenko beam model presented in \cite{2}.

![Fig. 1: Configurations of the Wilberforce pendulum: (left) pure vertical oscillations, (right) pure torsional oscillation of the pendulum bob.](image)

![Fig. 2: Kinematics of the precurved spatial Timoshenko beam.](image)

![Fig. 3: Deflection and first Cardan angle of the pendulum bob.](image)

\[ \begin{align*}
& \mathbf{r}_0 : I \rightarrow \mathbb{E}^3 \quad \text{and the corresponding orthonormal director triads are } \mathbf{D}_1 : I \rightarrow \mathbb{E}^3. \quad \text{While } \mathbf{D}_1 \text{ is identified with the unit tangent} \\
& \quad \text{to the reference centerline } \mathbf{r}_0, \text{ i.e., } \mathbf{D}_1 = r'_0/\|r'_0\| \text{ with } r'_0 := \frac{\partial \mathbf{r}_0}{\partial \xi}, \text{ the vectors } \mathbf{D}_2 \text{ and } \mathbf{D}_3 \text{ are identified with the geometric principal axes of the cross sections, see Figure 2. Since a helix is a transcendental curve that can be represented exactly by} \\
& \quad r'_0(\xi) = R \cos \phi(\xi) \mathbf{e}_1 + R \sin \phi(\xi) \mathbf{e}_2 + c \xi \mathbf{e}_3, \quad \phi(\xi) = 2\pi n \xi, \\
& \text{there is no exact representation for it by means of rational functions, e.g., B-splines. Thus, in order to obtain a helicoidal beam reference configuration, a fitting procedure is required. This was done by minimizing the quadratic error } e_k = \sum_{i=1}^k \|r_0(\xi_i) - \]

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Next, by introducing the arc length coordinate $s = \int_0^\xi J(\xi) \, d\xi$ of the reference centerline at $\xi$, together with $J(\xi) := \|\partial_0(\xi)/\partial \xi\|$, the respective differential measures $ds$ and $d\xi$ of the arc length and the parameter domain can be related by $ds = J(\xi) \, d\xi$. Let $f$ be an arbitrary mapping $f : I \times \mathbb{R} \to \mathbb{R}^3$, $(\xi, t) \mapsto f(\xi, t)$, its derivative with respect to the arc length of the reference centerline is defined as $\partial f(\xi, t) := \partial_0 f(\xi, t)/J(\xi)$.

Further, by defining an appropriate strain energy density $W(\Gamma_i, K_i, \xi)$, see [2], and application of Einstein summation convention on repeated indices, the beam’s material resistance is modeled by the internal virtual work

$$\delta W^{\text{int}} = -\int_I \left\{ \delta \mathbf{r}^i \cdot n_i d\mathbf{l}_i + \delta d_i \cdot (n_i \mathbf{r}^i + \varepsilon_{kji} m_k \mathbf{d}_j^i) + \delta d_j \cdot \varepsilon_{ijk} m_k \mathbf{d}_k \right\} \, d\xi,$$

where the contact forces $n_i(\xi, t) = \partial W/\partial \Gamma_i(\Gamma_i, K_i(\xi), \xi)$ and contact couples $m_i(\xi, t) = \partial W/\partial K_i(\Gamma_i, K_i(\xi), \xi)$ depend on the objective strain measures $\Gamma_i(\xi, t) = \mathbf{r}^i(\xi, t) \cdot d_i(\xi, t)/J(\xi)$, $K_i(\xi, t) = \varepsilon_{ijk} d_k(\xi, t) \cdot d_j^i(\xi, t)/J(\xi)$ and possibly on the material point $\xi$ itself.

Gravitational effects are incorporated by the external virtual work $\delta W^{\text{ext}} = \int_I \delta \mathbf{r} \cdot \mathbf{\Pi} J \, d\xi$, where the vector of gravitational forces $\mathbf{\Pi} = - \rho_0 \omega g \mathbf{e}_z$, the beam’s cross sectional area $A = \pi (d/2)^2$ and the gravitational acceleration $g$ were introduced.

In order to formulate the virtual work contributions of inertia effects, the beam is assumed to be a constrained three-dimensional continuous body [4,5] whose points in the current configuration are restricted to the coupling of bending and torsion of the deformed spring an additional torsional oscillation of the cylinder is induced to the given values for the moment of inertia in order to achieve an almost perfect phase shift shown in Fig. 3.

Acknowledgements This research has been funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Grant No. 405032572 as part of the priority program 2100 Soft Material Robotic Systems. Open access funding enabled and organized by Projekt DEAL.

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