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MECHANICAL REPRESENTATION AND STABILITY OF DYNAMICAL SYSTEMS CONTAINING FRACTIONAL SPRINGPOT ELEMENTS

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ABSTRACT

In this article we consider the Lyapunov stability of mechanical systems containing fractional springpot elements. We obtain the potential energy of a springpot by an infinite dimensional mechanical analogue model. Furthermore, we consider a simple dynamical system containing a springpot as a functional differential equation and use the potential energy of the springpot in a Lyapunov functional to prove uniform stability and discuss asymptotic stability of the equilibrium with the help of an invariance theorem.

INTRODUCTION

Fractional calculus is a mathematical discipline dealing with derivatives and integrals of arbitrary (non-integer) order. Among many other applications, the associated theory has been used in rheology to describe viscoelastically damped systems [1, 2]. Time-fractional elements, so called "springpots", are used together with classical springs and dashpots to describe the viscoelastic behavior on both short and long time-scales. To develop a Lyapunov stability theory for mechanical systems modelled by springpot-elements it is necessary to determine their conserved and dissipated energy. Therefore, we have to understand the physical meaning of fractional elements and their initialization depending on the applied definition of fractional derivative. In this paper, we propose representations for springpots consisting of an infinite number of springs and dashpots in

the fashion of [3, 4] as well as [5] and associate them to Caputo and Riemann-Liouville derivatives. Additionally, we consider the stored energy of the infinite spring-dashpot representation and derive the potential energy of a springpot. All observations are based on the frequency distributed model (or diffusive / infinite state representation) of fractional integrators introduced by Montseny [6], Matignon [7] and intensely elaborated by Trigeassou et al. [8–11]. The obtained energy terms will then be used to find Lyapunov functionals for mechanical systems containing springpots. We consider the system equations as time-delay (or functional) differential equations and use an associated stability theorem to prove uniform stability of a simple mass-spring-springpot-system. This approach can be generalized to more complex and nonlinear systems.

FRACTIONAL CALCULUS AND INFINITE STATE REPRESENTATION

We will consider the two classical definitions of fractional derivative by Riemann-Liouville and Caputo. Both concepts depend on the fractional Riemann-Liouville integral, which is defined for an integrable function $x = x(t)$ and scalar values $t_0 \leq t$, $\alpha > 0$ as

$$I_{t_0+}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} x(\tau)(t - \tau)^{\alpha-1} \mathrm{d}\tau, \tag{1}$$

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where $\Gamma(\alpha)$ is the Gamma function. For $\alpha = 0$ we set $I_{t_0+}^0 x := x$ and it can be seen directly that the choice $\alpha = 1$ leads to the classical integral. In this paper, we will describe this integral operator by the infinite state representation [9]

$$\begin{cases} \dot{z}(\omega,t) & = -\omega z(\omega,t) + x(t), \\ I_{t_0+}^\alpha x(t) & = \int_0^\infty \mu_\alpha(\omega)z(\omega,t)\mathrm{d}\omega. \end{cases} \quad (2)$$

The infinite state $z(\omega,t)$ fulfills the above differential equation $\forall \omega \geq 0$ and the fractional integral is obtained by integrating all contributions $z(\omega,t)$ weighted by the function

$$\mu_\alpha(\omega) := \frac{\sin(\alpha\pi)}{\pi}\omega^{-\alpha}. \quad (3)$$

We will have a close look at the correspondence between Eqns. (1) and (2) to emphasize the significance of the initial time instant t_0. Thus, consider Eqn. (1) as a convolution of functions x and

$$y_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}. \quad (4)$$

The Laplace transform of y_α and substitution $u := st$ yield

$$\begin{aligned} \mathscr{L}\{y_\alpha(t)\}(s) & = \int_0^\infty \frac{t^{\alpha-1}}{\Gamma(\alpha)}\mathrm{e}^{-st}\mathrm{d}t \\ & = \frac{1}{\Gamma(\alpha)}\underbrace{\int_0^\infty u^{\alpha-1}\mathrm{e}^{-u}\mathrm{d}u}_{=\Gamma(\alpha)} \cdot s^{-\alpha} = s^{-\alpha}. \end{aligned} \quad (5)$$

Therefore, we obtain an integral representation of Eqn. (4)

$$\begin{aligned} y_\alpha(t) & = \frac{1}{\Gamma(\alpha)}\mathscr{L}\{y_{1-\alpha}\}\Big|_{s=t} \\ & = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^\infty \omega^{-\alpha}\mathrm{e}^{-\omega t}\mathrm{d}\omega \\ & = \int_0^\infty \mu_\alpha(\omega)\mathrm{e}^{-\omega t}\mathrm{d}\omega \end{aligned} \quad (6)$$

where we used Eqn. (3) and the property

$$\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin(\alpha\pi)}. \quad (7)$$

of the Gamma function. This representation will now be used to derive the infinite state $z(\omega,t)$ corresponding to Eqn. (2). To

correctly initialize the fractional integral, we consult a past time interval $[t_0, 0]$, where $t_0 < 0$ or even $t_0 = -\infty$. This leads to a fractional integral with a negative lower bound. This approach, called "history-function based initialization" was elaborated in [12] and has been shown to be equivalent to the infinite state model [13]. In the following derivation, we can see this equivalence explicitly. Substituting Eqns. (4) and (6) in Eqn. (1), we obtain

$$\begin{aligned} I_{t_0+}^\alpha x(t) & = \int_{t_0}^t y_\alpha(t-\tau)x(\tau)\mathrm{d}\tau \\ & = \int_{t_0}^t \int_0^\infty \mu_\alpha(\omega)\mathrm{e}^{-\omega(t-\tau)}\mathrm{d}\omega x(\tau)\mathrm{d}\tau \\ & = \int_0^\infty \mu_\alpha(\omega)\underbrace{\int_{t_0}^t \mathrm{e}^{-\omega(t-\tau)}x(\tau)\mathrm{d}\tau}_{=:z(\omega,t)}\mathrm{d}\omega. \end{aligned} \quad (8)$$

The function $z(\omega,t)$, as defined in Eqn. (8), is a particular solution of the differential equation in Eqn. (2). Therefore, we have to initialize Eqn. (2) by

$$z(\omega,t_0) = 0 \quad (9)$$

to obtain equivalence of Eqns. (1) and (2). The interpretation of this condition and the choice of the initial integration time t_0 will be discussed in the next section. First we want to define the fractional Riemann-Liouville derivative of an integrable function $x = x(t)$ for $0 < \alpha < 1$, which will be the case in our application, as

$$^{RL}D_{t_0+}^\alpha x(t) = \frac{\mathrm{d}}{\mathrm{d}t}I_{t_0+}^{1-\alpha}x(t) \quad (10)$$

and analogously the fractional Caputo derivative for an absolutely continuous function $x = x(t)$ as

$$^{C}D_{t_0+}^\alpha x(t) = I_{t_0+}^{1-\alpha}\dot{x}(t). \quad (11)$$

MECHANICAL REPRESENTATION OF SPRINGPOTS

In this section we consider a springpot as an abstract mechanical element (Fig. 1) with coefficient $c > 0$, initial time $t_0 < 0$ and differentiation order $\alpha \in (0,1)$. The force f acting on a springpot with elongation q is given by the constitutive relation

$$f(t) = cD_{t_0+}^\alpha q(t), \quad (12)$$

where D will be the Riemann-Liouville or Caputo derivative. The time interval $[t_0, 0]$ represents the entire significant history of the springpot, i.e. for earlier times $t \leq t_0$ we assume $q(t) = 0$ and $f(t) = 0$.
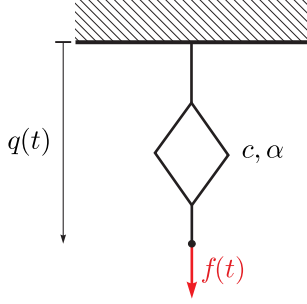
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FIGURE 1.   Force acting on a springpot.

Kelvin Representation

For the Riemann-Liouville derivative $^{RL}D_{t_0+}^{\alpha}$, we consider the equation

$$f(t) = c\,^{RL}D_{t_0+}^{\alpha}q(t) = c\frac{\mathrm{d}}{\mathrm{d}t}I_{t_0+}^{1-\alpha}q(t). \tag{13}$$

Applying $I_{t_0+}^{\alpha}$ on both sides leads to

$$\begin{aligned} I_{t_0+}^{\alpha}f(t) &= cI_{t_0+}^{\alpha}\,^{RL}D_{t_0+}^{\alpha}q(t) \\ &= c\left(q(t) - \frac{(t-t_0)^{\alpha-1}}{\Gamma(\alpha)}\lim_{\tau\to t_0+}I_{t_0+}^{1-\alpha}q(\tau)\right) \end{aligned} \tag{14}$$

as long as $f(t)$ is integrable for $t \geq t_0$ [14, Thm. 2.23]. Analogously, for the Caputo derivative $^{C}D_{t_0+}^{\alpha}$ the equation

$$f(t) = c\,^{C}D_{t_0+}^{\alpha}q(t) = cI_{t_0+}^{1-\alpha}\dot{q}(t) \tag{15}$$

may be reformulated applying $I_{t_0+}^{\alpha}$ on both sides as

$$\begin{aligned} I_{t_0+}^{\alpha}f(t) &= cI_{t_0+}^{\alpha}I_{t_0+}^{1-\alpha}\dot{q}(t) = cI_{t_0+}^{1}\dot{q}(t) = c\int_{t_0}^{t}\dot{q}(\tau)\mathrm{d}\tau \\ &= c(q(t) - q(t_0)). \end{aligned} \tag{16}$$

Due to our assumptions $f(t) = q(t) = 0 \quad \forall t \leq t_0$, the last terms on the right side of both Eqns. (14) and (16) vanish. Hence, considering Eqn. (2), we can represent a springpot for both kinds of fractional derivatives by

$$\begin{cases} \dot{z}(\omega,t) = -\omega z(\omega,t) + f(t), & z(\omega,t_0) = 0, \\ q(t) = \frac{1}{c}\int_0^{\infty}\mu_{\alpha}(\omega)z(\omega,t)\mathrm{d}\omega. \end{cases} \tag{17}$$

We will now propose a mechanical interpretation of Eqn. (17) similar as in [5]. Thus, consider an uncountably infinite-order
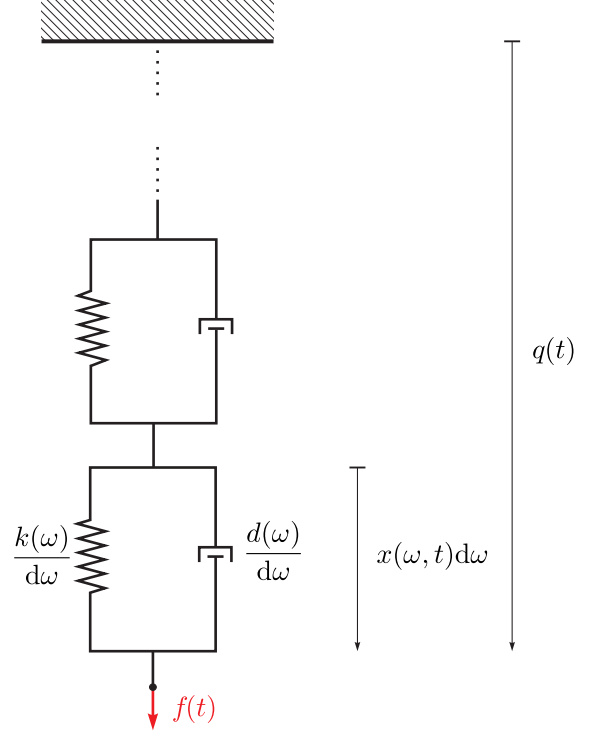

FIGURE 2.   Schematic Kelvin representation of a springpot.

series connection of Kelvin elements (Fig. 2) such that the elongation of the entire system is given by

$$q(t) = \int_0^{\infty}x(\omega,t)\mathrm{d}\omega, \tag{18}$$

where $x(\omega,t)$ is a density such that the elongation of the Kelvin elements of the system at time t is related to $x(\omega,t)\mathrm{d}\omega$. For a force f acting on the system, the relation

$$f(t) = k(\omega)x(\omega,t) + d(\omega)\dot{x}(\omega,t) \tag{19}$$

holds $\forall \omega \geq 0$ with distributed spring and dashpot parameters $\frac{k(\omega)}{\mathrm{d}\omega}$ and $\frac{d(\omega)}{\mathrm{d}\omega}$, respectively. Comparison of Eqns. (18) and (19) to Eqn. (17) leads to the identification

$$\begin{aligned} x(\omega,t) &= \frac{1}{c}\mu_{\alpha}(\omega)z(\omega,t), \quad k(\omega) = \frac{c\omega}{\mu_{\alpha}(\omega)}, \\ d(\omega) &= \frac{c}{\mu_{\alpha}(\omega)}, \qquad \omega = \frac{k(\omega)}{d(\omega)}. \end{aligned} \tag{20}$$

From this representation, we obtain an interpretation of the infinite state z as

$$\omega z(\omega,t) = k(\omega)x(\omega,t) =: f_s(\omega,t), \tag{21}$$
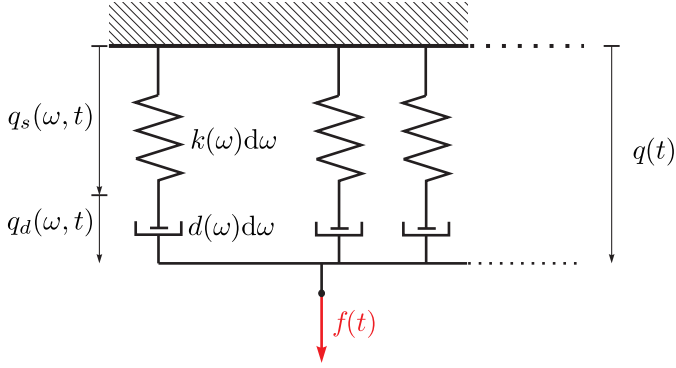
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FIGURE 3. Schematic Maxwell representation of a springpot.

where $f_s(\omega,t)$ is the force acting on the springs of the Kelvin elements. Moreover, we retrieve the energy of the springpot as the energy stored in the springs of the Kelvin elements

$$E(t) = \frac{1}{2} \int_0^\infty k(\omega)x^2(\omega,t)\mathrm{d}\omega \qquad (22)$$

or, with the identification in Eqn. (20), as

$$E(t) = \frac{1}{2c} \int_0^\infty \omega\mu_\alpha(\omega)z^2(\omega,t)\mathrm{d}\omega. \qquad (23)$$

Maxwell Representation

We can directly apply the distributed representation of the fractional integral, i.e. Eqn. (2), on Eqns. (13) and (15) to obtain

$$\begin{cases} \dot{y}_{RL}(\omega,t) = -\omega y_{RL}(\omega,t) + q(t), & y_{RL}(\omega,t_0) = 0, \\ f(t) = c\int_0^\infty \mu_{1-\alpha}(\omega)\dot{y}_{RL}(\omega,t)\mathrm{d}\omega \end{cases} \qquad (24)$$

for a Riemann-Liouville springpot and

$$\begin{cases} \dot{y}_C(\omega,t) = -\omega y_C(\omega,t) + \dot{q}(t), & y_C(\omega,t_0) = 0, \\ f(t) = c\int_0^\infty \mu_{1-\alpha}(\omega)y_C(\omega,t)\mathrm{d}\omega \end{cases} \qquad (25)$$

for a Caputo springpot, respectively. Again we examine an analogue system consisting of uncountably infinite springs and dashpots, this time a parallel arrangement of Maxwell elements (Fig. 3) as in [5], where the forces $g(\omega,t)\mathrm{d}\omega$ of the Maxwell elements are integrated to the resulting force

$$f(t) = \int_0^\infty g(\omega,t)\mathrm{d}\omega \qquad (26)$$

of the system. The springs of the Maxwell elements are characterized by their elongation $q_s(\omega,t)$ and spring constant $k(\omega)\mathrm{d}\omega$

and the dashpots by elongation $q_d(\omega,t)$ and constant $d(\omega)\mathrm{d}\omega$ such that the elongation of the system $q(t)$ appears as

$$q(t) = q_s(\omega,t) + q_d(\omega,t) \quad \forall \omega \geq 0 \qquad (27)$$

with

$$g(\omega,t)\mathrm{d}\omega = k(\omega)\mathrm{d}\omega\, q_s(\omega,t) = d(\omega)\mathrm{d}\omega\, \dot{q}_d(\omega,t). \qquad (28)$$

Derivation of Eqn. (27) and inserting Eqn. (28) leads to

$$\dot{q}(t) = \frac{\dot{g}(\omega,t)}{k(\omega)} + \frac{g(\omega,t)}{d(\omega)}. \qquad (29)$$

Comparison of Eqns. (26) and (29) to Eqns. (24) and (25) results in the identification

$$g(\omega,t) = c\mu_{1-\alpha}(\omega)\dot{y}_{RL}(\omega,t) = c\mu_{1-\alpha}(\omega)y_C(\omega,t),$$
$$k(\omega) = c\mu_{1-\alpha}(\omega), \quad d(\omega) = \frac{c\mu_{1-\alpha}(\omega)}{\omega}, \quad \omega = \frac{k(\omega)}{d(\omega)}. \qquad (30)$$

Furthermore, we obtain an interpretation of the infinite states y_{RL} and y_C of the Riemann-Liouville and Caputo springpot as

$$\omega y_{RL}(\omega,t) = q(t) - \dot{y}_{RL}(\omega,t) = q(t) - \frac{g(\omega,t)}{c\mu_{1-\alpha}(\omega)}$$
$$= q(t) - q_s(\omega,t) = q_d(\omega,t),$$
$$y_C(\omega,t) = \dot{y}_{RL}(\omega,t) = \frac{g(\omega,t)}{c\mu_{1-\alpha}(\omega)} = \frac{g(\omega,t)}{k(\omega)} = q_s(\omega,t). \qquad (31)$$

Again, we consider the energy of the mechanical equivalent system which in this situation is the potential energy stored in the springs of the Maxwell elements, i.e.

$$E(t) = \frac{1}{2} \int_0^\infty k(\omega)q_s^2(\omega,t)\mathrm{d}\omega, \qquad (32)$$

which can be reformulated with Eqns. (30) and (31) as

$$E(t) = \frac{c}{2} \int_0^\infty \mu_{1-\alpha}(\omega)y_C^2(\omega,t)\mathrm{d}\omega = \frac{c}{2} \int_0^\infty \mu_{1-\alpha}(\omega)\dot{y}_{RL}^2(\omega,t)\mathrm{d}\omega. \qquad (33)$$

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Relationship between different representations

From the two mechanical representations of the springpot we obtain equivalent energy equations (23) and (33). While Eqn. (23) is expressed in terms of the force quantity z, we obtain (33) in terms of the elongation quantities $\dot{y}_{RL}$ and y_C. Furthermore we derive the following correspondence of these quantities from the energy equations

$$z(\omega,t) = c\omega^{\alpha-1}\dot{y}_{RL}(\omega,t) = c\omega^{\alpha-1}y_C(\omega,t) \quad \text{f. a. a. } \omega. \quad (34)$$

Moreover, we note that the difference in modelling the Maxwell representation for the Riemann-Liouville and Caputo derivative lies only in the choice of coordinates (see Eqn. (31)). Therefore, both definitions can be used to model the same physical relation.

STABILITY CONSIDERATIONS
Introduction

Our long-term aim is to develop a Lyapunov stability theory for dynamical systems containing fractional springpot elements. Therefore, we want to use the energy terms derived above as Lyapunov functional candidates. In contrast to many other investigations in this field [8, 15, 16], we do not consider explicit systems of fractional differential equations. Instead, we want to describe springpots in dynamical systems by their infinite state representation and apply stability criteria similar to the well-known results of Lyapunov stability theory for ordinary differential equations. Hence, we investigate the stability theory for *functional differential equations* [17–20]. We regard the example of a single degree-of-freedom mass-spring-springpot-system with a Caputo-type springpot (Fig. 4), which can be described by the equation of motion

$$m\ddot{q}(t) = -kq(t) - c^C D_{t_0+}^{\alpha}q(t), \quad t \geq 0 \quad (35)$$

with mass m, elongation $q(t)$, spring coefficient k, springpot coefficient c and differentiation order $\alpha \in (0,1)$ and given initial functions $\varphi_1, \varphi_2 \in CB((-\infty,0];\mathbb{R})$ such that

$$\begin{aligned} q(t) &= \varphi_1(t), \quad t \leq 0, \\ \dot{q}(t) &= \varphi_2(t), \quad t \leq 0 \end{aligned} \quad (36)$$

where $\varphi_i(t) = 0$ for $t \leq t_0$, $i = 1,2$. Using Eqn. (25), we reformulate Eqn. (35) as a system of first order differential equations

$$\begin{cases} \dot{q}(t) = v(t), \\ \dot{v}(t) = -\frac{k}{m}q(t) - \frac{c}{m}\int_{t_0}^{t}\int_0^{\infty}\mu_{1-\alpha}(\omega)e^{-\omega(t-\tau)}d\omega\,v(\tau)d\tau, \end{cases} \quad (37)$$

where $v(t)$ is the velocity of the mass. Using the definition
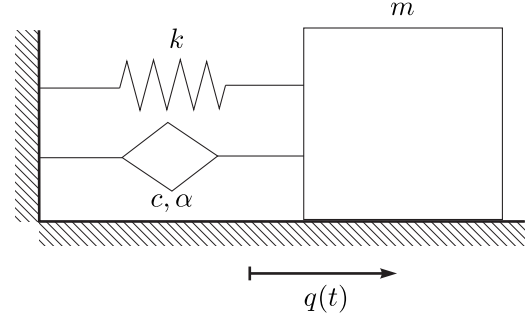
FIGURE 4. Single degree-of-freedom mass-spring-springpot system.

$$v_t(s) = v(t+s), \quad s \in (-\infty,0], \quad (38)$$

we can reformulate Eqn. (37) as

$$\begin{cases} \dot{q}(t) = v(t), \\ \dot{v}(t) = -\frac{k}{m}q(t) - \frac{c}{m}\int_{t_0-t}^{0}\int_0^{\infty}\mu_{1-\alpha}(\omega)e^{\omega s}d\omega\,v_t(s)ds, \end{cases} \quad (39)$$

which is a functional differential equation (FDE) with unbounded delay. In the following, we want to introduce the theory of FDEs to prove uniform stability of the trivial solution of Eqn. (39) with given initial functions as in Eqn. (36). Additionally, as solutions of Eqn. (39) all seem to approach the trivial solution, we discuss the autonomous case, i.e. we consider Eqn. (39) with bounded delay to apply an invariance theorem and describe problems using the energy functional of the unbounded case to prove asymptotic stability.

Stability of FDEs

Consider an FDE of the form

$$\dot{x}(t) = f(t,x_t), \quad t \geq t_1 \geq 0 \quad (40)$$

with

$$\begin{aligned} x &\in C((-\infty,T);\mathbb{R}^n), \quad T > t_1, \\ x_t(s) &= x(t+s), \quad s \in (-\infty,0], \\ f &: [0,\infty) \times CB((-\infty,0];\mathbb{R}^n) \supset S \to \mathbb{R}^n. \end{aligned} \quad (41)$$

For Eqn. (40) we want to state Lyapunov functionals $V(t,\varphi)$ defined on the space $[0,\infty) \times CB((-\infty,0];\mathbb{R}^n)$ together with the norm

$$\|\varphi\|_{\infty} = \sup_{s \in (-\infty,0]} \|\varphi(s)\|_n, \quad (42)$$

where $\|.\|_n$ is the Euclidean norm. The subset S in Eqn. (41) on which f is defined will be considered as

$$S = [0, \infty) \times Q_H,$$
$$Q_H := \{ \varphi \in CB((-\infty, 0]; \mathbb{R}^n) \mid \|\varphi\|_\infty < H \}, \quad H > 0 \quad (43)$$

and we assume f locally Lipschitzian in Q_H which ensures local existence and uniqueness of a solution $x(t_1, \varphi)$ of Eqn. (40) for a given initial function φ and intitial time t_1 [18, 19]. Moreover, let $f(t, 0) = 0 \; \forall t \geq t_1$, such that the trivial solution $x(t_1, \varphi)(t) = 0$ is an equilibrium of the system. In the following theorem, we formulate sufficient conditions for uniform stability of the trivial solution of Eqn. (40) by a Lyapunov theorem adapted to the space $CB((-\infty, 0]; \mathbb{R}^n)$ as in [18–20]. Therefore, we need the following definitions.

Definition 1:
A *solution* of Eqn. (40) with initial function $\varphi \in CB((-\infty, 0]; \mathbb{R}^n)$ and initial time $t_1 \geq 0$ is a function $x(t_1, \varphi)$ defined and continuous on an interval $(-\infty, T)$, $T > t_1$, such that $x_t(t_1, \varphi) \in S$ for $t \in [t_1, T)$, $x_{t_1}(t_1, \varphi) = \varphi$ and $x(t_1, \varphi)(t)$ satisfies Eqn. (40) for $t \in [t_1, T)$.

Definition 2 (Stability)**:**
The trivial solution $x(t_1, \varphi)(t) = 0$ of system (40) together with (41), (43), initial function φ and initial time t_1 is called

(a) *stable*, if for all $\varepsilon \in (0, H]$ there exists a $\delta = \delta(t_1, \varepsilon) > 0$ such that $\|x(t_1, \varphi)(t)\|_n \leq \varepsilon$ for $t \geq t_1$ if $\varphi \in Q_\delta$.
(b) *uniformly stable*, if for all $\varepsilon \in (0, H]$ there exists a $\delta = \delta(\varepsilon) > 0$ independent from t_1, such that $\|x(t_1, \varphi)(t)\|_n \leq \varepsilon$ for $t \geq t_1$ if $\varphi \in Q_\delta$.
(c) *asymptotically stable*, if it is stable and for all t_1 there exists a $\delta = \delta(t_1) > 0$ such that $\lim_{t \to 0} \|x(t_1, \varphi)(t)\|_n = 0$ if $\varphi \in Q_\delta$.

Theorem 1 (Stability [18])**:**
Let $f : [0, \infty) \times Q_H \to \mathbb{R}^n$ such that $f(t, 0) = 0 \; \forall t \geq t_1$ and denote $u_i : [0, \infty) \to \mathbb{R}$, $i = 1, 2$ some scalar, continuous, non-decreasing functions such that $u_i(0) = 0$ and $u_i(r) > 0$ for $r > 0$. Let there exist a continuous functional $V : [0, \infty) \times Q_H \to \mathbb{R}$ such that

$$u_1(\|\varphi(0)\|_n) \leq V(t, \varphi) \leq u_2(\|\varphi\|_\infty), \quad (44)$$
$$\dot{V}(t, x_t) \leq 0, \quad (45)$$

then the trivial solution of Eqn. (40) is uniformly stable.

Proof. Let $\varepsilon > 0$ be given with $\varepsilon < H$ and find $\delta > 0$ with $u_2(\delta) < u_1(\varepsilon)$. For $\varphi \in Q_H$ with $\|\varphi\|_\infty < \delta$, and since

$\dot{V}(t, x_t(t_1, \varphi)) \leq 0$, we have

$$u_1(\|x_t(t_1, \varphi)(0)\|_n) \leq V(t, x_t(t_1, \varphi))$$
$$\leq V(t_1, \varphi) \leq u_2(\|\varphi\|_\infty) \leq u_2(\delta) < u_1(\varepsilon).$$

Therefore we obtain

$$\|x(t_1, \varphi)(t)\|_n < \varepsilon,$$

which proves that the trivial solution of (40) is uniformly stable. $\square$

Mass-Spring-Springpot System

We will apply the above theory on the two-dimensional system (39). Therefore, we introduce the energy functional

$$V(t, q_t, v_t) = \frac{1}{2} m v_t^2(0) + \frac{1}{2} k q_t^2(0)$$
$$+ \frac{1}{2} c \int_0^\infty \mu_{1-\alpha}(\omega) \left(\int_{t_0 - t}^0 e^{\omega s} v_t(s) \mathrm{d}s \right)^2 \mathrm{d}\omega, \quad (46)$$

where the last addend is deduced from Eqn. (33). Again, we will use the variable y_C, which in this case has the form

$$y_C(\omega, t) = \int_{t_0 - t}^0 e^{\omega s} v_t(s) \mathrm{d}s. \quad (47)$$

The derivative along solutions of Eqn. (39) of this candidate Lyapunov functional is the rate of dissipation

$$\dot{V}(t, q_t, v_t) = k q_t(0) v_t(0) + m v_t(0) \dot{v}_t(0)$$
$$+ c \int_0^\infty \mu_{1-\alpha}(\omega) y_C(\omega, t) (v_t(0) - \omega y_C(\omega, t)) \mathrm{d}\omega$$
$$= v_t(0) \left(m \ddot{q}_t(0) + k q_t(0) + c \int_0^\infty \mu_{1-\alpha}(\omega) y_C(\omega, t) \mathrm{d}\omega \right) \quad (48)$$
$$- c \int_0^\infty \omega \mu_{1-\alpha}(\omega) y_C^2(\omega, t) \mathrm{d}\omega$$
$$= -c \int_0^\infty \omega \mu_{1-\alpha}(\omega) y_C^2(\omega, t) \mathrm{d}\omega \leq 0$$

which satisfies Eqn. (45). Therefore, we state the following assertion.

Proposition 1:
The trivial solution of Eqn. (39) is uniformly stable.

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Proof. Eqn. (45) is satisfied, but we still have to prove (44). As all addends in Eqn. (46) are non-negative, we can estimate

$$
\begin{aligned}
V(t,q_t,v_t) &\geq \frac{1}{2}\min\{k,m\}\left(q^2(t)+v^2(t)\right) \\
&= \frac{1}{2}\min\{k,m\}\|(q(t)\quad v(t))^\mathrm{T}\|_2^2 \\
&= u_1(\|(q(t)\quad v(t))^\mathrm{T}\|_2)
\end{aligned}
\tag{49}
$$

with $u_1(r) := \frac{1}{2}\min\{k,m\}r^2$. To show the second inequality in (44), we consider the inner squared integral in the last addend of (46) first and evaluate

$$
\left(\int\limits_{t_0-t}^{0}\mathrm{e}^{\omega s}v_t(s)\mathrm{d}s\right)^2 = \left|\int\limits_{t_0-t}^{0}\mathrm{e}^{\omega s}v_t(s)\mathrm{d}s\right|^2 \leq \left(\int\limits_{t_0-t}^{0}\mathrm{e}^{\omega s}|v_t(s)|\mathrm{d}s\right)^2.
\tag{50}
$$

Subsequently, we split the interval of integration of the outer integral in two parts and use (50). For $(1,\infty)$ we obtain

$$
\begin{aligned}
&\int_1^\infty \mu_{1-\alpha}(\omega)\left(\int_{t_0-t}^0 \mathrm{e}^{\omega s}v_t(s)\mathrm{d}s\right)^2\mathrm{d}\omega \\
&\leq \int_1^\infty \mu_{1-\alpha}(\omega)\left(\int_{-\infty}^0 \mathrm{e}^{\omega s}\mathrm{d}s\right)^2\mathrm{d}\omega\|v_t\|_\infty^2 \\
&= \frac{\sin((1-\alpha)\pi)}{\pi}\int_1^\infty \omega^{\alpha-3}\mathrm{d}\omega\|v_t\|_\infty^2 \\
&= \frac{\sin((1-\alpha)\pi)}{\pi(2-\alpha)}\|v_t\|_\infty^2.
\end{aligned}
\tag{51}
$$

For the integration over $(0,1)$, we achieve by partial integration

$$
\begin{aligned}
&\int_0^1 \mu_{1-\alpha}(\omega)\left(\int_{t_0-t}^0 \mathrm{e}^{\omega s}\dot{q}_t(s)\mathrm{d}s\right)^2\mathrm{d}\omega \\
&= \int_0^1 \mu_{1-\alpha}(\omega)\left([\mathrm{e}^{\omega s}q_t(s)]_{t_0-t}^0 - \omega\int_{t_0-t}^0 \mathrm{e}^{\omega s}q_t(s)\mathrm{d}s\right)^2\mathrm{d}\omega \\
&\leq \int_0^1 \mu_{1-\alpha}(\omega)\left(2(q(t))^2+2\omega^2\left(\int_{t_0-t}^0 \mathrm{e}^{\omega s}q_t(s)\mathrm{d}s\right)^2\right)\mathrm{d}\omega \\
&\leq \int_0^1 \mu_{1-\alpha}(\omega)\left(2\|q_t\|_\infty^2+2\omega^2\left(\int_{-\infty}^0 \mathrm{e}^{\omega s}\mathrm{d}s\right)^2\|q_t\|_\infty^2\right)\mathrm{d}\omega \\
&= 4\frac{\sin((1-\alpha)\pi)}{\pi\alpha}\|q_t\|_\infty^2.
\end{aligned}
\tag{52}
$$

Thus, we conclude that there exists a constant $C > 0$, such that $V(t,q_t,v_t) \leq C\|(q_t\quad v_t)^\mathrm{T}\|_\infty^2$. $\square$

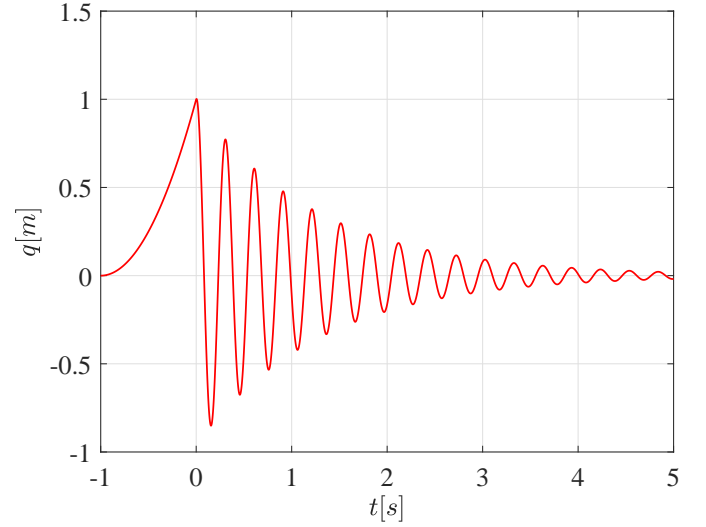**FIGURE 5**. Numerical solution of (35) using Grünwald-Letnikov approximation for $m=1kg$, $k=400\frac{N}{m}$, $\alpha=0.5$, $c=10\frac{Ns^\alpha}{m}$, $t_0=-1s$ and initial functions $q(t)=(t-t_0)^2\frac{m}{s^2}$, $v(t)=2(t-t_0)\frac{m}{s^2}$ for $t\in[t_0,0]$.

Bounded Delay and Asymptotic Stability

Considering numerical solutions of Eqn. (35) together with given initial functions lead to the conjecture, that the trivial solution of Eqn. (35) is asymptotically stable (Fig. 5). Similar to classical Lyapunov theory, a sufficient condition for asymptotic stability of an FDE with unbounded delay may be formulated as in Thm. 1 with the additional assumptions

$$
\exists L > 0: \|f(t,\varphi)\|_n < L, \quad t\geq t_1, \quad \varphi\in Q_H
\tag{53}
$$

and

$$
\dot{V} \leq -u_3\left(\|x(t_1,\varphi)(t)\|_n\right)
\tag{54}
$$

for a scalar, continuous non-decreasing function $u_3 : [0,\infty) \to \mathbb{R}$ such that $u_3(0) = 0$ and $u_3(r) > 0$ for $r > 0$ [18]. Condition (53) can be proved similar to the second inequality in the proof of Prop. 1 but Eqn. (54) is not fulfilled by the energy functional (46). The same situation occurs in the case of a single degree-of-freedom mass-spring-system with linear damping. The associated energy function is not sufficient to prove asymptotic stability of the trivial solution. However, in the time-autonomous case the problem can be solved using LaSalle's invariance principle [21, 22]. To apply this approach to Eqn. (35) we need to consider an autonomous version of Eqn. (39). Therefore we use the short-memory principle for fractional derivatives [23, Chap. 7.3], i.e. we neglect the history of the springpots near the time

$t = t_0$ and only consider a time interval $[t-h,t]$ $\forall t > 0$ with memory length $h > 0$, which leads to the differential equation

$$m\ddot{q}(t) = -kq(t) - c^C D_{t-h+}^\alpha q(t), \quad t \geq 0 \qquad (55)$$

or the autonomous FDE

$$\begin{cases} \dot{q}(t) = v(t), \\ \dot{v}(t) = -\frac{k}{m}q(t) - \frac{c}{m}\int_{-h}^0 \int_0^\infty \mu_{1-\alpha}(\omega)e^{\omega s}d\omega\, v_t(s)ds, \end{cases} \qquad (56)$$

i.e. we consider an FDE of the form

$$\dot{x}(t) = f(x_t), \quad t \geq 0 \qquad (57)$$

where f is defined on a subset of $C([-h,0];\mathbb{R}^n)$ and the solution $x(\varphi) \in C([-h,T);\mathbb{R}^n)$, $T > 0$ of Eqn. (57) only depends on the initial function

$$\varphi \in \tilde{Q}_H := \{\varphi \in C([-h,0];\mathbb{R}^n) \mid \|\varphi\|_\infty < H\}, \quad H > 0.$$

For this case, we can formulate an invariance theorem in the following way.

Definition 3:
(i) An element $\psi \in C([-h,0];\mathbb{R}^n)$ belongs to the ω-*limit set* $\Omega(\varphi)$ of φ if $x(\varphi)$ is defined on $[-h,\infty)$ and there exists a non-negative sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$ such that $\|x_{t_n}(\varphi) - \psi\|_\infty \to 0$ as $n \to \infty$.
(ii) A set $Q \subset C([-h,0];\mathbb{R}^n)$ is called *invariant* if $x_t(\varphi) \in Q$ for any $\varphi \in Q$ and $t \in [0,\infty)$.

Theorem 2 (Asymptotic Stability [17,18])**:**
Let $f : \tilde{Q}_H \to \mathbb{R}^n$ such that $f(0) = 0$ and for all $H_1 < H$, $\varphi \in \tilde{Q}_{H_1}$ there is a constant $L > 0$ such that $\|f(\varphi)\|_n < L$. Denote $u_i : [0,\infty) \to \mathbb{R}$, $i = 1,2$ some scalar, continuous, non-decreasing functions such that $u_i(0) = 0$ and $u_i(r) > 0$ for $r > 0$ and $u_1(r) \to \infty$ as $r \to \infty$. Let there exist a continuous functional $V : \tilde{Q}_H \to \mathbb{R}$ such that

$$u_1(\|\varphi(0)\|_n) \leq V(\varphi) \leq u_2(\|\varphi\|_\infty), \qquad (58)$$
$$\dot{V}(x_t) \leq 0 \qquad (59)$$

and let $\{0\}$ be the largest invariant set in $\{\varphi | \dot{V}(\varphi) = 0\}$, then the trivial solution of Eqn. (57) is asymptotically stable.

For the proof of this theorem we need a lemma on properties of the ω-limit set of the initial function φ as in the finite dimensional case.

Lemma 1:
Let $x(\varphi)$ a solution of Eqn. (57) with initial function φ and assume $\{x_t(\varphi)|t \geq 0\}$ belongs to a compact set in $\tilde{Q}_H$, then $\Omega(\varphi)$ is non-empty, compact, invariant and

$$\text{dist}(x_t(\varphi),\Omega(\varphi)) \to 0 \text{ as } t \to \infty.$$

The proof of this lemma is similar to the corresponding one in the finite dimensional case [22, C3]. The interesting part is the assertion that $\{x_t|t \geq 0\}$ belongs to a compact subset of $C([-h,0];\mathbb{R}^n)$.

Lemma 2:
Let $x(\varphi)$ a solution of Eqn. (57) with initial function φ and f under conditions of the above theorem. If $x(\varphi)$ is defined on $[-h,\infty)$ and $\|x_t(\varphi)\|_\infty \leq H_1 < H$ $\forall t \geq 0$, then the family of functions $\{x_t(\varphi)|t \geq 0\}$ belongs to a compact subset of $C([-h,0];\mathbb{R}^n)$.

Proof. By the assumptions and the Arzelà-Ascoli theorem, it still has to be shown that $\{x_t(\varphi)|t \geq 0\}$ is equicontinuous on $[-h,0]$. This is a consequence of the boundedness of f for bounded inputs, see [21, Chap. 3, Lemma 4.8]. $\square$

Proof of Thm. 2. Stability is proved as in Thm. 1. For asymptotic stability, consider the open level sets $\Omega_c = \{\varphi|V(\varphi) < c\}$. For $\varphi \in \Omega_c$, we have $u_1(\|\varphi(0)\|_n) < c$ and as $u_1(r) \to \infty$ for $r \to \infty$, we can choose $c > 0$ such that $\|\varphi(0)\|_n \leq H_1$. As $\dot{V} \leq 0$ on Ω_c, $x_t(\varphi) \in \Omega_c$ for $t \geq 0$ and hence $\|x(\varphi)(t)\|_n \leq H_1$ for $t \geq 0$ which implies $\|x_t(\varphi)\|_\infty \leq H_1$. By Lemma 2, $\{x_t(\varphi)|t \geq 0\}$ belongs to a compact subset of $C([-h,0];\mathbb{R}^n)$ so that $V(x_t(\varphi))$, being non-increasing, is bounded from below and therefore has a limit $\lim_{t\to\infty} V(x_t(\varphi)) = a$. By Lemma 1, $\{x_t(\varphi)|t \geq 0\}$ has a non-empty, compact, invariant ω-limit set $\Omega(\varphi)$ which belongs to the aforementioned compact set. This implies, as V is continuous for $\psi \in \Omega(\varphi)$, that $V(\psi) = \lim_{t\to\infty} V(x_{t_n}(\varphi)) = a$. Hence, $V \equiv a$ on $\Omega(\varphi)$ and as $\Omega(\varphi)$ is invariant, we obtain $\dot{V} = 0$ in $\Omega(\varphi)$ and $\Omega(\varphi) \subset \{0\}$. Again, by Lemma 1, every solution approaches its ω-limit set for $t \to \infty$. Hence, x_t approaches $\{0\}$ for $t \to \infty$. $\square$

To apply Thm. 2 to Eqn. (56), we tried to use a Lyapunov functional similar to the energy functional (46), i.e.

$$\tilde{V}(q_t,v_t) = \frac{1}{2}mv_t^2(0) + \frac{1}{2}kq_t^2(0) + \frac{1}{2}c\int_0^\infty \mu_{1-\alpha}(\omega)\left(\int_{-h}^0 e^{\omega s}v_t(s)ds\right)^2 d\omega. \qquad (60)$$

For this functional, the inequalities in (58) are satisfied, which may be proved as in Prop. 1. However, the derivative of Eqn. (60)

along solutions of (56) has the form

$$\dot{V}(q_t, v_t) = -c \int_0^\infty \omega \mu_{1-\alpha}(\omega) \left(\int_{-h}^0 e^{\omega s} v_t(s) \mathrm{d}s \right)^2 \mathrm{d}\omega$$
$$- c v_t(-h) \int_0^\infty e^{-\omega h} \mu_{1-\alpha}(\omega) \int_{-h}^0 e^{\omega s} v_t(s) \mathrm{d}s \mathrm{d}\omega. \tag{61}$$

Unfortunately, the second term in Eqn. (61) leads to the question, if $\tilde{V}$ still satisfies the inequality (59), which we could not answer yet. It is obvious that the influence of the second term decreases with growing memory length h but we still need further estimation to prove asymptotic stability of the trivial solution.

CONCLUDING REMARKS

In this paper, the uniform stability of the trivial solution of the investigated system with a fractional springpot element was proved with the help of a Lyapunov theorem similar to the finite dimensional case. Our proof and results are constructive in the sense that we can even modify Eqn. (35) substituting the linear spring by a cubic one and change the second term in Eqn. (46) to a fourth order monomial and obtain uniform stability in the same way. Hence, the approach applies to nonlinear systems as well. Moreover, we utilized the short memory principle, which allows the formulation of an invariance principle for FDEs and seems to be a promising approach to prove asymptotic stability.

REFERENCES

[1] Bagley, R. L., and Torvik, P. J., 1985. "Fractional calculus in the transient analysis of viscoelastically damped structures". *AIAA Journal, 23*(6), pp. 918 – 925.

[2] Schmidt, A., and Gaul, L., 2002. "Parameter Identification and FE Implementation of a Viscoelastic Constitutive Equation Using Fractional Derivatives". *Proc. Appl. Math. Mech., 1*(1).

[3] Heymans, N., and Bauwens, J. C., 1994. "Fractal rheological models and fractional differential equations for viscoelastic behavior". *Rheologica Acta, 33*(3), May, pp. 210–219.

[4] Schiessel, H., and Blumen, A., 1993. "Hierarchical analogues to fractional relaxation equations". *Journal of Physics A: Mathematical and General, 26*(19), pp. 5057–5069.

[5] Papoulia, K. D., Panoskaltsis, V. P., Kurup, N. V., and Korovajchuk, I., 2010. "Rheological representation of fractional order viscoelastic material models". *Rheologica Acta, 49*(4), Apr, pp. 381–400.

[6] Montseny, G., 1998. "Diffusive representation of pseudo-differential time-operators". *ESAIM: Proc., 5*, pp. 159–175.

[7] Matignon, D., 1998. "Stability properties for generalized fractional differential systems". *ESAIM: Proc., 5*, pp. 145–158.

[8] Trigeassou, J.-C., Maamri, N., Sabatier, J., and Oustaloup, A., 2011. "A Lyapunov approach to the stability of fractional differential equations". *Signal Processing, 91*(3), pp. 437–445.

[9] Trigeassou, J.-C., Maamri, N., Sabatier, J., and Oustaloup, A., 2012. "Transients of fractional-order integrator and derivatives". *Signal, Image and Video Processing*, pp. 1–14.

[10] Trigeassou, J.-C., Maamri, N., and Oustaloup, A., 2016. "Lyapunov stability of commensurate fractional order systems: a physical interpretation". *Journal of Computational and Nonlinear Dynamics, 11*(5), p. 051007.

[11] Trigeassou, J., Maamri, N., Sabatier, J., and Oustaloup, A., 2012. "State variables and transients of fractional order differential systems". *Computers & Mathematics with Applications, 64*(10), pp. 3117 – 3140. Advances in FDE, III.

[12] Lorenzo, C. F., and Hartley, T. T., 2008. "Initialization of fractional-order operators and fractional differential equations". *Journal of Computational and Nonlinear Dynamics, 3*(2), p. 021101.

[13] Hartley, T. T., Lorenzo, C. F., Trigeassou, J.-C., and Maamri, N., 2013. "Equivalence of history-function based and infinite-dimensional-state initializations for fractional-order operators". *Journal of Computational and Nonlinear Dynamics, 8*(4), p. 041014.

[14] Diethelm, K., 2010. *The Analysis of Fractional Differential Equations : An Application-Oriented Exposition Using Differential Operators of Caputo Type*. Springer, Berlin.

[15] Li, Y., Chen, Y. Q., and Podlubny, I., 2010. "Stability of fractional-order nonlinear dynamic systems: Lyapunov direct method and generalized Mittag–Leffler stability". *Computers & Mathematics with Applications, 59*(5), pp. 1810–1821.

[16] Lakshmikantham, V., Leela, S., and Devi, J. V., 2009. *Theory of Fractional Dynamic Systems*. Cambridge Scientific Publ.

[17] Hale, J. K., 1977. *Theory of Functional Differential Equations*, 2. ed. Springer, New York [u.a.].

[18] Kolmanovskii, V. B., and Nosov, V. R., 1986. *Stability of Functional Differential Equations*, Vol. 180. Elsevier.

[19] Burton, T. A., 1985. *Stability and Periodic Solutions of Ordinary and Functional Differential Equations*, Vol. 178. Acad. Pr.

[20] Burton, T. A., 2005. *Volterra Integral and Differential Equations*, Vol. 202. Elsevier.

[21] LaSalle, J. P., and Artstein, Z., 1976. *The Stability of Dynamical Systems*. Regional Conference Series in Applied Mathematics; 25. Society for Industrial and Applied Mathematics, Philadelphia.

[22] Khalil, H. K., 2000. *Nonlinear Systems*, 3. internat. ed. Pearson Education International, Upper Saddle River, NJ.

[23] Podlubny, I., c 1999. *Fractional Differential Equations. An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of their Solution and some of their Applications*. Academic Press, San Diego.