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Bifurcations of equilibria in non-smooth continuous systems

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Abstract

The aim of this paper is to show a variety of bifurcation phenomena of equilibria that can be observed in non-smooth continuous systems. In non-smooth systems so-called 'multiple crossing bifurcations' can occur, for which the eigenvalues jump more than once over the imaginary axis, and which do not have a classical bifurcation as counterpart. Novel theoretical results are given for a class of planar systems but no general theory is available for the multi-dimensional case. A number of well chosen examples of multiple crossing bifurcations are discussed in detail.

Keywords: Non-smooth systems; Non-smooth vector fields; Piecewise linear non-smooth continuous systems; Hopf bifurcation

1. Introduction

Dynamical systems can possess stationary states or equilibria which can be stable or unstable. It is often desirable to know how the equilibria of a system change when a parameter of the system is changed. The number and stability of equilibria can change at a certain critical parameter value. Loosely speaking, this qualitative change in the structural behaviour of the system is called *bifurcation*.

The theory of bifurcations of equilibria in smooth vector fields is well understood [9,10,17,18]. However, much less is known about bifurcations of equilibria in non-smooth continuous vector fields. Bifurcations in non-smooth continuous systems, i.e. differential equations with a continuous but non-smooth right-hand side, have been studied in a previous paper [12,14] of the author. It has been shown that a bifurcation in a non-smooth continuous system can be accompanied by a jump of an eigenvalue over the imaginary axis under the variation of a parameter. The analysis of non-smooth continuous systems played the role of a stepping stone in [12,14] for the analysis of the larger class of Filippov systems. The current paper focuses on non-smooth continuous systems and presents novel results which have been gained since [12,14]. New bifurcation theorems for a class of planar systems are proven. More importantly, the distinction between single crossing bifurcations and multiple crossing bifurcations is essential and new. The present paper shows a number of very complicated multiple crossing bifurcations in very simple systems. These examples put our present knowledge on bifurcations in non-smooth systems in a new perspective. Single crossing bifurcations have been discussed in [12,14] and show the behaviour of a 'classical bifurcation', i.e. a bifurcation known from classical bifurcation theory. The behaviour of multiple crossing bifurcations, however, is far more complicated as will be shown by the examples presented in the current paper. Smooth approximations of these non-smooth examples show how these multiple crossing bifurcations unfold in a number of classical bifurcations.

Bifurcations of equilibria of non-smooth continuous systems are related to bifurcations of fixed points of non-smooth continuous maps. Nusse and York [16] study so-called 'border-collision bifurcations' of two-dimensional non-smooth discrete maps. Feigin [7,8] considers periodic solutions of Filippov systems (i.e. differential equations with a discontinuous right-hand side or differential inclusions), of which the Poincaré maps are locally non-smooth continuous maps. He introduced the term 'C-bifurcation' for the non-conventional bifurcations which occur in periodic solutions of Filippov systems. The work of Feigin has been extended by di Bernardo et al. [3,4]. Bifurcations of periodic solutions in Filippov systems and their

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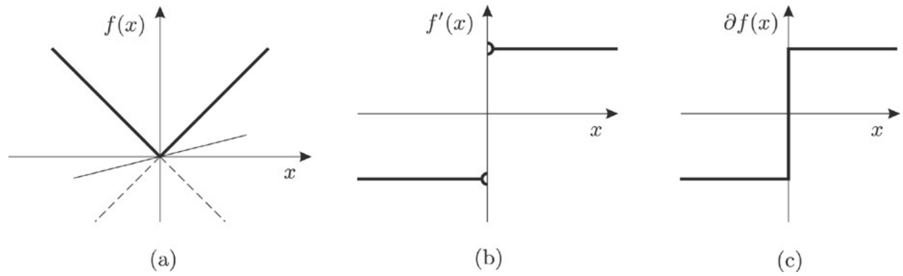


Fig. 1. Function (a), classical derivative (b) and generalized derivative (c).

resulting Poincaré maps are furthermore discussed in [5,11]. Bifurcations in impacting systems are studied in [15]. The aim of the current paper is to study bifurcations of *equilibria* in non-smooth continuous systems, i.e. *differential equations* with a non-smooth continuous right-hand side, and the paper does not study non-smooth discrete mappings which arise from Filippov systems or impacting systems. We refer the reader to [13] for a literature review of bifurcations in general non-smooth systems.

The generalized differential, which is needed for the stability analysis of equilibria of non-smooth continuous systems, will be briefly discussed in Section 2. Attention will be paid in Section 3 to the definition of bifurcation adopted in this paper. Section 4 introduces the type of non-smooth systems which will be considered. Subsequently, the basic idea of a discontinuous bifurcation is presented in Section 5. New results on bifurcations in non-smooth continuous systems with a piecewise linear right-hand side and a single switching boundary are presented and proved in Sections 6 and 7. The general case is far more complex and no rigorous results on bifurcations will be given for general non-smooth continuous systems with more than one switching boundary. The complications of bifurcations in non-smooth continuous systems will be demonstrated through a number of novel examples in Section 8. Concluding remarks are given in Section 9.

2. Generalized differential of continuous functions

The classical derivative of smooth functions, i.e. functions which are continuous and differentiable up to any order in their arguments, will be extended in this section to the generalized derivative (and differential) of Clarke for non-smooth continuous functions.

Consider a scalar continuous piecewise differentiable function $f(x)$ with a kink (i.e. non-smooth point) at one value of x, such as $f(x) = |x|$ (Fig. 1). The derivative $f'(x)$ is defined by the tangent line to the graph of f when the graph is smooth at x

$$f'(x) = \frac{\partial f}{\partial x}(x) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x}. \tag{1}$$

Although the function is not absolutely differentiable at every point x, it possesses at each x a left and right derivative defined as

$$f'_-(x) = \lim_{y \uparrow x} \frac{f(y) - f(x)}{y - x}, \qquad f'_+(x) = \lim_{y \downarrow x} \frac{f(y) - f(x)}{y - x}. \tag{2}$$

The *generalized derivative* of f at x is declared as *any* value $f'_q(x)$ included between its left and right derivatives [1,2]. Such an intermediate value can be expressed as a convex combination of the left and right derivatives.

$$f'_q(x) = (1 - q) f'_-(x) + q f'_+(x), \quad 0 \le q \le 1. \tag{3}$$

Geometrically, a generalized derivative is the slope of *any* line drawn through the point $(x, f(x))$ and between the left and right tangent lines (drawn as dashed lines in Fig. 1(a)). The set of all the generalized derivatives of f at x, more generally the convex hull of the derivative extremes, is called the *generalized differential* of f at x

$$\begin{aligned} \partial f(x) = \overline{\mathrm{co}}\{f'_-(x), f'_+(x)\} = \{f'_q(x) \mid f'_q(x) \\ = (1 - q) f'_-(x) + q f'_+(x), 0 \le q \le 1\}. \end{aligned} \tag{4}$$

The generalized differential of Clarke at x is the set of the slopes of all the lines included in the cone bounded by the left and right tangent lines and is a closed convex set (Fig. 1(b), (c)). In non-smooth analysis, the generalized differential is for instance used to define a local extremum of f at x by $0 \in \partial f$, which is the generalized form of $f'(x) = 0$ in smooth analysis [1].

Infinitely many directional derivatives exist for functions in $\mathbb{R}^n$, whereas only two directional derivatives exist for scalar functions (the left and right derivative). For $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^m$, differentiable almost everywhere, we define the *generalized differential of Clarke* as

$$\partial \boldsymbol{f}(\boldsymbol{x}) = \bigcap_{\delta > 0} \overline{\mathrm{co}}\{\nabla \boldsymbol{f}(\boldsymbol{y}) \mid \boldsymbol{y} \in \boldsymbol{x} + B_\delta(\boldsymbol{0})\} \subset \mathbb{R}^{n \times m}, \tag{5}$$

with the gradient

$$\nabla \boldsymbol{f}(\boldsymbol{x}) = \left(\frac{\partial \boldsymbol{f}(\boldsymbol{x})}{\partial \boldsymbol{x}} \right)^{\mathrm{T}} \in \mathbb{R}^{n \times m}. \tag{6}$$

The generalized differential (5) simplifies to (4) for the scalar case. We define the *generalized Jacobian of Clarke* as the transpose of the generalized differential

$$\boldsymbol{J}(\boldsymbol{x}) = (\partial \boldsymbol{f}(\boldsymbol{x}))^{\mathrm{T}} \subset \mathbb{R}^{m \times n}, \tag{7}$$

which reduced for a smooth function $\boldsymbol{f}$ to $\boldsymbol{J}(\boldsymbol{x}) = \nabla \boldsymbol{f}(\boldsymbol{x})^{\mathrm{T}}$. Note that $\boldsymbol{f}(\boldsymbol{x})$ can be convex or non-convex in the above definitions. The image of the generalized differential $\partial \boldsymbol{f}(\boldsymbol{x})$ is for continuous functions always a closed convex set.

3. Definition of bifurcation

In this paper, we consider bifurcations of equilibria of autonomous systems which depend on a scalar parameter $\mu \in \mathbb{R}$:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \mu). \tag{8}$$

Let n denote the dimension of the system, i.e. $\boldsymbol{x} \in \mathbb{R}^n$. The system (8) is called *smooth* if $\boldsymbol{f}(\boldsymbol{x}, \mu)$ is differentiable up to any order in both $\boldsymbol{x}$ and μ. Equilibria of (8) are solutions of the algebraic equations

$$\boldsymbol{0} = \boldsymbol{f}(\boldsymbol{x}, \mu). \tag{9}$$

Several different definitions of bifurcation exist and are also applied to non-smooth systems [13]. In this paper we adopt the following definition of a *bifurcation point*:

Definition 1 (*Geometric Definition of a Bifurcation [17]*). A bifurcation point (with respect to μ) is a solution $(\boldsymbol{x}^*, \mu^*)$, where the number of equilibria or (quasi-)periodic solutions changes when μ passes μ^*.

Remark. A quasi-periodic solution is a solution that can be expressed as a countable sum of time-periodic functions with an incommensurate set of base frequencies [9,17].

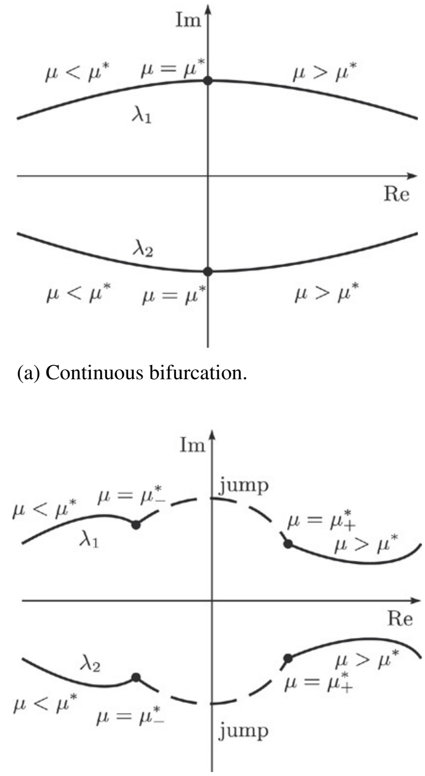
In other words, if $n_E(\mu)$, $n_P(\mu)$ and $n_Q(\mu)$ denote the number of coexisting equilibria, periodic solutions and quasi-periodic solutions of (8) respectively, then a bifurcation occurs at $\mu = \mu^*$ if

$$\left(\lim_{\mu \uparrow \mu^*} n_X(\mu) \neq n_X(\mu^*)\right) \bigvee \left(\lim_{\mu \downarrow \mu^*} n_X(\mu) \neq n_X(\mu^*)\right),$$

where X stands for E, P or Q. If the branches of equilibria or (quasi-)periodic solutions, that are created/destroyed, are connected to the equilibrium point $\boldsymbol{x}^*$, in the sense that two or more branches intersect or that a branch folds at $(\boldsymbol{x}^*, \mu^*)$, then the point $(\boldsymbol{x}^*, \mu^*)$ is called a *bifurcation point*.

Definition 1 is to be understood as saying that not only do the number of equilibria and (quasi-)periodic solutions near the point under consideration have to be taken into account, but so also do those at the point under consideration. Consider for instance the normal form $\dot{x} = f(x, \mu) = \mu x - x^2$ of the transcritical bifurcation. In this case there are two equilibria for $\mu < 0$, one equilibrium for $\mu = 0$ (which is the point under consideration) and two equilibria for $\mu > 0$. The point $(x, \mu) = (0, 0)$ is therefore a bifurcation point because the number of equilibria changes at this point for varying μ (the change is: 2–1–2). We conclude that if branches intersect, then their intersection point must be a bifurcation point.

Likewise, the system $\dot{x} = \mu x$ has one equilibrium for $\mu < 0$, an infinite number of equilibria for $\mu = 0$ (which is the point under consideration) and one equilibrium for $\mu > 0$ (the change is: 1–∞–1). The point $(x, \mu) = (0, 0)$ is therefore a bifurcation point. Definition 1 is a purely geometric definition of a bifurcation, which does not use any knowledge about the stability of the limit sets. Stability might be exchanged at a bifurcation point but this is not necessary for higher dimensional systems.



(a) Continuous bifurcation.

(b) Discontinuous bifurcation.

Fig. 2. Eigenvalue paths at a continuous bifurcation (a) and a discontinuous single crossing bifurcation (b) (see Definition 3).

A bifurcation of an equilibrium branch in a higher dimensional system causes the equilibrium to gain or lose stability within an eigenspace (not necessarily implying an exchange of stability of the equilibrium). Bifurcations of equilibria are therefore directly associated with an eigenvalue (or pair of complex conjugate eigenvalues) that moves from the left complex half-plane to the right complex half-plane (or vice versa) under variation of a parameter. The Jacobian matrices of smooth systems are smooth functions of the state vector and parameter. The eigenvalues of the Jacobian matrix will therefore also depend continuously (but not necessarily smoothly) on the parameter. A bifurcation of an equilibrium point of a smooth system occurs when one eigenvalue (or a pair of them) passes the imaginary axis when a parameter is varied. The scenario is depicted in Fig. 2(a) where a pair of complex conjugate eigenvalues passes the imaginary axis when a parameter μ is varied and a Hopf bifurcation occurs at some critical value $\mu = \mu^*$. The bifurcations occurring in smooth systems are called *continuous bifurcations* in this paper because the eigenvalues behave continuously.

4. Non-smooth continuous systems

An autonomous dynamical system of the form (8), dependent on a parameter μ, is called a *non-smooth continuous system* if $\boldsymbol{f}(\boldsymbol{x}, \mu)$ is continuous in $\boldsymbol{x}$ but non-smooth on one or more switching boundaries Σ. The system $\dot{x} = |x + \mu| + 3x^2$ is for instance a non-smooth continuous system due to the absolute value operator. Non-smooth continuous systems are therefore a subclass of Filippov systems (see [13]). Non-smooth continuous systems with a single switching boundary can generally be put in the form

$$\dot{\boldsymbol{x}} = \begin{cases} \boldsymbol{f}_-(\boldsymbol{x}, \mu), & \boldsymbol{x} \in \mathcal{V}_- \bigcup \Sigma, \\ \boldsymbol{f}_+(\boldsymbol{x}, \mu), & \boldsymbol{x} \in \mathcal{V}_+ \end{cases} \tag{10}$$

depending on a parameter μ and being continuous on the switching boundary Σ. The functions $\boldsymbol{f}_\pm$ are smooth functions in $\boldsymbol{x}$ and μ. The switching boundary function $h(\boldsymbol{x}, \mu)$ defines the subspaces

$$\begin{aligned} \mathcal{V}_- &= \{\boldsymbol{x} \in \mathbb{R}^n \mid h(\boldsymbol{x}, \mu) < 0\}, \\ \Sigma &= \{\boldsymbol{x} \in \mathbb{R}^n \mid h(\boldsymbol{x}, \mu) = 0\}, \\ \mathcal{V}_+ &= \{\boldsymbol{x} \in \mathbb{R}^n \mid h(\boldsymbol{x}, \mu) > 0\}. \end{aligned} \tag{11}$$

Continuity of (10) requires that $\boldsymbol{f}_+$ and $\boldsymbol{f}_-$ agree on Σ

$$\boldsymbol{f}_-(\boldsymbol{x}, \mu) = \boldsymbol{f}_+(\boldsymbol{x}, \mu), \quad \forall \boldsymbol{x} \in \Sigma. \tag{12}$$

If the non-smooth continuous system is piecewise linear in $\boldsymbol{x}$ within the subspaces $\mathcal{V}_-$ and $\mathcal{V}_+$, then system (10) can be written in the form

$$\dot{\boldsymbol{x}} = \begin{cases} \boldsymbol{J}_-(\mu)\boldsymbol{x} + \boldsymbol{b}_-(\mu), & \boldsymbol{x} \in \mathcal{V}_- \bigcup \Sigma, \\ \boldsymbol{J}_+(\mu)\boldsymbol{x} + \boldsymbol{b}_+(\mu), & \boldsymbol{x} \in \mathcal{V}_+. \end{cases} \tag{13}$$

Moreover, if we assume the switching boundary to be a hyperplane $h(\boldsymbol{x}, \mu) = \boldsymbol{w}^{\mathrm{T}}(\mu)\boldsymbol{x} + w(\mu)$, then the continuity condition (12) yields

$$-(\boldsymbol{J}_+(\mu) - \boldsymbol{J}_-(\mu))\boldsymbol{x} = \boldsymbol{b}_+(\mu) - \boldsymbol{b}_-(\mu), \quad \forall \boldsymbol{x} \in \Sigma, \tag{14}$$

or, for non-singular $(\boldsymbol{J}_+(\mu) - \boldsymbol{J}_-(\mu))$,

$$-\boldsymbol{w}^{\mathrm{T}}(\mu)\,(\boldsymbol{J}_+(\mu) - \boldsymbol{J}_-(\mu))^{-1}\,(\boldsymbol{b}_+(\mu) - \boldsymbol{b}_-(\mu)) + w(\mu) = \boldsymbol{0}. \tag{15}$$

Non-smooth continuous systems are nonlinear and generally not piecewise linear. If the equilibrium is located on a switching boundary, then the nonlinear system can locally be approximated by a piecewise linear system of the form (13).

5. Discontinuous bifurcation: The basic idea

Some aspects and definitions of bifurcations of equilibria in smooth systems have been briefly recalled in Section 3. Bifurcations of equilibria in smooth systems are associated with an eigenvalue (or pair of complex conjugate eigenvalues) that passes the imaginary axis under variation of a parameter. The bifurcation analysis of dynamical systems therefore hinges on the dependency of the Jacobian matrix (and its eigenvalues) on system parameters. Non-smooth continuous systems possess switching boundaries on which the vector field is non-smooth and for which the classical Jacobian matrix cannot be obtained. In this section we will try to enlarge the concept of bifurcation to the class of non-smooth continuous systems.

Consider an autonomous non-smooth continuous system of the form (10), having a single switching boundary Σ. Let $\boldsymbol{x}_\mu$ be an equilibrium point of (10) for some value of μ, i.e. $\boldsymbol{f}_-(\boldsymbol{x}_\mu, \mu) = \boldsymbol{0}$ for $\boldsymbol{x}_\mu \in \mathcal{V}_- \cup \Sigma$ or $\boldsymbol{f}_+(\boldsymbol{x}_\mu, \mu) = \boldsymbol{0}$ for $\boldsymbol{x}_\mu \in \mathcal{V}_+$. If $\boldsymbol{x}_\mu$ is not on Σ, then we can find a single-valued Jacobian matrix $\boldsymbol{J}(\boldsymbol{x}_\mu, \mu)$

$$\boldsymbol{J}(\boldsymbol{x}_\mu, \mu) = \begin{cases} \boldsymbol{J}_-(\boldsymbol{x}_\mu, \mu) = \left.\dfrac{\partial \boldsymbol{f}_-(\boldsymbol{x}, \mu)}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}=\boldsymbol{x}_\mu}, & \boldsymbol{x}_\mu \in \mathcal{V}_-, \\ \boldsymbol{J}_+(\boldsymbol{x}_\mu, \mu) = \left.\dfrac{\partial \boldsymbol{f}_+(\boldsymbol{x}, \mu)}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}=\boldsymbol{x}_\mu}, & \boldsymbol{x}_\mu \in \mathcal{V}_+, \end{cases}$$

that locally defines the vector field around the equilibrium point $\boldsymbol{x}_\mu$ if the matrix $\boldsymbol{J}(\boldsymbol{x}_\mu, \mu)$ does not have eigenvalues on the imaginary axis. The matrices $\boldsymbol{J}_-(\boldsymbol{x}, \mu)$ and $\boldsymbol{J}_+(\boldsymbol{x}, \mu)$ are the Jacobian matrices on either side of Σ associated with the vector field in $\mathcal{V}_-$ and $\mathcal{V}_+$. If $\boldsymbol{x}_\mu \in \Sigma$, then the local vector field is determined by *two* Jacobian matrices, i.e. $\boldsymbol{J}_-(\boldsymbol{x}_\mu, \mu)$ and $\boldsymbol{J}_+(\boldsymbol{x}_\mu, \mu)$. Assume that we vary μ such that the equilibrium point $\boldsymbol{x}_\mu$ moves from $\mathcal{V}_-$ to $\mathcal{V}_+$ via Σ, i.e. transversally through Σ. Let $\boldsymbol{x}_\Sigma$ denote the unique equilibrium on Σ for $\mu = \mu_\Sigma$:

$$\begin{aligned} &\boldsymbol{x}_\mu \in \mathcal{V}_-, \quad \mu < \mu_\Sigma, \\ &\boldsymbol{x}_\mu = \boldsymbol{x}_\Sigma \in \Sigma, \quad \mu = \mu_\Sigma, \\ &\boldsymbol{x}_\mu \in \mathcal{V}_+, \quad \mu > \mu_\Sigma. \end{aligned}$$

The Jacobian matrix $\boldsymbol{J}(\boldsymbol{x}_\mu, \mu)$ varies as μ is varied and is discontinuous at $\mu = \mu_\Sigma$ for which $\boldsymbol{x}_\mu = \boldsymbol{x}_\Sigma$. Loosely speaking, we say that $\boldsymbol{J}(\boldsymbol{x}_\mu, \mu)$ 'jumps' at $\mu = \mu_\Sigma$ from $\boldsymbol{J}_-(\boldsymbol{x}_\Sigma, \mu_\Sigma)$ to $\boldsymbol{J}_+(\boldsymbol{x}_\Sigma, \mu_\Sigma)$. A jump of the Jacobian matrix under the influence of a parameter implies a jump of the eigenvalues. In Section 2 we elaborated on how we can define a generalized differential $\partial \boldsymbol{f}$. Similarly, a *generalized Jacobian* $\boldsymbol{J}(\boldsymbol{x}, \mu)$ was defined in (7) as the transpose of the generalized differential of $\boldsymbol{f}$ with respect to $\boldsymbol{x}$

$$\boldsymbol{J}(\boldsymbol{x}, \mu) = (\partial_{\boldsymbol{x}} \boldsymbol{f}(\boldsymbol{x}, \mu))^{\mathrm{T}}, \tag{16}$$

which is set-valued at $(\boldsymbol{x}_\Sigma, \mu_\Sigma)$. The generalized Jacobian at $(\boldsymbol{x}_\Sigma, \mu_\Sigma)$ is therefore the closed convex hull of $\boldsymbol{J}_-(\boldsymbol{x}_\Sigma, \mu_\Sigma)$ and $\boldsymbol{J}_+(\boldsymbol{x}_\Sigma, \mu_\Sigma)$

$$\begin{aligned} \boldsymbol{J}(\boldsymbol{x}_\Sigma, \mu_\Sigma) &= \overline{\mathrm{co}}\{\boldsymbol{J}_-(\boldsymbol{x}_\Sigma, \mu_\Sigma), \boldsymbol{J}_+(\boldsymbol{x}_\Sigma, \mu_\Sigma)\} \\ &= \{(1-q)\boldsymbol{J}_-(\boldsymbol{x}_\Sigma, \mu_\Sigma) \\ &\quad + q\boldsymbol{J}_+(\boldsymbol{x}_\Sigma, \mu_\Sigma), \forall q \in [0, 1]\}. \end{aligned} \tag{17}$$

In fact, (17) defines *how* the Jacobian 'jumps' at Σ. The generalized Jacobian is, for a system with a single switching boundary, a convex combination of two matrices $\boldsymbol{J}_-(\boldsymbol{x}, \mu)$ and $\boldsymbol{J}_+(\boldsymbol{x}, \mu)$ if $\boldsymbol{x} \in \Sigma$. To be more precise, (17) gives the set of values which the generalized Jacobian can attain on Σ. From the set-valued generalized Jacobian we can obtain the set-valued eigenvalues. We can look upon $\mathrm{eig}(\boldsymbol{J}(\boldsymbol{x}_\Sigma, \mu_\Sigma))$ together with (17) as if it gives a *unique* path of eigenvalues 'during' the jump as q is varied from 0 to 1.

Instead of a transversal intersection of $\boldsymbol{x}_\mu$ through Σ, it can also be possible that the equilibrium slides along Σ under variation of μ. Also in this case we can speak of a generalized

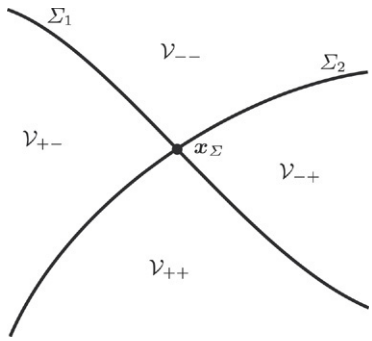


Fig. 3. State space with two switching boundaries.

Jacobian, but the Jacobian will be set-valued for an interval of μ. In the following, we will for simplicity focus on a transversal intersection of Σ.

Systems with multiple switching boundaries can possess equilibria located on the crossing of two or more switching boundaries. Two switching boundaries Σ_1 and Σ_2 divide the state space into four subspaces $\mathcal{V}_{++}$, $\mathcal{V}_{+-}$, $\mathcal{V}_{-+}$ and $\mathcal{V}_{--}$ (Fig. 3). The generalized Jacobian of an equilibrium point $\boldsymbol{x}_\Sigma \in \Sigma_1 \bigcap \Sigma_2$ located on the crossing of Σ_1 and Σ_2 is the convex hull of four Jacobian matrices

$$\begin{aligned}\boldsymbol{J}(\boldsymbol{x}_\Sigma, \mu_\Sigma) &= \overline{\text{co}}\{\boldsymbol{J}_{++}(\boldsymbol{x}_\Sigma, \mu_\Sigma), \boldsymbol{J}_{+-}(\boldsymbol{x}_\Sigma, \mu_\Sigma),\\ &\qquad \boldsymbol{J}_{-+}(\boldsymbol{x}_\Sigma, \mu_\Sigma), \boldsymbol{J}_{++}(\boldsymbol{x}_\Sigma, \mu_\Sigma)\}\\ &= \{q_1(\boldsymbol{J}_{-+} - \boldsymbol{J}_{--}) + q_2(\boldsymbol{J}_{+-} - \boldsymbol{J}_{--})\\ &\qquad + \boldsymbol{J}_{--}, \forall q_1, q_2 \in [0, 1]\},\end{aligned} \tag{18}$$

where use has been made of the continuity condition $\boldsymbol{J}_{+-} - \boldsymbol{J}_{--} = \boldsymbol{J}_{++} - \boldsymbol{J}_{-+}$ and in which the dependency of $(\boldsymbol{x}_\Sigma, \mu_\Sigma)$ has been omitted for brevity. Consequently, a generalized Jacobian of a point on k switching boundaries is dependent on k auxiliary variables q_i, $i = 1 \ldots k$. If the equilibrium point $\boldsymbol{x}_\mu$ of a system with two switching boundaries moves under the variation of μ from one subspace to another, e.g.

$$\begin{aligned}&\boldsymbol{x}_\mu \in \mathcal{V}_{--}, \quad \mu < \mu_\Sigma,\\ &\boldsymbol{x}_\mu = \boldsymbol{x}_\Sigma \in \Sigma_1 \bigcap \Sigma_2, \quad \mu = \mu_\Sigma,\\ &\boldsymbol{x}_\mu \in \mathcal{V}_{++}, \quad \mu > \mu_\Sigma,\end{aligned}$$

then the generalized Jacobian (18) does not define a *unique path* of eigenvalues 'during' the jump as q_2 can be varied independently from q_1. The values of q_1 and q_2 both vary from 0 to 1 as x_Σ moves from $\mathcal{V}_{--}$ to $\mathcal{V}_{++}$ but (q_1, q_2) is unknown. Hence, (18) defines a jump of the eigenvalues of which only the start and end points are known.

Example 1. Consider the equilibrium point $(x_1, x_2) = (0, 0)$ for $\mu = 0$ of the system

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= -|x_1 + 4\mu| - |x_2 - \mu| - \frac{1}{2}x_2 + 5\mu,\end{aligned} \tag{19}$$

which is located on the crossing of two switching boundaries $\Sigma_1 = \{x_1 = -4\mu\}$ and $\Sigma_2 = \{x_2 = \mu\}$. The Jacobian matrices in the four subspaces surrounding the equilibrium point are

$$\begin{aligned}\boldsymbol{J}_{++} &= \begin{bmatrix} 0 & 1 \\ -1 & -\frac{3}{2} \end{bmatrix}, \qquad \boldsymbol{J}_{+-} = \begin{bmatrix} 0 & 1 \\ -1 & \frac{1}{2} \end{bmatrix},\\ \boldsymbol{J}_{-+} &= \begin{bmatrix} 0 & 1 \\ 1 & -\frac{3}{2} \end{bmatrix}, \qquad \boldsymbol{J}_{--} = \begin{bmatrix} 0 & 1 \\ 1 & \frac{1}{2} \end{bmatrix}.\end{aligned} \tag{20}$$

The generalized Jacobian of the equilibrium point $\boldsymbol{x}_\Sigma = \boldsymbol{0}$ is set-valued and is the convex hull of four Jacobian matrices

$$\begin{aligned}\boldsymbol{J}(\boldsymbol{x}_\Sigma, 0) &= \overline{\text{co}}\{\boldsymbol{J}_{++}, \boldsymbol{J}_{+-}, \boldsymbol{J}_{-+}, \boldsymbol{J}_{++}\}\\ &= \{q_1(\boldsymbol{J}_{-+} - \boldsymbol{J}_{--}) + q_2(\boldsymbol{J}_{+-} - \boldsymbol{J}_{--})\\ &\qquad + \boldsymbol{J}_{--}, \forall q_1, q_2 \in [0, 1]\}.\end{aligned} \tag{21}$$

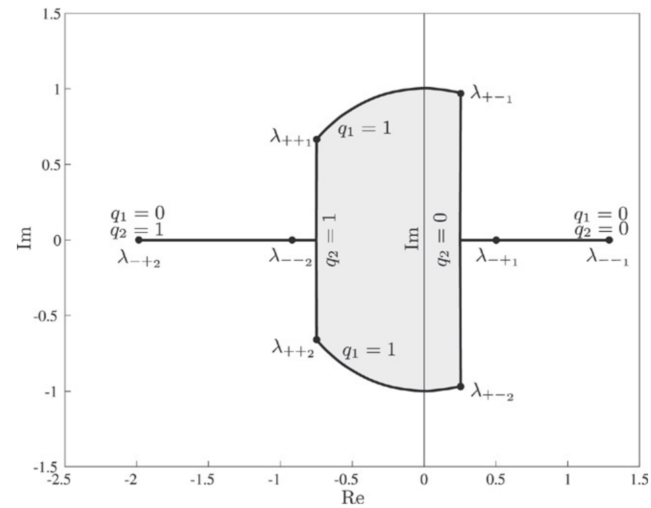


Fig. 4. Set of eigenvalues of the generalized Jacobian (21).

The set-valued generalized Jacobian has set-valued eigenvalues. The set is two-dimensional in the sense that it is dependent on two auxiliary variables q_1 and q_2. The set of eigenvalues is therefore not a one-dimensional path but a two-dimensional subspace in the complex plane (see Fig. 4). The bifurcation point of system (19) will be analyzed in Example 5.

It is important to realize that for smooth systems the eigenvalues are single-valued functions of the parameter μ and that the eigenvalues are set-valued functions in μ for non-smooth continuous systems. An eigenvalue can pass the imaginary axis while varying μ, leading to a classical bifurcation, but it can also cross the imaginary axis during its jump defined by the generalized Jacobian. Examples will be given in the following sections where jumps of eigenvalues over the imaginary axis lead to non-classical bifurcations.

We will name a bifurcation associated with a jump of an eigenvalue (or a pair of them) over the imaginary axis a *discontinuous bifurcation* (Definition 2). If the system possesses (locally) only one switching boundary, then one can speak of a *path* of the eigenvalues. A typical scenario of a discontinuous bifurcation is depicted in Fig. 2(b) where the unique path of a pair of complex conjugate eigenvalues on the jump is indicated by the dashed lines. The eigenvalue path 'during' the jump is determined by the eigenvalues of

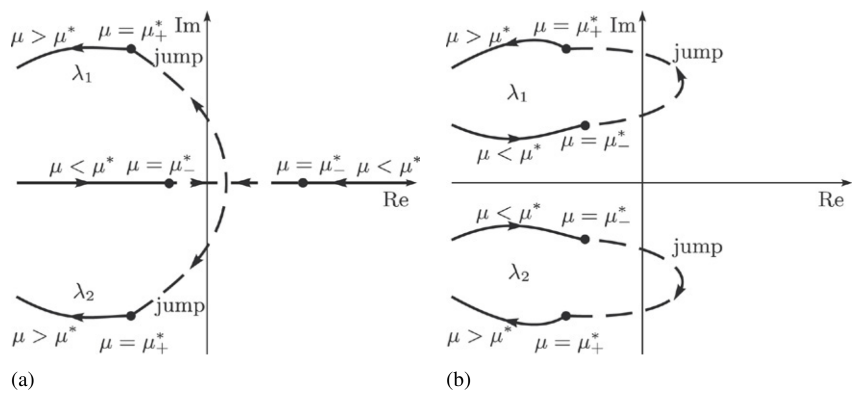


Fig. 5. Eigenvalue paths with multiple crossings at a bifurcation (arrows for increasing μ).

the convex hull of the Jacobian matrices $\boldsymbol{J}_-(\boldsymbol{x}_\Sigma, \mu_\Sigma)$ and $\boldsymbol{J}_+(\boldsymbol{x}_\Sigma, \mu_\Sigma)$.

The possibility of the eigenvalues for becoming set-valued greatly complicates the bifurcation behaviour as the eigenvalue(s) can also cross the imaginary axis multiple times during its jump. The bifurcations associated with eigenvalues that cross the imaginary axis multiple times will be called *multiple crossing bifurcations* (Definition 3). Two examples (with a one-dimensional path of eigenvalues) are depicted in Fig. 5(a) and (b). Fig. 5(a) shows two real-valued eigenvalues that jump to a pair of complex conjugate eigenvalues for increasing μ. The imaginary axis is crossed twice during the jump, first through the origin by one eigenvalue, and a second time as a complex conjugate pair. The scenario depicted in Fig. 5(a) corresponds to a discontinuous bifurcation which is a combination of a classical Hopf bifurcation and a classical turning point bifurcation, as we will see in Section 8. A pair of complex conjugate eigenvalues crosses the imaginary axis twice during the jump in Fig. 5(b). The scenarios depicted in Fig. 5(a) and (b) are multiple crossing bifurcations of set-valued eigenvalues which form a one-dimensional path in the complex plane. The bifurcation point of system (19) in Example 1 has set-valued eigenvalues that form an area in the complex plane (see Fig. 4). The area of eigenvalues contains an interval of the imaginary axis. The set of eigenvalues of Example 1 has therefore more than one intersection point with the imaginary axis and the associated bifurcation will also be denoted by the term multiple crossing bifurcation. A discontinuous bifurcation can therefore be a single crossing bifurcation, which behaves very much like a classical bifurcation, or it can be a multiple crossing bifurcation, being far more complex.

We call the type of bifurcation at which set-valued eigenvalues cross the imaginary axis a discontinuous bifurcation because the eigenvalues behave discontinuously at the bifurcation point. A bifurcation point, as defined by Definition 1, is called a *discontinuous bifurcation point* if the eigenvalues at the bifurcation point are set-valued and contain a value on the imaginary axis.

Definition 2 (*Discontinuous Bifurcation*). Let $\boldsymbol{x}_\mu$ be an equilibrium, depending on $\mu \in \mathbb{R}$, of a non-smooth continuous system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \mu)$ which has a finite number of switching boundaries Σ_j, $j = 1, \dots, k$. Let $\boldsymbol{x}_\mu = \boldsymbol{x}_\Sigma$ for $\mu = \mu_\Sigma$ be an equilibrium located on one or more switching boundaries, i.e. $\boldsymbol{x}_\Sigma \in \Sigma_1 \bigcap \Sigma_2 \cdots \bigcap \Sigma_l$, $1 \le l \le k$. Let $(\boldsymbol{x}_\Sigma, \mu_\Sigma)$ be a bifurcation point in the sense of Definition 1. A bifurcation point $(\boldsymbol{x}_\Sigma, \mu_\Sigma)$ is a discontinuous bifurcation point if the generalized Jacobian $\boldsymbol{J}(\boldsymbol{x}_\Sigma, \mu_\Sigma)$ is set-valued and if there exists an i such that

$$\mathrm{Re}(\lambda_i) \ni 0, \quad \lambda = \mathrm{eig}(\boldsymbol{J}(\boldsymbol{x}_\Sigma, \mu_\Sigma)).$$

A bifurcation point which is not discontinuous is called a continuous bifurcation point. All bifurcations in smooth dynamical systems are continuous bifurcation points. A discontinuous bifurcation can be a single or a multiple crossing bifurcation. Let $\mathcal{S}_\lambda = \mathrm{eig}(\boldsymbol{J}(\boldsymbol{x}_\Sigma, \mu_\Sigma))$ denote the subspace in the complex plane of the set-valued eigenvalues and let Im^+ be the subspace of purely imaginary numbers with positive imaginary part containing the origin.

Definition 3 (*Single/Multiple Crossing Bifurcation*). If $\mathcal{S}_\lambda \bigcap \mathrm{Im}^+$ comprises only one element, then the bifurcation is a single crossing bifurcation, whereas it is called a multiple crossing bifurcation if $\mathcal{S}_\lambda \bigcap \mathrm{Im}^+$ comprises more than one element.

All continuous bifurcation points are single crossing bifurcations if one eigenvalue or one pair of complex conjugate eigenvalues participates in the bifurcation. A special class of non-smooth continuous piecewise linear systems will be studied in Sections 6 and 7. The restriction to this special class of systems allows us to rigorously prove some bifurcation theorems. A number of multiple crossing bifurcations will be studied in Section 8.

6. Coexisting equilibria for a single switching boundary

A bifurcation point has been defined in Section 3 as a point on which the number of equilibria and (quasi-)periodic solutions changes under the influence of a parameter (Definition 1). The creation (and destruction) of the coexistence of equilibria under variation of a parameter (e.g. a turning point bifurcation or a pitchfork bifurcation in smooth dynamical systems) is therefore by definition a bifurcation. This kind of bifurcation can also occur in non-smooth systems. In this section we will derive criteria for the coexistence of equilibria in certain piecewise linear systems with a single switching boundary.

The coexistence of equilibria *for a subclass* of piecewise linear non-smooth continuous systems with only one switching boundary can conveniently be analyzed with a method developed by Feigin [6–8]. The method was developed to analyze coexistence of fixed points in piecewise linear mappings used to study periodic solutions [3]. The method can, with a little adjustment, be applied to the analysis of equilibria of a special class of non-smooth continuous differential equations with a piecewise linear right-hand side and a single switching boundary, as is shown in [4] and in the remainder of this section.

Consider a piecewise linear non-smooth continuous system of the following special form

$$\dot{\boldsymbol{x}} = \begin{cases} \boldsymbol{J}_- \boldsymbol{x} + \boldsymbol{c}\mu, & \boldsymbol{x} \in \mathcal{V}_- \bigcup \Sigma, \\ \boldsymbol{J}_+ \boldsymbol{x} + \boldsymbol{c}\mu, & \boldsymbol{x} \in \mathcal{V}_+, \end{cases} \tag{22}$$

depending on a parameter $\mu \in \mathbb{R}$ and being continuous on the switching boundary Σ. Let the switching function be given by $h(\boldsymbol{x}) = x_n$, not being dependent on μ, which defines the subspaces $\mathcal{V}_-$, Σ and $\mathcal{V}_+$ (11). Note that the class of piecewise linear non-smooth continuous systems of the form (22) is very special. The Jacobian matrices do not depend on μ and it holds that $\boldsymbol{b}_-(\mu) = \boldsymbol{b}_+(\mu) = \boldsymbol{c}\mu$, which is linear in μ. The system has only a single switching boundary which is the hyperplane $x_n = 0$. A system of the form (22) with an arbitrary switching boundary can always be transformed to have $x_n = 0$ as switching boundary. We explicitly state that the non-smooth continuous systems considered in the following sections are generally not of the form (22) but belong to the more general class of piecewise linear non-smooth continuous systems (13) with one or more switching boundaries.

Continuity at the switching boundary of system (22) implies that the first $n-1$ columns of the Jacobians agree, i.e. $\boldsymbol{J}_{-ik} = \boldsymbol{J}_{+ik}, \forall i = 1 \dots n, k = 1 \dots n-1$. Assuming that $\boldsymbol{J}_-$ and $\boldsymbol{J}_+$ are non-singular, at most one equilibrium can exist in each of the subspaces $\mathcal{V}_-$ and $\mathcal{V}_+$ because the system is linear within these subspaces. Let $\boldsymbol{x}_- \in \mathcal{V}_- \bigcup \Sigma$ and $\boldsymbol{x}_+ \in \mathcal{V}_+$ be equilibria of system (22), i.e. $\boldsymbol{0} = \boldsymbol{J}_- \boldsymbol{x}_- + \boldsymbol{c}\mu$ and $\boldsymbol{0} = \boldsymbol{J}_+ \boldsymbol{x}_+ + \boldsymbol{c}\mu$. If $\boldsymbol{J}_-$ and $\boldsymbol{J}_+$ are non-singular, then we can solve for $\boldsymbol{x}_-$ and $\boldsymbol{x}_+$

$$\begin{aligned} \boldsymbol{x}_- &= -\boldsymbol{J}_-^{-1} \boldsymbol{c}\mu = -\frac{\operatorname{adj}(\boldsymbol{J}_-)}{\det(\boldsymbol{J}_-)} \boldsymbol{c}\mu, \\ \boldsymbol{x}_+ &= -\boldsymbol{J}_+^{-1} \boldsymbol{c}\mu = -\frac{\operatorname{adj}(\boldsymbol{J}_+)}{\det(\boldsymbol{J}_+)} \boldsymbol{c}\mu, \end{aligned} \tag{23}$$

where $\operatorname{adj}(\boldsymbol{A})$ denotes the adjoint matrix of $\boldsymbol{A}$. It follows from (23) that the elements of $\boldsymbol{x}_-$ and $\boldsymbol{x}_+$ can be expressed as

$$x_{-_k} = \frac{b_{-_k}}{\det(\boldsymbol{J}_-)} \mu, \qquad x_{+_k} = \frac{b_{+_k}}{\det(\boldsymbol{J}_+)} \mu, \tag{24}$$

with $b_{-_k} = [-\operatorname{adj}(\boldsymbol{J}_-)\boldsymbol{c}]_k$ and $b_{+_k} = [-\operatorname{adj}(\boldsymbol{J}_+)\boldsymbol{c}]_k$. It holds that $b_{-_n} = b_{+_n} =: b_n$ because the matrices $\boldsymbol{J}_-$ and $\boldsymbol{J}_+$ differ only in the nth column (see also [3]). The nth elements of the equilibria $\boldsymbol{x}_-$ and $\boldsymbol{x}_+$

$$x_{-_n} = \frac{b_n}{\det(\boldsymbol{J}_-)} \mu, \qquad x_{+_n} = \frac{b_n}{\det(\boldsymbol{J}_+)} \mu, \tag{25}$$

are therefore only functions of b_n, the determinant of the Jacobian and the parameter μ. We have to require that $x_{-_n} \le 0$ and $x_{+_n} > 0$ in order to let the equilibria be admissible $\boldsymbol{x}_- \in \mathcal{V}_- \bigcup \Sigma$ and $\boldsymbol{x}_+ \in \mathcal{V}_+$. Only one of the equilibria exists for $\mu = 0$, i.e. $\boldsymbol{x}_{-,\mu=0} = \boldsymbol{0}$, being located at the switching boundary. The two equilibria coexist for $\mu < 0$ *or* $\mu > 0$ if the elements x_{-_n} and x_{+_n} have opposite signs. Coexistence of the equilibria for $\mu < 0$ ($\mu > 0$) implies non-existence of equilibria for $\mu > 0$ ($\mu < 0$). A necessary and sufficient condition for coexistence of equilibria of system (22) is

$$\det(\boldsymbol{J}_-) \det(\boldsymbol{J}_+) < 0. \tag{26}$$

The equilibria exist for opposite signs of μ if

$$\det(\boldsymbol{J}_-) \det(\boldsymbol{J}_+) > 0. \tag{27}$$

The coexistence of equilibria of system (22) is therefore determined by the signs of the determinants of the Jacobian matrices. Moreover, the sign of the determinant of the Jacobian depends solely on the number of negative real-valued eigenvalues, because $\det(\boldsymbol{J}) = \lambda_1 \lambda_2 \dots \lambda_n$ and $\lambda\bar{\lambda} \ge 0$ (in which $\bar{\lambda}$ is the complex conjugate of λ). If the number of negative real-valued eigenvalues is odd, then the determinant of the non-singular Jacobian is negative. If the number of negative real-valued eigenvalues is even, then the determinant is positive. A non-singular Jacobian with only complex conjugate eigenvalues has therefore a positive determinant.

The coexistence conditions of Feigin (i.e. (26) and (27)) have direct consequences for the bifurcation behaviour of system (22). If condition (26) is satisfied, then a bifurcation in the sense of Definition 1 must exist for $\mu = 0$ because the number of equilibria for $\mu < 0$ is different from the number of equilibria for $\mu > 0$. The branch of equilibria turns around at $\mu = 0$ and the bifurcation point can appropriately be named as a turning point bifurcation. If condition (27) is satisfied, then the branch of equilibria for $\mu < 0$ continues for $\mu > 0$. Bifurcations of fixed points of piecewise linear non-smooth continuous mappings were (partly) classified by means of conditions (26) and (27) in the work of Feigin and di Bernardo [3,4,6–8].

7. Planar systems with a single switching boundary

In this section we will rigorously prove some results on bifurcations in planar piecewise linear non-smooth continuous systems with a single switching boundary. Consider a planar piecewise linear non-smooth continuous system of the special form (22)

$$\dot{\boldsymbol{x}} = \begin{cases} \boldsymbol{J}_- \boldsymbol{x} + \boldsymbol{c}\mu, & \boldsymbol{x} \in \mathcal{V}_- \bigcup \Sigma, \\ \boldsymbol{J}_+ \boldsymbol{x} + \boldsymbol{c}\mu, & \boldsymbol{x} \in \mathcal{V}_+, \end{cases} \tag{28}$$

with $\boldsymbol{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^{\mathrm{T}}$ being continuous on the switching boundary $\Sigma = \{\boldsymbol{x} \in \mathbb{R}^2 \mid x_2 = 0\}$. The continuity conditions on the switching boundary require that $J_{-11} = J_{+11}$ and

$J_{-21} = J_{+21}$. Assuming that the Jacobian matrices $\boldsymbol{J}_-$ and $\boldsymbol{J}_+$ are non-singular, at most one equilibrium can exist in each of the subspaces $\mathcal{V}_-$ and $\mathcal{V}_+$ because the system is linear and hyperbolic within these subspaces. In Section 6 it has been proven that the equilibria coexist for $\mu < 0$ or for $\mu > 0$ if (26) holds and one equilibrium exists for all μ if (27) holds. The determinant/trace of a convex combination of $\boldsymbol{J}_-$ and $\boldsymbol{J}_+$ is a convex combination of the determinants/traces of $\boldsymbol{J}_-$ and $\boldsymbol{J}_+$ as follows from the following proposition.

Proposition 1. *For a planar continuous piecewise linear non-smooth continuous system of the form (28) it holds that*

$$\det(q\boldsymbol{J}_+ + (1-q)\boldsymbol{J}_-) = q\det(\boldsymbol{J}_+) + (1-q)\det(\boldsymbol{J}_-),$$

and

$$\operatorname{trace}(q\boldsymbol{J}_+ + (1-q)\boldsymbol{J}_-) = q\operatorname{trace}(\boldsymbol{J}_+) + (1-q)\operatorname{trace}(\boldsymbol{J}_-).$$

Proof. It follows from the continuity conditions that

$$q\boldsymbol{J}_+ + (1-q)\boldsymbol{J}_- = \begin{bmatrix} J_{-11} & qJ_{+12} + (1-q)J_{-12} \\ J_{-21} & qJ_{+22} + (1-q)J_{-22} \end{bmatrix}.$$

Evaluation of the determinant and the trace completes the proof. □

The coexistence criterion (26) suggests that a turning point bifurcation can occur in the system (28).

Theorem 1 (*Turning Point Bifurcation Theorem*). *If a planar continuous piecewise linear non-smooth continuous system of the form (28) with a single switching boundary undergoes a turning point bifurcation, then the path of the set-valued eigenvalues crosses the imaginary axis through the origin.*

Proof. If there exists a turning point bifurcation, then it follows from (26) that $\det(\boldsymbol{J}_-)\det(\boldsymbol{J}_+) < 0$. The turning point $\boldsymbol{x}^*$ must be located on the switching boundary Σ. The generalized Jacobian $\boldsymbol{J}$ of the equilibrium on Σ is the set $\boldsymbol{J} = \overline{\text{co}}(\boldsymbol{J}_-, \boldsymbol{J}_+) = \{\boldsymbol{J}_q \mid \boldsymbol{J}_q = q\boldsymbol{J}_+ + (1-q)\boldsymbol{J}_-,\ q \in [0, 1]\}$. It therefore must hold, using Proposition 1, that there exists a $q \in [0, 1]$ for which $\det(\boldsymbol{J}_q) = 0$. Consequently, there exists a set-valued eigenvalue λ_i of the generalized Jacobian $\boldsymbol{J}$ containing the origin, i.e. $0 \in \lambda_i$ with $\lambda = \operatorname{eig}(\boldsymbol{J})$. The path of the set-valued eigenvalues of the generalized Jacobian at the discontinuous turning point bifurcation therefore passes the origin. □

A pitchfork bifurcation cannot occur in a system of the form (28) because at most two equilibria can coexist for one value of μ. Similarly, a transcritical bifurcation of (28) is impossible because the equilibria $\boldsymbol{x}_- = -\boldsymbol{J}_-^{-1}\boldsymbol{c}\mu$, $\boldsymbol{x}_+ = -\boldsymbol{J}_+^{-1}\boldsymbol{c}\mu$ cannot exist for $\mu < 0$ *and* $\mu > 0$. Another possibility for bifurcation in (28) is a Hopf bifurcation, at which a limit cycle is created/destroyed.

Proposition 2. *If a planar continuous piecewise linear non-smooth continuous system of the form (28) has a limit cycle then it must hold that*

$$\operatorname{trace}(\boldsymbol{J}_+)\operatorname{trace}(\boldsymbol{J}_-) \leq 0.$$

Proof. The proof follows from Bendixson's criterion [9], which states if the trace of the Jacobian matrix does not change sign and is not identical to zero in a simply connected region D, then there does not exist a periodic solution which lies entirely in D. Consequently, if a continuous piecewise linear non-smooth continuous system with a single switching boundary has a periodic solution, then the trace of the Jacobian matrix must change or be equal to zero. Bendixson's criterion can be derived from Green's theorem and was originally stated for smooth dynamical systems. Green's theorem relates a line integral to a surface integral. A system of the form (28) is non-smooth on a switching boundary Σ. A line integral along a periodic solution Γ is not affected by the non-smoothness on Σ because the set of points of Γ which are on Σ is of measure zero. Similarly, the surface integral over the interior of Γ is not affected by the non-smoothness on Σ because the area of Σ is zero. Bendixson's criterion can therefore be used for non-smooth continuous systems. The trace of the Jacobian in a piecewise linear non-smooth continuous system of the form (28) is constant in each of the subspaces $\mathcal{V}_-$ and $\mathcal{V}_+$. For a periodic solution it must therefore hold that $\operatorname{trace}(\boldsymbol{J}_+)\operatorname{trace}(\boldsymbol{J}_-) \leq 0$. □

Theorem 2 (*Hopf Bifurcation Theorem*). *If a planar continuous piecewise linear non-smooth continuous system with a single switching boundary of the form (28) undergoes a Hopf bifurcation at $\mu = 0$, such that a path of equilibria exists for $\mu < 0$ and $\mu \geq 0$ and a limit cycle exists for $\mu < 0$ or for $\mu > 0$, then the path of the set-valued generalized Jacobian at the bifurcation point must pass the imaginary axis with a complex conjugate pair of eigenvalues.*

Proof. If an equilibrium exists for $\mu < 0$ as well as for $\mu > 0$ then condition (27) must hold, i.e. $\det(\boldsymbol{J}_+)\det(\boldsymbol{J}_-) > 0$. If a limit cycle exists in a planar system, then there must be at least one equilibrium located in the interior of this limit cycle. As there is only one equilibrium, it follows from index theory [9] that the equilibrium cannot be a saddle point. Consequently, it must hold that if the equilibrium is located in $\mathcal{V}_+$ (or $\mathcal{V}_-$) then it holds that $\det(\boldsymbol{J}_+) > 0$ (or $\det(\boldsymbol{J}_-) > 0$). It follows from $\det(\boldsymbol{J}_+)\det(\boldsymbol{J}_-) > 0$ that $\det(\boldsymbol{J}_+) > 0$ and $\det(\boldsymbol{J}_-) > 0$. The determinant of every convex combination $\boldsymbol{J}_q = q\boldsymbol{J}_+ + (1-q)\boldsymbol{J}_-$ is due to Proposition 1 also positive $\det(\boldsymbol{J}_q) > 0$ for $q \in [0, 1]$. Furthermore, the existence of a limit cycle implies that $\operatorname{trace}(\boldsymbol{J}_+)\operatorname{trace}(\boldsymbol{J}_-) \leq 0$ (see Proposition 2). It therefore must hold, using Proposition 1, that there exists a $q \in [0, 1]$ for which $\operatorname{trace}(\boldsymbol{J}_q) = 0$. The eigenvalues of $\boldsymbol{J}_q$ are zeros of the characteristic equation

$$\lambda^2 - \operatorname{trace}(\boldsymbol{J}_q)\lambda + \det(\boldsymbol{J}_q) = 0.$$

Consequently, there exists a $q \in [0, 1]$ for which there is a complex conjugate pair of eigenvalues $\lambda_{1,2} = \pm \mathrm{i}\sqrt{\det(\boldsymbol{J}_q)}$ on the imaginary axis which belongs to the matrix $\boldsymbol{J}_q$. The path of the set-valued generalized Jacobian must therefore pass the imaginary axis with a complex conjugate pair of eigenvalues. □

Planar piecewise linear non-smooth continuous systems with a single switching boundary can therefore show two types of single crossing bifurcations: a discontinuous turning point bifurcation and a discontinuous Hopf bifurcation. Theorems 1 and 2 are new results, but rely on the coexistence results of Section 6 which are due to Feigin [6]. A discontinuous multiple crossing bifurcation of a system of type (28) will be studied in Example 7 of Section 8.

8. Multiple crossing bifurcations

A number of single crossing bifurcations have been treated in [12] and are characterized by a single crossing of the eigenvalue(s) through the imaginary axis. If the eigenvalues are set-valued, which is the case for discontinuous bifurcations, then the set of eigenvalues at a single crossing bifurcation forms a one-dimensional path in the complex plane. The eigenvalue(s) either move continuously through the imaginary axis under the variation of a parameter (being a continuous bifurcation) or a one-dimensional path of eigenvalues crosses the imaginary axis during a jump (leading to a discontinuous bifurcation). Non-smooth continuous systems can also exhibit bifurcations of equilibria for which a one-dimensional path of eigenvalue(s) crosses multiple times the imaginary axis (see Example 7), as was already pointed out in Section 5. Equilibria of non-smooth continuous systems with multiple switching boundaries can have set-valued eigenvalues which form not a one-dimensional path but a two-dimensional area in the complex plane (see Example 5). Such a set of eigenvalues, which forms an area in the complex plane, can contain part of the imaginary axis leading to a multiple crossing bifurcation. Multiple crossing bifurcations are much more complex than single crossing bifurcations and do not have a smooth counterpart. In the following, we will discuss a number of two-dimensional systems showing multiple crossing bifurcations. First a multiple crossing bifurcation will be studied (Example 2) of which the behaviour is the combination of two single crossing bifurcations. Examples 2 and 4 are smooth approximations of the non-smooth system in Example 2. Subsequently, examples of a multiple crossing bifurcations are studied (Examples 5 and 7) which show a bifurcation behaviour which cannot directly be understood as the combination of two single crossing bifurcations. Examples 6 and 8, which are smooth approximations of Examples 5 and 7 respectively, demonstrate how complicated the bifurcation behaviours of these multiple crossing bifurcations are.

Example 2 (*Combined Hopf and Pitchfork Behaviour*). Consider the two-dimensional non-smooth continuous system

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= -x_1 + |x_1+\mu| - |x_1-\mu|\\ &\quad - x_2 - |x_2+\mu| + |x_2-\mu|.\end{aligned} \tag{29}$$

The system has three equilibria for $\mu > 0$: $x_1 = 0, x_2 = 0$ and $x_1 = \pm 2\mu$, $x_2 = 0$. For $\mu \le 0$, the only equilibrium is the trivial equilibrium point $x_1 = 0, x_2 = 0$. The point $(\boldsymbol{x}, \mu) = (\boldsymbol{0}, 0)$ is a bifurcation point according to Definition 1. The generalized Jacobian matrix of the system

$$\boldsymbol{J}(\boldsymbol{x},\mu) = \begin{bmatrix} 0 & 1 \\ J_{21} & J_{22} \end{bmatrix}, \tag{30}$$

with

$$\begin{aligned}J_{21} &= -1 + \mathrm{Sign}(x_1+\mu) - \mathrm{Sign}(x_1-\mu),\\ J_{22} &= -1 - \mathrm{Sign}(x_2+\mu) + \mathrm{Sign}(x_2-\mu),\end{aligned} \tag{31}$$

is set-valued at four different switching boundaries. The bifurcation point is located at the crossing of the four switching boundaries, which causes the generalized Jacobian at the bifurcation point to be dependent on four auxiliary variables q_i

$$\boldsymbol{J}(\boldsymbol{0}, 0) = \{\boldsymbol{J}_q,\ q_i \in [0,1],\ i = 1,\ldots,4\}, \tag{32}$$

with

$$\boldsymbol{J}_q = \begin{bmatrix} 0 & 1 \\ -1-2q_1+2q_2 & -1-2q_3+2q_4 \end{bmatrix}. \tag{33}$$

The set-valued generalized Jacobian at the bifurcation point defines a set of eigenvalues in the complex plane. This set of eigenvalues is spanned in the complex plane by four auxiliary variables. It is therefore not possible to speak of a 'path' of eigenvalues. The set of eigenvalues is only a path if the bifurcation point is located on only one switching boundary or if the system is one-dimensional, which forces the set of eigenvalues to be on the real axis. The fact that the set of eigenvalues at the bifurcation point is higher dimensional tremendously complicates the analysis of the system.

The Jacobian matrix at the trivial branch ($x_1 = x_2 = 0$) is $\boldsymbol{J}_-^{\mathrm{tr}}$ for $\mu < 0$ with

$$\boldsymbol{J}_-^{\mathrm{tr}} = \boldsymbol{J}(\boldsymbol{0}, \mu<0) = \begin{bmatrix} 0 & 1 \\ -3 & 1 \end{bmatrix}, \qquad \lambda_{1,2} = \frac{1}{2} \pm \mathrm{i}\frac{1}{2}\sqrt{11}, \tag{34}$$

and $\boldsymbol{J}_+^{\mathrm{tr}}$ for $\mu > 0$ with

$$\begin{aligned}&\boldsymbol{J}_+^{\mathrm{tr}} = \boldsymbol{J}(\boldsymbol{0}, \mu>0) = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix},\\ &\lambda_{1,2} = -\frac{3}{2} \pm \frac{1}{2}\sqrt{13} \approx \{0.30, -3.30\}.\end{aligned} \tag{35}$$

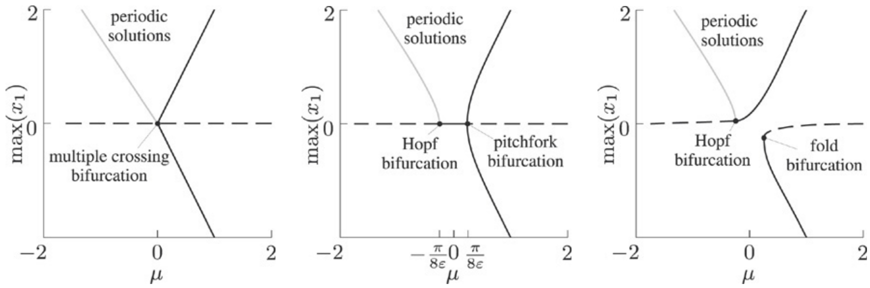
The trivial equilibrium is therefore an unstable focus for $\mu < 0$ and a saddle point for $\mu > 0$. The Jacobian matrix on the non-trivial branches is

$$\begin{aligned}&\boldsymbol{J}^{\mathrm{non}} = \boldsymbol{J}([\pm 2\mu, 0]^{\mathrm{T}}, \mu>0) = \begin{bmatrix} 0 & 1 \\ -1 & -3 \end{bmatrix},\\ &\lambda_{1,2} = -\frac{3}{2} \pm \frac{1}{2}\sqrt{5} \approx \{-0.38, -2.62\}.\end{aligned} \tag{36}$$

Equilibria on the non-trivial branches are therefore stable nodes.

Using the transformation

$$y_1 = \frac{x_1}{\mu}, \qquad y_2 = \frac{x_2}{\mu}, \tag{37}$$



(a) System (29). (b) System (40). (c) System (43).

Fig. 6. Bifurcation diagrams of the non-smooth system (29) (left) and the smooth approximating systems (40) (middle) and (43) (right).

we can transform system (29) for $\mu < 0$ into

$$
\begin{aligned}
\dot{y}_1 &= y_2, \\
\dot{y}_2 &= -y_1 - |y_1 + 1| + |y_1 - 1| - y_2 + |y_2 + 1| - |y_2 - 1|,
\end{aligned}
\tag{38}
$$

and for $\mu > 0$ into

$$
\begin{aligned}
\dot{y}_1 &= y_2, \\
\dot{y}_2 &= -y_1 + |y_1 + 1| - |y_1 - 1| - y_2 - |y_2 + 1| + |y_2 - 1|.
\end{aligned}
\tag{39}
$$

The transformed systems are independent of μ for $\mu \neq 0$. Equilibria and periodic solutions of (38) and (39) are after an inverse transformation with (37) also equilibria and periodic solutions of system (29). The locations of the equilibria of system (29) scale therefore with μ. But also all periodic solutions of system (29) scale with μ. This means that the shape of a periodic solution of (29) does not change for varying μ, but the size of the periodic solution scales with μ. The period time is independent of μ. The bifurcation diagram of system (29) is depicted in Fig. 6(a). Branches of equilibria are indicated by black lines and periodic branches by grey lines. Stable branches are indicated by solid lines and unstable branches by dashed lines. The periodic solution and the period time have been found by numerical simulation. The point $(x_1, x_2, \mu) = (0, 0, 0)$ is a bifurcation point where two branches of equilibria bifurcate from the trivial branch, similar to a pitchfork bifurcation, and also a periodic solution is created at the bifurcation point. The magnitude $\max(x_1)$ varies linearly in μ for all branches, as was expected from the transformation. The period time of the periodic solution is $T = 4.03$ s and is independent of μ. The phase plane of system (29) is shown in Fig. 7 for three different values of μ. The system has for $\mu < 0$ an unstable focus and a stable periodic solution. The periodic solution disappears for $\mu = 0$ and the trivial equilibrium point turns into a stable focus. The trivial equilibrium point becomes a saddle point for $\mu > 0$ and two additional stable nodes appear at $\boldsymbol{x} = \begin{bmatrix} \pm 2\mu & 0 \end{bmatrix}^{\mathrm{T}}$. The stable and unstable invariant manifolds of the saddle point $\boldsymbol{x} = \boldsymbol{0}$ are depicted by thick lines in Fig. 7 and the unstable invariant manifolds form heteroclinic connections between the saddle point and the stable nodes. The multiple crossing bifurcation has the following bifurcation structure:

$$
\left.\begin{array}{l}\text{unstable focus}\\ \text{periodic solution}\end{array}\right\} \xrightarrow{\text{multiple crossing bifurcation of (29)}} \left\{\begin{array}{l}\text{stable node}\\ \text{saddle}\\ \text{stable node}\end{array}\right.
$$

The bifurcation structure is different from any known bifurcation structure of (co-dimension 1) bifurcations in smooth systems.

Example 3 (*Symmetric Smooth Approximation of* (29)). The multiple crossing bifurcation in the previous example shows behaviour which is the combination of a Hopf and a pitchfork bifurcation. We now investigate whether we obtain a Hopf and a pitchfork bifurcation if we smooth the system. We therefore study the smooth system

$$
\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -x_1 + \frac{2}{\pi}\arctan(\varepsilon(x_1+\mu))(x_1+\mu) \\
&\quad - \frac{2}{\pi}\arctan(\varepsilon(x_1-\mu))(x_1-\mu) \\
&\quad - x_2 - \frac{2}{\pi}\arctan(\varepsilon(x_2+\mu))(x_2+\mu) \\
&\quad + \frac{2}{\pi}\arctan(\varepsilon(x_2-\mu))(x_2-\mu),
\end{aligned}
\tag{40}
$$

which is a smooth approximation of system (29). The smooth system (40) can be expanded in a Taylor series around $x_1 = x_2 = 0$ and $\mu \ll 1$

$$
\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &\approx \left(-1 + \frac{8}{\pi}\varepsilon\mu\right)x_1 - \frac{64}{3\pi}\varepsilon^3\mu x_1^3 \\
&\quad + \left(-1 - \frac{8}{\pi}\varepsilon\mu\right)x_2 + \frac{64}{3\pi}\varepsilon^3\mu x_2^3.
\end{aligned}
\tag{41}
$$

The chosen regularization of the non-smooth terms in (40) is symmetric in the sense that it preserves the symmetry of the system. The smooth approximating system (40), therefore, also has the trivial branch of equilibria $(x_1, x_2) = (0, 0)$. The locations of the eigenvalues on the trivial branch have been computed numerically (with $\varepsilon = 10$) for varying μ in the range -0.2 to 0.2 and are plotted (indicated by '$*$') in the complex plane in Fig. 8(a). The eigenvalue path of the convex combination of the Jacobian matrices (35) of the non-smooth system (29)

$$
\boldsymbol{J}^{\mathrm{tr}} = \{(\boldsymbol{J}_+^{\mathrm{tr}} - \boldsymbol{J}_-^{\mathrm{tr}})q + \boldsymbol{J}_-^{\mathrm{tr}},\ q \in [0, 1]\},
\tag{42}
$$

which is a subset of $\boldsymbol{J}(\boldsymbol{0}, 0)$ and uses only a *single* auxiliary variable q, is indicated by a solid line in Fig. 8(a) and (b).

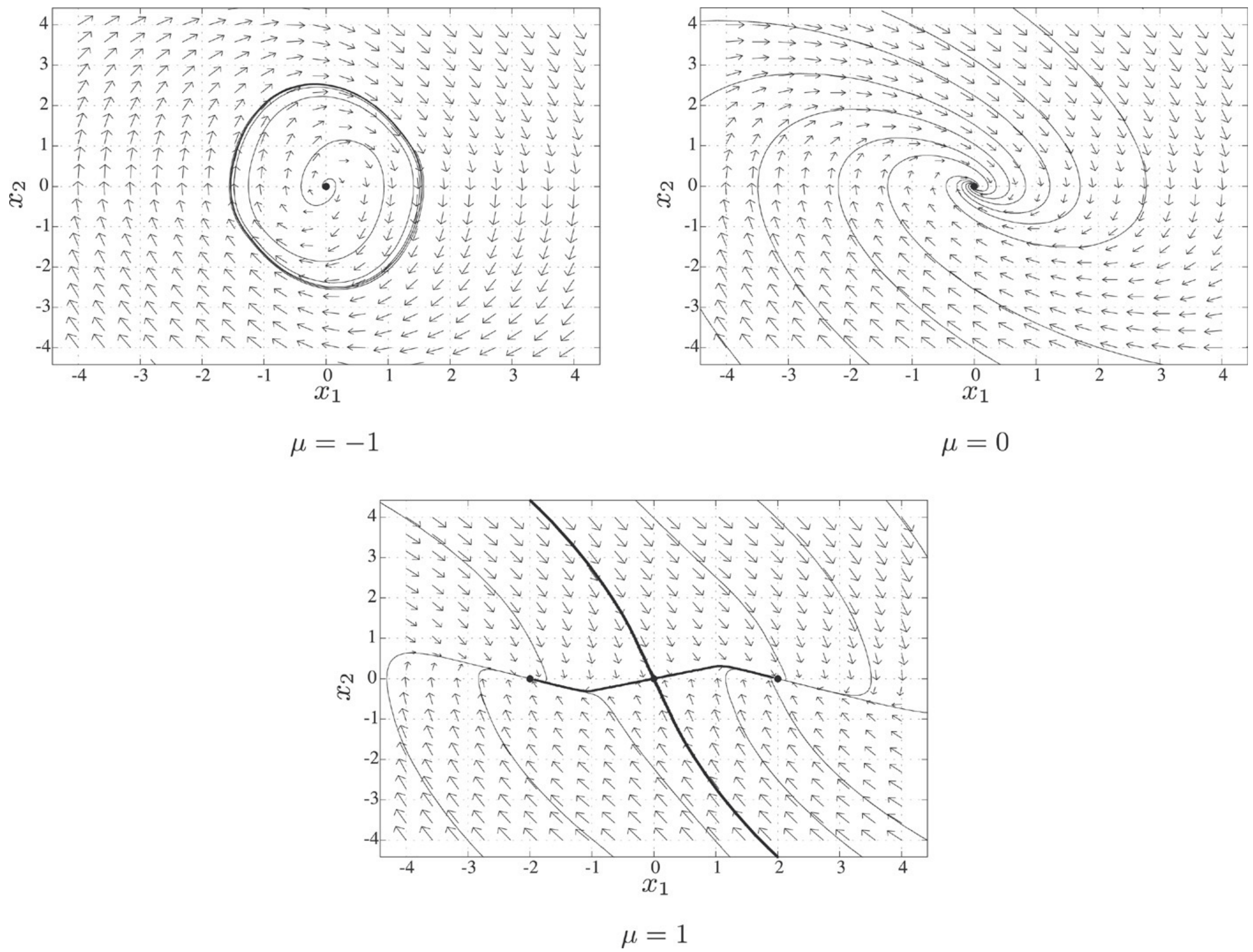
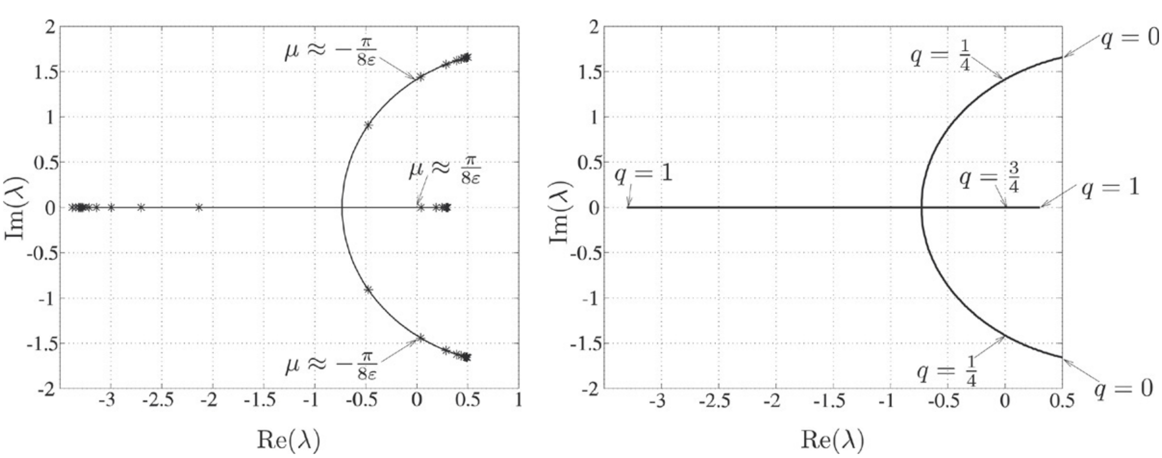


Fig. 7. Multiple crossing bifurcation of system (29).



(a) Eigenvalues of the approximating system.

(b) Path of the set-valued eigenvalues of $\boldsymbol{J}(0, 0)^{\mathrm{tr}}$.

Fig. 8. Eigenvalue path of system (29).

The eigenvalues of the smooth approximating system seem to be almost located on the eigenvalue path of the above convex combination of $\boldsymbol{J}_+^{\mathrm{tr}}$ and $\boldsymbol{J}_-^{\mathrm{tr}}$ of the non-smooth system (29) (the $*$ signs are located on the solid line in Fig. 8(a)). We observe that the eigenvalues of the convex combination of $\boldsymbol{J}_+^{\mathrm{tr}}$ and $\boldsymbol{J}_-^{\mathrm{tr}}$ cross the imaginary axis twice. At $q = \frac{1}{4}$ a pair of complex conjugate eigenvalues passes the imaginary axis and at $q = \frac{3}{4}$ a single eigenvalue passes the origin. The coincidence of the eigenvalue path of the smooth approximating system with the eigenvalue path of the convex combination (42) suggests that the bifurcation behaviour of this particular system might be studied via the eigenvalue path of $\boldsymbol{J}^{\mathrm{tr}}$, which uses only a *single* auxiliary variable. The set $\boldsymbol{J}^{\mathrm{tr}}$ is a subset of $\boldsymbol{J}(\mathbf{0}, 0)$, being dependent on *four* auxiliary variables. The reason why this bifurcation can be analyzed with a single auxiliary variable can be sought in the symmetries of the system. Bifurcation points on multiple switching boundaries can generally not be studied by a convex combination of two Jacobian matrices as will be demonstrated in Example 5.

The trivial branch of the smooth approximating system (40) also undergoes a Hopf bifurcation and a pitchfork bifurcation but at different values of μ. The bifurcation diagram of the smooth approximating system (40) is sketched in Fig. 6(b). A Hopf bifurcation destroys a periodic solution and turns an unstable focus into a stable focus whereafter the stable focus is transformed into a stable node:

$$\left.\begin{array}{r}\text{unstable focus}\\ \text{periodic solution}\end{array}\right\}\xrightarrow[\text{bifurcation}]{\text{Hopf}}\text{stable focus}\xrightarrow[\text{transition}]{\text{focus–node}}\text{stable node}$$

The stable node subsequently undergoes a pitchfork bifurcation:

$$\text{stable node}\xrightarrow[\text{bifurcation}]{\text{pitchfork}}\left\{\begin{array}{l}\text{stable node}\\ \text{saddle}\\ \text{stable node}\end{array}\right.$$

The Hopf bifurcation is approximately located at $\mu = -\frac{\pi}{8\varepsilon}$ and the pitchfork bifurcation approximately at $\mu = \frac{\pi}{8\varepsilon}$. The two bifurcations approach each other for increasing ε. The two bifurcations seem to occur simultaneously in the non-smooth system (29).

Example 4 (*Non-Symmetric Smooth Approximation of* (29)). We now study another smooth approximating system of system (29) using a non-symmetric regularization:

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= -x_1 + \frac{2}{\pi}\arctan(\varepsilon(x_1+\mu))(x_1+\mu)\\ &\quad - \frac{2}{\pi}\arctan(\varepsilon(x_1-\mu))(x_1-\mu)\\ &\quad - x_2 - \frac{2}{\pi}\arctan(\varepsilon(x_2+\mu))(x_2+\mu)\\ &\quad + \frac{2}{\pi}\arctan(\varepsilon(x_2-\mu))(x_2-\mu) + \frac{1}{\varepsilon}.\end{aligned}\tag{43}$$

The non-symmetric regularization does not preserve the symmetry of the system. The bifurcation diagram of the smooth approximating system (40) is sketched in Fig. 6(c). A Hopf bifurcation destroys a periodic solution and creates a stable focus:

$$\left.\begin{array}{r}\text{unstable focus}\\ \text{periodic solution}\end{array}\right\}\xrightarrow[\text{bifurcation}]{\text{Hopf}}\text{stable focus}$$

The stable focus is transformed into a stable node. Two other branches are created by a turning point bifurcation:

$$\begin{array}{lll}\text{stable focus}\xrightarrow[\text{transition}]{\text{focus–node}}\text{stable node} & & \text{stable node}\\ \qquad\qquad\emptyset & \xrightarrow[\text{bifurcation}]{\text{turning point}} & \left\{\begin{array}{l}\text{saddle}\\ \text{stable node}\end{array}\right.\end{array}$$

The multiple crossing bifurcation of Example 2 shows a similarity with a Hopf and a pitchfork bifurcation. Different smooth approximating systems (Examples 3 and 4), however, give a different sequence of continuous bifurcations. Consequently, we cannot name the multiple crossing bifurcation in the non-smooth system (29) with the term 'Hopf–pitchfork bifurcation'. The terminology for multiple crossing bifurcation points will become even more problematic in the following examples.

Example 5 (*Combined Hopf and Turning Point Behaviour*). Consider the two-dimensional non-smooth continuous system

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= -|x_1+4\mu| - |x_2-\mu| - \frac{1}{2}x_2 + 5\mu,\end{aligned}\tag{44}$$

which is piecewise linear with two switching boundaries $\Sigma_1 = \{x_1 + 4\mu = 0\}$ and $\Sigma_2 = \{x_2 - \mu = 0\}$. The system (44) has for $\mu > 0$ two distinct equilibria

$$\begin{array}{llll}\text{equilibrium 1:} & x_1 = 0, & x_2 = 0, & \text{unstable focus,}\\ \text{equilibrium 2:} & x_1 = -8\mu, & x_2 = 0, & \text{saddle point.}\end{array}\tag{45}$$

The two equilibria agree at $\mu = 0$ and the system has no equilibria for $\mu < 0$. The phase plane of the system is shown in Fig. 9 for $\mu = -1$, $\mu = 0$ and $\mu = 1$. The stable and unstable invariant manifolds of the saddle point are depicted with thick solid lines. A single equilibrium exists for $\mu = 0$ and has only two invariant half-manifolds (stable and unstable). An equilibrium having only two invariant half-manifolds is a peculiarity of non-smooth continuous systems. The phase plane for $\mu = 1$ also shows a stable periodic solution. All trajectories, invariant manifolds and periodic solutions in Fig. 9 are obtained by numerical integration. The corresponding bifurcation diagram (lower right corner Fig. 9) reveals that two unstable equilibrium branches and one stable periodic branch meet each other at $\mu = 0$.

Apparently, a bifurcation (in the sense of Definition 1) occurs at $\mu = 0$. The bifurcation point exhibits the behaviour of a turning point, as an equilibrium branch turns around, as well as the behaviour of a Hopf bifurcation, because a branch of periodic solutions is created at the bifurcation point. The discontinuous bifurcation has the structure:

$$\emptyset\xrightarrow{\text{multiple crossing bifurcation}}\left\{\begin{array}{l}\text{saddle}\\ \text{unstable focus}\\ \text{stable periodic solution}\end{array}\right.$$

The system is piecewise linear and has the generalized Jacobian matrix

$$\boldsymbol{J}(\boldsymbol{x}, \mu) = \begin{bmatrix} 0 & 1\\ -\mathrm{Sign}(x_1+4\mu) & -\mathrm{Sign}(x_2-\mu) - \frac{1}{2}\end{bmatrix},\tag{46}$$

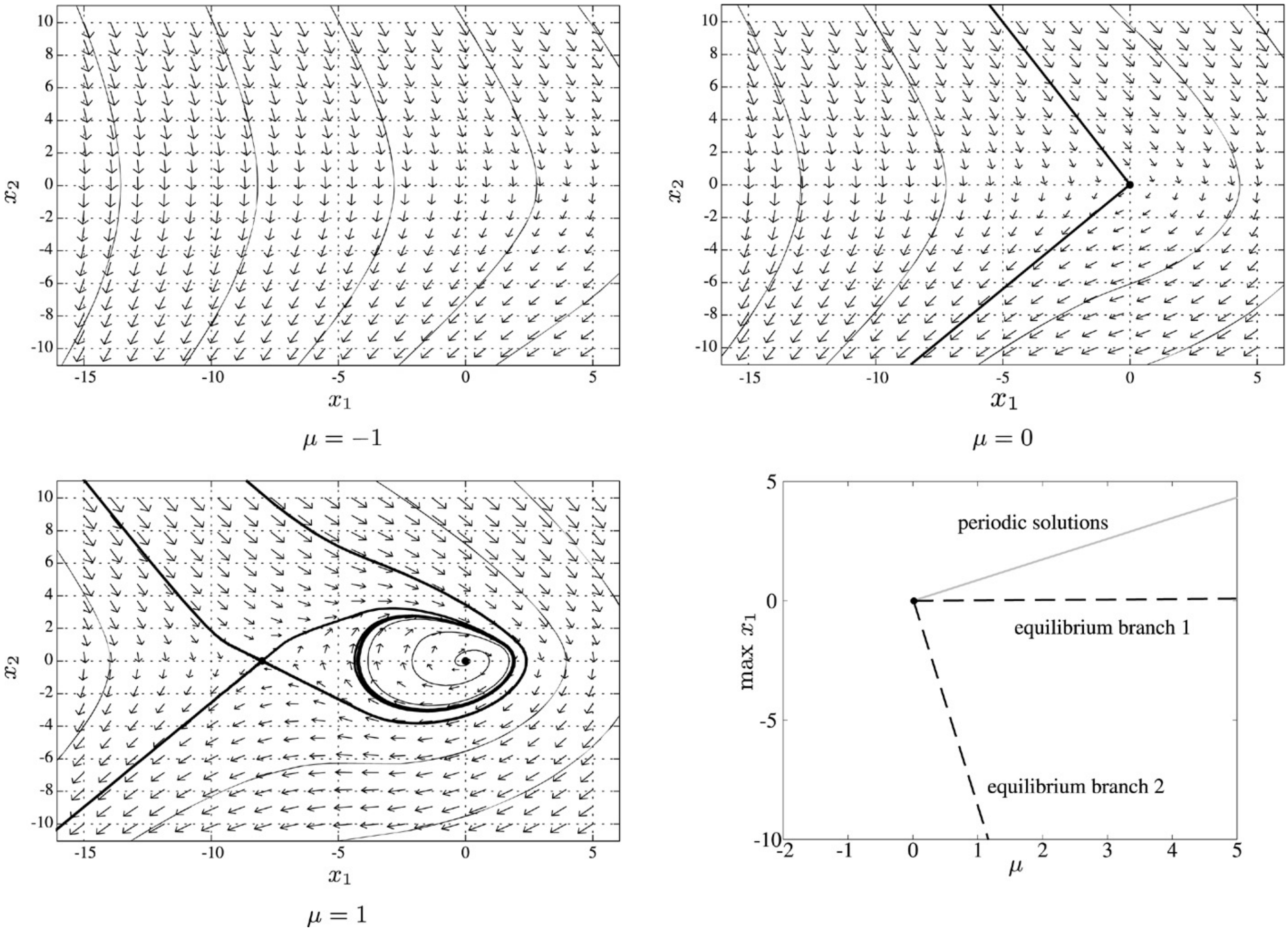


Fig. 9. Phase planes and bifurcation diagram of system (44).

which gives the set-valued Jacobian matrix at the bifurcation point

$$\boldsymbol{J}(\mathbf{0}, 0) = \{\boldsymbol{J}_q,\ q_1, q_2 \in [0, 1]\},$$

$$\text{where } \boldsymbol{J}_q = \begin{bmatrix} 0 & 1 \\ -2q_1 + 1 & -2q_2 + \frac{1}{2} \end{bmatrix}. \tag{47}$$

The fact that the bifurcation point is located on *two* switching boundaries makes the eigenvalues dependent on two auxiliary variables (q_1 and q_2). The set-valued eigenvalues of the set-valued Jacobian $\boldsymbol{J}(\mathbf{0}, 0)$ have been studied in Example 1. The set of eigenvalues of $\boldsymbol{J}(\mathbf{0}, 0)$ forms an area in the complex plane which intersects the imaginary axis as is depicted in Fig. 4. The discontinuous bifurcation of system (44) is therefore a multiple crossing bifurcation, showing the behaviour of both a Hopf bifurcation and a turning point bifurcation.

Example 6 (*Smooth Approximation of System* (44)). Consider the following smooth approximation of system (44):

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\frac{2}{\pi} \arctan\left(\varepsilon(x_1 + 4\mu)\right)(x_1 + 4\mu) \\ &\quad - \frac{2}{\pi} \arctan\left(\varepsilon(x_2 - \mu)\right)(x_2 - \mu) - \frac{1}{2}x_2 + 5\mu. \end{aligned} \tag{48}$$

The smooth system (48) will be studied in the neighbourhood of the discontinuous bifurcation point ($x_1 = 0, x_2 = 0$) at $\mu = 0$ of the non-smooth system (44) using Taylor series

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\epsilon(x_1 + 4\mu)^2 - \epsilon(x_2 - \mu)^2 - \frac{1}{2}x_2 + 5\mu, \end{aligned} \tag{49}$$

for $\epsilon|x_1 + 4\mu| \ll 1,\ \epsilon|x_2 - \mu| \ll 1$ with the abbreviation $\epsilon = \frac{2}{\pi}\varepsilon$. System (49) has for $0 < \mu < \frac{5}{\epsilon}$ two equilibria $x_1 = -4\mu \pm \sqrt{5\mu/\epsilon - \mu^2}$, $x_2 = 0$. Apparently, the smoothed system exhibits (for this particular smoothing) a turning point bifurcation at ($x_1 = 0, x_2 = 0$) with $\mu = 0$. The Jacobian matrix on the equilibrium branches of system (49) is

$$\boldsymbol{J} = \begin{bmatrix} 0 & 1 \\ \pm 2\sqrt{5\mu\epsilon - \mu^2\epsilon^2} & 2\mu\epsilon - \frac{1}{2} \end{bmatrix}, \tag{50}$$

with the characteristic equation

$$\lambda^2 - \left(2\mu\epsilon - \frac{1}{2}\right)\lambda \pm 2\sqrt{5\mu\epsilon - \mu^2\epsilon^2} = 0. \tag{51}$$

The eigenvalues cross the imaginary axis for two values of μ (only considering $\mu < \frac{5}{\epsilon}$)

$$\begin{aligned} &\mu = 0: \quad \lambda_1 = 0,\ \lambda_2 = -\frac{1}{2}, \quad \text{turning point bifurcation},\\ &\mu = \frac{1}{4\epsilon}: \quad \lambda_{1,2} = \pm \mathrm{i}\sqrt[4]{\frac{19}{4}}, \\ &\qquad \text{Hopf bifurcation on equilibrium branch 1.} \end{aligned} \tag{52}$$

The path of the eigenvalues gives rise to a turning point bifurcation at $\mu = 0$, causing the equilibrium branch to turn around, and a Hopf bifurcation at $\mu = \frac{1}{4\epsilon}$, creating a branch of periodic solutions. The Hopf bifurcation point approaches the turning point bifurcation for increasing values of ϵ. The logical structure of the bifurcation diagram of the smooth approximating system shows two continuous single crossing bifurcations. A turning point bifurcation creates a saddle and a stable node, which is transformed into a stable focus:

$$\emptyset \xrightarrow{\text{turning point bifurcation}} \begin{cases} \text{saddle} \\ \text{stable node} \xrightarrow[\text{transition}]{\text{node–focus}} \text{stable focus} \end{cases}$$

Subsequently, the stable focus undergoes a Hopf bifurcation:

$$\text{stable focus} \xrightarrow{\text{Hopf bifurcation}} \begin{cases} \text{unstable focus} \\ \text{stable periodic solution} \end{cases}$$

The discontinuous bifurcation point of the non-smooth system (44) exhibits the bifurcation behaviour of both a turning point bifurcation and a Hopf bifurcation. The branches of a periodic solution, a saddle point and a focus meet each other at the discontinuous bifurcation point. The particular smooth approximating system studied in this example shows a turning point bifurcation and a Hopf bifurcation. Other smooth approximating systems of (44) may have a different sequence of continuous bifurcations.

The systems considered in Examples 2 and 5 show multiple crossing bifurcations. The bifurcation points of systems (29) and (44) are located on more than one switching boundary. As a result, the generalized Jacobian matrix at the bifurcation point is dependent on multiple auxiliary variables and a (unique) 'path' of the eigenvalues cannot be obtained. A multiple crossing bifurcation can however also occur in a system with a single switching boundary as will be shown in the following example.

Example 7 (*Multiple Crossing Bifurcation With a Turning Point*). Consider the two-dimensional non-smooth continuous system

$$\begin{aligned} \dot{x}_1 &= x_1 + 2|x_1| + x_2, \\ \dot{x}_2 &= x_1 + 2|x_1| + \frac{1}{2}x_2 + \mu, \end{aligned} \tag{53}$$

which is piecewise linear and has a single switching boundary $\Sigma = \{\boldsymbol{x} \in \mathbb{R}^2 \mid x_1 = 0\}$. Note that system (53) is of the type (28) (after a simple coordinate transformation $y_1 = x_2$, $y_2 = x_1$). The system (53) has for $\mu < 0$ two distinct equilibria

$$\begin{aligned} &\text{equilibrium 1:} \quad x_1 = -\frac{2}{3}\mu, \quad x_2 = 2\mu, \\ &\text{equilibrium 2:} \quad x_1 = 2\mu, \quad x_2 = 2\mu, \end{aligned} \tag{54}$$

and has no equilibria for $\mu > 0$. The generalized Jacobian matrix of the system is

$$\boldsymbol{J}(x_1) = \begin{bmatrix} 1 + 2\,\mathrm{Sign}(x_1) & 1 \\ 1 + 2\,\mathrm{Sign}(x_1) & \frac{1}{2} \end{bmatrix}, \tag{55}$$

which takes a constant value at each side of the switching boundary

$$\begin{aligned} &\boldsymbol{J}_- = \begin{bmatrix} -1 & 1 \\ -1 & \frac{1}{2} \end{bmatrix} \quad \text{for } x_1 < 0, \\ &\lambda_{1,2} = -\frac{1}{4} \pm \mathrm{i}\sqrt{\frac{7}{16}}, \end{aligned} \tag{56}$$

$$\begin{aligned} &\boldsymbol{J}_+ = \begin{bmatrix} 3 & 1 \\ 3 & \frac{1}{2} \end{bmatrix} \quad \text{for } x_1 > 0, \\ &\lambda_{1,2} = \frac{7}{4} \pm \sqrt{\frac{73}{16}} \approx \{-0.386, 3.886\}. \end{aligned} \tag{57}$$

The generalized Jacobian at the bifurcation point $\boldsymbol{x} = \boldsymbol{0}$ is the closed convex hull of the Jacobians on each side of the switching boundary

$$\boldsymbol{J}(\boldsymbol{0}) = \overline{\mathrm{co}}(\boldsymbol{J}_-, \boldsymbol{J}_+) = \{(1-q)\boldsymbol{J}_- + q\boldsymbol{J}_+, \forall q \in [0, 1]\}. \tag{58}$$

The eigenvalues $\lambda_{1,2}$ of $\boldsymbol{J}(\boldsymbol{0})$ are set-valued and form a *path* in the complex plane with q as path parameter. The path of eigenvalues of the generalized Jacobian at $\boldsymbol{x} = \boldsymbol{0}$ is depicted in Fig. 10 (lower right corner). The eigenvalues of $\boldsymbol{J}_q$ are purely complex for $q = \frac{1}{8}$ and one eigenvalue crosses the origin for $q = \frac{1}{4}$. The path of the eigenvalues of $\boldsymbol{J}(\boldsymbol{0})$ shows that the discontinuous bifurcation point is a multiple crossing bifurcation. With the previous examples in mind, one might suggest that the behaviour of this multiple crossing bifurcation is the combination of two single crossing bifurcations, a Hopf bifurcation and a turning point bifurcation. Fig. 10 depicts the phase plane of (53) for $\mu = -1$, $\mu = 0$ and $\mu = 1$. The two equilibria (54) are present for $\mu = -1$, of which one equilibrium is a stable focus and the other equilibrium is a saddle point. The invariant manifolds of the saddle point show an interesting behaviour. An unstable invariant half-manifold of the saddle point is spiralling towards the stable focus while one of the stable invariant half-manifolds is folded to the other stable invariant half-manifold. The two equilibria collide to one equilibrium for $\mu = 0$ and only two invariant half-manifolds (stable and unstable) remain (see also Example 5). No equilibrium or periodic solution exists for $\mu = 1$. The multiple crossing bifurcation has the structure:

$$\left.\begin{matrix} \text{stable focus} \\ \text{saddle} \end{matrix}\right\} \xrightarrow{\text{multiple crossing bifurcation of (53)}} \emptyset$$

Clearly, the behaviour of a turning point bifurcation is present in the bifurcation scenario observed in Fig. 10 and Theorem 1 applies. The bifurcation scenario does not show a creation (or destruction) of a periodic solution under variation of μ, i.e. no behaviour of a Hopf bifurcation. The multiple crossing bifurcation is therefore not simply the combination of two single crossing bifurcations.

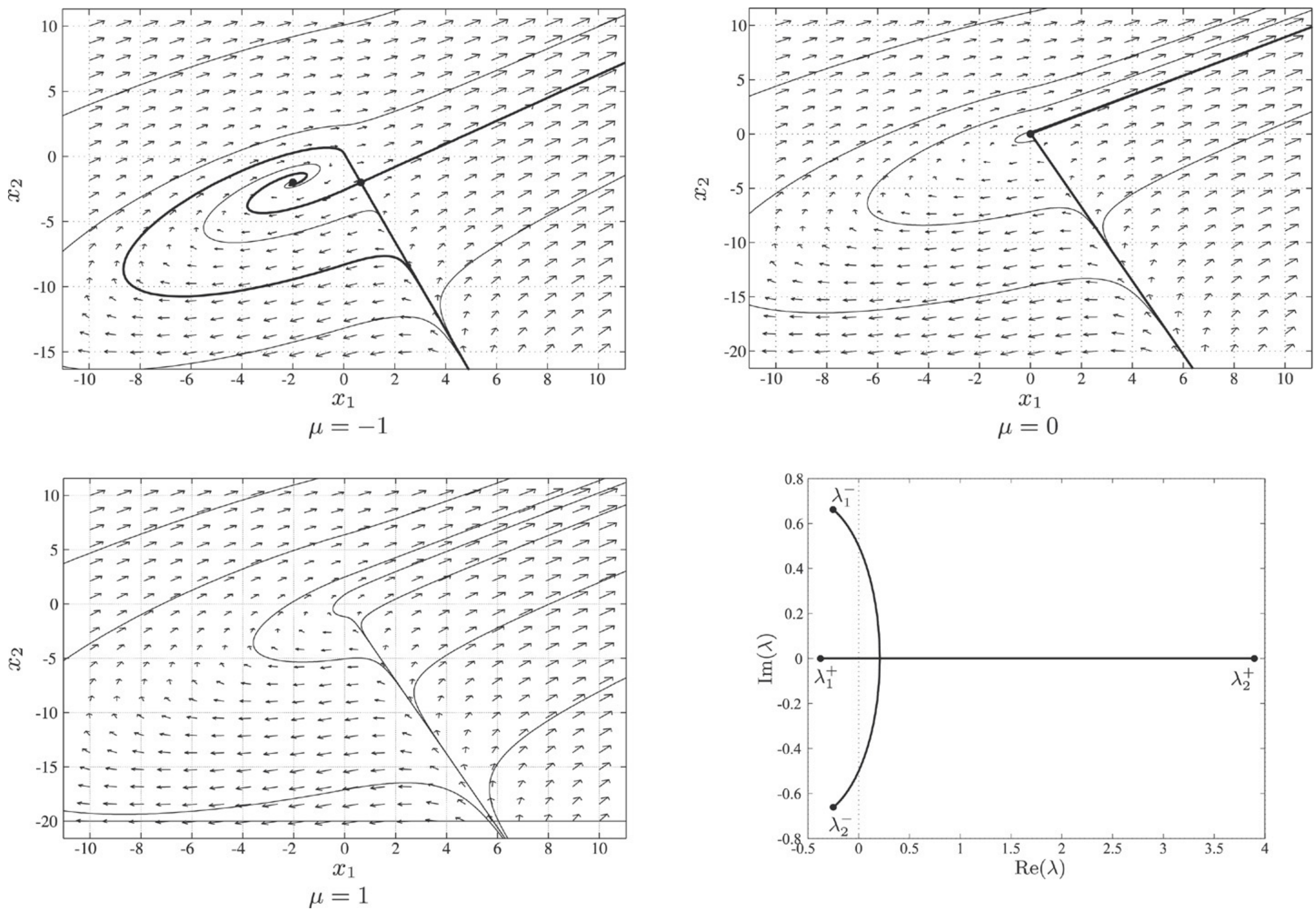


Fig. 10. Phase planes of system (53) and the path of eigenvalues of $\boldsymbol{J}(\boldsymbol{0})$ (58).

Example 8 (*Smooth Approximation of System* (53)). Some insight into the bifurcation behaviour depicted in Fig. 10 can be obtained by considering a smooth approximation of system (53)

$$\begin{aligned}\dot{x}_1 &= x_1 + \frac{4}{\pi}\arctan(\varepsilon x_1)x_1 + x_2,\\ \dot{x}_2 &= x_1 + \frac{4}{\pi}\arctan(\varepsilon x_1)x_1 + \frac{1}{2}x_2 + \mu.\end{aligned} \tag{59}$$

Of course, we have to keep in mind that (59) is just one particular smooth approximation of (53). Fig. 11 shows the phase planes of (59) for six different values of μ in the neighbourhood of $\mu = 0$ (using $\varepsilon = 20$). Two equilibria exist for $\mu = -0.005$ and the phase plane is very similar to the phase plane in Fig. 10 for $\mu = -1$. An unstable invariant half-manifold of the saddle point is spiralling towards the stable focus. A stable invariant half-manifold of the saddle point is very close to this spiralling unstable invariant half-manifold, turns around it and is folded towards the other stable invariant half-manifold. The stable and unstable invariant half-manifolds, already close for $\mu = -0.05$, collide for $\mu = -0.0013987$. The collision of the two invariant half-manifolds causes a homoclinic trajectory, i.e. a trajectory that connects an equilibrium point with itself. The homoclinic trajectory only exists for $\mu = -0.0013987$ and is immediately destroyed if μ is further increased. The destruction of the homoclinic trajectory causes the stable invariant half-manifold to spiral (in reverse time) around the equilibrium point, as can been seen in the phase plane for $\mu = 0$. The unstable invariant half-manifold is folded towards the other unstable invariant half-manifold of the saddle point. The behaviour of the stable and unstable invariant half-manifold is therefore inverted. This type of global bifurcation is called *homoclinic bifurcation*. The homoclinic bifurcation creates (or destroys) a periodic solution. The periodic solution can be seen in the phase plane for $\mu = 0$ and forms the boundary of the two-dimensional region of attraction of the stable focus. The periodic solution is therefore an unstable limit cycle. The structure of the bifurcation scenario of Fig. 11 (left side) is summarized in the following diagram:

$$\left.\begin{array}{l}\text{stable focus}\\ \\ \text{saddle}\end{array}\right\}\xrightarrow[\text{bifurcation}]{\text{homoclinic}}\begin{array}{l}\text{stable focus}\\ \text{unstable limit cycle}\\ \text{saddle}\end{array}$$

A further increase of the parameter μ diminishes the size of the periodic solution and a Hopf bifurcation occurs at $\mu = 0.002556$ (Fig. 11, right side). The equilibrium, which is a stable focus for $\mu < 0.002556$, becomes an unstable focus after the Hopf bifurcation and turns into an unstable node after a focus–node transition. Finally, the two equilibria, being a saddle

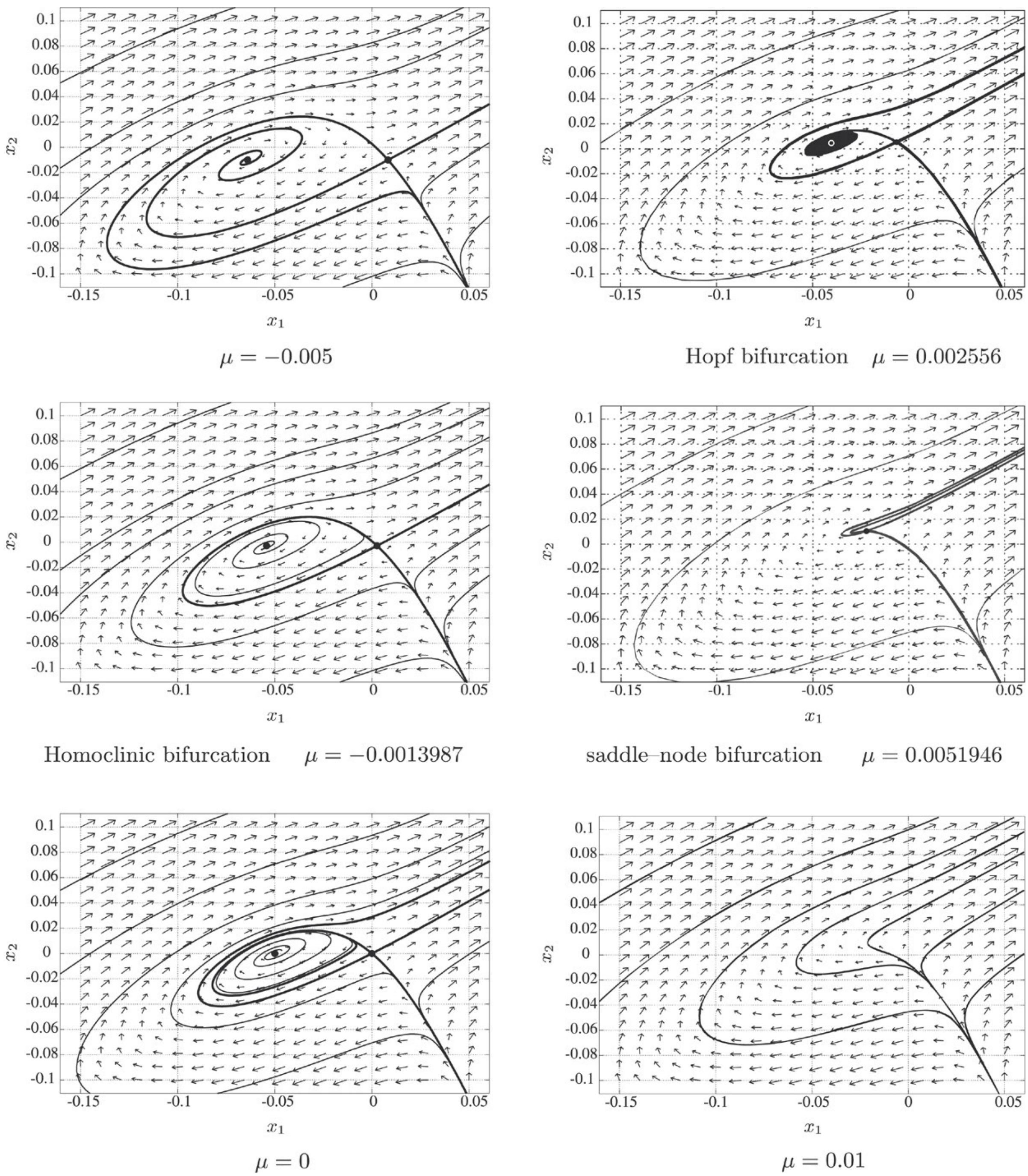


Fig. 11. Phase planes of system (59), $\varepsilon = 20$.

point and a node, collide and a turning point bifurcation takes place for $\mu = 0.0051946$. No equilibrium or periodic solution is present in the phase plane for $\mu = 0.01$. The structure of the bifurcation scenario of Fig. 11 (right side) continues the bifurcation scenario of Fig. 11 (left side). A Hopf bifurcation destroys a limit cycle and transforms a stable focus into an unstable focus, which coexists with a saddle:

$$\left.\begin{array}{l}\text{stable focus}\\ \text{unstable limit cycle}\end{array}\right\} \xrightarrow[]{\text{Hopf bifurcation}} \text{unstable focus}$$
$$\text{saddle} \qquad\qquad\qquad\qquad \text{saddle}$$

Subsequently, the unstable focus is transformed into an unstable node and is destroyed together with the coexisting saddle by a turning point bifurcation:

$$\left.\begin{array}{lcl}\text{unstable focus} & \xrightarrow{\text{focus–node transition}} & \text{unstable node}\\ \text{saddle} & & \text{saddle}\end{array}\right\} \xrightarrow{\text{turning point bifurcation}} \emptyset$$

The smoothing of the non-smooth terms causes the eigenvalues to be a single-valued function of the parameter μ. The multiple crossing bifurcation of (53) is therefore, for this particular choice of the smoothing function, torn apart in two single crossing bifurcations (a Hopf bifurcation and a turning point bifurcation) and a global bifurcation (a homoclinic bifurcation). The bifurcation structure of the non-smooth continuous system shows only one discontinuous multiple crossing bifurcation:

$$\left.\begin{array}{l}\text{stable focus}\\ \text{saddle}\end{array}\right\} \xrightarrow[\text{bifurcation}]{\text{multiple crossing}} \emptyset$$

which replaces the complex structure of the smooth approximating system. Again, the question arises of how to name this particular multiple crossing bifurcation. The multiple crossing bifurcation is basically a discontinuous turning point bifurcation for which Theorem 1 applies. However, the behaviour of the discontinuous bifurcation is much more complex than the behaviour of just a turning point, which is reflected by the complex structure of single crossing bifurcations of the smooth approximating system (59). It has become clear that the problem of terminology is becoming extremely difficult when studying more complex bifurcations.

9. Concluding remarks

The current paper makes a clear distinction between single and multiple crossing bifurcations of equilibria in non-smooth continuous systems. Single crossing bifurcations were analyzed in [12,14]. For each of the classical continuous bifurcations (turning point, transcritical and Hopf bifurcation) a discontinuous single crossing bifurcation as the non-smooth counterpart has been found. A continuous bifurcation is in fact a special case of a single crossing bifurcation, for which the set-valuedness of the eigenvalues reduces to a singleton. In the current paper a number of discontinuous multiple crossing bifurcations have been discussed in detail and have been shown to behave in a much more complex way than single crossing bifurcations.

Multiple crossing bifurcations *can* show the behaviour of the combination of two (or more) continuous bifurcations. For instance, Example 2 of Section 8 shows a discontinuous bifurcation which can be looked upon as a combined Hopf and pitchfork bifurcation. However, multiple crossing bifurcations can also be much more complex and have a qualitative behaviour that is not just the combination of two continuous bifurcations. The bifurcation diagram might still show a very classical bifurcation phenomenon, but a smooth system which locally approximates the non-smooth system reveals the underlying complex bifurcation structure. The multiple crossing bifurcation encountered in Example 7, for instance, shows a bifurcation behaviour similar to that of a turning point. A smooth approximating system (Example 8) unfolds the multiple crossing bifurcation into a homoclinic bifurcation, a Hopf bifurcation and a turning point bifurcation. This complex structure of bifurcations in the smooth unfolding could not have been anticipated by looking at the multiple crossing bifurcation. Different smooth approximating systems might of course show different unfoldings of the multiple crossing bifurcations. We can therefore not use the unfolding to classify a multiple crossing bifurcation. Still, the crossing points of the eigenvalues with the imaginary axis *may* give a hint as to the behaviour of the multiple crossing bifurcation.

Although the classification of multiple crossing bifurcations is still a major problem, it seems intuitively correct for *single* crossing bifurcations that we can make a classification based on the crossing of the eigenvalue(s) with the imaginary axis. We do not have a rigorous proof that we can make this classification of single crossing bifurcations for general non-smooth continuous systems. The subclass of planar systems of the form (22) are an exception for which we are able to prove the turning point bifurcation Theorem 1 and the Hopf bifurcation Theorem 2, which are novel results. These results prove that, at least for this class of systems, the type of bifurcation is determined by the set-valued path of the eigenvalues at the bifurcation point.

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