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**ORIGINAL**

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# Experimental and theoretical investigation of the energy dissipation of a rolling disk during its final stage of motion

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**Abstract** This paper is concerned with the dominant dissipation mechanism for a rolling disk in the final stage of its motion. The aim of this paper is to present the various dissipation mechanisms for a rolling disk which are used in the literature in a unified framework. Furthermore, new experiments on the 'Euler disk' using a high-speed video camera and a novel image analysis technique are presented. The combined experimental/theoretical approach of this paper sheds some more light on the dominant dissipation mechanism on the time-scale of several seconds.

**Keywords** Euler disk · Spinning · Finite-time singularity · Rolling friction · Non-smooth dynamics

## 1 Introduction

If a coin is spun on a table, then we observe a peculiar kind of motion. After a brief initial phase, the coin wobbles/spins while remaining on more or less the same spot. Very slowly the coin loses height. This motion is accompanied by a ringing noise of which the frequency is rapidly increasing and tends to infinity before the motion and sound abruptly stop. This phenomenon is exemplified by the 'Euler disk', a scientific toy consisting of a heavy metal disk on a slightly concave mirror.

The abrupt halt of a spinning disk is often called the 'finite-time singularity' in literature [8,18]. There exists a tremendous amount of literature on the dynamics of the rolling disk, e.g. [1–8,11–13,17–24] but this list is far from complete. Here, we will only give an overview of the literature on dissipation mechanisms which explain the finite-time singularity and of the literature reporting measurements of this phenomenon.

In a brief article of *Nature*, Moffatt [18] proposed a dissipation mechanism due to viscous drag of the layer of air between the disk and the table. Moffatt showed that, according to this dissipation model, the inclination $\theta(t)$ and precession rate $\dot{\alpha}(t)$ of the disk vary with time according to the power-law

$$\theta(t) \propto (t_f - t)^n, \quad \dot{\alpha} \propto (t_f - t)^{-\frac{1}{2}n}, \tag{1}$$

with the exponent $n = \frac{1}{3}$. The viscous air drag model of Moffatt was extended by Bildsten [3] to account for boundary layer effects which are expected to occur for larger values of the inclination angle. The derivations of Bildsten reveal an exponent of $n = \frac{4}{9}$. Observations of spinning coins in vacuum led van den Engh et al. [23] to suppose that air viscosity is not the dominant dissipation mechanism during the final stage of motion. Moffatt [19] replies that air viscosity is rather insensitive to the pressure and, therefore, that these observations

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are inconclusive. Moreover, he points out that air drag has a smaller value of $n$ than other dissipation mechanisms and will therefore finally dominate. The article of Moffatt led to an increased interest in the finite-time singularity of the rolling disk and opened the scientific discussion on the responsible dissipation mechanism.

McDonald and McDonald [17] present a dissipation mechanism for rolling friction for which $n = \frac{1}{2}$. Furthermore, the precession rate of a rolling disk is determined experimentally using a flashlight and a phototransistor (5 kS/s) during 10 s. The experimental results of [17] agree well with $n = \frac{1}{2}$.

Stanislavsky and Weron [22] recorded the sound of a rolling disk and analysed the change in the spectrum of the sound between the various stages of motion. No definite conclusions can be drawn from these measurements.

Kessler and O'Reilly [11] study the dynamics of a rolling disk under the influence of sliding, rolling and pivoting dissipation. The sliding friction model in [11] has a static and a dynamic friction coefficient and the numerical results therefore show stick-slip-like behaviour. The numerical simulations show an asymptotic energy decrease, i.e. the disk does not stop in finite time.

Easwar et al. [8] report measurements of the precession rate with a high-speed video camera but do not discuss the details of their measurement technique. The experimental results of [8] agree well with $n = \frac{2}{3}$ which the authors attribute to rolling friction.

Petrie et al. [21] conducted measurements of the 'Euler disk' using a normal video camera (30 fps) during 140 s. A strip with markers was glued on top of the disk and the top view of the motion of the disk was recorded. The precession rate and angular velocity around the axis of symmetry where retrieved from image analysis. The inclination angle $\theta$ was determined from the variation of the apparent length of the strip, which resulted in a large experimental error for the inclination angle $\theta$. In [21] it is concluded that the disk rolls without slip during the first 90 s. The measurements are inconclusive for the last 50 s because of the low frame rate.

Caps et al. [5] present a rather detailed experimental study of various rolling disks using a high-speed video camera (125–500 fps) and a laser beam during about 10 s. The inclination angle, precession rate and angular velocity around the axis of symmetry of the disk are each measured with a different experimental setup during a different run and, therefore, have not been obtained simultaneously. The experimental results agree with values of $n$ between $\frac{1}{2}$ and $\frac{2}{3}$. Measurements on a torus are believed in [5] to confirm the supposition of van den Engh et al. [23] that air drag is only of minor importance.

In Le Saux et al. [13] and Leine et al. [15], being previous papers of the author, a detailed numerical study has been carried out of a rolling disk under the influence of combined sliding, rolling and pivoting friction. The presented modelling technique includes impact and stick–slip transitions and is able to numerically simulate the transition of the disk from motion to rest and onwards, i.e. the finite-time singularity is within the simulation time-interval.

From the above literature overview we can draw a number of conclusions. Apparently, the general opinion in the scientific community is tending to believe that rolling friction is the dominant dissipation mechanism during the final stage of motion. At this point we have to ask ourselves on which time-scale the final stage of motion is considered. The viscous air drag dissipation might (for highly polished surfaces) be dominant during the last milliseconds, whereas rolling friction can be dominant if we consider the final stage of motion on the time-scale of seconds. The current state-of-the-art experimental results of [5] are only partially satisfactory. The inclination angle, precession rate and angular velocity around the axis of symmetry of the disk are measured, but not simultaneously. Some analytical work exists on the exponent $n$ for various dissipation models, but the results are scattered over the literature and are presented in different notation.

The aim of this paper is twofold. Firstly, the various existing dissipation mechanisms are discussed in a unified framework. This allows for a better comparison of the dissipation mechanisms. Secondly, new experiments on the 'Euler disk' are presented in this paper. The experiments have been conducted with a high-speed video camera (1,000 fps) during 10 s. An image analysis technique is presented with which the inclination angle $\theta$ and precession rate $\dot{\alpha}$ are obtained simultaneously. The combined experimental/theoretical approach of this paper gives more insight into the dominant dissipation mechanism on the time-scale of several seconds.

The paper is organised as follows. The equations of motion of a rolling disk are reviewed in Sect. 2. A theoretical analysis of the dissipation-free dynamics of the disk is given in Sect. 3 and it is shown that the dissipation-free dynamics has a manifold of stationary states for which the inclination remains constant. The stability of these stationary states is analysed using the method of Lyapunov functions by exploiting the integrable structure of the system. Subsequently, all dissipation mechanisms for the rolling disk, which are used in the literature, are discussed in Sect. 4. The effect of these dissipation mechanisms on the dynamics of the rolling disk is discussed in Sect. 5. The exponent of the power-law (1) is determined for each dissipation mechanism and an overview of the energy decay for the various dissipation mechanisms is given. The experimental setup and experimental results are presented in Sect. 6. Finally, conclusions are given in Sect. 7 and a discussion of the results of this paper in comparison to the results of the existing literature is given.

## 2 Rolling disk model

In this section, we give a model for a thick rolling disk under the assumption of pure rolling, i.e. rolling without slip, see also [2,20,22]. The rigid-body kinematics of a rolling disk are presented in Sect. 2.1 and the equations of motion are briefly derived in Sect. 2.2. Finally, the contact forces are discussed in Sect. 2.3.

### 2.1 Kinematics

The kinematical model, presented here, describes the mechanical system under consideration as a thick disk submitted to a bilateral geometric constraint at the contact point $C$ (Fig. 1a).

An absolute coordinate frame $I = (O, \mathbf{e}_x^I, \mathbf{e}_y^I, \mathbf{e}_z^I)$ is attached to the table. We introduce the frame $R = (O, \mathbf{e}_x^R, \mathbf{e}_y^R, \mathbf{e}_z^R)$ which is obtained by rotating the frame $I$ over an angle $\alpha$ around $\mathbf{e}_z^I$, i.e. $\mathbf{e}_x^R = \cos\alpha\,\mathbf{e}_x^I + \sin\alpha\,\mathbf{e}_y^I$, $\mathbf{e}_y^R = -\sin\alpha\,\mathbf{e}_x^I + \cos\alpha\,\mathbf{e}_y^I$ and $\mathbf{e}_z^R = \mathbf{e}_z^I$. Furthermore, we introduce the frame $K = (B, \mathbf{e}_x^K, \mathbf{e}_y^K, \mathbf{e}_z^K)$ which is obtained by rotating the frame $R$ over an angle $\beta$ around $\mathbf{e}_x^R$, i.e. $\mathbf{e}_x^K = \mathbf{e}_x^R$, $\mathbf{e}_y^K = \cos\beta\,\mathbf{e}_y^R + \sin\beta\,\mathbf{e}_z^R$ and $\mathbf{e}_z^K = -\sin\beta\,\mathbf{e}_y^R + \cos\beta\,\mathbf{e}_z^R$. Note that frame $K$ is not body-fixed, but moves along with the disk such that $\mathbf{e}_y^K$ is the axis of revolution and the $\mathbf{e}_x^K$-axis remains parallel to the table. The components of a vector $\boldsymbol{r}$ in frame $I$ are expressed as ${}_I\boldsymbol{r}$.

We consider a disk with an (outer) radius $r_0$ and height $2h$ (Fig. 1b). The disk's bottom surface, with which the disk is in contact with the table, has a rounded edge. The contact point $C$ between the disk and the table therefore runs for small inclination on a tread with constant radius $r$ being slightly smaller than $r_0$. The geometrical centre of the bottom surface is denoted with $B$. The centre of mass $S$ is located on the $\mathbf{e}_y^K$ axis at a distance $h$ from $B$. The disk has mass $m$ and the principal moments of inertia $I_1 = I_2 = \frac{1}{4}mr_0^2 + \frac{1}{12}mh^2$ and $I_3 = \frac{1}{2}mr_0^2$ with respect to the centre of mass $S$. The inertia tensor in frame $K$ therefore reads as ${}_K\boldsymbol{\Theta}_S = \mathrm{diag}(I_1, I_3, I_1)$. The gravitational acceleration is $g$ in the negative $\mathbf{e}_z^I$ direction. We define a parametrisation of the disk $(x,\ y,\ \alpha,\ \beta,\ \gamma)$, as illustrated in Fig. 1a, which is a minimal set of coordinates with respect to the geometric constraint at the contact point $C$. In this section, we derive the equations of motion using the coordinates $(x,\ y,\ \alpha,\ \beta,\ \gamma)$ and the angular velocity vector ${}_K\boldsymbol{\Omega} = \left[{}_K\omega_x\ {}_K\omega_y\ {}_K\omega_z\right]^{\mathrm{T}}$ expressed in frame $K$. We will often write the components ${}_K\omega_x$, ${}_K\omega_y$ and ${}_K\omega_z$ as $\omega_x$, $\omega_y$ and $\omega_z$ and omit the subscript $K$ unless confusion exists.

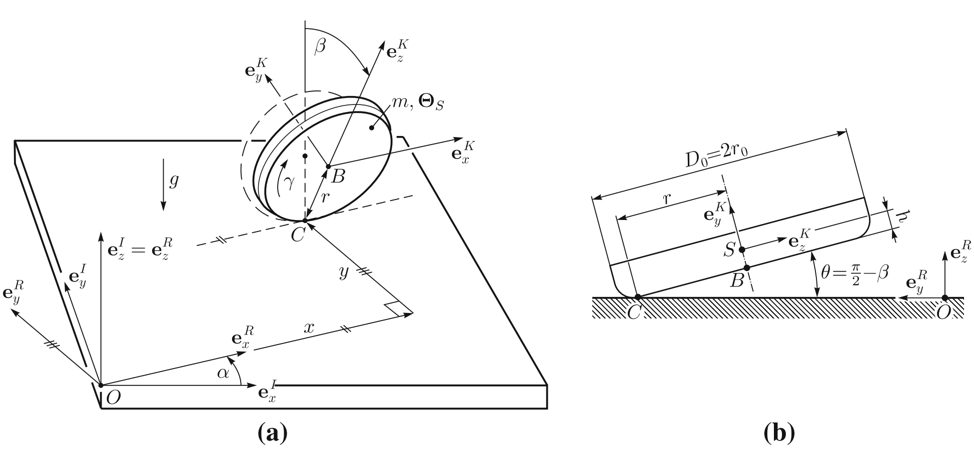



**Fig. 1** Rolling disk model: **a** parameterisation and **b** dimensions

First, we derive the angular velocity ${}_K\boldsymbol{\Omega}$ and relate it to the derivatives of the rotational coordinates $(\alpha,\ \beta,\ \gamma)$:

$$ {}_K\boldsymbol{\Omega} = \dot{\alpha}\ {}_K\mathbf{e}_z^R + \dot{\beta}\ {}_K\mathbf{e}_x^K + \dot{\gamma}\ {}_K\mathbf{e}_y^K = \begin{bmatrix} \dot{\beta} \\ \dot{\alpha}\sin\beta + \dot{\gamma} \\ \dot{\alpha}\cos\beta \end{bmatrix}. \tag{2} $$

Equating the components of ${}_K\boldsymbol{\Omega}$ gives the expressions $\dot{\beta} = \omega_x$, $\dot{\alpha} = \omega_z \sec\beta$ and $\dot{\gamma} = \omega_y - \omega_z \tan\beta$. The rotational velocity of frame $K$ with respect to the inertial frame $I$ can therefore be expressed in frame $K$ as

$$ {}_K\boldsymbol{\omega}_{IK} = \dot{\alpha}\ {}_K\mathbf{e}_z^R + \dot{\beta}\ {}_K\mathbf{e}_x^K = \begin{bmatrix} \omega_x \\ \omega_z\tan\beta \\ \omega_z \end{bmatrix}. \tag{3} $$

Similarly, we obtain the rotational velocity of frame $R$ with respect to frame $I$ expressed in frame $R$ as ${}_R\boldsymbol{\omega}_{IR} = \dot{\alpha}\ {}_R\mathbf{e}_z^R = \omega_z \sec\beta\ {}_R\mathbf{e}_z^R$. The angular acceleration $\boldsymbol{\Psi}$ is obtained from $\boldsymbol{\Omega}$ by using Euler's differentiation rule

$$ {}_K\boldsymbol{\Psi} = {}_K\dot{\boldsymbol{\Omega}} + {}_K\boldsymbol{\omega}_{IK} \times {}_K\boldsymbol{\Omega} = \begin{bmatrix} \dot{\omega}_x - \omega_y\omega_z + \omega_z^2\tan\beta \\ \dot{\omega}_y \\ \dot{\omega}_z - \omega_x\omega_z\tan\beta + \omega_x\omega_y \end{bmatrix}. \tag{4} $$

The point $A$ is a body-fixed point which is momentarily located at the contact point $C$ and, therefore, momentarily has the coordinates ${}_R\boldsymbol{r}_{OA} = {}_R\boldsymbol{r}_{OC} = \begin{bmatrix} x & y & 0 \end{bmatrix}^{\mathrm{T}}$. The velocity of the body-fixed point $A$, denoted by $\boldsymbol{v}_A$, momentarily vanishes if pure rolling is assumed. However, the vanishing of the velocity $\boldsymbol{v}_A = \mathbf{0}$ does not imply a vanishing of the acceleration of point $A$, i.e. $\boldsymbol{a}_A \neq \mathbf{0}$. The point $A$ is therefore not a fixed point with respect to the inertial frame $I$. Using the distance vector ${}_K\boldsymbol{r}_{AS} = \begin{bmatrix} 0 & h & r \end{bmatrix}^{\mathrm{T}}$, the position of the centre of mass $S$ can be found to be

$$ {}_K\boldsymbol{r}_{OS} = {}_K\boldsymbol{r}_{OA} + {}_K\boldsymbol{r}_{AS} = \begin{bmatrix} x \\ h + y\cos\beta \\ r - y\sin\beta \end{bmatrix}, \quad {}_R\boldsymbol{r}_{OS} = \begin{bmatrix} x \\ y + h\cos\beta - r\sin\beta \\ h\sin\beta + r\cos\beta \end{bmatrix}, \tag{5} $$

expressed in frame $K$ and $R$, respectively. We calculate the velocity $\boldsymbol{v}_S$ of the centre of mass from (5) using Euler's differentiation rule:

$$ {}_R\boldsymbol{v}_S = {}_R\dot{\boldsymbol{r}}_{OS} + {}_R\boldsymbol{\omega}_{IR} \times {}_R\boldsymbol{r}_{OS} = \begin{bmatrix} \dot{x} - \omega_z\sec\beta(y + h\cos\beta - r\sin\beta) \\ \dot{y} - h\omega_x\sin\beta - r\omega_x\cos\beta + \omega_z\sec\beta\, x \\ h\omega_x\cos\beta - r\omega_x\sin\beta \end{bmatrix}. \tag{6} $$

Subsequently, we calculate the velocity of the point $A$ by using the rigid-body equation $\boldsymbol{v}_A = \boldsymbol{v}_S + \boldsymbol{\Omega} \times \boldsymbol{r}_{SA}$. The velocity of the point $A$ has the form $\boldsymbol{v}_A = \gamma_{Tx}\,\mathbf{e}_x^R + \gamma_{Ty}\,\mathbf{e}_y^R$ with

$$ \gamma_{Tx} = \dot{x} - \omega_z\sec\beta(y - r\sin\beta) - \omega_y r, \tag{7} $$

$$ \gamma_{Ty} = \dot{y} + \omega_z\sec\beta\, x, \tag{8} $$

being the relative sliding velocities of the contact point in $\mathbf{e}_x^R$ and $\mathbf{e}_y^R$ direction respectively. The pure rolling condition leads to the two velocity constraints $\gamma_{Tx} = 0$ and $\gamma_{Ty} = 0$. With these constraints the velocity $\boldsymbol{v}_S$ becomes

$$ {}_K\boldsymbol{v}_S = {}_K\boldsymbol{\Omega} \times {}_K\boldsymbol{r}_{AS} = \begin{bmatrix} r\omega_y - h\omega_z \\ -r\omega_x \\ h\omega_x \end{bmatrix}, \tag{9} $$

from which we obtain the acceleration $\boldsymbol{a}_S$ of the centre of mass $S$

$$ {}_K\boldsymbol{a}_S = {}_K\dot{\boldsymbol{v}}_S + {}_K\boldsymbol{\omega}_{IK} \times {}_K\boldsymbol{v}_S = \begin{bmatrix} r\dot{\omega}_y - h\dot{\omega}_z + \omega_x\omega_z(r + \tan\beta) \\ -r\dot{\omega}_x + \omega_z(r\omega_y - h\omega_z) - h\omega_x^2 \\ h\dot{\omega}_x - \omega_z\tan\beta(r\omega_y - h\omega_z) - r\omega_x^2 \end{bmatrix}. \tag{10} $$

The velocity of the contact point $C$ over the table equals ${}_R\boldsymbol{v}_C = {}_R\dot{\boldsymbol{r}}_{OC} + {}_R\boldsymbol{\omega}_{IR} \times {}_R\boldsymbol{r}_{OC} = r(\omega_y - \omega_z \tan\beta)\, {}_R\mathbf{e}_x^R$. The velocity of the point $C$ over the rim of the disk equals (under the assumption of pure rolling) the velocity in $\mathbf{e}_x^R$ direction of the point $C$ over the table, i.e.

$$\gamma_{\text{cont}} = r(\omega_y - \omega_z \tan\beta). \tag{11}$$

## 2.2 Equations of motion

Let $\boldsymbol{p} = m\boldsymbol{a}_S$ denote the linear momentum of the system. The only fixed point in the system, which can be used to set up the angular momentum, is the origin $O$. The angular momentum with respect to $O$, denoted by $\boldsymbol{L}_O$, can be expressed as

$$\begin{aligned} \boldsymbol{L}_O &= \boldsymbol{r}_{OS} \times \boldsymbol{p} + \boldsymbol{\Theta}_S \boldsymbol{\Omega} \\ &= \boldsymbol{r}_{OA} \times \boldsymbol{p} + \boldsymbol{r}_{AS} \times \boldsymbol{p} + \boldsymbol{\Theta}_S \boldsymbol{\Omega}. \end{aligned} \tag{12}$$

The dynamics of the system is governed by the balance of linear and angular momentum

$$\begin{aligned} \dot{\boldsymbol{p}} &= \boldsymbol{F}, \\ \dot{\boldsymbol{L}}_O &= \boldsymbol{M}_O, \end{aligned} \tag{13}$$

where $\boldsymbol{F}$ is the applied force and $\boldsymbol{M}_O$ is the applied moment with respect to the origin $O$. The unknown contact forces, which impose the constraints, contribute to $\boldsymbol{F}$ and $\boldsymbol{M}_O$. These constraint forces do not contribute to the applied moment $\boldsymbol{M}_A$ with respect to the contact point $A = C$. The latter can be decomposed into a gravitational moment $\boldsymbol{M}_A^{\text{grav}} = mgr \sin\beta\, \mathbf{e}_x^K$ and a moment $\boldsymbol{M}^{\text{diss}}$ due to some kind of dissipation, i.e. $\boldsymbol{M}_A = \boldsymbol{M}_A^{\text{grav}} + \boldsymbol{M}^{\text{diss}}$. In order to free the equations from the unknown contact forces we set up the applied moment around the contact point $A$

$$\boldsymbol{M}_A = \boldsymbol{M}_O + \boldsymbol{r}_{AO} \times \boldsymbol{F} = \dot{\boldsymbol{L}}_O + \boldsymbol{r}_{AO} \times \dot{\boldsymbol{p}}. \tag{14}$$

Differentiation of (12) gives

$$\begin{aligned} \dot{\boldsymbol{L}}_O &= \dot{\boldsymbol{r}}_{OA} \times \boldsymbol{p} + \boldsymbol{r}_{OA} \times \dot{\boldsymbol{p}} + \dot{\boldsymbol{r}}_{AS} \times \boldsymbol{p} + \boldsymbol{r}_{AS} \times \dot{\boldsymbol{p}} + \boldsymbol{\Theta}_S \boldsymbol{\Psi} + \boldsymbol{\Omega} \times \boldsymbol{\Theta}_S \boldsymbol{\Omega} \\ &= m\boldsymbol{v}_A \times \boldsymbol{v}_S + \boldsymbol{r}_{OA} \times \dot{\boldsymbol{p}} + m(\boldsymbol{v}_S - \boldsymbol{v}_A) \times \boldsymbol{v}_S + m\boldsymbol{r}_{AS} \times \boldsymbol{a}_S + \boldsymbol{\Theta}_S \boldsymbol{\Psi} + \boldsymbol{\Omega} \times \boldsymbol{\Theta}_S \boldsymbol{\Omega} \\ &= \boldsymbol{r}_{OA} \times \dot{\boldsymbol{p}} + m\boldsymbol{r}_{AS} \times \boldsymbol{a}_S + \boldsymbol{\Theta}_S \boldsymbol{\Psi} + \boldsymbol{\Omega} \times \boldsymbol{\Theta}_S \boldsymbol{\Omega}. \end{aligned} \tag{15}$$

Substitution of (15) into (14) gives the equation

$$m\boldsymbol{r}_{AS} \times \boldsymbol{a}_S + \boldsymbol{\Theta}_S \boldsymbol{\Psi} + \boldsymbol{\Omega} \times \boldsymbol{\Theta}_S \boldsymbol{\Omega} = \boldsymbol{M}_A, \tag{16}$$

from which we obtain three equations of motion for the three components of the angular velocity vector ${}_K\boldsymbol{\Omega}$

$$\left(k_1 + 1 + \epsilon^2\right) \dot{\omega}_x - (k_2 + 1 + \epsilon \tan\beta)\, \omega_y \omega_z + \left((k_1 + \epsilon^2) \tan\beta + \epsilon\right) \omega_z^2 = \tilde{g}(\sin\beta - \epsilon \cos\beta) + f_x^{\text{diss}}, \tag{17}$$

$$(k_2 + 1)\dot{\omega}_y - \epsilon \dot{\omega}_z + (1 + \epsilon \tan\beta)\omega_x \omega_z = f_y^{\text{diss}}, \tag{18}$$

$$(k_1 + \epsilon^2)\dot{\omega}_z - \epsilon \dot{\omega}_y - \left((k_1 + \epsilon^2) \tan\beta + \epsilon\right) \omega_x \omega_z + k_2 \omega_x \omega_y = f_z^{\text{diss}}, \tag{19}$$

with the constants

$$k_1 = \frac{I_1}{mr^2}, \quad k_2 = \frac{I_3}{mr^2}, \quad \epsilon = \frac{h}{r}, \quad \tilde{g} = \frac{g}{r}, \tag{20}$$

and the generalised forces

$$f_x^{\text{diss}} = \frac{1}{mr^2}\, {}_K M_x^{\text{diss}}, \quad f_y^{\text{diss}} = \frac{1}{mr^2}\, {}_K M_y^{\text{diss}}, \quad f_z^{\text{diss}} = \frac{1}{mr^2}\, {}_K M_z^{\text{diss}}. \tag{21}$$

These equations agree for the dissipation-free case ($\boldsymbol{M}^{\text{diss}} = \mathbf{0}$) with those of [1,2] and for an infinitely thin disk ($\epsilon = 0$) with those of [20,22].

The kinetic energy in the system is given by

$$\begin{aligned} E_{\text{kin}} &= \frac{1}{2} m_K \boldsymbol{v}_S^{\mathrm{T}} {}_K \boldsymbol{v}_S + \frac{1}{2} {}_K \boldsymbol{\Omega}^{\mathrm{T}} {}_K \boldsymbol{\Theta}_{S\,K} \boldsymbol{\Omega} \\ &= \frac{1}{2}\left(m(r\omega_y - h\omega_z)^2 + m(r^2 + h^2)\omega_x^2 + I_1\omega_x^2 + I_3\omega_y^2 + I_1\omega_z^2\right) \\ &= \frac{1}{2} m r^2 \left((\omega_y - \epsilon\omega_z)^2 + (1+\epsilon^2)\omega_x^2 + k_1\omega_x^2 + k_2\omega_y^2 + k_1\omega_z^2\right). \end{aligned} \tag{22}$$

The potential energy of the system is only due to gravity:

$$E_{\text{pot}} = mg(h\sin\beta + r\cos\beta - h) = mr^2\tilde{g}(\epsilon\sin\beta + \cos\beta - \epsilon). \tag{23}$$

In the absence of dissipation it holds that $E = E_{\text{kin}} + E_{\text{pot}} = \text{const}$.

### 2.3 Contact forces

In the derivation of the equations of motion (17)–(19) the disk is assumed to fulfill the geometric constraint $g_N = z_S - r\cos\beta = 0$, i.e. the disk is in contact with the table, as well as the pure rolling conditions $\gamma_{Tx} = 0$ and $\gamma_{Ty} = 0$. These constraints are induced by the normal contact force $\lambda_N$ and tangential forces $\lambda_{Tx}$ and $\lambda_{Ty}$ in $\mathbf{e}_x^R$ and $\mathbf{e}_y^R$ direction respectively. The constraint forces can be found from the balance of linear momentum

$$m\, {}_R\boldsymbol{a}_S = \begin{bmatrix} \lambda_{Tx} \\ \lambda_{Ty} \\ \lambda_N - mg \end{bmatrix}. \tag{24}$$

The tangential constraint forces $\lambda_{Tx}$ and $\lambda_{Ty}$ are due to Coulomb friction between the disk and the table. Hence, if the pure rolling conditions are assumed to hold, then the constraint forces $\lambda_{Tx}$ and $\lambda_{Ty}$ have to fulfill the Coulomb sticking condition

$$\sqrt{\lambda_{Tx}^2 + \lambda_{Ty}^2} < \mu\lambda_N, \tag{25}$$

where $\mu$ is the sliding friction coefficient.

## 3 Theoretical analysis for the dissipation-free case

The dynamics of the disk in the absence of dissipation is not only of importance in its own right, but also largely determines the dynamics when the dissipation is small. The three equations of motion (17)–(19) with $\boldsymbol{M}^{\text{diss}} = \mathbf{0}$ form together with $\dot{\beta} = \omega_x$ a four-dimensional autonomous set of differential equations:

$$\dot{\beta} = \omega_x, \tag{26}$$

$$\left(k_1 + 1 + \epsilon^2\right)\dot{\omega}_x - (k_2 + 1 + \epsilon\tan\beta)\,\omega_y\omega_z + \left((k_1 + \epsilon^2)\tan\beta + \epsilon\right)\omega_z^2 = \tilde{g}(\sin\beta - \epsilon\cos\beta), \tag{27}$$

$$(k_2 + 1)\dot{\omega}_y - \epsilon\dot{\omega}_z + (1 + \epsilon\tan\beta)\omega_x\omega_z = 0, \tag{28}$$

$$(k_1 + \epsilon^2)\dot{\omega}_z - \epsilon\dot{\omega}_y - \left((k_1 + \epsilon^2)\tan\beta + \epsilon\right)\omega_x\omega_z + k_2\omega_x\omega_y = 0. \tag{29}$$

The equilibria of these differential equations are studied in Sect. 3.1 and their stability is addressed in Sect. 3.2.

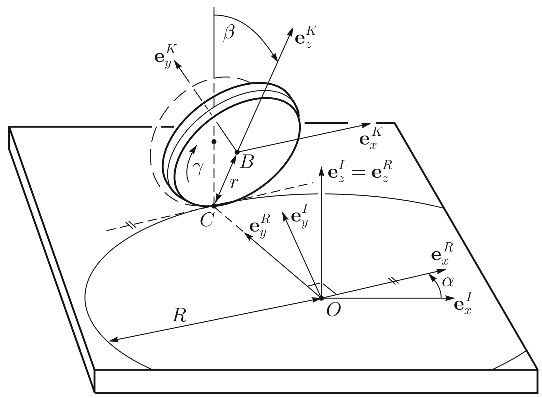



**Fig. 2** Circular rolling motion

### 3.1 Circular rolling motion

In this section, we analyse a particular type of rolling motion in the absence of dissipation. We consider the type of motion $(x_0(t), y_0(t), \alpha_0(t), \beta_0(t), \gamma_0(t))$ for which $x_0 = 0$ and $\beta_0 =$ const. in time $(0 < \beta_0 < \frac{\pi}{2})$. It follows that $\omega_{x0} = 0$ and from (28) and (29) that $\omega_{y0}$ = const. and $\omega_{z0}$ = const. Furthermore, the sticking constraint $\gamma_{Ty} = 0$ with (8) yields $\dot{y}_0 + \omega_{z0} \sec\beta_0\, x_0 = 0$ from which follows with $x_0 = 0$ that $\dot{y}_0 = 0$ and therefore $y_0 = R =$ const. Similarly, the constraint $\gamma_{Tx} = 0$ with (7) gives $\dot{x}_0 - \omega_{z0} \sec\beta(y_0 - r\sin\beta_0) - \omega_{y0} r = 0$, or, using $\dot{x}_0 = 0$ and $y_0 = R$,

$$\omega_{y0} = \omega_{z0} \sec\beta_0 \left(\sin\beta_0 - \frac{R}{r}\right). \tag{30}$$

Equation (30) is the condition for pure rolling, which means that, for a given time-interval of motion, the arc lengths covered by the contact point $C$ on both the perimeter of the circle $(O, R)$ and the perimeter of the disk are equal. During such a motion, the inclination of the disk $\beta_0$ with respect to the vertical $\mathbf{e}_z^I$ and the height of the centre of mass are constant in time. As the contact point $C$ moves on the contour of the disk, it describes on the table a circular trajectory $(O, R)$ of radius $R$ around the origin $O$ of the inertial frame (Fig. 2). In the following we refer to such a type of motion as *circular rolling motion*. A kind of gyroscopic balancing occurs during circular rolling motion. Substitution of $\dot{\omega}_{x0} = 0$ in (27) gives

$$-(k_2 + 1 + \epsilon\tan\beta_0)\,\omega_{y0}\omega_{z0} + \big((k_1 + \epsilon^2)\tan\beta_0 + \epsilon\big)\,\omega_{z0}^2 = \tilde{g}(\sin\beta_0 - \epsilon\cos\beta_0), \tag{31}$$

or by using (30)

$$\omega_{z0}^2 = \frac{\tilde{g}(\sin\beta_0 - \epsilon\cos\beta_0)}{(k_1 + \epsilon^2)\tan\beta_0 + \epsilon - (k_2 + 1 + \epsilon\tan\beta_0)\sec\beta_0\left(\sin\beta_0 - \frac{R}{r}\right)}, \tag{32}$$

which is the balance between the gyroscopic moment and the gravitational moment. We see from (32) that gyroscopic balancing can only occur if the denominator in (32) is positive and if $\epsilon = \frac{h}{r} < \tan\beta_0$. Furthermore, the friction forces $\lambda_{Tx}$ and $\lambda_{Ty}$, introduced in Sect. 2.3, have to fulfill the Coulomb sticking condition (25). The friction forces for circular rolling motion are $\lambda_{Tx} = 0$ and $\lambda_{Ty} = mr\omega_{z0}(\omega_{y0} - \epsilon\omega_{z0})/\cos^2\beta_0$. The four-dimensional system (26)-(29) has embedded in its four-dimensional state-space a two-dimensional manifold $\boldsymbol{q} = (\beta, \omega_x, \omega_y, \omega_z) \in \mathcal{M}$ with boundary, where

$$\mathcal{M} = \left\{\boldsymbol{q} \in \mathbb{R}^4 \mid \omega_x = 0, \omega_z^2 = \frac{(k_2 + 1 + \epsilon\tan\beta)\,\omega_y\omega_z + \tilde{g}(\sin\beta - \epsilon\cos\beta)}{(k_1 + \epsilon^2)\tan\beta + \epsilon}, |\lambda_{Ty}| < \mu m g\right\}. \tag{33}$$

Each point $(\beta_0, \omega_{x0}, \omega_{y0}, \omega_{z0}) \in \mathcal{M}$ is, in the absence of dissipation, an equilibrium of the four-dimensional system (26)–(29) and is what we named a circular rolling motion. As $\mathcal{M}$ consists of equilibria, it is (in the absence of dissipation) an invariant manifold.

Subsequently, we study a particular type of circular rolling motion for which, as the disk is rolling on the table, the centre of mass $S$ remains on the axis $(O, \mathbf{e}_z^I)$ and is therefore immobile with respect to the inertial frame. We call this type of motion *stationary rolling motion*, being characterised by (see (9))

$$_K\boldsymbol{v}_S = \mathbf{0} \quad \Rightarrow \quad r\omega_{y0} - h\omega_{z0} = 0 \quad \Rightarrow \quad \omega_{y0} = \epsilon\omega_{z0}. \tag{34}$$

The gyroscopic balance equation (32) can be written for stationary rolling motion as

$$\omega_{z0}^2 = \frac{\tilde{g}(\sin\beta_0 - \epsilon\cos\beta_0)}{k_1\tan\beta_0 - \epsilon k_2}. \tag{35}$$

The velocity of the contact point $\gamma_{\text{cont}} = r(\omega_{y0} - \omega_{z0}\tan\beta_0)$, given by (11), yields for stationary rolling motion $\gamma_{\text{cont}} = r(\epsilon - \tan\beta_0)\omega_{z0}$. In the limit of $\beta_0 \uparrow \frac{\pi}{2}$ it holds that $\omega_{z0}^2 \to 0$ and $\gamma_{\text{cont}}^2 \to \infty$. The contact point $C$ therefore moves infinitely fast on the circle $(O, R)$ with radius $R \to r$, and moves infinitely fast on the contour of the disk, while the disk does practically not rotate. Stationary rolling motion is a one-dimensional invariant sub-manifold $\mathcal{S} \subset \mathcal{M}$:

$$\mathcal{S} = \left\{(\beta, \omega_x, \omega_y, \omega_z) \in \mathbb{R}^4 \mid \omega_x = 0, \omega_y = \epsilon\omega_z, \omega_z^2 = \frac{\tilde{g}(\sin\beta - \epsilon\cos\beta)}{k_1\tan\beta - \epsilon k_2}\right\}. \tag{36}$$

The friction forces $\lambda_{Tx}$ and $\lambda_{Ty}$, introduced in Section 2.3, vanish for stationary rolling motion because the centre of mass $S$ does not accelerate for this kind of motion ($\boldsymbol{v}_S = \boldsymbol{a}_S = \mathbf{0}$).

### 3.2 Lyapunov stability analysis of circular rolling motion

The four-dimensional state space, described by (26)–(29), has an integrable structure. The integrability of the equations of motion of a disk rolling without slip on a rough horizontal surface (no dissipation) was first studied by Chaplygin [7], Appell [1] and Korteweg [12], see [20] for a short overview. The closed form solutions for a rolling disk without dissipation has been used by [2,4,20] to study the bifurcations of the stationary motions. Here, we will use the integrability result to study the stability of the stationary motions of the disk using a Lyapunov function.

We will use the notation $(\cdot)' = \mathrm{d}(\cdot)/\mathrm{d}\beta$. The prime derivatives are related to the time-derivatives through

$$\omega_y' = \frac{\mathrm{d}\omega_y}{\mathrm{d}\beta} = \frac{\dot{\omega}_y}{\omega_x}, \quad \omega_z' = \frac{\mathrm{d}\omega_z}{\mathrm{d}\beta} = \frac{\dot{\omega}_z}{\omega_x}. \tag{37}$$

Following [20], the differential equations (28) and (29) are divided by $\omega_x$ and yield a set of differential equations in $\beta$ for $\omega_y$ and $\omega_z$:

$$(k_2 + 1)\omega_y' - \epsilon\omega_z' + (1 + \epsilon\tan\beta)\omega_z = 0, \tag{38}$$

$$(k_1 + \epsilon^2)\omega_z' - \epsilon\omega_y' - \left((k_1 + \epsilon^2)\tan\beta + \epsilon\right)\omega_z + k_2\omega_y = 0. \tag{39}$$

Equations (38) and (39) can be combined in a second-order differential equation for $\omega_z(\beta)$:

$$\omega_z'' - \tan\beta\,\omega_z' - \left(\frac{1}{\cos^2\beta} + \frac{k_2(1 + \epsilon\tan\beta)}{k_1(k_2 + 1) + k_2\epsilon^2}\right)\omega_z = 0. \tag{40}$$

The parameters $\omega_y(t_0)$ and $\omega_z(t_0)$ define the initial conditions $\omega_y(\beta(t_0))$ and $\omega_z(\beta(t_0))$ and the values of $\omega_y$ and $\omega_z$ are therefore completely determined by the value of $\beta$. Consequently, we can write $\omega_y = \omega_y(\beta)$ and $\omega_z = \omega_z(\beta)$ as they are functions of $\beta$. The four-dimensional state space therefore reduces to a two-parameter family of second-order systems for $\beta(t)$ [20] and the equation of motion (27) for $\dot{\omega}_x = \ddot{\beta}$ yields

$$\left(k_1 + 1 + \epsilon^2\right)\ddot{\beta} - (k_2 + 1 + \epsilon\tan\beta)\,\omega_y\omega_z + \left((k_1 + \epsilon^2)\tan\beta + \epsilon\right)\omega_z^2 = \tilde{g}(\sin\beta - \epsilon\cos\beta), \tag{41}$$

with $\omega_y = \omega_y(\beta)$ and $\omega_z = \omega_z(\beta)$. We rewrite this autonomous second-order differential equation for $\beta$ as

$$\left(k_1 + 1 + \epsilon^2\right)\ddot{\beta} + \frac{\partial U}{\partial\beta} = 0, \tag{42}$$

using the potential function

$$U(\beta) = \frac{1}{2}\left((\omega_y - \epsilon\omega_z)^2 + k_2\omega_y^2 + k_1\omega_z^2\right) + \tilde{g}(\epsilon \sin\beta + \cos\beta - \epsilon). \tag{43}$$

The second-order system (41) has equilibria $\beta = \beta_0$ which have to fulfill $U'(\beta_0) = \partial U/\partial\beta|_{\beta=\beta_0} = 0$, and which are circular rolling motions of the four-dimensional system (26)–(29). The stability of these equilibria can be studied with a Lyapunov function

$$V(\bar{\beta}, \dot{\bar{\beta}}) = \frac{1}{2}\left(k_1 + 1 + \epsilon^2\right)\dot{\bar{\beta}}^2 + U(\beta_0 + \bar{\beta}) - U(\beta_0), \tag{44}$$

with $\bar{\beta} = \beta - \beta_0$ being the difference between $\beta$ and the equilibrium position $\beta_0$. The Lyapunov function $V$ equals the scaled total energy $E = E_{\text{kin}} + E_{\text{pot}}$, given by (22) and (23), shifted with the constant value $U(\beta_0)$, i.e. $V = \frac{1}{mr^2}E - U(\beta_0)$. Hence, the value of $V$ does not change along solution curves of the system because $\dot{V} = 0$. The potential $U(\beta_0 + \bar{\beta})$ allows for a Taylor series expansion around $\beta_0$

$$U(\beta_0 + \bar{\beta}) = U(\beta_0) + U'(\beta_0)\bar{\beta} + \frac{1}{2}U''(\beta_0)\bar{\beta}^2 + \mathcal{O}(\bar{\beta}^3), \tag{45}$$

in which the first-order term vanishes due to the equilibrium condition $U'(\beta_0) = 0$. The Lyapunov function $V$ can therefore be approximated around the origin by

$$V = \frac{1}{2}\left(k_1 + 1 + \epsilon^2\right)\dot{\bar{\beta}}^2 + \frac{1}{2}U''(\beta_0)\bar{\beta}^2 + \mathcal{O}(\bar{\beta}^3). \tag{46}$$

Hence, the Lyapunov function $V$ is locally positive definite if $U''(\beta_0) > 0$ and the equilibrium position $\beta_0$ is therefore Lyapunov stable if $U''(\beta_0) > 0$ is fulfilled. For small $\bar{\beta}$ it holds that

$$\left(k_1 + 1 + \epsilon^2\right)\ddot{\bar{\beta}} + U''(\beta_0)\bar{\beta} = 0, \tag{47}$$

from which we see that the disk swings for small amplitudes with a nutational frequency

$$\omega_{\text{nutation}}^2 = \frac{U''(\beta_0)}{k_1 + 1 + \epsilon^2}. \tag{48}$$

The second derivative of the potential $U$ can tediously be obtained by solving $\omega_y'$ and $\omega_z'$ from (38) and (39) and substitution into

$$\begin{aligned} U''(\beta_0) = &-\left(\omega_z'(\omega_y - \epsilon\omega_z) + \omega_z(\omega_y' - \epsilon\omega_z')\right)(1 + \epsilon\tan\beta) + 2k_1\omega_z\omega_z'\tan\beta \\ &+ \left(k_1\omega_z^2 - \epsilon\omega_z(\omega_y - \epsilon\omega_z)\right)\sec^2\beta - k_2(\omega_y\omega_z' + \omega_y'\omega_z) - \tilde{g}(\epsilon\sin\beta + \cos\beta), \end{aligned} \tag{49}$$

where $\beta = \beta_0$, $\omega_y = \omega_{y0}$ and $\omega_z = \omega_{z0}$. The derivation greatly simplifies for $\epsilon = 0$, i.e. if the disk is infinitely thin, and the result is

$$U''(\beta_0) = \left(k_1(1 + 3\tan^2\beta_0) + 1\right)\omega_{z0}^2 - (3k_2 + 1)\tan\beta_0\omega_{y0}\omega_{z0} + \frac{k_2}{k_1}(k_2 + 1)\omega_{y0}^2 - \tilde{g}\cos\beta_0. \tag{50}$$

If in addition stationary rolling motion is assumed, then we obtain using (35) with $\epsilon = 0$ and $\omega_{y0} = 0$

$$U''(\beta_0) = \left(k_1(1 + 3\tan^2\beta_0) + 1\right)\omega_{z0}^2 - \tilde{g}\cos\beta_0 = \left(3\tan^2\beta_0 + \frac{1}{k_1}\right)\tilde{g}\cos\beta_0, \tag{51}$$

which is positive $(0 < \beta < \pi/2)$. Consequently, stationary rolling motion is stable for an infinitely thin disk and has a nutational frequency (48) given by

$$\omega_{\text{nutation}}^2 = \frac{\left(3k_1\tan^2\beta_0 + 1\right)\tilde{g}\cos\beta_0}{k_1(k_1 + 1)}. \tag{52}$$

## 4 Dissipation mechanisms

In this section, we discuss a number of dissipation mechanisms of a rolling disk. First, we discuss two types of rolling friction and pivoting friction (drilling friction). Subsequently, we pay some attention to sliding friction of the disk over the table. Finally, viscous air drag models are addressed. For the formulation of dry friction laws as set-valued force-laws (i.e. inclusions) we refer to [9,16].

### 4.1 Classical rolling friction

Bodies in contact can experience a resistance against rolling over each other. At this point we have to ask ourselves what we exactly mean when we say that bodies 'roll' over each other [16]. We may call 'rolling' the movement of the contact point over the surface of one of the bodies (here already lies some ambiguity). A resistance against such a type of movement will be called contour friction and is discussed in Sect. 4.2. Usually, the term rolling is associated with resistance against a difference in angular velocity components of the contacting bodies which are tangential to the contact plane [10]. This will be called classical rolling friction. Contour friction and classical rolling friction may be identical to each other or be essentially different, depending on the type of system. For instance, if a planar wheel rolling over a flat table is considered, then the two types of rolling friction yield the same kind of dissipation mechanism, because the velocity of the contact point over the contour of the wheel is directly proportional to the angular velocity of the wheel. However, the two types of rolling friction are essentially different if we consider a three-dimensional disk rolling on a table.

The classical rolling friction law, applied to the rolling disk, describes a frictional moment in the horizontal plane of the table as a function of the projection of the angular velocity on the horizontal plane. The angular velocity $\mathbf{\Omega}$ of the disk has the components ${}_R\omega_x = \omega_x$ and ${}_R\omega_y = \omega_y \cos\beta - \omega_z \sin\beta$ around the $\mathbf{e}_x^R$ and $\mathbf{e}_y^R$ axes. For the motion of a rolling disk we can assume that the frictional moment is much smaller around the $\mathbf{e}_x^R$ axis than around the $\mathbf{e}_y^R$ axis. A dry classical rolling friction law (for the $\mathbf{e}_y^R$ axis) therefore reads as

$$M_{\text{roll}} \in -\mu_{\text{roll}} \lambda_N r \,\text{Sign}(\omega_{\text{roll}}), \tag{53}$$

with the friction coefficient $\mu_{\text{roll}}$ and the rolling angular velocity $\omega_{\text{roll}} = \mathbf{\Omega} \cdot \mathbf{e}_y^R = {}_R\omega_y = \omega_y \cos\beta - \omega_z \sin\beta$. Similarly, we can consider a viscous classical rolling friction model, described by $M_{\text{roll}} = -c_{\text{roll}}\omega_{\text{roll}}$. The classical rolling friction moment $M_{\text{roll}}$ induces a generalised moment $\boldsymbol{M}^{\text{diss}} = M_{\text{roll}} \mathbf{e}_y^R$ in the equations of motion (17)–(19).

### 4.2 Contour friction

Contour friction is a resisting moment against the movement of the contact point $C$ over the rim of the disk [13,16]. We prefer to consider a contour angular velocity $\omega_{\text{cont}} = \frac{\gamma_{\text{cont}}}{r}$. A dry contour friction law therefore reads as

$$M_{\text{cont}} \in -\mu_{\text{cont}} \lambda_N r \,\text{Sign}(\omega_{\text{cont}}), \tag{54}$$

where $\mu_{\text{cont}}$ is a dimensionless friction coefficient. Similarly, we can consider a viscous contour friction model, described by

$$M_{\text{cont}} = -c_{\text{cont}}\omega_{\text{cont}}. \tag{55}$$

The contour friction moment $M_{\text{cont}}$ induces a virtual power $\delta\omega_{\text{cont}} M_{\text{cont}} = (\delta\omega_y - \delta\omega_z \tan\beta) M_{\text{cont}}$ which equals the virtual power $\delta\omega_{x\,K} M_x^{\text{diss}} + \delta\omega_{y\,K} M_y^{\text{diss}} + \delta\omega_{z\,K} M_z^{\text{diss}}$ of the generalised forces. Considering arbitrary variations $\delta\omega_x$, $\delta\omega_y$ and $\delta\omega_z$ we conclude that the generalised moment due to contour friction reads as $\boldsymbol{M}^{\text{diss}} = M_{\text{cont}} \mathbf{e}_y^K - M_{\text{cont}} \tan\beta\, \mathbf{e}_z^K$, i.e.

$${}_K\boldsymbol{M}^{\text{diss}} = \begin{bmatrix} 0 \\ M_{\text{cont}} \\ -M_{\text{cont}} \tan\beta \end{bmatrix}. \tag{56}$$

### 4.3 Pivoting friction

Pivoting friction [14] is a frictional moment which resists a pivoting angular velocity $\omega_{\text{pivot}}$ of the disk around the contact point $C$. Pivoting friction for the rolling disk has been studied in [13]. If the pivoting angular velocity is large, then a coupling with sliding friction can exist. This coupling is modelled by the Coulomb-Contensou friction law [16], which is not of importance in this context and will not be considered here. A dry pivoting friction law reads as

$$M_{\text{pivot}} \in -\mu_{\text{pivot}}\lambda_N r \,\text{Sign}(\omega_{\text{pivot}}), \tag{57}$$

with the pivoting velocity $\omega_{\text{pivot}} = {}_I\omega_z = \sin\beta\omega_y + \cos\beta\omega_z$. Similarly, we can consider a viscous pivoting friction model $M_{\text{pivot}} = -c_{\text{pivot}}\omega_{\text{pivot}}$. The pivoting friction moment $M_{\text{pivot}}$ induces a generalised moment $\boldsymbol{M}^{\text{diss}} = M_{\text{pivot}}\mathbf{e}_z^I = M_{\text{pivot}} \sin\beta\, \mathbf{e}_y^K + M_{\text{pivot}} \cos\beta\, \mathbf{e}_z^K$.

### 4.4 Sliding friction

The equations of motion (17)–(19) have been derived under the assumption that the disk purely rolls over the table ($\gamma_{Tx} = \gamma_{Ty} = 0$), i.e. there is no sliding in the $\mathbf{e}_x^R$ and $\mathbf{e}_y^R$ direction of the contact point. The dissipation due to a resistance against sliding of the contact point over the table, which is called radial slippage in [17], can therefore not be studied with the equations of motion (17)–(19). A detailed numerical model of a rolling disk which also includes sliding friction has been presented in [13]. In order to study the effect of sliding friction analytically one would have to consider the equations of motion of a 'sliding disk', as have been discussed in [20], together with friction forces $\lambda_{Tx}$ and $\lambda_{Ty}$ [2] and a Coulomb or viscous friction law. However, the friction forces $\lambda_{Tx}$ and $\lambda_{Ty}$ vanish for stationary rolling motion. It is therefore concluded in [17] that sliding friction is not able to dissipate energy if the disk is in a state of stationary rolling motion.

### 4.5 Viscous air drag

Moffatt [18] proposes a dissipation mechanism due to viscous drag of the layer of air between the disk and the table. During the final stage of motion the inclination $\theta(t) = \pi/2 - \beta(t)$ is very small and the air is squeezed between the almost parallel surfaces of the disk and table. The maximal gap between the disk and the table has a height being proportional to $\sin\theta \approx \theta$. Moffatt assumes that the horizontal velocity of the air $u_H$ is proportional to the precession speed $\dot{\alpha}$. Furthermore, assuming a no-slip condition for the layer of air on the table and disk, he deduces that the layer of air has a shear proportional to $\frac{\partial u_H}{\partial z} \propto \dot{\alpha}/\theta$. Hence, assuming linear viscosity of the air, the disk experiences a moment $M_{\text{drag}} = -c_{\text{drag}}\dot{\alpha}/\theta$ around the $\mathbf{e}_z^I$ axis. The coefficient $c_{\text{drag}}$ depends on the viscosity of the air and the radius of the disk. The viscous air drag model induces a generalised moment $\boldsymbol{M}^{\text{diss}} = M_{\text{drag}}\mathbf{e}_z^I$.

The viscous air drag model of Moffatt has been extended by Bildsten [3] to account for boundary layer effects which occur for larger values of the inclination angle. Bildsten [3] argues that the viscous dissipation does not extend over the whole gap for larger values of the inclination $\theta$ but only occurs in boundary layers on the disk and table. The width of these boundary layers is proportional to $\delta \propto \dot{\alpha}^{-\frac{1}{2}}$ and the shear is therefore proportional to $M_{\text{drag}} \propto \frac{\partial u_H}{\partial z} \propto \dot{\alpha}/\delta \propto \dot{\alpha}^{\frac{3}{2}}$.

## 5 The finite-time singularity

The dissipation mechanisms presented in Sect. 4 lead to a monotonous decay of the energy and therefore ultimately to a decay of the inclination angle $\theta(t)$. The question of interest now is: which of these dissipation mechanisms predicts a finite-time singularity, or, in other words, an abrupt halt of the motion in a finite time? Furthermore, we would like to know how the time-history of $\theta(t)$ looks like for each of these dissipation mechanisms, e.g. the power-law relationship (1) with a certain exponent $n$. The most natural approach perhaps is to study the equations of motion for each of these dissipation mechanisms. This approach is followed in Sect. 5.1 for dry contour friction. A quicker approach, although less insightful, is to study the dissipative power of the dissipation mechanism as a function of the energy. The latter approach is followed in Sect. 5.2.

### 5.1 Analysis of the equations of motion with dry contour friction

In this section we study the equations of motion (17)–(19) for dry contour friction, i.e. the generalised moment $\boldsymbol{M}^{\text{diss}}$ is given by (56) with $M_{\text{cont}} = \mu_{\text{cont}}\lambda_N$ for $\omega_{\text{cont}} < 0$. The normal contact force $\lambda_N$ is during the final stage of motion approximately equal to the weight $mg$ (at least on a time-scale of several seconds). For small angles of $\theta = \frac{\pi}{2} - \beta$ it holds that $\theta \approx \sin\theta = \cos\beta$, $\sin\beta = \cos\theta \approx 1$ and $\tan\beta = \tan^{-1}\theta \approx \theta^{-1}$. Neglecting terms of order $\mathcal{O}(\theta^2)$, the equations of motion (17)–(19) can be written as

$$-\left(k_1 + 1 + \epsilon^2\right)\ddot{\theta} - \left(k_2 + 1 + \epsilon\theta^{-1}\right)\omega_y\omega_z + \left((k_1 + \epsilon^2)\theta^{-1} + \epsilon\right)\omega_z^2 = \tilde{g}(1 - \epsilon\theta), \tag{58}$$

$$(k_2 + 1)\dot{\omega}_y - \epsilon\dot{\omega}_z - (1 + \epsilon\theta^{-1})\dot{\theta}\omega_z = \mu_{\text{cont}}\tilde{g}, \tag{59}$$

$$(k_1 + \epsilon^2)\dot{\omega}_z - \epsilon\dot{\omega}_y + \left(k_1\theta^{-1} + \epsilon(1 + \epsilon\theta^{-1})\right)\dot{\theta}\omega_z - k_2\dot{\theta}\omega_y = -\mu_{\text{cont}}\tilde{g}\theta^{-1}. \tag{60}$$

Furthermore, we assume that $\omega_y$ is of order $\mathcal{O}(\epsilon\omega_z)$, which holds if the disk is approximately in a state of stationary rolling motion. Multiplying with $\theta$ and neglecting terms of order $\mathcal{O}(\theta^2)$, $\mathcal{O}(\epsilon\theta)$ and $\mathcal{O}(\epsilon^2)$ we can write the equations of motion (58) and (60) as

$$-(k_1 + 1)\ddot{\theta}\theta + k_1\omega_z^2 = \tilde{g}\theta, \tag{61}$$

$$k_1\dot{\omega}_z\theta + k_1\dot{\theta}\omega_z = -\mu_{\text{cont}}\tilde{g}, \tag{62}$$

whereas Eq. (59), which describes the $\omega_y$ dynamics, becomes obsolete. Using the chain rule $\frac{\mathrm{d}}{\mathrm{d}t}(\omega_z\theta) = \dot{\omega}_z\theta + \dot{\theta}\omega_z$, Eq. (62) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}(\omega_z\theta) = -\frac{1}{k_1}\mu_{\text{cont}}\tilde{g}. \tag{63}$$

Under the above assumptions, it therefore holds that

$$\omega_z(t)\,\theta(t) = \omega_z(t_0)\,\theta(t_0) - \frac{1}{k_1}\mu_{\text{cont}}\tilde{g}\,(t - t_0), \tag{64}$$

which reveals that there exists a final time $t_f$ such that $\omega(t_f)\,\theta(t_f) = 0$ with

$$t_f = t_0 + \frac{k_1\omega_z(t_0)\,\theta(t_0)}{\mu_{\text{cont}}\tilde{g}}. \tag{65}$$

It is convenient to define an inverse time $\tau = t_f - t$ for which $\omega_z\theta = \frac{1}{k_1}\mu_{\text{cont}}\tilde{g}\tau$, i.e. the finite-time singularity occurs at $\tau = 0$ while $\tau > 0$ denote time instants $t$ prior to $t = t_f$. Using $\frac{\mathrm{d}}{\mathrm{d}\tau}(\cdot) = -\frac{\mathrm{d}}{\mathrm{d}t}(\cdot)$ we arrive at a nonlinear non-autonomous second-order differential equation for $\theta(\tau)$:

$$(k_1 + 1)\frac{\mathrm{d}^2\theta}{\mathrm{d}\tau^2}\theta^3 = \frac{\mu_{\text{cont}}^2\tilde{g}^2}{k_1}\tau^2 - \tilde{g}\theta^3. \tag{66}$$

The right-hand side in (66) vanishes if $\theta(\tau) = \theta_{\text{balance}}(\tau)$ with

$$\theta_{\text{balance}}(\tau) = \left(\frac{\mu_{\text{cont}}^2\tilde{g}}{k_1}\right)^{\frac{1}{3}}\tau^{\frac{2}{3}}. \tag{67}$$

The quasi-stationary state $\theta_{\text{balance}}(\tau)$ is not a solution of the differential equation (66), but if $|\mathrm{d}^2\theta_{\text{balance}}(\tau)/\mathrm{d}\tau^2|$ is small compared to $\tilde{g}/(k_1 + 1)$, then it is a good approximative solution. It holds that

$$\frac{\mathrm{d}^2\theta_{\text{balance}}(\tau)}{\mathrm{d}\tau^2} = -\frac{2}{9}\left(\frac{\mu_{\text{cont}}^2\tilde{g}}{k_1}\right)^{\frac{1}{3}}\tau^{-\frac{4}{3}}. \tag{68}$$

Hence, the motion $\theta(\tau)$ is well approximated by $\theta_{\text{balance}}(\tau)$ if

$$\tau \gg \tau_c = \left(\frac{2}{9}(k_1 + 1)\right)^{\frac{3}{4}} k_1^{-\frac{1}{4}}\left(\frac{\mu_{\text{cont}}}{\tilde{g}}\right)^{\frac{1}{2}}. \tag{69}$$

The ratio $\mu_{\text{cont}}/\tilde{g}$ is usually much smaller than unity ($\tau_c \ll 1$). If $\tau < \tau_c$, then the disk can no longer keep up the gyroscopic balancing and the disk will 'fall'. The assumption $\lambda_N \approx mg$ does no longer hold for $\tau < \tau_c$.

We conclude that the dry contour friction dissipation mechanism predicts the power law $\theta(t) \propto (t_f - t)^n$ with $n = \frac{2}{3}$. This approximation is valid during the final stage of motion (stationary rolling motion) but only if we do not consider the very last fraction of a second before the motion stops. That is to say, for dry contour friction, the power law (1) with $n = \frac{2}{3}$ is valid during the final stage of motion on the time-scale of several seconds.

## 5.2 Energy decay for various dissipation mechanisms

In this section, we give analytical approximations for the energy decay of a rolling disk for the dissipation mechanisms presented in Sect. 4. The energy decay is studied during the final stage of motion which motivates the following standing assumptions for the type of motion

- A.1 The centre of mass is assumed to be almost immobile, i.e. stationary rolling motion holds and $\omega_y = \epsilon\omega_z$.
- A.2 We assume that the kinetic energy associated with $\omega_x$ is small compared to the potential energy, i.e. $\frac{1}{2}(1 + k_1)\dot{\omega}_x^2 \ll \tilde{g}\theta$.
- A.3 We assume $\theta = \pi/2 - \beta$ to be close to 0 and neglect terms of order $\mathcal{O}(\theta^2)$ with respect to terms of order $\mathcal{O}(1)$.
- A.4 We neglect terms of order $\mathcal{O}(\epsilon^2)$ and $\mathcal{O}(\epsilon\theta)$ with respect to terms of order $\mathcal{O}(1)$.

Once an approximation for the energy decay is found, one has to check whether the above assumptions are met. Using the assumptions A.1—A.4, we approximate the total energy $E = E_{\text{kin}} + E_{\text{pot}}$, see (22) and (23), by the expression

$$E = \frac{1}{2}mr^2\left(k_1\omega_z^2 + 2\tilde{g}\theta\right), \tag{70}$$

in which only the major terms have been taken into account. Using the assumptions A.1, A.2 and A.4 we approximate $\omega_z$ with the gyroscopic balance equation (35) and only retain leading terms

$$\omega_z^2 = \frac{\tilde{g}(\sin\beta - \epsilon\cos\beta)}{k_1\tan\beta - \epsilon k_2} \approx \frac{\tilde{g}(1 - \epsilon\theta)}{k_1\theta^{-1} - \epsilon k_2} \approx \frac{\tilde{g}}{k_1}\theta. \tag{71}$$

The assumption of gyroscopic balancing for quasi-stationary motion is called the 'adiabatic approximation' in [18]. By substitution of (71) in (70), the total energy of the system can be approximated with

$$E = \frac{3}{2}mr^2\tilde{g}\theta, \tag{72}$$

from which we see that the energy $E$ is proportional to the inclination $\theta$. In the following, we express for the various kinds of dissipation mechanisms the dissipative power $\dot{E}$ as a function of energy, i.e. $\dot{E} = f(E)$ for $E > 0$. The power–energy relation gives a scalar differential equation which approximates the time-evolution of the system during the final stage of motion.

If we choose a dry contour friction law, as introduced in Sect. 4.2, then the dissipation rate reads as

$$\dot{E} = -\mu_{\text{cont}}\lambda_N r|\omega_{\text{cont}}|. \tag{73}$$

The assumptions A.2 and A.3 allow us to approximate the normal contact force with $\lambda_N = mg$. We now have to express $\omega_{\text{cont}}$ as a function of $E$. Using (11), (71) and (72) it holds that

$$\omega_{\text{cont}}^2 = \left(\omega_y - \omega_z\tan\beta\right)^2 \approx \left(\epsilon - \frac{1}{\theta}\right)^2\omega_z^2 \approx \frac{\tilde{g}}{k_1}\frac{1}{\theta} \approx \frac{3mr^2\tilde{g}^2}{2k_1}\frac{1}{E}. \tag{74}$$

The dissipation rate $\dot{E}$ for dry contour friction, see (73), can therefore be expressed as

$$\dot{E} = -\frac{a}{\sqrt{E}}, \quad \text{with } a = \mu_{\text{cont}}\left(\frac{3\tilde{g}}{2k_1}\right)^{\frac{1}{2}}\left(mr^2\tilde{g}\right)^{\frac{3}{2}}. \tag{75}$$

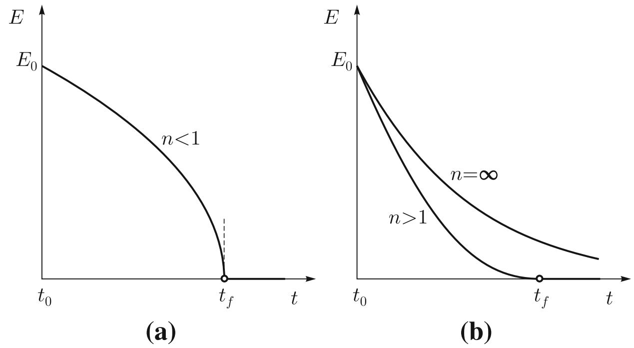



**Fig. 3** Energy decay for **a** $n<1$ and **b** $n>1$

For an arbitrary initial condition $E(t_0)=E_0>0$, the differential equation (75) obeys the solution

$$E(t)=\left(E_0^{\frac{3}{2}}-\frac{3}{2}a(t-t_0)\right)^{\frac{2}{3}}\quad \text{for } t_0\le t\le t_f, \tag{76}$$

which shows a decrease to zero in a finite time $t_f-t_0=2E_0^{\frac{3}{2}}/(3a)$ similar to Fig. 3a. From the energy $E(t)$ we can calculate $\theta(t)$ using (72) which gives

$$\theta(t)=\left(\theta_0^{\frac{3}{2}}-\sqrt{\frac{\tilde{g}}{k_1}}\mu_{\text{cont}}(t-t_0)\right)^{\frac{2}{3}}, \tag{77}$$

which agrees with (67) using the inverse time $\tau=t_f-t$. Now that the solution $\theta(t)$ is known, we can check the validity of the assumption A.2. Evaluation of the condition $\frac{1}{2}(1+k_1)\dot{\omega}_x^2\ll\tilde{g}\theta$ by substitution of (77) gives, again, the critical inverse time $\tau_c$ (69). Hence, the assumption A.2 fulfills the same role as the assumption $|\mathrm{d}^2\theta_{\text{balance}}(\tau)/\mathrm{d}\tau^2|\ll\tilde{g}/(k_1+1)$ in Sect. 5.1.

If we consider a viscous contour friction model $M_{\text{cont}}=-c_{\text{cont}}\omega_{\text{cont}}$ (55), then the dissipation rate reads as $\dot{E}=-c_{\text{cont}}\omega_{\text{cont}}^2$. Using the approximation (74), similar to the above analysis, we deduce that

$$\dot{E}=-\frac{a}{E},\quad \text{with } a=\frac{3}{2}\frac{m\tilde{g}^2r^2}{k_1}c_{\text{cont}}. \tag{78}$$

For an arbitrary initial condition $E(t_0)=E_0$, the differential equation (78) obeys the solution

$$E(t)=\left(E_0^2-2a(t-t_0)\right)^{\frac{1}{2}}\quad \text{for } t_0\le t\le t_f, \tag{79}$$

which shows a decrease to zero in a finite time $t_f-t_0=E_0^2/(2a)$. The critical inverse time $\tau_c$ for viscous contour friction yields

$$\tau_c=\left(\frac{c_{\text{cont}}}{4k_1mr^2}\right)^{\frac{1}{3}}\left(\frac{k_1+1}{\tilde{g}}\right)^{\frac{2}{3}}. \tag{80}$$

The power–energy relations can be deduced for all other dissipation mechanisms presented in Sect. 4: dry/viscous classical rolling friction, dry/viscous pivoting friction, sliding friction and the viscous air drag models of Moffatt [18] and Bildsten [3]. The results are summarised in Table 1 and the exponent $n$ is given in the last column. A sketch of the energy profiles depending on $n$ are shown in Fig. 3. Viscous classical rolling friction and viscous pivoting friction have an exponent $n=\infty$ as can be seen from the property $e^x=\lim_{n\to\infty}(1+\frac{x}{n})^n$ of the exponential function. In Sect. 4.4, we concluded that sliding friction is not able to dissipate energy if the disk is in a state of stationary rolling motion and it therefore holds that $\dot{E}=0$ under

**Table 1** Power–energy relations for various friction models

| Friction type | Differential equation | Energy profile | Exponent |
|---|---|---|---|
| Contour friction | | | |
| Dry | $\dot{E} = -aE^{-\frac{1}{2}}$ | $E(t) = \left(E_0^{\frac{3}{2}} - \frac{3}{2}a(t-t_0)\right)^{\frac{2}{3}}$ | $n = \frac{2}{3}$ |
| Viscous | $\dot{E} = -aE^{-1}$ | $E(t) = \left(E_0^2 - 2a(t-t_0)\right)^{\frac{1}{2}}$ | $n = \frac{1}{2}$ |
| Class. rolling friction | | | |
| Dry | $\dot{E} = -aE^{\frac{1}{2}}$ | $E(t) = \left(\sqrt{E_0} - \frac{a}{2}(t-t_0)\right)^2$ | $n = 2$ |
| Viscous | $\dot{E} = -aE$ | $E(t) = E_0 e^{-a(t-t_0)}$ | $n = \infty$ |
| Pivoting friction | | | |
| Dry | $\dot{E} = -aE^{\frac{1}{2}}$ | $E(t) = \left(\sqrt{E_0} - \frac{a}{2}(t-t_0)\right)^2$ | $n = 2$ |
| Viscous | $\dot{E} = -aE$ | $E(t) = E_0 e^{-a(t-t_0)}$ | $n = \infty$ |
| Sliding friction | $\dot{E} = 0$ | $E(t) = E_0$ | $n = 1$ |
| Viscous air drag | | | |
| Moffatt | $\dot{E} = -aE^{-2}$ | $E(t) = \left(E_0^3 - 3a(t-t_0)\right)^{\frac{1}{3}}$ | $n = \frac{1}{3}$ |
| Bildsten | $\dot{E} = -aE^{-\frac{5}{4}}$ | $E(t) = \left(E_0^{\frac{9}{4}} - \frac{9}{4}a(t-t_0)\right)^{\frac{4}{9}}$ | $n = \frac{4}{9}$ |

assumption A.1. The dissipation for sliding friction can be put in the form $\dot{E} = -aE^0$ with $a = 0$ and sliding friction therefore has a (theoretical) exponent $n = 1$.

We conclude that viscous classical rolling friction and viscous pivoting friction predict an asymptotic behaviour of the energy profile whereas sliding friction predicts no energy dissipation at all. All other dissipation mechanisms, discussed here, lead to a decrease of the energy in finite time. However, dry classical rolling friction and dry pivoting friction predict a parabolic decay of the energy ($n = 2$) and therefore not an abrupt halt of the motion. The viscous air drag model of Moffatt predicts the smallest value of the exponent $n$.

## 6 Experimental analysis

Experiments have been conducted on the 'Euler disk' (a scientific toy of Tangent Toy) using a high-speed video camera. The experimental setup and measurement technique are presented in Sect. 6.1. The experimental results are discussed in Sect. 6.2 and a comparison is given with the theoretical results of the previous sections. The geometric data and inertia properties of the 'Euler disk' are listed below:

| | |
|---|---|
| $D_0 = 2r_0 = 75.5$ mm | outer diameter of the disk |
| $D = 2r = 70.0$ mm | diameter of the tread |
| $H = 2h = 13$ mm | height of the disk |
| $m = 0.4499$ kg | measured mass |
| $I_1 = \frac{1}{4}mr_0^2 + \frac{1}{12}mh^2$ | principle moment of inertia |
| $I_3 = \frac{1}{2}mr_0^2$ | principle moment of inertia |
| $g = 9.81$ m/s$^2$ | gravitation |
| $\epsilon = \frac{h}{r} = 0.1857$ | relative thickness |
| $k_1 = \frac{I_1}{mr^2} = 0.2937$ | relative inertia |
| $k_2 = \frac{I_3}{mr^2} = 0.5817$ | relative inertia |
| $\tilde{g} = \frac{g}{r} = 280.28$ s$^{-2}$ | relative gravitation |

### 6.1 Experimental setup and measurement method

The experimental setup (Fig. 4) consists the disk which is spun on a glass (or aluminium) base-plate being fixed to the supporting table. A high-speed camera is positioned such that it records the side-view of the spinning disk. Two lamps with softboxes provide diffuse light in order to avoid shadows. The top and side of the disk have been painted white for better reflection. The high-speed video camera (NAC, Hi-Dcam II) has been used at a framerate of 1,000 fps with a shutter time of 1/1000 s and a resolution of $1{,}060 \times 348$ pixels. The disk is put in motion by hand and the measurement is stopped manually when the motion of the disk has ceased. The last 10,000 frames, which corresponds to 10 s recording time, are stored on the computer board.

The data is analysed frame-by-frame in a post-processing phase using a dedicated MatLab program written by the author. First, the frame is converted to a black-and-white image using an edge-detection algorithm (Image Processing Toolbox). The rim of the top surface of the disk is in this image visible as an ellipse if the surface is in view of the camera, or as the upper segment of an ellipse if the surface is not in view of the camera. Figure 5 shows a frame before and after post-processing. In a second step, a number of points on the rim of the disk are located. An ellipse is fitted on these points, which is a linear least-square problem. This leads to a first estimate for the semi-major and semi-minor axes of the top surface of the disk and for the position of its geometric centre. This first estimation for the parameters of the ellipse is already very good if the top surface is in view of the camera and the whole ellipse is visible. However, if the top surface is not in view of the camera, then only the upper part of the rim is visible and the fitted ellipse can be poor. The semi-major axis should be equal to the diameter of the disk of which the size in pixels is known at forehand. A second fitting procedure is carried out with a pre-specified semi-major axis. This leads to a nonlinear least-square problem, which is solved using the first estimation as initial guess. This final estimation of the ellipse is satisfactory for

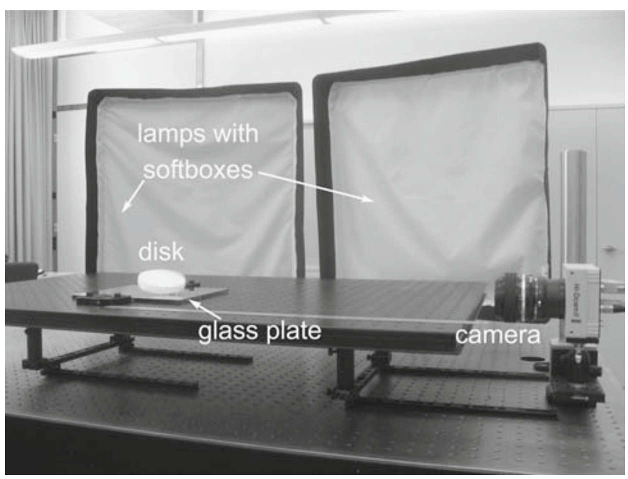



**Fig. 4** Experimental setup

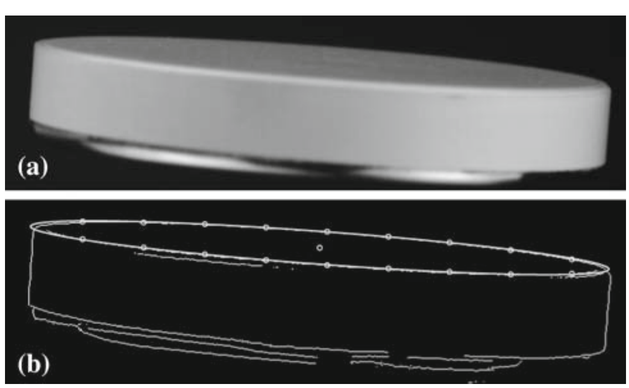



**Fig. 5** Frame before processing (**a**) and after processing (**b**)

all frames. The parameters of the ellipse provide two pieces of information: the inclination angle $\theta$ of the disk and the precession angle $\alpha$. No information is obtained about the angle $\gamma$.

Figure 5b shows the post-processed frame. Eighteen points (indicated with small circles) have been found on the rim of the top surface. The fitted ellipse is shown with a larger line-thickness. The described post-processing technique of the images which fits an ellipse on the top surface has the advantage that it gives a reasonably accurate result even if the contour of the disk is blurred, because the method uses a number of points, which averages out uncertainties. The blurring in the images prohibits techniques which simply determine the inclination angle by finding the highest point of the disk.

## 6.2 Experimental results

A number of measurements have been done with both glass and aluminum flat base-plates. The results were always qualitatively similar, with the distinction that the spinning times on an aluminum plate were much smaller. Here, we discuss only one measurement with a glass base-plate.

Figure 6a shows the time-history of the measured inclination angle $\theta(t)$. We observe that the disk comes to an abrupt halt at $t = t_f = 9.61$ s after which $\theta(t) = 0$, i.e. the disk lies flat on the base-plate. We also see that the motion of the disk consists of a 'slow motion' with a superimposed high-frequency oscillation. The slow motion of the inclination $\theta$ shows a kind of 'square-root' behaviour, i.e. the slope tends to minus infinity just before the motion ceases. This is often called the 'finite-time singularity' in literature.

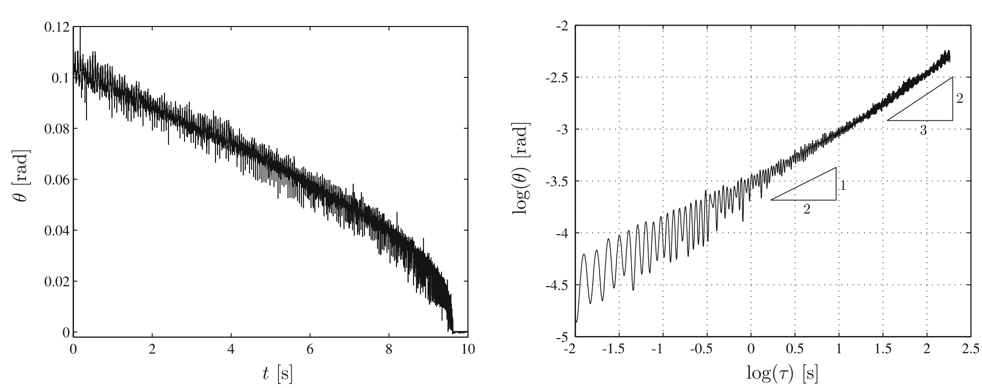
Figure 6b shows $\log(\theta)$ as a function of $\log(\tau)$, where $\tau = t_f - t$ is the inverse time and $\log(x)$ denotes the natural logarithm of $x$. The 'finite-time singularity' occurs for $\tau = 0$ s. The slope of the curve in Fig. 6b varies from $\frac{2}{3}$ for large $\tau$ to $\frac{1}{2}$ for small $\tau$. We therefore read from Fig. 6b that for different time-intervals it holds that

$$\theta(\tau) \propto \tau^n, \tag{81}$$

with $n = \frac{2}{3}$ or $n = \frac{1}{2}$. Furthermore, the curve in Fig. 6(b) crosses the vertical axis $\log \tau = 0$ at the value $-3.5$ and it therefore approximately holds that $\theta(\tau) = 0.0302 \times \tau^n$. Assuming dry contour friction and using (67) and (69), we obtain the estimate $\mu_{\text{cont}} = 1.7 \times 10^{-4}$ for the contour friction coefficient and $\tau_c = 4.2 \times 10^{-4}$ s for the critical time. Similarly, assuming viscous contour friction and using (80), we obtain the estimate $c_{\text{cont}} = 1.1 \times 10^{-7}$ Nms and $\tau_c = 1.5 \times 10^{-3}$ s. Hence, according to the theoretical analysis of Sect. 5, the assumption of gyroscopic balancing can no longer be expected to hold during the last milliseconds before the end of motion.

The angular velocity $\omega_z = \dot{\alpha} \cos \beta$ is obtained from $\beta(t) = \frac{\pi}{2} - \theta(t)$ and numerical differentiation of $\alpha(t)$ with a low-pass filtering (50 Hz cut-off frequency). Furthermore, a semi-analytical theoretical prediction of $\omega_z$ is obtained using Eq. (35) and the measured values of $\beta(t)$. Equation (35) assumes that the disk is in a state of stationary rolling motion. Figure 7a shows the 'measured' angular velocity $\omega_z$ as a solid line and the theoretical prediction with Eq. (35) by small circles. We see that the two estimations of $\omega_z$ agree very well.

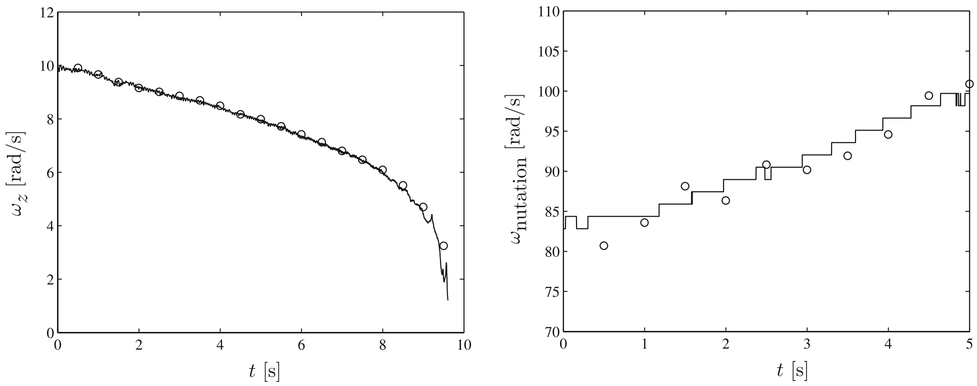
The high-frequency content of the signal $\theta(t)$ is analysed using a moving-window discrete fast Fourier transform. At each discrete time-instant a window of 2000 samples is taken (2 s), centred around that time-instant.




**(a)** Time-history

**(b)** Double-logarithmic plot versus $\tau = t_f - t$

**Fig. 6** Measured inclination angle $\theta$




**(a)** Angular velocity $\omega_z$ versus time **(b)** Nutation frequency versus time

**Fig. 7** Measured angular velocity and nutation frequency (*solid lines*) and semi-analytical theoretical predictions (*circles*)

Each window is analysed by a $2^{12}$ point FFT and the frequency corresponding to the highest peak in the spectral density curve is determined. This frequency, which is shown in Fig. 7b by a solid line, is a measured estimate for the nutational frequency $\omega_{\text{nutation}}$. The described method has a resolution of $2\pi \cdot 1000/2^{12} = 1.53$ rad/s, which explains the stairs of the solid line in Fig. 7b. Only the first 5 s have been shown because the measured estimate for $\omega_{\text{nutation}}$ becomes unreliable when the slope of $\theta(t)$ is too large and varies too much during 2 s. A semi-analytical theoretical prediction of $\omega_{\text{nutation}}$ is obtained using Eq. (48) together with (38), (39), $\omega_y(t) = \epsilon\omega_z(t)$, (49) and the measured values of $\beta(t)$ and $\omega_z(t)$. The theoretical prediction of $\omega_{\text{nutation}}(t)$ assumes that the disk in a state of stationary rolling motion. The theoretical prediction of $\omega_{\text{nutation}}(t)$ is shown in Fig. 7b as small circles. The two estimates of $\omega_{\text{nutation}}(t)$ agree reasonably well.

From this experiment we can draw a number of conclusions about the motion and dissipation mechanism during the final stage of motion of a rolling disk. We first discuss the conclusions about the type of motion and then discuss the dissipation mechanism.

The good agreement between experimentally obtained values for $\omega_z$ and $\omega_{\text{nutation}}$ with theoretically estimates under the assumption of stationary rolling motion indicates that the disk is approximately in a state of stationary rolling motion. That is to say, the disk is during the final stage of motion in the neighbourhood of a quasi-equilibrium state for which the centre of mass is almost immobile. The dissipation in the system causes the quasi-equilibrium state to slowly change over time. Apparently, the state of the disk slides almost along the one-dimensional sub-manifold $\mathcal{S}$ of stationary rolling motion equilibrium states given by (36). The motion of the disk consists of the superposition of the slowly varying quasi-equilibrium state and a high frequency nutational oscillation.

We now come to conclusions about the energy decay and the responsible dissipation mechanism. The theoretical results in Sect. 5.2 have been derived under the standing assumptions A.1–A.4. The discussion in the previous paragraph indicates that assumption A.1 is fulfilled. The measured decay of the inclination angle $\theta$ over time shown in Fig. 6 indicates that the inclination is proportional to the fractional power of the inverse time $\tau$. Assumption A.2 is therefore not fulfilled for very small $\tau$, i.e. for $t$ close to $t_f$, because $\omega_x = -\dot{\theta}$ tends to infinity. However, if we do not consider the last fraction of a second before the finite-time singularity and consider the final stage of motion of the disk on a time-scale of seconds, then we can reasonable say that assumption A.2 is fulfilled. Clearly, also assumptions A.3 and A.4 are fulfilled, because $\epsilon = 0.1875$ and $\theta < 0.12$ rad. Hence, under the validity of these assumptions, Eq. (72) expresses the proportionality between the total energy $E$ and the inclination $\theta$, which in turns leads to the proportionality

$$E(\tau) \propto \tau^n, \tag{82}$$

with $n = \frac{2}{3}$ for large $\tau$ and $n = \frac{1}{2}$ for small $\tau$.

## 7 Conclusions

A literature overview of experiments on the rolling disk has been given in Sect. 1. The publications which give an experimental value for the exponent $n$ are listed in Table 2. It can be seen that all publications, including the results of Sect. 6, report the exponents $n = \frac{1}{2}$ and/or $n = \frac{2}{3}$.

**Table 2** Experimental results on the exponent $n$

| | |
|---|---|
| McDonald and McDonald [17] | $n = \frac{1}{2}$ |
| Easwar et al. [8] | $n = \frac{2}{3}$ |
| Caps et al. [5] | $n = \frac{1}{2}, \frac{2}{3}$ |
| This paper | $n = \frac{1}{2}, \frac{2}{3}$ |

Various dissipation mechanisms for the rolling disk have been discussed in Sect. 5.2 and the corresponding energy profiles and exponents are listed in Table 1. The dry contour friction dissipation mechanism leads to the exponent $n = \frac{2}{3}$, whereas a viscous contour friction dissipation mechanism has the exponent $n = \frac{1}{2}$. It is therefore tempting to make the quick conclusion that dry contour friction prevails at the beginning of the stationary rolling phase and that viscous contour friction prevails during the last one or two seconds before the motion stops. The contour velocity $\gamma_{\text{cont}}$ tends to infinity if $\tau$ approaches zero, which can explain why viscous contour friction prevails for small $\tau$ (large contour velocity) and dry contour friction prevails for large $\tau$ (small contour velocity). However, we should be careful with definite statements about the nature of the dissipation mechanism. All we can really say is that a dissipation mechanism of dry and viscous contour friction can well explain the observed experimental results, but other dissipation mechanisms might exist which lead to the same exponent $n$ in the energy decay relationship.

In Sect. 1, it was mentioned that the considered time-scale is of importance when speaking about *the* dominant dissipation mechanism for the rolling disk. Moffatt [19] suggests that viscous air drag has to prevail at the very end of the motion as the associated exponent is smaller than, for instance, the exponent of contour or classical rolling friction. However, the exponent for the various dissipation mechanisms, including viscous air drag, have been derived under the assumption of gyroscopic balancing. This puts a lower bound $\tau_c$ on the inverse time $\tau$ in the order of milliseconds, i.e. at the very end of the motion. Moreover, the effect of the surface roughness and contamination may play a role when the inclination angle and gap between disk and table become very small. It is therefore questionable whether viscous air drag will finally become dominant if the surfaces are not highly polished. Furthermore, the experimental results do not have a sufficient resolution to reveal the dynamics at extremely small inclination angles ($\theta < 0.01$ rad). The question of the dominant dissipation mechanism during the last fraction of a second, which is perhaps of less practical interest, therefore remains unanswered.

Consequently, the experimental evidence and theoretical analysis presented in this paper do not prove but strongly suggest that dry and viscous contour friction are the dominant dissipation mechanisms for the finite-time singularity of the 'Euler disk' on a time-scale of several seconds, i.e. the time-scale on which the measurements have been performed with a reasonable accuracy.

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