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ORIGINAL ARTICLE

Stability properties of equilibrium sets of non-linear mechanical systems with dry friction and impact

R. I. Leine · N. van de Wouw

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Abstract In this paper, we will give conditions under which the equilibrium set of multi-degree-of-freedom *non-linear* mechanical systems with an arbitrary number of frictional *unilateral* constraints is attractive. The theorems for attractivity are proved by using the framework of measure differential inclusions together with a Lyapunov-type stability analysis and a generalisation of LaSalle's invariance principle for non-smooth systems. The special structure of mechanical multi-body systems allows for a natural Lyapunov function and an elegant derivation of the proof. Moreover, an instability theorem for assessing the instability of equilibrium sets of non-linear mechanical systems with frictional bilateral constraints is formulated. These results are illustrated by means of examples with both unilateral and bilateral frictional constraints.

Keywords Attractivity · Lyapunov stability · Measure differential inclusion · Unilateral constraint

R. I. Leine (✉)
IMES – Center of Mechanics, ETH Zürich, CH-8092 Zürich, Switzerland
e-mail: remco.leine@imes.mavt.ethz.ch

N. van de Wouw
Department of Mechanical Engineering, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands
e-mail: N.v.d.Wouw@tue.nl

1 Introduction

Dry friction can seriously affect the performance of a wide range of systems. More specifically, the stiction phenomenon in friction can induce the presence of equilibrium sets, see for example [46]. The stability properties of such equilibrium sets is of major interest when analysing the global dynamic behaviour of these systems.

The aim of the paper is to present a number of theoretical results that can be used to rigourously prove the conditional attractivity of the equilibrium set for non-linear mechanical systems with frictional unilateral constraints (including impact) using Lyapunov stability theory and LaSalle's invariance principle.

The dynamics of mechanical systems with set-valued friction laws are described by differential inclusions of Filippov-type (so-called Filippov systems), see [27, 31] and references therein. Filippov systems, describing systems with friction, can exhibit equilibrium sets, which correspond to the stiction behaviour of those systems. Many publications deal with stability and attractivity properties of (sets of) equilibria in differential inclusions [1–3, 6, 21, 26, 43, 47]. For example, in [2, 43] the attractivity of the equilibrium set of a dissipative one-degree-of-freedom friction oscillator with one switching boundary (i.e. one dry friction element) is discussed. Moreover, in [3, 6, 43] the Lyapunov stability of an equilibrium point in the equilibrium set is shown. Most papers are limited to either one-degree-of-freedom systems or to systems exhibiting only one switching boundary. Very often, the stability properties of an equilibrium point in the equilibrium set is investigated and not the stability properties of the set itself. In this context, it is worth mentioning that in the more general scope of discontinuous systems (without impulsive loads), a range of results regarding stability conditions for *isolated* equilibria are available, see for example [23] in which conditions for stability are formulated in terms of the existence of common quadratic or piece-wise quadratic Lyapunov functions. Yakubovich et al. [47] discuss the stability and dichotomy (a form of attractivity) of equilibrium sets in differential inclusions within the framework of absolute stability. In [9], extensions are given of the absolute stability problem and the Lagrange–Dirichlet theorem for systems with monotone multi-valued mappings (such as, for example Coulomb friction and unilateral contact constraints). In the absolute stability framework, strict passivity properties of a linear part of the system are required for proving the asymptotic stability of an isolated equilibrium point, which may be rather restrictive for mechanical systems in general. Adly et al. [1, 21] study stability properties of equilibrium sets of differential inclusions describing mechanical systems with friction. It is assumed that the non-smoothness is stemming from a maximal monotone operator (e.g. friction with a constant normal force). Existence and uniqueness of solutions is therefore always fulfilled. A basic Lyapunov theorem for stability and attractivity is given in [1, 21] for first-order differential inclusions with maximal monotone operators. The results are applied to linear mechanical systems with friction. It is assumed in [1] that the relative sliding velocity of the frictional contacts depends linearly on the generalised velocities. Conditions for the attractivity of an equilibrium set are given. The results are generalised in [1] to conservative systems with an arbitrary potential energy function. In a previous publication [45], we provided conditions under which the equilibrium set of multi-degree-of-freedom *linear* mechanical systems with an arbitrary number of Coulomb friction elements is attractive using Lyapunov-type stability analysis and a generalisation of LaSalle's invariance principle for non-smooth systems. Moreover, dissipative as well as non-dissipative linear systems have been considered. The analysis was restricted to bilateral frictional constraints and linear systems.

Unilateral contact between rigid bodies does not only lead to the possible separation of contacting bodies but can also lead to impact when bodies collide. Systems with impact between rigid bodies undergo instantaneous changes in the velocities of the bodies. Impact systems, with or without friction, can be properly described by measure differential inclusions as introduced by Moreau [32, 34] (see also [8, 18, 31]), which allow for discontinuities in the state of the system. Measure differential inclusions, being more general than Filippov systems, can exhibit equilibrium sets as well. The results in [9] on the absolute stability problem and the Lagrange–Dirichlet theorem apply also to systems with unilateral contact and impact. However, once more only the stability of isolated equilibria is addressed. In [12], stability conditions of isolated equilibria for a class of discontinuous systems (with state-jumps), formulated as cone-complementarity systems, are posed using a passivity-based approach.

The stability of hybrid systems with state-discontinuities is addressed by a vast number of researchers in the field of control theory. The book of Bainov and Simeonov [7] focuses on systems with impulsive effects and gives many useful Russian references. Lyapunov stability theorems, instability theorems and theorems for boundedness are given by Ye et al. [48]. Pettersson and Lennartson [38] propose stability theorems using multiple Lyapunov functions. By using piecewise quadratic Lyapunov function candidates and replacing the regions where the different stability conditions have to be valid by quadratic inequality functions (and exploiting the S-procedure), the problem of verifying stability is turned into a Linear Matrix Inequality (LMI) problem. See also the review article of Davrazos and Koussoulas [15]. Many publications focus on the control of mechanical systems with frictionless unilateral contacts by means of Lyapunov functions. See, for instance Brogliato et al. [11] and Tornambè [44] and the book [8] for further references.

The Lagrange–Dirichlet stability theorem is extended by Brogliato [9] to measure differential inclusions describing mechanical systems with frictionless impact. The idea to use Lyapunov functions involving indicator functions associated with unilateral constraints is most probably due to [9]. More generally, the work of Chareyron and Wieber [13, 14] is concerned with a Lyapunov stability framework for measure differential inclusions describing mechanical systems with frictionless impact. It is clearly explained in [14] why the Lyapunov function has to be globally positive definite, in order to prove stability in the presence of state-discontinuities (when no further assumptions on the system or the form of the Lyapunov function are made). The importance of this condition has also been stated in [7, 48] for hybrid systems and in [11, 44] for mechanical systems with frictionless unilateral constraints. Moreover, LaSalle's invariance principle is generalised in [10] to differential inclusions and in [13, 14] to measure differential inclusions describing mechanical systems with frictionless impact. The proof of LaSalle's invariance principle strongly relies on the positive invariance of limit sets. It is assumed in [13, 14] that the system enjoys continuity of the solution with respect to the initial condition which is a sufficient condition for positive invariance of limit sets. In [14], an extension of LaSalle invariance principle to systems with unilateral constraints is presented (more specifically, it is applied to mechanical systems with frictionless unilateral contacts). In [10], an extension of LaSalle invariance principle for a class of unilateral dynamical systems, the so-called evolution variational inequalities, is presented.

Instability results for finite-dimensional variational inequalities can be found in the work of Goeleven and Brogliato [20, 21], whereas Quittner [40] gives instability results for a class of parabolic variational inequalities in Hilbert spaces.

In this paper, we will give conditions under which the equilibrium set of multi-degree-of-freedom *non-linear* mechanical systems with an arbitrary number of frictional *unilateral* constraints (i.e. systems with friction and impact) is attractive. The theorems for attractivity are proved by using the framework of measure differential inclusions together with a Lyapunov-type stability analysis and a generalisation of LaSalle's invariance principle for non-smooth systems, which is based on the assumption that every limit set is positively invariant (see also [28]). The special structure of mechanical multi-body systems allows for a natural choice of the Lyapunov function and a systematic derivation of the proof for this large class of systems.

In Sections 2 and 3, the constitutive laws for frictional unilateral contact and impact are formulated as set-valued force laws. The modelling of mechanical systems with dry friction and impact by measure differential inclusions is discussed in Section 4. Subsequently, the attractivity properties of the equilibrium set of a system with frictional unilateral contact are studied in Section 5. Non-linear mechanical systems with bilateral frictional constraints form an important sub-class of systems and are studied in Section 6, where also instability conditions for equilibrium sets are proposed. Moreover, for both classes of systems the attractivity analysis provides an estimate for the region of attraction of the equilibrium sets. In Section 7, a number of examples are studied in order to illustrate the theoretical results of Sections 5 and 6. Moreover, an example is given that shows the conservativeness of the estimated region of attraction. Finally, a discussion of the results and concluding remarks are given in Section 8.

2 Frictional contact laws in the form of set-valued force laws

In this section, we formulate the constitutive laws for frictional unilateral contact formulated as set-valued force laws (see [18] for an extensive treatise on the subject). Normal contact between rigid bodies is described by a set-valued force law called Signorini's law. Consider two convex rigid bodies at a relative distance g_{N} from each other (Fig. 1). The normal contact distance g_{N} is uniquely defined for convex bodies and is such, that the points 1 and 2 have parallel tangent planes (shown as dashed lines in Fig. 1). The normal

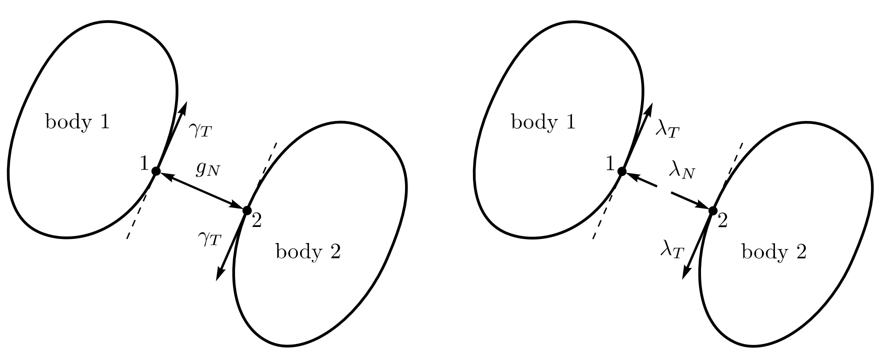


Fig. 1 Contact distance g_{N} and tangential velocity γ_{T} between two rigid bodies

contact distance g_{N} is non-negative because the bodies do not penetrate into each other. The bodies touch when $g_{\mathrm{N}} = 0$. The normal contact force λ_{N} between the bodies is non-negative because the bodies can exert only repelling forces on each other, i.e. the constraint is unilateral. The normal contact force vanishes when there is no contact, i.e. $g_{\mathrm{N}} > 0$, and can only be positive when contact is present, i.e. $g_{\mathrm{N}} = 0$. Under the assumption of impenetrability, expressed by $g_{\mathrm{N}} \geq 0$, only two situations may occur:

$$\begin{aligned} &g_{\mathrm{N}} = 0 \wedge \lambda_{\mathrm{N}} \geq 0 \quad \text{contact}, \\ &g_{\mathrm{N}} > 0 \wedge \lambda_{\mathrm{N}} = 0 \quad \text{no contact}. \end{aligned} \tag{1}$$

From Equation (1), we see that the normal contact law shows a complementarity behaviour: the product of the contact force and normal contact distance is always zero, i.e. $g_{\mathrm{N}}\lambda_{\mathrm{N}} = 0$. The relation between the normal contact force and the normal contact distance is therefore described by

$$g_{\mathrm{N}} \geq 0, \quad \lambda_{\mathrm{N}} \geq 0, \quad g_{\mathrm{N}}\lambda_{\mathrm{N}} = 0, \tag{2}$$

which is the inequality complementarity condition between g_{N} and λ_{N}. The inequality complementarity behaviour of the normal contact law is depicted in the left figure of Fig. 2 and shows a set-valued graph of admissible combinations of g_{N} and λ_{N}. The magnitude of the contact force is denoted by λ_{N} and the direction of the contact force is normal to the bodies, i.e. along the line 1–2 in Fig. 1.

The normal contact law can also be expressed by the subdifferential (see Equation (149) in Appendix B) of a non-smooth conjugate potential $\Psi^*_{C_{\mathrm{N}}}(g_{\mathrm{N}})$

$$-\lambda_{\mathrm{N}} \in \partial \Psi^*_{C_{\mathrm{N}}}(g_{\mathrm{N}}), \tag{3}$$

where C_{N} is the admissible set of negative contact forces $-\lambda_{\mathrm{N}}$,

$$C_{\mathrm{N}} = \{-\lambda_{\mathrm{N}} \in \mathbb{R} \mid \lambda_{\mathrm{N}} \geq 0\} = \mathbb{R}^-, \tag{4}$$

and $\Psi_{C_{\mathrm{N}}}$ is the indicator function of C_{N}. In Appendix B, several results from convex analysis are reviewed. Alternatively, we can formulate the contact law in a compact form by means of the normal cone of C_{N} (see Equation (150) in Appendix B):

$$g_{\mathrm{N}} \in N_{C_{\mathrm{N}}}(-\lambda_{\mathrm{N}}). \tag{5}$$

The potential $\Psi_{C_{\mathrm{N}}}$ is depicted in the upper left figure of Fig. 3 and is the indicator function of $C_{\mathrm{N}} = \mathbb{R}^-$. Taking the subdifferential of the indicator function gives the set-valued relation $g_{\mathrm{N}} \in \partial \Psi_{C_{\mathrm{N}}}(-\lambda_{\mathrm{N}})$, depicted in the lower left figure. Interchanging the axis gives the lower right figure which expresses Equation (3) and is equivalent to the left graph of Fig. 2. Integration of the latter relation gives the support function $\Psi^*_{C_{\mathrm{N}}}(g_{\mathrm{N}})$, which is the conjugate of the indicator function on C_{N}.

The normal contact law, also called Signorini's law, expresses impenetrability of the contact and can formally be stated for a number of contact points $i = 1, \ldots, n_C$ as

$$\boldsymbol{g}_{\mathrm{N}} \in N_{C_{\mathrm{N}}}(-\boldsymbol{\lambda}_{\mathrm{N}}), \quad C_{\mathrm{N}} = \{-\boldsymbol{\lambda}_{\mathrm{N}} \in \mathbb{R}^n \mid \boldsymbol{\lambda}_{\mathrm{N}} \geq \mathbf{0}\}, \tag{6}$$

where $\boldsymbol{\lambda}_{\mathrm{N}}$ is the vector containing the normal contact forces $\lambda_{\mathrm{N}i}$ and $\boldsymbol{g}_{\mathrm{N}}$ is the vector of normal contact distances $g_{\mathrm{N}i}$. Signorini's law, which is a set-valued law for normal contact on displacement level, can for closed contacts with $g_{\mathrm{N}} = 0$ be expressed on velocity level:

$$\gamma_{\mathrm{N}} \in N_{C_{\mathrm{N}}}(-\lambda_{\mathrm{N}}), \quad g_{\mathrm{N}} = 0, \tag{7}$$

where γ_{N} is the relative normal contact velocity, i.e. $\gamma_{\mathrm{N}} = \dot{g}_{\mathrm{N}}$ for non-impulsive motion.

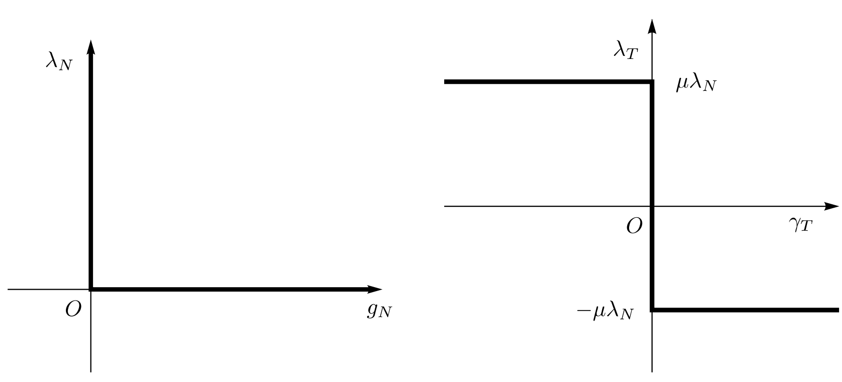
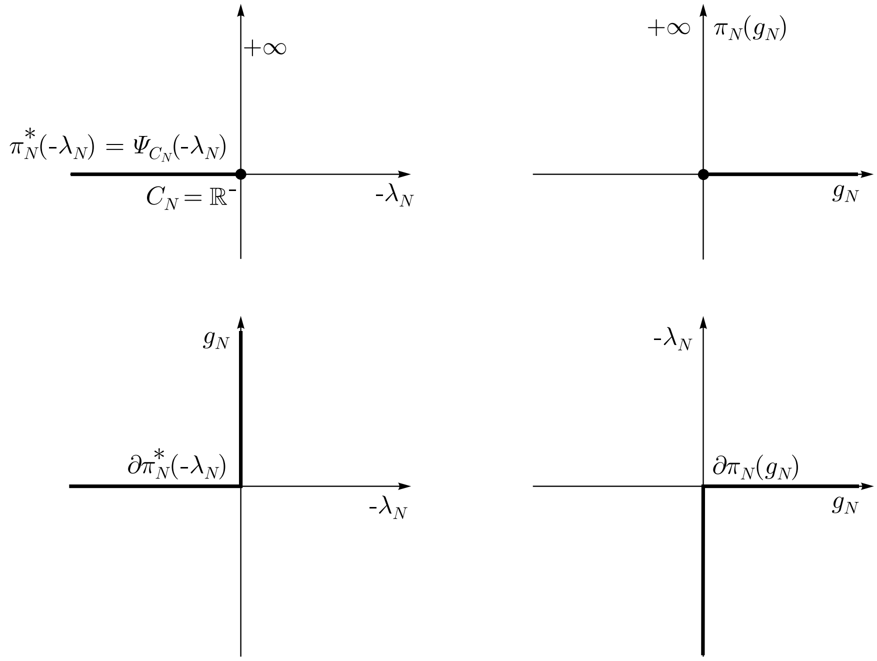


Fig. 2 Signorini's normal contact law and Coulomb's friction law

Fig. 3 Potential, conjugate potential and subdifferential of the normal contact problem $C = C_{\mathrm{N}} = \mathbb{R}^-$



Coulomb's friction law is another classical example of a force law that can be described by a non-smooth potential. Consider two bodies as depicted in Fig. 1 with Coulomb friction at the contact point. We denote the relative velocity of point 1 with respect to point 2 along their tangent plane by γ_{T}. If contact is present between the bodies, i.e. $g_{\mathrm{N}} = 0$, then the friction between the bodies imposes a force λ_{T} along the tangent plane of the contact point. If the bodies are sliding over each other, then the friction force λ_{T} has the magnitude $\mu\lambda_{\mathrm{N}}$ and acts in the direction of $-\gamma_{\mathrm{T}}$

$$-\lambda_{\mathrm{T}} = \mu\lambda_{\mathrm{N}}\,\mathrm{sign}(\gamma_{\mathrm{T}}), \qquad \gamma_{\mathrm{T}} \neq 0, \tag{8}$$

where μ is the friction coefficient and λ_{N} is the normal contact force. If the relative tangential velocity vanishes, i.e. $\gamma_{\mathrm{T}} = 0$, then the bodies purely roll over each other without slip. Pure rolling, or no slip for locally flat objects, is denoted by *stick*. If the bodies stick, then the friction force must lie in the interval $-\mu\lambda_{\mathrm{N}} \le \lambda_{\mathrm{T}} \le \mu\lambda_{\mathrm{N}}$. For unidirectional friction, i.e. for planar contact problems, the following three cases are possible:

$$\begin{array}{llll}
\gamma_{\mathrm{T}} = 0 & \Rightarrow & |\lambda_{\mathrm{T}}| \le \mu\lambda_{\mathrm{N}} & \text{sticking,} \\
\gamma_{\mathrm{T}} < 0 & \Rightarrow & \lambda_{\mathrm{T}} = +\mu\lambda_{\mathrm{N}} & \text{negative sliding,} \\
\gamma_{\mathrm{T}} > 0 & \Rightarrow & \lambda_{\mathrm{T}} = -\mu\lambda_{\mathrm{N}} & \text{positive sliding.}
\end{array} \tag{9}$$

We can express the friction force by a potential $\pi_{\mathrm{T}}(\gamma_{\mathrm{T}})$, which we mechanically interpret as a dissipation function,

$$-\lambda_{\mathrm{T}} \in \partial\pi_{\mathrm{T}}(\gamma_{\mathrm{T}}), \qquad \pi_{\mathrm{T}}(\gamma_{\mathrm{T}}) = \mu\lambda_{\mathrm{N}}|\gamma_{\mathrm{T}}|, \tag{10}$$

from which follows the set-valued force law

$$-\lambda_{\mathrm{T}} \in \begin{cases} \mu\lambda_{\mathrm{N}}, & \gamma_{\mathrm{T}} > 0, \\ [-1, 1]\mu\lambda_{\mathrm{N}}, & \gamma_{\mathrm{T}} = 0, \\ -\mu\lambda_{\mathrm{N}}, & \gamma_{\mathrm{T}} < 0. \end{cases} \tag{11}$$

A non-smooth convex potential therefore leads to a maximal monotone set-valued force law. The admissible values of the negative tangential force λ_{T} form a convex set C_{T} that is bounded by the values of the normal force [39]:

$$C_{\mathrm{T}} = \{-\lambda_{\mathrm{T}} \mid -\mu\lambda_{\mathrm{N}} \le \lambda_{\mathrm{T}} \le +\mu\lambda_{\mathrm{N}}\}. \tag{12}$$

Coulomb's law can be expressed with the aid of the indicator function of C_{T} as

$$\gamma_{\mathrm{T}} \in \partial\Psi_{C_{\mathrm{T}}}(-\lambda_{\mathrm{T}}) \Leftrightarrow \gamma_{\mathrm{T}} \in N_{C_{\mathrm{T}}}(-\lambda_{\mathrm{T}}), \tag{13}$$

where the indicator function $\Psi_{C_{\mathrm{T}}}$ is the conjugate potential of the support function $\pi_{\mathrm{T}}(\gamma_{\mathrm{T}}) = \Psi^*_{C_{\mathrm{T}}}(\gamma_{\mathrm{T}})$ [18], see Fig. 4.

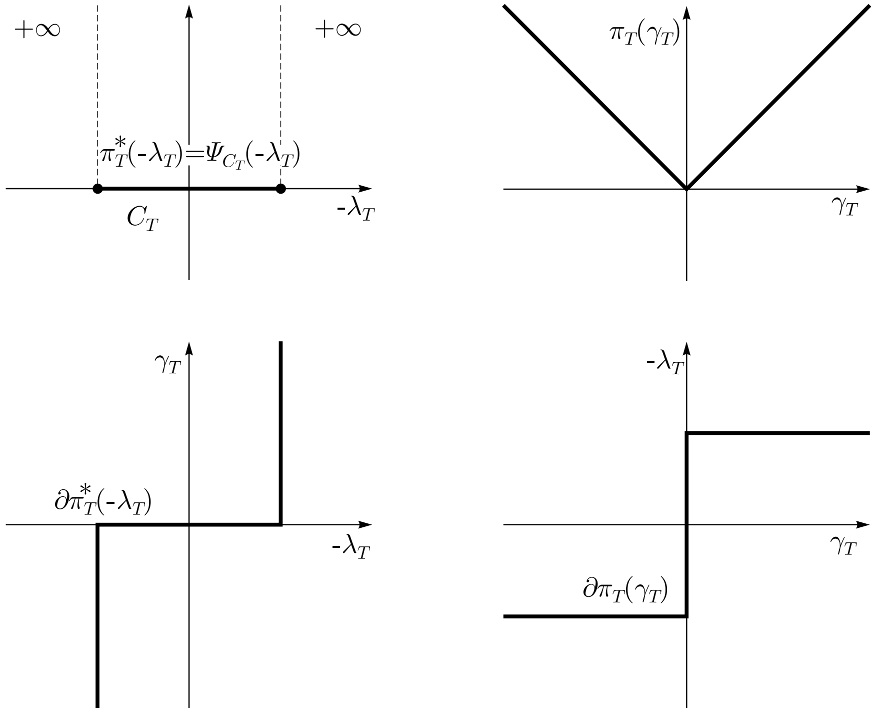


Fig. 4 Potential, conjugate potential and subdifferential of the tangential contact problem $C = C_\mathrm{T}$

The classical Coulomb's friction law for spatial contact formulates a two-dimensional friction force $\boldsymbol{\lambda}_\mathrm{T} \in \mathbb{R}^2$ which lies in the tangent-plane of the contacting bodies. The set of negative admissible friction forces is a convex set $C_\mathrm{T} \subset \mathbb{R}^2$ that is a disk for isotropic Coulomb friction:

$$C_\mathrm{T} = \{-\boldsymbol{\lambda}_\mathrm{T} \mid \|\boldsymbol{\lambda}_\mathrm{T}\| \le \mu \lambda_\mathrm{N}\}. \tag{14}$$

Using the set C_T, the spatial Coulomb friction law can be formulated as

$$\begin{aligned} \boldsymbol{\gamma}_\mathrm{T} \in \partial \Psi_{C_\mathrm{T}}(-\boldsymbol{\lambda}_\mathrm{T}) &\Longleftrightarrow -\boldsymbol{\lambda}_\mathrm{T} \in \partial \Psi^*_{C_\mathrm{T}}(\boldsymbol{\gamma}_\mathrm{T}) \\ &\Longleftrightarrow \boldsymbol{\gamma}_\mathrm{T} \in N_{C_\mathrm{T}}(-\boldsymbol{\lambda}_\mathrm{T}), \end{aligned} \tag{15}$$

in which $\boldsymbol{\gamma}_\mathrm{T} \in \mathbb{R}^2$ is the relative sliding velocity. Similarly, an elliptic choice of C_T would result in an orthotropic Coulomb friction law.

A combined friction law, which takes into account sliding friction as well as pivoting (or drilling) friction, can be formulated using a three-dimensional set of admissible (generalised) friction forces and is called the spatial Coulomb–Contensou friction law [30]. The function $\boldsymbol{\gamma}_\mathrm{T} \in \mathbb{R}^p$ is the relative velocity of the bodies at the contact point. For planar Coulomb friction, it holds that $p = 1$, while $p = 2$ for spatial Coulomb friction and $p = 3$ for Coulomb–Contensou friction. A combined spatial sliding–pivoting–rolling friction law would result in $p = 5$ (two forces, three torques).

A one-way clutch is another example of a non-smooth force law on velocity level and can also be derived from a non-smooth velocity potential (support function):

$$-\lambda_c \in \partial \Psi^*_{C_c}(\gamma_c), \tag{16}$$

see [18] for details. The set of negative admissible forces of a one-way clutch is $C_c = \mathbb{R}^-$. Note that $\mathbf{0} \neq \operatorname{int} C_c$.

The friction law of Coulomb (or Coulomb–Contensou), as defined earlier, assumes the friction forces to be a function of the unilateral normal forces. Both the normal contact forces and the friction forces have to be determined. However, in many applications the situation is less complicated as the normal force is constant or at least a given function of time. A known normal contact force allows for a simplified contact law. The tangential friction forces are assumed to obey either one of the following friction laws:

- *Associated Coulomb's law* for which the normal force is known in advance. The set of admissible negative contact forces is given by

$$C_\mathrm{T}(F_\mathrm{N}) = \{-\boldsymbol{\lambda}_\mathrm{T} \mid \|\boldsymbol{\lambda}_\mathrm{T}\| \le \mu F_\mathrm{N}\}, \tag{17}$$

which is dependent on the known normal forces F_N and friction coefficient μ. This friction law is described by a maximal monotone set-valued operator (the Sign-function in the planar case) on the relative sliding velocity γ_T.

- *Non-associated Coulomb's law* for which the normal force is dependent on the generalised coordinates $\boldsymbol{q}$ and/or generalised velocities $\boldsymbol{u}$ and therefore not known in advance. The set of admissible negative contact forces is given by

$$C_\mathrm{T}(\lambda_\mathrm{N}) = \{-\boldsymbol{\lambda}_\mathrm{T} \mid \|\boldsymbol{\lambda}_\mathrm{T}\| \le \mu\lambda_\mathrm{N}\}, \tag{18}$$

which is dependent on the normal contact forces λ_N and friction coefficient μ. Non-associated Coulomb friction is not described by a maximal monotone operator on γ_T, since the normal contact force λ_N varies in time.

3 Impact laws

Signorini's law and Coulomb's friction law are set-valued force laws for non-impulsive forces. In order to describe impact, we need to introduce impact laws for the contact impulses. We will consider a Newton-type of restitution law,

$$\gamma_\mathrm{N}^+ = -e_\mathrm{N}\gamma_\mathrm{N}^-, \quad g_\mathrm{N} = 0, \tag{19}$$

which relates the post-impact velocity γ_N^+ of a contact point to the pre-impact velocity γ_N^- by Newton's coefficient of restitution e_N. The case $e_\mathrm{N} = 1$ corresponds to a completely elastic contact, whereas $e_\mathrm{N} = 0$ corresponds to a completely inelastic contact. The impact, which causes the sudden change in relative velocity, is accompanied by a normal contact impulse $\Lambda_\mathrm{N} > 0$. Following [17], suppose that, for any reason, the contact does not participate in the impact, i.e. that the value of the normal contact impulse Λ_N is zero, although the contact is closed. This happens normally for multi-contact situations. For this case, we allow the post-impact relative velocities to be higher than the value prescribed by Newtons impact law, $\gamma_\mathrm{N}^+ > -e_\mathrm{N}\gamma_\mathrm{N}^-$, in order to express that the contact is superfluous and could be removed without changing the contact-impact process. We can therefore express the impact law as an inequality complementarity on velocity–impulse level:

$$\Lambda_\mathrm{N} \ge 0, \quad \xi_\mathrm{N} \ge 0, \quad \Lambda_\mathrm{N}\xi_\mathrm{N} = 0, \tag{20}$$

with $\xi_\mathrm{N} = \gamma_\mathrm{N}^+ + e_\mathrm{N}\gamma_\mathrm{N}^-$ (see [17]). Similarly to Signorini's law on velocity level, we can write the impact law in normal direction as

$$\xi_\mathrm{N} \in N_{C_\mathrm{N}}(-\Lambda_\mathrm{N}), \quad g_\mathrm{N} = 0, \tag{21}$$

or by using the support function

$$-\Lambda_\mathrm{N} \in \partial\Psi_{C_\mathrm{N}}^*(\xi_\mathrm{N}), \quad g_\mathrm{N} = 0. \tag{22}$$

A normal contact impulse Λ_N at a frictional contact leads to a tangential contact impulse $\boldsymbol{\Lambda}_\mathrm{T}$ with $\|\boldsymbol{\Lambda}_\mathrm{T}\| \le \mu\Lambda_\mathrm{N}$. We therefore have to specify a tangential impact law as well. The tangential impact law can be formulated in a similar way as has been done for the normal impact law:

$$-\boldsymbol{\Lambda}_\mathrm{T} \in \partial\Psi_{C_\mathrm{T}(\Lambda_\mathrm{N})}^*(\boldsymbol{\xi}_\mathrm{T}), \quad g_\mathrm{N} = 0, \tag{23}$$

with $\boldsymbol{\xi}_\mathrm{T} = \boldsymbol{\gamma}_\mathrm{T}^+ + e_\mathrm{T}\boldsymbol{\gamma}_\mathrm{T}^-$. This impact law involves a tangential restitution coefficient e_T. This restitution coefficient, which is normally considered to be zero, can be used to model the tangential velocity reversal as observed in the motion of the Super Ball, being a very elastic ball used on play grounds. More information on the physical meaning of the tangential restitution coefficient can be found in [37].

4 Modelling of non-linear mechanical systems with dry friction and impact

In this section, we will define the class of non-linear time-autonomous mechanical systems with unilateral frictional contact for which the stability results will be derived in Section 5. We first derive a measure differential inclusion that describes the temporal dynamics of mechanical systems with discontinuities in the velocity. Subsequently, we study the equilibrium set of the measure differential inclusion.

4.1 The measure differential inclusion

We assume that these mechanical systems exhibit only bilateral holonomic frictionless constraints and unilateral constraints in which dry friction can be present. Furthermore, we assume that a set of independent generalised coordinates, $\boldsymbol{q} \in \mathbb{R}^n$, for which these bilateral constraints are eliminated from the formulation of the dynamics of the system, is known. The generalised coordinates $\boldsymbol{q}(t)$ are assumed to be absolutely continuous functions of time t. Also, we assume the generalised velocities, $\boldsymbol{u}(t) = \dot{\boldsymbol{q}}(t)$ for almost all t, to be functions of locally bounded variation. At each time-instance it is therefore possible to define a left limit $\boldsymbol{u}^-$ and a right limit $\boldsymbol{u}^+$ of the velocity. The generalised accelerations $\dot{\boldsymbol{u}}$ are therefore not for all t defined. The set of discontinuity points $\{t_j\}$ for which $\dot{\boldsymbol{u}}$ is not defined is assumed to be Lebesgue negligible. We formulate the dynamics of the system using a Lagrangian approach, resulting in[1]

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}(T_{,\boldsymbol{u}}) - T_{,\boldsymbol{q}} + U_{,\boldsymbol{q}}\right)^{\mathrm{T}} = \boldsymbol{f}^{\mathrm{nc}}(\boldsymbol{q},\boldsymbol{u}) + \boldsymbol{W}_{\mathrm{N}}(\boldsymbol{q})\boldsymbol{\lambda}_{\mathrm{N}} + \boldsymbol{W}_{\mathrm{T}}(\boldsymbol{q})\boldsymbol{\lambda}_{\mathrm{T}}, \tag{24}$$

or, alternatively,

$$\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{u}} - \boldsymbol{h}(\boldsymbol{q},\boldsymbol{u}) = \boldsymbol{W}_{\mathrm{N}}(\boldsymbol{q})\boldsymbol{\lambda}_{\mathrm{N}} + \boldsymbol{W}_{\mathrm{T}}(\boldsymbol{q})\boldsymbol{\lambda}_{\mathrm{T}}, \tag{25}$$

which is a differential equation for the non-impulsive part of the motion. Herein, $\boldsymbol{M}(\boldsymbol{q}) = \boldsymbol{M}^{\mathrm{T}}(\boldsymbol{q}) > 0$ is the mass-matrix. The scalar T represents kinetic energy and it is assumed that it can be written as $T = \frac{1}{2}\boldsymbol{u}^{\mathrm{T}}\boldsymbol{M}(\boldsymbol{q})\boldsymbol{u}$. Moreover, U denotes the potential energy. The column-vector $\boldsymbol{f}^{\mathrm{nc}}$ in Equation (24) represents all smooth generalised non-conservative forces. The state-dependent column-vector $\boldsymbol{h}(\boldsymbol{q},\boldsymbol{u})$ in Equation (25) contains all differentiable forces (both conservative and non-conservative), such as spring forces, gravitation, smooth damper forces and gyroscopic terms.

We introduce the following index sets:

$$I_G = \{1,\ldots,n_C\} \quad \text{the set of all contacts,}$$
$$I_{\mathrm{N}} = \{i \in I_G \mid g_{\mathrm{N}i}(\boldsymbol{q}) = 0\} \quad \text{the set of all closed contacts,} \tag{26}$$

and set up the force laws and impact laws of each contact as has been elaborated in Sections 2 and 3. The normal contact distances $g_{\mathrm{N}i}(\boldsymbol{q})$ depend on the generalised coordinates $\boldsymbol{q}$ and are gathered in a vector $\boldsymbol{g}_{\mathrm{N}}(\boldsymbol{q})$.

During a non-impulsive part of the motion, the normal contact force $-\lambda_{\mathrm{N}i} \in C_{\mathrm{N}}$ and friction force $-\boldsymbol{\lambda}_{\mathrm{T}i} \in C_{\mathrm{T}i} \subset \mathbb{R}^p$ of each closed contact $i \in I_{\mathrm{N}}$, are assumed to be associated with a non-smooth potential, being the support function of a convex set, i.e.

$$-\lambda_{\mathrm{N}i} \in \partial\Psi^*_{C_{\mathrm{N}}}(\gamma_{\mathrm{N}i}), \quad -\boldsymbol{\lambda}_{\mathrm{T}i} \in \partial\Psi^*_{C_{\mathrm{T}i}}(\boldsymbol{\gamma}_{\mathrm{T}i}), \tag{27}$$

where $C_{\mathrm{N}} = \mathbb{R}^-$ and the set $C_{\mathrm{T}i}$ can be dependent on the normal contact force $\lambda_{\mathrm{N}i} \geq 0$. The normal and tangential contact forces of all n_C contacts are gathered in columns $\boldsymbol{\lambda}_{\mathrm{N}} = \{\lambda_{\mathrm{N}i}\}$ and $\boldsymbol{\lambda}_{\mathrm{T}} = \{\boldsymbol{\lambda}_{\mathrm{T}i}\}$ and the corresponding normal and tangential relative velocities are gathered in columns $\boldsymbol{\gamma}_{\mathrm{N}} = \{\gamma_{\mathrm{N}i}\}$ and $\boldsymbol{\gamma}_{\mathrm{T}} = \{\boldsymbol{\gamma}_{\mathrm{T}i}\}$, for $i \in I_G$. We assume that these contact velocities are related to the generalised velocities through:

$$\boldsymbol{\gamma}_{\mathrm{N}}(\boldsymbol{q},\boldsymbol{u}) = \boldsymbol{W}_{\mathrm{N}}^{\mathrm{T}}(\boldsymbol{q})\boldsymbol{u}, \quad \boldsymbol{\gamma}_{\mathrm{T}}(\boldsymbol{q},\boldsymbol{u}) = \boldsymbol{W}_{\mathrm{T}}^{\mathrm{T}}(\boldsymbol{q})\boldsymbol{u}. \tag{28}$$

It should be noted that $\boldsymbol{W}_X^{\mathrm{T}}(\boldsymbol{q}) = \frac{\partial\boldsymbol{\gamma}_X}{\partial\boldsymbol{u}}$ for $X = N, T$. This assumption is very important as it excludes rheonomic contacts.

Equation (25) together with the set-valued force laws (27) form a differential inclusion

$$\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{u}} - \boldsymbol{h}(\boldsymbol{q},\boldsymbol{u}) \in -\sum_{i\in I_{\mathrm{N}}} \boldsymbol{W}_{\mathrm{N}i}(\boldsymbol{q})\partial\Psi^*_{C_{\mathrm{N}}}(\gamma_{\mathrm{N}i}) - \boldsymbol{W}_{\mathrm{T}i}(\boldsymbol{q})\partial\Psi^*_{C_{\mathrm{T}i}}(\boldsymbol{\gamma}_{\mathrm{T}i}), \quad \text{for almost all } t. \tag{29}$$

Differential inclusions of this type are called Filippov systems [16]. The differential inclusion (29) only holds for impact free motion.

Subsequently, we define for each contact point the constitutive impact laws

$$-\Lambda_{\mathrm{N}i} \in \partial\Psi^*_{C_{\mathrm{N}}}(\xi_{\mathrm{N}i}), \quad -\boldsymbol{\Lambda}_{\mathrm{T}i} \in \partial\Psi^*_{C_{\mathrm{T}i}(\Lambda_{\mathrm{N}i})}(\boldsymbol{\xi}_{\mathrm{T}i}), \quad i \in I_{\mathrm{N}}, \tag{30}$$

with

$$\xi_{\mathrm{N}i} = \gamma^+_{\mathrm{N}i} + e_{\mathrm{N}i}\gamma^-_{\mathrm{N}i}, \quad \boldsymbol{\xi}_{\mathrm{T}i} = \boldsymbol{\gamma}^+_{\mathrm{T}i} + e_{\mathrm{T}i}\boldsymbol{\gamma}^-_{\mathrm{T}i}, \tag{31}$$

in which $e_{\mathrm{N}i}$ and $e_{\mathrm{T}i}$ are the normal and tangential restitution coefficients, respectively. The inclusions (30) form very complex set-valued mappings representing the contact laws at the impulse level. The force laws for non-impulsive motion can be put in the same form because $\boldsymbol{u}^+ = \boldsymbol{u}^-$ holds in the absence of impacts and because of the positive homogeneity of the support function (see Appendix B):

$$-\lambda_{\mathrm{N}i} \in \partial\Psi^*_{C_{\mathrm{N}}}(\xi_{\mathrm{N}i}), \quad -\boldsymbol{\lambda}_{\mathrm{T}i} \in \partial\Psi^*_{C_{\mathrm{T}i}(\lambda_{\mathrm{N}i})}(\boldsymbol{\xi}_{\mathrm{T}i}). \tag{32}$$

[1] Note that the sub-script $_{,x}$ indicates a partial derivative operation $\partial/\partial x$.

We now replace the differential inclusion (29), which holds for almost all t, by an equality of measures

$$\begin{aligned}&\boldsymbol{M}(\boldsymbol{q})\,\mathrm{d}\boldsymbol{u}-\boldsymbol{h}(\boldsymbol{q},\boldsymbol{u})\,\mathrm{d}t\\&\quad=\boldsymbol{W}_{\mathrm{N}}(\boldsymbol{q})\,\mathrm{d}\boldsymbol{\Lambda}_{\mathrm{N}}+\boldsymbol{W}_{\mathrm{T}}(\boldsymbol{q})\,\mathrm{d}\boldsymbol{\Lambda}_{\mathrm{T}}\quad\forall t,\end{aligned}\tag{33}$$

which holds for all time-instances t. The differential measure of the contact impulsions $\mathrm{d}\boldsymbol{\Lambda}_{\mathrm{N}}$ and $\mathrm{d}\boldsymbol{\Lambda}_{\mathrm{T}}$ contains a Lebesgue measurable part $\boldsymbol{\lambda}\,\mathrm{d}t$ and an atomic part $\boldsymbol{\Lambda}\,\mathrm{d}\eta$

$$\mathrm{d}\boldsymbol{\Lambda}_{\mathrm{N}}=\boldsymbol{\lambda}_{\mathrm{N}}\,\mathrm{d}t+\boldsymbol{\Lambda}_{\mathrm{N}}\,\mathrm{d}\eta,\quad \mathrm{d}\boldsymbol{\Lambda}_{\mathrm{T}}=\boldsymbol{\lambda}_{\mathrm{T}}\,\mathrm{d}t+\boldsymbol{\Lambda}_{\mathrm{T}}\,\mathrm{d}\eta,\tag{34}$$

which can be expressed as inclusions

$$\begin{aligned}&-\mathrm{d}\Lambda_{\mathrm{N}i}\in\partial\Psi^*_{C_{\mathrm{N}}}(\xi_{\mathrm{N}i})(\mathrm{d}t+\mathrm{d}\eta),\\&\quad-\mathrm{d}\boldsymbol{\Lambda}_{\mathrm{T}i}\in\partial\Psi^*_{C_{\mathrm{T}i}(\lambda_{\mathrm{N}i})}(\boldsymbol{\xi}_{\mathrm{T}i})\,\mathrm{d}t+\partial\Psi^*_{C_{\mathrm{T}i}(\Lambda_{\mathrm{N}i})}(\boldsymbol{\xi}_{\mathrm{T}i})\,\mathrm{d}\eta.\end{aligned}\tag{35}$$

As an abbreviation we write

$$\boldsymbol{M}(\boldsymbol{q})\,\mathrm{d}\boldsymbol{u}-\boldsymbol{h}(\boldsymbol{q},\boldsymbol{u})\,\mathrm{d}t=\boldsymbol{W}(\boldsymbol{q})\,\mathrm{d}\boldsymbol{\Lambda}\quad\forall t,\tag{36}$$

using short-hand notation

$$\begin{aligned}&\boldsymbol{\lambda}=\begin{bmatrix}\boldsymbol{\lambda}_{\mathrm{N}}\\\boldsymbol{\lambda}_{\mathrm{T}}\end{bmatrix},\quad \boldsymbol{\Lambda}=\begin{bmatrix}\boldsymbol{\Lambda}_{\mathrm{N}}\\\boldsymbol{\Lambda}_{\mathrm{T}}\end{bmatrix},\\&\boldsymbol{W}=[\,\boldsymbol{W}_{\mathrm{N}}\quad\boldsymbol{W}_{\mathrm{T}}\,],\quad \gamma=\begin{bmatrix}\gamma_{\mathrm{N}}\\\gamma_{\mathrm{T}}\end{bmatrix}.\end{aligned}\tag{37}$$

Furthermore we introduce the quantities

$$\boldsymbol{\xi}\equiv\gamma^++\boldsymbol{E}\gamma^-,\quad \boldsymbol{\delta}\equiv\gamma^+-\gamma^-,\tag{38}$$

with $\boldsymbol{E}=\mathrm{diag}(\{e_{\mathrm{N}i},e_{\mathrm{T}i}\})$ from which we deduce

$$\begin{aligned}&\gamma^+=(\boldsymbol{I}+\boldsymbol{E})^{-1}(\boldsymbol{\xi}+\boldsymbol{E}\boldsymbol{\delta}),\\&\gamma^-=(\boldsymbol{I}+\boldsymbol{E})^{-1}(\boldsymbol{\xi}-\boldsymbol{\delta}).\end{aligned}\tag{39}$$

The equality of measures (36) together with the set-valued force laws (35) form a measure differential inclusion that describes the time-evolution of a mechanical system with discontinuities in the generalised velocities. Such a measure differential inclusion does not necessarily have existence and uniqueness of solutions for all admissible initial conditions. Indeed, if the friction coefficient is large, then the coupling between motion normal to the constraint and tangential to the constraint can cause existence and uniqueness problems (known as the Painlevé problem [8, 29]). In the following, we will assume existence and uniqueness of solutions in forward time. The contact laws guarantee that the generalised positions $\boldsymbol{q}(t)$ are such that penetration is avoided ($g_{\mathrm{N}i}\geq 0$) and the generalised positions therefore remain within the admissible set

$$\mathcal{K}=\{\boldsymbol{q}\in\mathbb{R}^n\mid g_{\mathrm{N}i}(\boldsymbol{q})\geq 0\quad\forall i\in I_G\},\tag{40}$$

for all t. The condition $\boldsymbol{q}(t)\in\mathcal{K}$ follows of course from the assumption of existence of solutions. We remark, however, that the following theorems can be relaxed to systems with non-uniqueness of solutions.

4.2 Equilibrium set

The measure differential inclusion described by Equations (36) and (35) exhibits an equilibrium set. Note that the assumption of scleronomic contacts implies that $\boldsymbol{\gamma}_{\mathrm{T}}=\boldsymbol{0}$ for $\boldsymbol{u}=\boldsymbol{0}$, see Equation (28). This means that every equilibrium implies sticking in all closed contact points. Every equilibrium position has to obey the equilibrium inclusion

$$\boldsymbol{h}(\boldsymbol{q},\boldsymbol{0})-\sum_{i\in I_{\mathrm{N}}}\big(\boldsymbol{W}_{\mathrm{N}i}(\boldsymbol{q})\partial\Psi^*_{C_{\mathrm{N}}}(\boldsymbol{0})+\boldsymbol{W}_{\mathrm{T}i}(\boldsymbol{q})\partial\Psi^*_{C_{\mathrm{T}i}}(\boldsymbol{0})\big)\ni\boldsymbol{0},\tag{41}$$

which, using $C=\partial\Psi^*_C(0)$, simplifies to

$$\boldsymbol{h}(\boldsymbol{q},\boldsymbol{0})-\sum_{i\in I_{\mathrm{N}}}(\boldsymbol{W}_{\mathrm{N}i}(\boldsymbol{q})C_{\mathrm{N}i}+\boldsymbol{W}_{\mathrm{T}i}(\boldsymbol{q})C_{\mathrm{T}i})\ni\boldsymbol{0}.\tag{42}$$

An equilibrium set, being a simply connected set of equilibrium points, is therefore given by ($C_{\mathrm{N}i}=-\mathbb{R}^+$)

$$\begin{aligned}\mathcal{E}\subset\Bigg\{&(\boldsymbol{q},\boldsymbol{u})\in\mathbb{R}^n\times\mathbb{R}^n\,|\,(\boldsymbol{u}=\boldsymbol{0})\wedge\boldsymbol{h}(\boldsymbol{q},\boldsymbol{0})\\&+\sum_{i\in I_{\mathrm{N}}}(\boldsymbol{W}_{\mathrm{N}i}(\boldsymbol{q})\mathbb{R}^+-\boldsymbol{W}_{\mathrm{T}i}(\boldsymbol{q})C_{\mathrm{T}i})\ni\boldsymbol{0}\Bigg\}\end{aligned}\tag{43}$$

and is positively invariant if we assume uniqueness of the solutions in forward time. With $\mathcal{E}$ we denote an equilibrium set of the measure differential inclusion in the state-space $(\boldsymbol{q},\boldsymbol{u})$, while $\mathcal{E}_{\boldsymbol{q}}$ is reserved for the union of equilibrium positions $\boldsymbol{q}^*$, i.e. $\mathcal{E} = \{(\boldsymbol{q}, \boldsymbol{u}) \in \mathbb{R}^n \times \mathbb{R}^n \mid \boldsymbol{q} \in \mathcal{E}_{\boldsymbol{q}}, \boldsymbol{u} = \boldsymbol{0}\}$. Note that non-linear mechanical systems without dry friction can exhibit multiple equilibria. Similarly, a system with dry friction may exhibit multiple equilibrium sets.

Let us now state some consequences of the assumptions made, which will be used in the next section. Due to the fact that the kinetic energy can be described by

$$T = \frac{1}{2}\boldsymbol{u}^{\mathrm{T}}\boldsymbol{M}(\boldsymbol{q})\boldsymbol{u} = \frac{1}{2}\sum_r\sum_s M_{rs}u^r u^s, \tag{44}$$

with $\boldsymbol{M}(\boldsymbol{q}) = \boldsymbol{M}^{\mathrm{T}}(\boldsymbol{q})$, we can write in tensorial language

$$\begin{aligned}
\frac{\partial T}{\partial q^k} &= \frac{1}{2}\sum_r\sum_s\left(\frac{\partial M_{rs}}{\partial q^k}\right)u^r u^s, \\
\frac{\partial T}{\partial u^k} &= \sum_r M_{kr}u^r, \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial u^k}\right) &= \sum_r M_{kr}\dot{u}^r + \sum_r\sum_s\left(\frac{\partial M_{kr}}{\partial q^s}\right)u^r u^s \\
&= \sum_r M_{kr}\dot{u}^r + 2\frac{\partial T}{\partial q^k} \\
&\quad + \sum_r\sum_s\left(\frac{\partial M_{kr}}{\partial q^s} - \frac{\partial M_{rs}}{\partial q^k}\right)u^r u^s \\
\frac{\mathrm{d}}{\mathrm{d}t}(T_{,\boldsymbol{u}}) &= \dot{\boldsymbol{u}}^{\mathrm{T}}\boldsymbol{M}(\boldsymbol{q}) + 2T_{,\boldsymbol{q}} - (\boldsymbol{f}^{\mathrm{gyr}})^{\mathrm{T}} \\
&\qquad\qquad \text{for almost all } t
\end{aligned} \tag{45}$$

with the gyroscopic forces [36]

$$\begin{aligned}
\boldsymbol{f}^{\mathrm{gyr}} &= \{f_k^{\mathrm{gyr}}\}, \\
f_k^{\mathrm{gyr}} &= -\sum_r\sum_s\left(\frac{\partial M_{kr}}{\partial q^s} - \frac{\partial M_{rs}}{\partial q^k}\right)u^r u^s.
\end{aligned} \tag{46}$$

In the next section, we will exploit that the gyroscopic forces $\boldsymbol{f}_{\mathrm{gyr}}$ have zero power [36]

$$\begin{aligned}
\boldsymbol{u}^{\mathrm{T}}\boldsymbol{f}^{\mathrm{gyr}} &= \sum_k u^k f_k^{\mathrm{gyr}} \\
&= -\sum_k\sum_r\sum_s\left(\frac{\partial M_{kr}}{\partial q^s} - \frac{\partial M_{rs}}{\partial q^k}\right)u^r u^s u^k = 0.
\end{aligned} \tag{47}$$

In the same way as before, we can write the differential measure of $T_{,\boldsymbol{u}}$ as

$$\mathrm{d}(T_{,\boldsymbol{u}}) = \mathrm{d}\boldsymbol{u}^{\mathrm{T}}\boldsymbol{M}(\boldsymbol{q}) + 2T_{,\boldsymbol{q}}\,\mathrm{d}t - (\boldsymbol{f}^{\mathrm{gyr}})^{\mathrm{T}}\,\mathrm{d}t \quad \forall t. \tag{48}$$

Comparison with Equations (25) and (24) yields

$$\boldsymbol{h} = \boldsymbol{f}^{\mathrm{nc}} + \boldsymbol{f}^{\mathrm{gyr}} - (T_{,\boldsymbol{q}} + U_{,\boldsymbol{q}})^{\mathrm{T}}, \tag{49}$$

or in index notation

$$\begin{aligned}
h_k &= f_k^{\mathrm{nc}} - \frac{\partial U}{\partial q^k} - \frac{\partial T}{\partial q^k} + f_k^{\mathrm{gyr}} \\
&= f_k^{\mathrm{nc}} - \frac{\partial U}{\partial q^k} - \frac{\partial T}{\partial q^k} - \sum_r\sum_s\left(\frac{\partial M_{kr}}{\partial q^s} - \frac{\partial M_{rs}}{\partial q^k}\right)u^r u^s \\
&= f_k^{\mathrm{nc}} - \frac{\partial U}{\partial q^k} - \frac{1}{2}\sum_r\sum_s\left(2\frac{\partial M_{kr}}{\partial q^s} - \frac{\partial M_{rs}}{\partial q^k}\right)u^r u^s \\
&= f_k^{\mathrm{nc}} - \frac{\partial U}{\partial q^k} - \frac{1}{2}\sum_r\sum_s\left(\frac{\partial M_{kr}}{\partial q^s} + \frac{\partial M_{ks}}{\partial q^r} - \frac{\partial M_{rs}}{\partial q^k}\right)u^r u^s \\
&= f_k^{\mathrm{nc}} - \frac{\partial U}{\partial q^k} - \sum_r\sum_s\Gamma_{k,rs}u^r u^s
\end{aligned} \tag{50}$$

in which we recognise the holonomic Christoffel symbols of the first kind [36]

$$\Gamma_{k,rs} = \Gamma_{k,sr} := \frac{1}{2}\left(\frac{\partial M_{kr}}{\partial q^s} + \frac{\partial M_{ks}}{\partial q^r} - \frac{\partial M_{rs}}{\partial q^k}\right). \tag{51}$$

5 Attractivity of equilibrium sets for non-linear systems

In this section, we will investigate the attractivity properties of the equilibrium sets defined in the previous section.

We define the following non-linear functionals $\mathbb{R}^n \to \mathbb{R}$ on $\boldsymbol{u} \in \mathbb{R}^n$:

- $D_{\boldsymbol{q}}^{\mathrm{nc}}(\boldsymbol{u}) := -\boldsymbol{u}^{\mathrm{T}}\boldsymbol{f}^{\mathrm{nc}}(\boldsymbol{q}, \boldsymbol{u})$ is the dissipation rate function of the smooth non-conservative forces.
- $D_{\boldsymbol{q}}^{\lambda_{\mathrm{T}}}(\boldsymbol{u}) := \sum_{i\in I_{\mathrm{N}}}\frac{1}{1+e_{\mathrm{T}i}}\Psi^*_{C_{\mathrm{T}i}(\lambda_{\mathrm{N}i})}(\boldsymbol{\xi}_{\mathrm{T}i}(\boldsymbol{q}, \boldsymbol{u}))$ is the dissipation rate function of the tangential contact forces.
- $D_{\boldsymbol{q}}^{\Lambda_{\mathrm{T}}}(\boldsymbol{u}) := \sum_{i\in I_{\mathrm{N}}}\frac{1}{1+e_{\mathrm{T}i}}\Psi^*_{C_{\mathrm{T}i}(\Lambda_{\mathrm{N}i})}(\boldsymbol{\xi}_{\mathrm{T}i}(\boldsymbol{q}, \boldsymbol{u}))$ is the dissipation rate function of the tangential contact impulses.

For non-impulsive motion it holds that $\gamma_T = \gamma_T^+ = \gamma_T^-$ and $\boldsymbol{\xi}_T = (1 + e_T)\gamma_T$. Due to the fact that the support function is positively homogeneous, it follows that

$$
\begin{aligned}
D_{\boldsymbol{q}}^{\lambda_T}(\boldsymbol{u}) &= \sum_{i\in I_N} \Psi^*_{C_{Ti}(\lambda_{Ni})}(\gamma_{Ti}(\boldsymbol{q},\boldsymbol{u})) \\
&= \sum_{i\in I_N} -\boldsymbol{\lambda}_{Ti}\gamma_{Ti}(\boldsymbol{q},\boldsymbol{u}),
\end{aligned} \tag{52}
$$

from which we see that the dissipation rate function of the tangential contact forces does not depend on the restitution coefficient e_T. The above dissipation rate functions are of course functions of $(\boldsymbol{q},\boldsymbol{u})$, but we write them as non-linear functionals on $\boldsymbol{u}$ for every fixed $\boldsymbol{q}$ so that we can speak of the zero set of the functional $D_{\boldsymbol{q}}(\boldsymbol{u})$:

$$
D_{\boldsymbol{q}}^{-1}(0) = \{\boldsymbol{u} \in \mathbb{R}^n \mid D_{\boldsymbol{q}}(\boldsymbol{u}) = 0\}. \tag{53}
$$

As stated before, the type of systems under investigation may exhibit multiple equilibrium sets. Here, we will study the attractivity properties of a specific given equilibrium set. By $\boldsymbol{q}_e$ we denote an equilibrium position of the system with unilateral *frictionless* contacts

$$
\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{u}} - \boldsymbol{h}(\boldsymbol{q},\boldsymbol{u}) - \boldsymbol{W}_N(\boldsymbol{q})\boldsymbol{\lambda}_N = \boldsymbol{0}, \tag{54}
$$

from which follows that the equilibrium position $\boldsymbol{q}_e$ is determined by the inclusion

$$
\boldsymbol{h}(\boldsymbol{q}_e,\boldsymbol{0}) - \sum_{i\in I_G} \boldsymbol{W}_{Ni}(\boldsymbol{q}_e)\partial\Psi^*_{C_N}(g_{Ni}(\boldsymbol{q}_e)) \ni \boldsymbol{0} \tag{55}
$$

or

$$
\boldsymbol{h}(\boldsymbol{q}_e,\boldsymbol{0}) - \sum_{i\in I_N} \boldsymbol{W}_{Ni}(\boldsymbol{q}_e)\partial\Psi^*_{C_N}(\underbrace{\gamma_{Ni}(\boldsymbol{q}_e,\boldsymbol{0})}_{=0}) \ni \boldsymbol{0}, \tag{56}
$$

which is equivalent to

$$
\boldsymbol{h}(\boldsymbol{q}_e,\boldsymbol{0}) + \boldsymbol{W}_N(\boldsymbol{q}_e)\mathbb{R}^+ \ni \boldsymbol{0}, \quad \boldsymbol{W}_N = \{\boldsymbol{W}_{Ni}\},\ i \in I_N. \tag{57}
$$

Let the potential $Q(\boldsymbol{q})$ be the *total* potential energy of the system

$$
Q(\boldsymbol{q}) = U(\boldsymbol{q}) + \sum_{i\in I_G} \Psi^*_{C_N}(g_{Ni}(\boldsymbol{q})), \tag{58}
$$

which is the sum of the potential energy of all smooth potential forces and the support functions of the normal contact forces. Moreover, we assume that the equilibrium position $\boldsymbol{q}_e$ is a local minimum of the total potential energy $Q(\boldsymbol{q})$, i.e.

$$
Q(\boldsymbol{q}) = \begin{cases} 0 & \boldsymbol{q} = \boldsymbol{q}_e \\ > 0 & \forall \boldsymbol{q} \in \mathcal{U}\backslash\{\boldsymbol{q}_e\} \end{cases}, \quad \boldsymbol{0} \notin \partial Q(\boldsymbol{q}), \forall \boldsymbol{q} \in \mathcal{U}\backslash\{\boldsymbol{q}_e\}. \tag{59}
$$

The sub-set $\mathcal{U}$ is assumed to enclose the equilibrium set $\mathcal{E}_q$ under investigation. Notice that the equilibrium point $\boldsymbol{q}_e$ of the system without friction is also an equilibrium point of the system with friction, $(\boldsymbol{q}_e, \boldsymbol{0}) \in \mathcal{E}$. In case the system does exhibit multiple equilibrium sets, the attractivity of $\mathcal{E}$ will be only local for obvious reasons. In the following, we will make use of the Lyapunov candidate function

$$
\begin{aligned}
V &= T(\boldsymbol{q},\boldsymbol{u}) + Q(\boldsymbol{q}) \\
&= T(\boldsymbol{q},\boldsymbol{u}) + U(\boldsymbol{q}) + \sum_{i\in I_G} \Psi^*_{C_N}(g_{Ni}(\boldsymbol{q})),
\end{aligned} \tag{60}
$$

being the sum of kinetic and total potential energy. The function $V: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is an extended lower semi-continuous function. Moreover, the function $V(t) = V(\boldsymbol{q}(t), \boldsymbol{u}(t))$ is of locally bounded variation in time (see Appendix D) because $\boldsymbol{q}(t)$ is absolutely continuous and remains in the admissible set $\mathcal{K}$ defined in (40), $\boldsymbol{u} \in \text{lbv}(I, \mathbb{R}^n)$, and T is a Lipschitz continuous function and Q is an extended lower semi-continuous function but only dependent on $\boldsymbol{q}(t)$. In the following, we will make use of the differential measure $\mathrm{d}V$ of $V(t)$. If it holds that $\mathrm{d}V \leq 0$, then it follows that

$$
V^+(t) - V^-(t_0) = \int_{[t_0,t]} \mathrm{d}V \leq 0, \tag{61}
$$

which means that $V(t)$ is non-increasing. Similarly, $\mathrm{d}V < 0$ implies a strict decrease of $V(t)$. We now formulate a technical result that states conditions under which the equilibrium set can be shown to be (locally) attractive.

Theorem 1 (Attractivity of the equilibrium set). *Consider an equilibrium set $\mathcal{E}$ of the system (36), with constitutive laws (27) and (35). If*

1. $T = \frac{1}{2}\boldsymbol{u}^T\boldsymbol{M}(\boldsymbol{q})\boldsymbol{u}$, with $\boldsymbol{M}(\boldsymbol{q}) = \boldsymbol{M}^T(\boldsymbol{q}) > 0$,

2. *the equilibrium position $\boldsymbol{q}_\mathrm{e}$ is a local minimum of the total potential energy $Q(\boldsymbol{q})$ and $Q(\boldsymbol{q})$ has a non-vanishing generalised gradient for all $\boldsymbol{q} \in \mathcal{U}\backslash\{\boldsymbol{q}_\mathrm{e}\}$, i.e. $\mathbf{0} \notin \partial Q(\boldsymbol{q})\,\forall \boldsymbol{q} \in \mathcal{U}\backslash\{\boldsymbol{q}_\mathrm{e}\}$, and the equilibrium set $\mathcal{E}_{\boldsymbol{q}}$ is contained in $\mathcal{U}$, i.e. $\mathcal{E}_{\boldsymbol{q}} \subset \mathcal{U}$,*
3. *$D^{\mathrm{nc}}_{\boldsymbol{q}}(\boldsymbol{u}) = -\boldsymbol{u}^\mathrm{T}\boldsymbol{f}^{\mathrm{nc}} \geq 0$, i.e. the smooth non-conservative forces are dissipative, and $\boldsymbol{f}^{\mathrm{nc}} = \mathbf{0}$ for $\boldsymbol{u} = \mathbf{0}$,*
4. *there exists a non-empty set $I_C \subset I_G$ and an open neighbourhood $\mathcal{V} \subset \mathbb{R}^n \times \mathbb{R}^n$ of the equilibrium set, such that $\dot{\gamma}_{\mathrm{N}i}(\boldsymbol{q},\boldsymbol{u}) < 0$ (a.e.) for $\forall i \in I_C\backslash I_\mathrm{N}$ and $(\boldsymbol{q},\boldsymbol{u}) \in \mathcal{V}$,*
5. *$D^{\mathrm{nc}\,-1}_{\boldsymbol{q}}(0) \cap D^{\lambda_{TC}\,-1}_{\boldsymbol{q}}(0) \cap \ker \boldsymbol{W}^\mathrm{T}_{NC}(\boldsymbol{q}) = \{\mathbf{0}\}\forall \boldsymbol{q} \in \mathcal{C}$ with*

$$\boldsymbol{g}_{NC} = \{\boldsymbol{g}_{\mathrm{N}i}\},\ \boldsymbol{W}_{NC} = \{\boldsymbol{w}_{\mathrm{N}i}\}\quad \text{for } i \in I_C \text{ as defined in 4.,}$$

$$\mathcal{C} = \{\boldsymbol{q} \mid \boldsymbol{g}_{NC}(\boldsymbol{q}) = \mathbf{0}\},$$

$$D^{\lambda_{TC}}_{\boldsymbol{q}} = \sum_{i \in I_C \cap I_\mathrm{N}} \Psi^*_{C_{\mathrm{T}i}(\lambda_{\mathrm{N}i})}(\gamma_{\mathrm{T}i}(\boldsymbol{q},\boldsymbol{u})),$$

6. *$0 \leq e_{\mathrm{N}i} < 1$, $|e_{\mathrm{T}i}| < 1\ \forall i \in I_G$,*
7. *one of the following conditions holds*
   a. *the restitution coefficients are small in the sense that $\frac{2e_{\max}}{1+e_{\max}} < \frac{1}{cond(\boldsymbol{G}(\boldsymbol{q}))}\ \forall \boldsymbol{q} \in \mathcal{C}$ where $\boldsymbol{G}(\boldsymbol{q}) := \boldsymbol{W}(\boldsymbol{q})^\mathrm{T}\boldsymbol{M}(\boldsymbol{q})^{-1}\boldsymbol{W}(\boldsymbol{q})$ and $e_{\max}$ is the largest restitution coefficient, i.e. $e_{\max} \geq \max(e_{\mathrm{N}i}, e_{\mathrm{T}i})\,\forall i \in I_G$,*
   b. *all restitution coefficients are equal, i.e. $e = e_{\mathrm{N}i} = e_{\mathrm{T}i}\forall i \in I_G$,*
   c. *friction is absent, i.e. $\mu_i = 0\,\forall i \in I_G$,*
8. *$\mathcal{E} \subset \mathcal{I}_{\rho^*}$ in which the set $\mathcal{I}_{\rho^*}$, with $\mathcal{I}_\rho = \{(\boldsymbol{q},\boldsymbol{u}) \in \mathbb{R}^n \times \mathbb{R}^n \mid V(\boldsymbol{q},\boldsymbol{u}) < \rho\}$, is the largest level set of V, given by (60), that is contained in $\mathcal{V}$ and $\mathcal{Q} = \{(\boldsymbol{q},\boldsymbol{u}) \in \mathbb{R}^n \times \mathbb{R}^n \mid \boldsymbol{q} \in \mathcal{U}\}$, i.e.*

$$\rho^* = \max_{\{\rho:\mathcal{I}_\rho \subset (\mathcal{V}\cap\mathcal{Q})\}} \rho, \tag{62}$$

9. *each limit set in $\mathcal{I}_{\rho^*}$ is positively invariant,*

then the equilibrium set $\mathcal{E}$ is locally attractive and $\mathcal{I}_{\rho^*}$ is a conservative estimate for the region of attraction.

Proof: Note that V is positive definite around the equilibrium point $(\boldsymbol{q},\boldsymbol{u}) = (\boldsymbol{q}_\mathrm{e},\mathbf{0})$ due to conditions 1 and 2 in the theorem. Classically, we seek the time-derivative of V in order to prove the decrease of V along solutions of the system. However, $\dot{\boldsymbol{u}}$ is not defined for all t and $\boldsymbol{u}$ can undergo jumps. We therefore compute the differential measure of V:

$$\mathrm{d}V = \mathrm{d}T + \mathrm{d}Q. \tag{63}$$

The total potential energy, being an extended lower semi-continuous function, is only a function of the generalised displacements $\boldsymbol{q}$, which are absolutely continuous in time, and it therefore holds that

$$\begin{aligned}\mathrm{d}Q &= \mathrm{d}Q(\boldsymbol{q})(\mathrm{d}\boldsymbol{q})\\ &= U_{,\boldsymbol{q}}\,\mathrm{d}\boldsymbol{q} + d\Psi_{\mathcal{K}}(\boldsymbol{q})(\mathrm{d}\boldsymbol{q}),\end{aligned} \tag{64}$$

where $d\,Q(\boldsymbol{q})(\mathrm{d}\boldsymbol{q})$ is the subderivative (see Appendix C) of Q at $\boldsymbol{q}$ in the direction $\mathrm{d}\boldsymbol{q} = \boldsymbol{u}\,\mathrm{d}t$. The subderivative $d\Psi_{\mathcal{K}}(\boldsymbol{q})(\mathrm{d}\boldsymbol{q})$ of the indicator function $\Psi_{\mathcal{K}}(\boldsymbol{q})$ equals the indicator function on the associated contingent cone $K_{\mathcal{K}}(\boldsymbol{q})$ (see Equation (164))

$$\mathrm{d}\Psi_{\mathcal{K}}(\boldsymbol{q})(\mathrm{d}\boldsymbol{q}) = \Psi_{K_{\mathcal{K}}(\boldsymbol{q})}(\mathrm{d}\boldsymbol{q}). \tag{65}$$

It holds that $\mathrm{d}\boldsymbol{q} = \boldsymbol{u}\,\mathrm{d}t$ with $\boldsymbol{u} \in K_{\mathcal{K}}(\boldsymbol{q})$ due to the consistency of the system and the indicator function on the contingent cone therefore vanishes, i.e. $\Psi_{K_{\mathcal{K}}(\boldsymbol{q})}(\boldsymbol{u}\,\mathrm{d}t) = 0$. Consequently, the differential measure of Q simplifies to

$$\begin{aligned}\mathrm{d}Q &= U_{,\boldsymbol{q}}\,\mathrm{d}\boldsymbol{q} + \Psi_{K_{\mathcal{K}}(\boldsymbol{q})}(\mathrm{d}\boldsymbol{q})\\ &= U_{,\boldsymbol{q}}\boldsymbol{u}\,\mathrm{d}t + \Psi_{K_{\mathcal{K}}(\boldsymbol{q})}(\boldsymbol{u}\,\mathrm{d}t), \quad \boldsymbol{u} \in K_{\mathcal{K}}(\boldsymbol{q})\\ &= U_{,\boldsymbol{q}}\boldsymbol{u}\,\mathrm{d}t.\end{aligned} \tag{66}$$

The kinetic energy $T(\boldsymbol{q},\boldsymbol{u}) = \frac{1}{2}\boldsymbol{u}^\mathrm{T}\boldsymbol{M}(\boldsymbol{q})\boldsymbol{u}$ is a symmetric quadratic form in $\boldsymbol{u}$. Using the results of Appendix E, we deduce that the differential measure of T is

$$\mathrm{d}T = \frac{1}{2}(\boldsymbol{u}^+ + \boldsymbol{u}^-)^\mathrm{T}\boldsymbol{M}(\boldsymbol{q})\,\mathrm{d}\boldsymbol{u} + T_{,\boldsymbol{q}}\,\mathrm{d}\boldsymbol{q}. \tag{67}$$

The differential measure of the Lyapunov candidate V becomes

$$\begin{aligned}\mathrm{d}V &\overset{(66)+(67)}{=} \frac{1}{2}(\boldsymbol{u}^+ + \boldsymbol{u}^-)^\mathrm{T}\boldsymbol{M}(\boldsymbol{q})\,\mathrm{d}\boldsymbol{u} + (T_{,\boldsymbol{q}} + U_{,\boldsymbol{q}})\,\mathrm{d}\boldsymbol{q}\\ &\overset{(36)}{=} \frac{1}{2}(\boldsymbol{u}^+ + \boldsymbol{u}^-)^\mathrm{T}(\boldsymbol{h}(\boldsymbol{q},\boldsymbol{u})\,\mathrm{d}t + \boldsymbol{W}\,\mathrm{d}\boldsymbol{\Lambda})\\ &\quad + (T_{,\boldsymbol{q}} + U_{,\boldsymbol{q}})\boldsymbol{u}\,\mathrm{d}t.\end{aligned} \tag{68}$$

A term $\frac{1}{2}(\boldsymbol{u}^+ + \boldsymbol{u}^-)^{\mathrm{T}}\,\mathrm{d}t$ in front of a Lebesgue measurable term equals $\boldsymbol{u}^{\mathrm{T}}\,\mathrm{d}t$. Together with Equation (49), i.e. $\boldsymbol{h} = \boldsymbol{f}^{\mathrm{nc}} + \boldsymbol{f}^{\mathrm{gyr}} - (T_{,\boldsymbol{q}} + U_{,\boldsymbol{q}})^{\mathrm{T}}$, and Equation (34) with Equation (37) we obtain

$$\begin{aligned}\mathrm{d}V &= \boldsymbol{u}^{\mathrm{T}}\boldsymbol{f}^{\mathrm{nc}}\,\mathrm{d}t + \boldsymbol{u}^{\mathrm{T}}\boldsymbol{f}^{\mathrm{gyr}}\,\mathrm{d}t + \boldsymbol{u}^{\mathrm{T}}\boldsymbol{W}\boldsymbol{\lambda}\,\mathrm{d}t\\ &\quad + \frac{1}{2}(\boldsymbol{u}^+ + \boldsymbol{u}^-)^{\mathrm{T}}\boldsymbol{W}\boldsymbol{\Lambda}\,\mathrm{d}\eta.\end{aligned} \tag{69}$$

The gyroscopic forces have zero power $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{f}^{\mathrm{gyr}} = 0$ (see Equation (47)). Moreover, the constraints are assumed to be scleronomic and according to Equation (28) it therefore holds that $\boldsymbol{\gamma} = \boldsymbol{W}^{\mathrm{T}}\boldsymbol{u}$, which gives

$$\begin{aligned}\mathrm{d}V &= \boldsymbol{u}^{\mathrm{T}}\boldsymbol{f}^{\mathrm{nc}}\,\mathrm{d}t + \boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{\lambda}\,\mathrm{d}t + \frac{1}{2}(\boldsymbol{\gamma}^+ + \boldsymbol{\gamma}^-)^{\mathrm{T}}\boldsymbol{\Lambda}\,\mathrm{d}\eta\\ &\overset{(39)}{=} \boldsymbol{u}^{\mathrm{T}}\boldsymbol{f}^{\mathrm{nc}}\,\mathrm{d}t + \boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{\lambda}\,\mathrm{d}t + \tfrac{1}{2}((\boldsymbol{I}+\boldsymbol{E})^{-1}(2\boldsymbol{\xi}\\ &\quad - (\boldsymbol{I}-\boldsymbol{E})\boldsymbol{\delta}))^{\mathrm{T}}\boldsymbol{\Lambda}\,\mathrm{d}\eta\\ &= \boldsymbol{u}^{\mathrm{T}}\boldsymbol{f}^{\mathrm{nc}}\,\mathrm{d}t + \boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{\lambda}\,\mathrm{d}t + \boldsymbol{\xi}^{\mathrm{T}}(\boldsymbol{I}+\boldsymbol{E})^{-1}\boldsymbol{\Lambda}\,\mathrm{d}\eta\\ &\quad - \frac{1}{2}\boldsymbol{\delta}^{\mathrm{T}}(\boldsymbol{I}-\boldsymbol{E})(\boldsymbol{I}+\boldsymbol{E})^{-1}\boldsymbol{\Lambda}\,\mathrm{d}\eta\\ &\overset{(34)+(38)}{=} \boldsymbol{u}^{\mathrm{T}}\boldsymbol{f}^{\mathrm{nc}}\,\mathrm{d}t + \boldsymbol{\xi}^{\mathrm{T}}(\boldsymbol{I}+\boldsymbol{E})^{-1}\,\mathrm{d}\boldsymbol{\Lambda}\\ &\quad - \frac{1}{2}\boldsymbol{\delta}^{\mathrm{T}}(\boldsymbol{I}-\boldsymbol{E})(\boldsymbol{I}+\boldsymbol{E})^{-1}\boldsymbol{\Lambda}\,\mathrm{d}\eta\\ &= \boldsymbol{u}^{\mathrm{T}}\boldsymbol{f}^{\mathrm{nc}}\,\mathrm{d}t + \sum_{i\in I_{\mathrm{N}}}\left(\frac{\xi_{\mathrm{N}i}\,\mathrm{d}\Lambda_{\mathrm{N}i}}{1+e_{\mathrm{N}i}} + \frac{\boldsymbol{\xi}_{\mathrm{T}i}^{\mathrm{T}}\,\mathrm{d}\boldsymbol{\Lambda}_{\mathrm{T}i}}{1+e_{\mathrm{T}i}}\right)\\ &\quad - \frac{1}{2}\boldsymbol{\delta}^{\mathrm{T}}(\boldsymbol{I}-\boldsymbol{E})(\boldsymbol{I}+\boldsymbol{E})^{-1}\boldsymbol{\Lambda}\,\mathrm{d}\eta.\end{aligned} \tag{70}$$

Using Equations (35) and (158), we obtain

$$\begin{aligned}\xi_{\mathrm{N}i}\,\mathrm{d}\Lambda_{\mathrm{N}i} &= -\Psi^*_{C_{\mathrm{N}}}(\xi_{\mathrm{N}i})(\mathrm{d}t + \mathrm{d}\eta) = 0\\ \boldsymbol{\xi}_{\mathrm{T}i}^{\mathrm{T}}\,\mathrm{d}\boldsymbol{\Lambda}_{\mathrm{T}i} &= -\Psi^*_{C_{\mathrm{T}i}(\lambda_{\mathrm{N}i})}(\boldsymbol{\xi}_{\mathrm{T}i})\,\mathrm{d}t\\ &\quad - \Psi^*_{C_{\mathrm{T}i}(\Lambda_{\mathrm{N}i})}(\boldsymbol{\xi}_{\mathrm{T}i})\,\mathrm{d}\eta \le 0,\end{aligned} \tag{71}$$

because of Equation (159) and $\Psi^*_{C_{\mathrm{N}}}(\xi_{\mathrm{N}i}) = \Psi_{\mathbb{R}^+}(\xi_{\mathrm{N}i}) = 0$ for admissible $\xi_{\mathrm{N}i} \ge 0$. Moreover, applying Equation (28) to Equation (38) gives

$$\boldsymbol{\delta} \coloneqq \boldsymbol{\gamma}^+ - \boldsymbol{\gamma}^- = \boldsymbol{W}^{\mathrm{T}}(\boldsymbol{u}^+ - \boldsymbol{u}^-) = \boldsymbol{W}^{\mathrm{T}}\boldsymbol{M}^{-1}\boldsymbol{W}\boldsymbol{\Lambda} = \boldsymbol{G}\boldsymbol{\Lambda}, \tag{72}$$

in which we used the abbreviation

$$\boldsymbol{G} := \boldsymbol{W}^{\mathrm{T}}\boldsymbol{M}^{-1}\boldsymbol{W}, \tag{73}$$

which is known as the Delassus matrix [34]. The matrix $\boldsymbol{G}$ is positive definite when $\boldsymbol{W}$ has full rank, because $\boldsymbol{M} > 0$. The matrix $\boldsymbol{G}$ is only positive semi-definite if the matrix $\boldsymbol{W}$ does not have full rank, meaning that the generalised force directions of the contact forces are linearly dependent. However, we assume that the matrix $\boldsymbol{W}$ only contains the generalised force directions of unilateral constraints, and that these unilateral constraints do not constitute a bilateral constraint. It therefore holds that there exists no $\boldsymbol{\Lambda}_{\mathrm{N}} \neq \boldsymbol{0}$ such that $\boldsymbol{W}_{\mathrm{N}}\boldsymbol{\Lambda}_{\mathrm{N}} = \boldsymbol{0}$. The impact law requires that $\boldsymbol{\Lambda}_{\mathrm{N}} \ge \boldsymbol{0}$. Hence, it holds that $\boldsymbol{\Lambda}_{\mathrm{N}}^{\mathrm{T}}\boldsymbol{W}_{\mathrm{N}}^{\mathrm{T}}\boldsymbol{M}^{-1}\boldsymbol{W}_{\mathrm{N}}\boldsymbol{\Lambda}_{\mathrm{N}} > 0$ for all $\boldsymbol{\Lambda}_{\mathrm{N}} \neq \boldsymbol{0}$ with $\boldsymbol{\Lambda}_{\mathrm{N}} \ge \boldsymbol{0}$, even if the unilateral constraints are linearly dependent. Moreover, $\boldsymbol{\Lambda}_{\mathrm{T}} \neq \boldsymbol{0}$ implies $\boldsymbol{\Lambda}_{\mathrm{N}} \neq \boldsymbol{0}$. The inequality $\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{G}\boldsymbol{\Lambda} > 0$ therefore holds for all $\boldsymbol{\Lambda} \neq \boldsymbol{0}$ which obey the impact law (22), even if dependent unilateral constraints are considered.

Using Equation (72), we can put the last term in Equation (70) in the following quadratic form

$$\begin{aligned}&\frac{1}{2}\boldsymbol{\delta}^{\mathrm{T}}(\boldsymbol{I}-\boldsymbol{E})(\boldsymbol{I}+\boldsymbol{E})^{-1}\boldsymbol{\Lambda}\,\mathrm{d}\eta\\ &\quad = \frac{1}{2}\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{G}(\boldsymbol{I}-\boldsymbol{E})(\boldsymbol{I}+\boldsymbol{E})^{-1}\boldsymbol{\Lambda}\,\mathrm{d}\eta.\end{aligned} \tag{74}$$

in which $\boldsymbol{G}(\boldsymbol{I}-\boldsymbol{E})(\boldsymbol{I}+\boldsymbol{E})^{-1}$ is a square matrix.

The matrix $(\boldsymbol{I}-\boldsymbol{E})(\boldsymbol{I}+\boldsymbol{E})^{-1}$ is a diagonal matrix which is positive definite if the contacts are not purely elastic, i.e. $0 \le e_{\mathrm{N}i} < 1$ and $0 \le e_{\mathrm{T}i} < 1$ for all i. The smallest diagonal element of $(\boldsymbol{I}-\boldsymbol{E})(\boldsymbol{I}+\boldsymbol{E})^{-1}$ is $\frac{1-e_{\max}}{1+e_{\max}}$. Using Proposition 4 in Appendix A, we deduce that if $\boldsymbol{G}$ is positive definite and if condition 7a holds, then the positive definiteness of $\boldsymbol{G}(\boldsymbol{I}-\boldsymbol{E})(\boldsymbol{I}+\boldsymbol{E})^{-1}$ implies

$$\frac{1}{2}\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{G}(\boldsymbol{I}-\boldsymbol{E})(\boldsymbol{I}+\boldsymbol{E})^{-1}\boldsymbol{\Lambda} > 0, \quad \forall\boldsymbol{\Lambda} \neq \boldsymbol{0}. \tag{75}$$

If the generalised force directions are linearly dependent, then the Delassus matrix $\boldsymbol{G}$ is singular and $\mathrm{cond}(\boldsymbol{G})$ is infinity. Condition 7a can therefore not hold.

If $\boldsymbol{G}$ is positive semi-definite (or even positive definite) and all restitution coefficients are equal to e (condition 7b), then the product $\frac{1}{2}\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{G}(\boldsymbol{I}-\boldsymbol{E})(\boldsymbol{I}+\boldsymbol{E})^{-1}\boldsymbol{\Lambda}$ simplifies to $\frac{1}{2}\frac{1-e}{1+e}\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{G}\boldsymbol{\Lambda}$ which is in general non-negative. Again, we can show that Equation (75) still holds for dependent unilateral constraints if we consider $\boldsymbol{\Lambda} \neq \boldsymbol{0}$ with $\boldsymbol{\Lambda} \ge \boldsymbol{0}$.

If $\boldsymbol{G}$ is positive semi-definite (or even positive definite) and friction is absent (condition 7c: $\mu_i = 0\,\forall i \in$

I_G), then it holds that

$$
\begin{aligned}
&\frac{1}{2}\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{G}(\boldsymbol{I}-\boldsymbol{E})(\boldsymbol{I}+\boldsymbol{E})^{-1}\boldsymbol{\Lambda} \\
&\quad=\frac{1}{2}(\gamma_{\mathrm{N}}^{+}-\gamma_{\mathrm{N}}^{-})^{\mathrm{T}}(\boldsymbol{I}-\boldsymbol{E})(\boldsymbol{I}+\boldsymbol{E})^{-1}\boldsymbol{\Lambda}_{\mathrm{N}} \\
&\quad=\sum_{i}\frac{1}{2}(\gamma_{\mathrm{N}i}^{+}-\gamma_{\mathrm{N}i}^{-})\frac{1-e_{\mathrm{N}i}}{1+e_{\mathrm{N}i}}\Lambda_{\mathrm{N}i}.
\end{aligned}
\tag{76}
$$

The impact law requires that $\gamma_{\mathrm{N}i}^{+}+e_{\mathrm{N}i}\gamma_{\mathrm{N}i}^{-}>0$ and $\Lambda_{\mathrm{N}i}\geq 0$. Moreover, the unilateral contacts did not penetrate before the impact and the pre-impact relative velocities $\gamma_{\mathrm{N}i}^{-}$ are therefore non-positive. The post-impact relative velocities $\gamma_{\mathrm{N}i}^{+}=-e_{\mathrm{N}i}\gamma_{\mathrm{N}i}^{-}$ are therefore non-negative for $0\leq e_{\mathrm{N}i}<1$. Furthermore, if $\Lambda_{\mathrm{N}i}>0$, then it must hold that $\gamma_{\mathrm{N}i}^{-}<0$. Hence, $\frac{1}{2}\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{G}(\boldsymbol{I}-\boldsymbol{E})(\boldsymbol{I}+\boldsymbol{E})^{-1}\boldsymbol{\Lambda}>0$ for all $\boldsymbol{\Lambda}\neq\boldsymbol{0}$ with $\boldsymbol{\Lambda}\geq\boldsymbol{0}$.

Looking again at the differential measure of the total energy (70), we realise that (under conditions 6 and 7) all terms related to the contact forces and impulses are dissipative or passive. Moreover, if we consider not purely elastic contacts, then nonzero contact impulses $\boldsymbol{\Lambda}$ strictly dissipate energy.

We can now decompose the differential measure $\mathrm{d}V$ in a Lebesgue part and an atomic part

$$
\mathrm{d}V=\dot{V}\,\mathrm{d}t+(V^{+}-V^{-})\,\mathrm{d}\eta,
\tag{77}
$$

with (see Equation (52) and above)

$$
\begin{aligned}
\dot{V}&=\boldsymbol{u}^{\mathrm{T}}\boldsymbol{f}^{\mathrm{nc}}-\sum_{i\in I_{\mathrm{N}}}\frac{1}{1+e_{\mathrm{T}i}}\Psi^{*}_{C_{\mathrm{T}i}(\lambda_{\mathrm{N}i})}(\boldsymbol{\xi}_{\mathrm{T}i}) \\
&=-D_{\boldsymbol{q}}^{\mathrm{nc}}(\boldsymbol{u})-D_{\boldsymbol{q}}^{\lambda_{\mathrm{T}}}(\boldsymbol{u}) \\
&\leq 0
\end{aligned}
\tag{78}
$$

and

$$
\begin{aligned}
V^{+}-V^{-}&=-\sum_{i\in I_{\mathrm{N}}}\left(\frac{1}{1+e_{\mathrm{T}i}}\Psi^{*}_{C_{\mathrm{T}i}(\Lambda_{\mathrm{N}i})}(\boldsymbol{\xi}_{\mathrm{T}i})\right) \\
&\quad-\frac{1}{2}\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{G}(\boldsymbol{I}+\boldsymbol{E})^{-1}(\boldsymbol{I}-\boldsymbol{E})\boldsymbol{\Lambda} \\
&=-D_{\boldsymbol{q}}^{\Lambda_{\mathrm{T}}}(\boldsymbol{u})-\frac{1}{2}\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{G}(\boldsymbol{I}+\boldsymbol{E})^{-1}(\boldsymbol{I}-\boldsymbol{E})\boldsymbol{\Lambda} \\
&\leq 0.
\end{aligned}
\tag{79}
$$

For positive differential measures $\mathrm{d}t$ and $\mathrm{d}\eta$, we deduce that the differential measure of V (77) is non-positive, $\mathrm{d}V\leq 0$. There are a number of cases for $\mathrm{d}V$ to distinguish:

- **Case $\boldsymbol{u}=\boldsymbol{0}$:** It directly follows that $\mathrm{d}V=0$.
- **Case** $g_{\mathrm{N}i}=0$ **and** $\gamma_{\mathrm{N}i}^{-}<0$ **for some** $i\in I_{\mathrm{N}}$**:** One or more contacts are closing, i.e. there are impacts. It follows from (75) that $V^{+}-V^{-}<0$ and therefore that $\mathrm{d}V<0$.
- **Case $\boldsymbol{g}_{NC}=\boldsymbol{0}$, $\boldsymbol{u}\in\ker\boldsymbol{W}_{NC}^{\mathrm{T}}$ and $\boldsymbol{u}=\boldsymbol{u}^{-}=\boldsymbol{u}^{+}$ with $\boldsymbol{g}_{NC}=\{g_{\mathrm{N}i}\}$ for $i\in I_C$:** It then holds that all contacts in I_C are closed and remain closed, $I_C\subset I_{\mathrm{N}}$. We now consider $\dot{V}$ as a non-linear operator on $\boldsymbol{u}$ and write

$$
\begin{aligned}
&\dot{V}=0, \quad \boldsymbol{u}\in\dot{V}_{\boldsymbol{q}}^{-1}(\boldsymbol{0}), \\
&\dot{V}<0, \quad \boldsymbol{u}\notin\dot{V}_{\boldsymbol{q}}^{-1}(\boldsymbol{0}),
\end{aligned}
\tag{80}
$$

with

$$
\begin{aligned}
\dot{V}_{\boldsymbol{q}}^{-1}(\boldsymbol{0})&=D_{\boldsymbol{q}}^{\mathrm{nc}\,-1}(0)\cap D_{\boldsymbol{q}}^{\lambda_{\mathrm{T}}\,-1}(0) \\
&\subset D_{\boldsymbol{q}}^{\mathrm{nc}\,-1}(0)\cap D_{\boldsymbol{q}}^{\lambda_{TC}\,-1}(0).
\end{aligned}
\tag{81}
$$

Condition 5 of the theorem states that, if the contacts in I_C are persistent ($\boldsymbol{W}_{NC}^{\mathrm{T}}\boldsymbol{u}=\boldsymbol{0}$), then dissipation can only vanish if $\boldsymbol{u}=\boldsymbol{0}$, i.e. $D_{\boldsymbol{q}}^{\mathrm{nc}\,-1}(0)\cap D_{\boldsymbol{q}}^{\lambda_{TC}\,-1}(0)=\{\boldsymbol{0}\}$. In other words, if all contacts in I_C are closed and remain closed and $\boldsymbol{u}\neq\boldsymbol{0}$ then dissipation is present. Using condition 5 and $\boldsymbol{u}\in\ker\boldsymbol{W}_{NC}^{\mathrm{T}}\setminus\{\boldsymbol{0}\}$, it follows that $\dot{V}_{\boldsymbol{q}}^{-1}(\boldsymbol{0})=\{\boldsymbol{0}\}$ and hence

$$
\begin{aligned}
&\dot{V}=0, \quad \boldsymbol{u}=\boldsymbol{0}, \\
&\dot{V}<0, \quad \boldsymbol{u}\neq\boldsymbol{0}.
\end{aligned}
\tag{82}
$$

Impulsive motion for this case is excluded. For a strictly positive differential measure $\mathrm{d}t$, we obtain the differential measure of V as given in Equation (77)

$$
\begin{aligned}
&\mathrm{d}V=0, \quad \boldsymbol{u}=\boldsymbol{0}, \\
&\mathrm{d}V<0, \quad \boldsymbol{u}\neq\boldsymbol{0}.
\end{aligned}
\tag{83}
$$

- **Case $\boldsymbol{g}_{NC}=\boldsymbol{0}$, $\boldsymbol{u}\notin\ker\boldsymbol{W}_{NC}^{\mathrm{T}}\setminus\{\boldsymbol{0}\}$ and $\boldsymbol{W}_{\mathrm{N}i}\boldsymbol{u}>0$ for some $i\in I_C$:** It then holds that one or more contacts will open. All we can say is that $\mathrm{d}V\leq 0$.
- **Case** $g_{\mathrm{N}i}>0$ **for some** $i\in I_C$**:** One or more contacts are open. All we can say is that $\mathrm{d}V\leq 0$.

We conclude that

$$
\begin{aligned}
&\mathrm{d}V=0 \quad \text{for } \boldsymbol{u}=\boldsymbol{0}, \\
&\mathrm{d}V\leq 0 \quad \text{for } \boldsymbol{g}_{NC}\neq\boldsymbol{0}, \\
&\mathrm{d}V<0 \quad \text{for } \boldsymbol{g}_{NC}=\boldsymbol{0},\boldsymbol{u}^{-}\neq\boldsymbol{0}.
\end{aligned}
\tag{84}
$$

We now apply a generalisation of LaSalle's invariance principle, which is valid when every limit set is a positively invariant set [14, 28]. A sufficient condition for the latter is continuity of the solution with respect to the initial condition. Non-smooth mechanical systems with multiple impacts do generally not possess continuity with respect to the initial condition. It is therefore explicitly stated in Condition 9 of Theorem 1 that every limit set in $\mathcal{I}_{\rho^*}$ is positively invariant. Hence, under this assumption, the generalisation of LaSalle's invariance principle can be applied.

Let us consider the set $\mathcal{I}_{\rho^*}$ where ρ^* is chosen such that $\mathcal{I}_{\rho^*} \subset (\mathcal{V} \cap \mathcal{Q})$, see Equation (62). Note that $\mathcal{I}_{\rho^*}$ is a positively invariant set due to the choice of V. Moreover, the set $\mathcal{S}$ is defined as

$$\mathcal{S} = \{(\boldsymbol{q}, \boldsymbol{u}) \mid \mathrm{d}V = 0\}, \tag{85}$$

which generally has a nonzero intersection with $\mathcal{P} = \{(\boldsymbol{q}, \boldsymbol{u}) \mid \boldsymbol{g}_{NC} \neq \boldsymbol{0}, \boldsymbol{g}_{NC} \geq \boldsymbol{0}\}$.

Consider a solution curve with an arbitrary initial condition in $\mathcal{P}$ for $t = t_0$. Due to condition 4 of the theorem, which requires that $\dot{\gamma}_{\mathrm{N}i} < 0$ (a.e.) for $\forall i \in I_C \backslash I_{\mathrm{N}}$, at least one impact will occur for some $t > t_0$. The impact does not necessarily occur at a contact in I_C. In any case, the impact will cause $\mathrm{d}V < 0$ at the impact time. Therefore, there exists no solution curve with initial condition in $\mathcal{P}$ that remains in the intersection $\mathcal{P} \cap \mathcal{S}$. Hence, it holds that the intersection $\mathcal{P} \cap \mathcal{S}$ does not contain any invariant sub-set. We therefore seek the largest invariant set in $\mathcal{T} = \{(\boldsymbol{q}, \boldsymbol{u}) \mid \boldsymbol{g}_{NC}(\boldsymbol{q}) = \boldsymbol{0}, \boldsymbol{u} = \boldsymbol{0}\}$. Using the fact that $\boldsymbol{u}$ should be zero, and that this implies that no impulsive forces can occur in the measure differential inclusion describing the dynamics of the system, yields:

$$\begin{aligned}
&\boldsymbol{M}(\boldsymbol{q})\,\mathrm{d}\boldsymbol{u} - \boldsymbol{h}(\boldsymbol{q}, \boldsymbol{0})\,\mathrm{d}t = \boldsymbol{W}_{\mathrm{N}}(\boldsymbol{q})\,\mathrm{d}\boldsymbol{\Lambda}_{\mathrm{N}} + \boldsymbol{W}_{\mathrm{T}}(\boldsymbol{q})\,\mathrm{d}\boldsymbol{\Lambda}_{\mathrm{T}} \\
&\quad\Rightarrow\ \boldsymbol{h}(\boldsymbol{q}, \boldsymbol{0})\,\mathrm{d}t + \boldsymbol{W}_{\mathrm{N}}(\boldsymbol{q})\boldsymbol{\lambda}_{\mathrm{N}}\,\mathrm{d}t + \boldsymbol{W}_{\mathrm{T}}(\boldsymbol{q})\boldsymbol{\lambda}_{\mathrm{T}}\,\mathrm{d}t = \boldsymbol{0} \\
&\quad\Rightarrow\ \boldsymbol{h}(\boldsymbol{q}, \boldsymbol{0}) + \boldsymbol{W}_{\mathrm{N}}(\boldsymbol{q})\boldsymbol{\lambda}_{\mathrm{N}} + \boldsymbol{W}_{\mathrm{T}}(\boldsymbol{q})\boldsymbol{\lambda}_{\mathrm{T}} = \boldsymbol{0} \\
&\quad\Rightarrow\ \boldsymbol{h}(\boldsymbol{q}, \boldsymbol{0}) - \sum_i \boldsymbol{W}_{N_i}(\boldsymbol{q})\partial\Psi^*_{C_{N_i}}(\boldsymbol{0}) \\
&\qquad - \sum_i \boldsymbol{W}_{T_i}(\boldsymbol{q})\partial\Psi^*_{C_{T_i}}(\boldsymbol{0}) \ni \boldsymbol{0} \\
&\quad\Rightarrow\ \boldsymbol{h}(\boldsymbol{q}, \boldsymbol{0}) + \sum_i \boldsymbol{W}_{N_i}(\boldsymbol{q})\mathbb{R}^+ - \sum_i \boldsymbol{W}_{T_i}(\boldsymbol{q})C_{T_i} \ni \boldsymbol{0}.
\end{aligned} \tag{86}$$

Consequently, we can conclude that the largest invariant set in $\mathcal{S}$ is the equilibrium set $\mathcal{E}$. Hence, it can be concluded from LaSalle's invariance principle that $\mathcal{E}$ is an attractive set. □

Remark. If no conditions on the restitution coefficients exist (other than $0 \leq e_{\mathrm{N}i} < 1$ and $|e_{\mathrm{T}i}| < 1 \forall i$) and if friction is present, then the impact laws (35) can, under circumstances, lead to an energy increase. Such an energetic inconsistency has been reported by Kane and Levinson [24]. In the proof of Theorem 1, we derived sufficient conditions for the energetical consistency (dissipativity) of the adopted impact laws.

In the following propositions we derive some sufficient conditions for conditions 3–5 of Theorem 1. These conditions are less general but easier to check.

Proposition 1 (Sufficient conditions for condition 4). *Let $\dot{\boldsymbol{\gamma}}_{No} = \{\dot{\gamma}_{\mathrm{N}i}\}$, $i \in I_G \backslash I_{\mathrm{N}}$, be the normal contact accelerations of the open contacts and $\dot{\boldsymbol{\gamma}}_{Nc} = \{\dot{\gamma}_{\mathrm{N}i}\}$, $i \in I_{\mathrm{N}}$, be the normal contact accelerations of the closed contacts. If the following conditions are fulfilled*

1. $\boldsymbol{W}_{No}^{\mathrm{T}}\boldsymbol{M}^{-1}(\boldsymbol{I} - \boldsymbol{W}_{Nc}(\boldsymbol{W}_{Nc}^{\mathrm{T}}\boldsymbol{M}^{-1}\boldsymbol{W}_{Nc})^{-1}\boldsymbol{W}_{Nc}^{\mathrm{T}}\boldsymbol{M}^{-1})\boldsymbol{h} < \boldsymbol{0}$ *with* $\boldsymbol{W}_{No} = \{\boldsymbol{W}_{\mathrm{N}i}\}, \boldsymbol{W}_{Nc} = \{\boldsymbol{W}_{Nj}\}$, $j \in I_{\mathrm{N}}$, $i \in I_G \backslash I_{\mathrm{N}}$ *for arbitrary sub-sets* $I_{\mathrm{N}} \subset I_G$,
2. $\boldsymbol{W}_{\mathrm{N}}^{\mathrm{T}}\boldsymbol{M}^{-1}\boldsymbol{W}_{\mathrm{T}} = \boldsymbol{O}$,

then it holds that $\dot{\boldsymbol{\gamma}}_{No} < 0$ for almost all t, which is equivalent to condition 4 of Theorem 1 with $I_C = I_G$.

Proof: Consider an arbitrary index set I_{N} of temporarily closed contacts. We consider the contacts to be closed for a nonzero time-interval. The normal contact accelerations of the closed contacts $\dot{\boldsymbol{\gamma}}_{Nc}$ are therefore zero:

$$\begin{aligned}
\dot{\boldsymbol{\gamma}}_{Nc} &= \boldsymbol{W}_{Nc}^{\mathrm{T}}\dot{\boldsymbol{u}} \\
\boldsymbol{0} &= \boldsymbol{W}_{Nc}^{\mathrm{T}}\boldsymbol{M}^{-1}(\boldsymbol{h} + \boldsymbol{W}_c\boldsymbol{\lambda}_c) \\
\boldsymbol{0} &= \boldsymbol{W}_{Nc}^{\mathrm{T}}\boldsymbol{M}^{-1}(\boldsymbol{h} + \boldsymbol{W}_{Nc}\boldsymbol{\lambda}_{Nc})
\end{aligned} \tag{87}$$

The normal contact forces $\boldsymbol{\lambda}_{Nc}$ of the closed contacts can therefore for almost all t be expressed as:

$$\boldsymbol{\lambda}_{Nc} = -\left(\boldsymbol{W}_{Nc}^{\mathrm{T}}\boldsymbol{M}^{-1}\boldsymbol{W}_{Nc}\right)^{-1}\boldsymbol{W}_{Nc}^{\mathrm{T}}\boldsymbol{M}^{-1}\boldsymbol{h}. \tag{88}$$

It therefore holds for the normal contact accelerations of the open contacts $\dot{\boldsymbol{\gamma}}_{No}$ that

$$\begin{aligned}
\dot{\boldsymbol{\gamma}}_{No} &= \boldsymbol{W}_{No}^{\mathrm{T}}\dot{\boldsymbol{u}} \\
&= \boldsymbol{W}_{No}^{\mathrm{T}}\boldsymbol{M}^{-1}(\boldsymbol{h} + \boldsymbol{W}_c\boldsymbol{\lambda}_c)
\end{aligned}$$

$$
\begin{aligned}
&= \boldsymbol{W}_{No}^{\mathrm{T}} \boldsymbol{M}^{-1}(\boldsymbol{h} + \boldsymbol{W}_{Nc}\boldsymbol{\lambda}_{Nc}) \\
&= \boldsymbol{W}_{No}^{\mathrm{T}} \boldsymbol{M}^{-1}\big(\boldsymbol{h} - \\
&\quad \boldsymbol{W}_{Nc}(\boldsymbol{W}_{Nc}^{\mathrm{T}}\boldsymbol{M}^{-1}\boldsymbol{W}_{Nc})^{-1}\boldsymbol{W}_{Nc}^{\mathrm{T}}\boldsymbol{M}^{-1}\boldsymbol{h}\big) \\
&< 0 \qquad (89)
\end{aligned}
$$

for almost all t. □

Proposition 2. *If $\boldsymbol{f}^{\mathrm{nc}} = -\boldsymbol{C}\boldsymbol{u}$, then it holds that $D_{\boldsymbol{q}}^{\mathrm{nc}-1}(0) = \ker \boldsymbol{C}$, i.e. the zero set of $D_{\boldsymbol{q}}^{\mathrm{nc}}(\boldsymbol{u})$ is the nullspace of $\boldsymbol{C}$.*

Proof: Substitution gives $D_{\boldsymbol{q}}^{\mathrm{nc}}(\boldsymbol{u}) = \boldsymbol{u}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{u}$. The proof is immediate. □

The forces $\boldsymbol{\lambda}_{\mathrm{T}i}$ (and impulses $\boldsymbol{\Lambda}_{\mathrm{T}i}$), which are derived from a support function on the set $C_{\mathrm{T}i}$, have in the above almost always been associated with friction forces, but can also be forces from a one-way clutch. Friction and the one-way clutch are described by the same inclusion on velocity level, but they are different in the sense that $\boldsymbol{0} \in \mathrm{bdry} C_{\mathrm{T}i}$ holds for the one-way clutch and $\boldsymbol{0} \in \mathrm{int}\, C_{\mathrm{T}i}$ holds for friction. The dissipation function of friction is a PDF, meaning that friction is dissipative when a relative sliding velocity is present, whereas no dissipation occurs in the one-way clutch. This insight leads to the following proposition:

Proposition 3. *If $\boldsymbol{0} \in \mathrm{int}\, C_{\mathrm{T}i}\ \forall i \in I_G$, then it holds that $D_{\boldsymbol{q}}^{\lambda_D\,-1}(0) = \ker \boldsymbol{W}_{\mathrm{T}}^{\mathrm{T}}(\boldsymbol{q})$, i.e. the zero set of $D_{\boldsymbol{q}}^{\lambda_{\mathrm{T}}}(\boldsymbol{u})$ is the nullspace of $\boldsymbol{W}_D^{\mathrm{T}}(\boldsymbol{q})$.*

Proof: Because of $\boldsymbol{0} \in \mathrm{int}\, C_{\mathrm{T}i}\ \forall i \in I_G$, it follows from Equation (160) that $\Psi_{C_{\mathrm{T}i}}^{*}(\boldsymbol{\gamma}_{\mathrm{T}i}) > 0$ for $\boldsymbol{\gamma}_{\mathrm{T}i} \neq \boldsymbol{0}$, i.e. $\Psi_{C_{\mathrm{T}i}}^{*}(\boldsymbol{\gamma}_{\mathrm{T}i}(\boldsymbol{q},\boldsymbol{u})) = 0 \Leftrightarrow \boldsymbol{\gamma}_{\mathrm{T}i}(\boldsymbol{q},\boldsymbol{u}) = 0$. Moreover, it follows from assumption (28) that $\boldsymbol{\gamma}_{\mathrm{T}i}(\boldsymbol{q},\boldsymbol{u}) = 0 \Leftrightarrow \boldsymbol{u} \in \ker \boldsymbol{W}_{\mathrm{T}i}^{\mathrm{T}}(\boldsymbol{q})$. The proof follows from the definition (52) of $D_{\boldsymbol{q}}^{\lambda_{\mathrm{T}}}(\boldsymbol{u})$. □

If Propositions 2 and 3 are fulfilled then we can simplify condition 3 and 5 of Theorem 1.

Corollary 1. *If $\boldsymbol{f}^{\mathrm{nc}} = -\boldsymbol{C}\boldsymbol{u}$ and $\boldsymbol{0} \in \mathrm{int}\, C_{\mathrm{T}i}\ \forall i \in I_G$, then condition 3 is equivalent to $\boldsymbol{C} > 0$ and condition 5 is equivalent to $\ker \boldsymbol{C} \cap \ker \boldsymbol{W}_{\mathrm{T}}^{\mathrm{T}}(\boldsymbol{q}) \cap \ker \boldsymbol{W}_{\mathrm{N}}^{\mathrm{T}}(\boldsymbol{q}) = \{\boldsymbol{0}\}$.*

Using Propositions 1–3 and Corollary 1, we can formulate the following corollary which is a special case of Theorem 1:

Corollary 2. *Consider an equilibrium set $\mathcal{E}$ of the system (36) with constitutive laws (27) and (35). If*

1. *$T = \frac{1}{2}\boldsymbol{u}^{\mathrm{T}}\boldsymbol{M}(\boldsymbol{q})\boldsymbol{u}$, with $\boldsymbol{M}(\boldsymbol{q}) = \boldsymbol{M}^{\mathrm{T}}(\boldsymbol{q}) > 0$,*
2. *the equilibrium position $\boldsymbol{q}_{\mathrm{e}}$ is a local minimum of the total potential energy $Q(\boldsymbol{q})$ and $Q(\boldsymbol{q})$ has a non-vanishing generalised gradient for all $\boldsymbol{q} \in \mathcal{U}\backslash\{\boldsymbol{q}_{\mathrm{e}}\}$, i.e. $\boldsymbol{0} \notin \partial Q(\boldsymbol{q})\, \forall \boldsymbol{q} \in \mathcal{U}\backslash\{\boldsymbol{q}_{\mathrm{e}}\}$, and the equilibrium set $\mathcal{E}_{\boldsymbol{q}}$ is contained in $\mathcal{U}$, i.e. $\mathcal{E}_{\boldsymbol{q}} \subset \mathcal{U}$,*
3. *$D_{\boldsymbol{q}}^{\mathrm{nc}} = -\boldsymbol{u}^{\mathrm{T}}\boldsymbol{f}^{\mathrm{nc}} = \boldsymbol{u}^{\mathrm{T}}\boldsymbol{C}(\boldsymbol{q})\boldsymbol{u} \geq 0$, i.e. the non-conservative forces are linear in $\boldsymbol{u}$ and dissipative,*
4. *$\boldsymbol{W}_{No}^{\mathrm{T}}\boldsymbol{M}^{-1}(\boldsymbol{I} - \boldsymbol{W}_{Nc}(\boldsymbol{W}_{Nc}^{\mathrm{T}}\boldsymbol{M}^{-1}\boldsymbol{W}_{Nc})^{-1}\boldsymbol{W}_{Nc}^{\mathrm{T}}\boldsymbol{M}^{-1})\boldsymbol{h} < \boldsymbol{0}$ with $\boldsymbol{W}_{No} = \{\boldsymbol{W}_{\mathrm{N}i}\}, \boldsymbol{W}_{Nc} = \{\boldsymbol{W}_{Nj}\}$, $j \in I_{\mathrm{N}}$, $i \in I_G\backslash I_{\mathrm{N}}$ for arbitrary sub-sets $I_{\mathrm{N}} \subset I_G$, and $\boldsymbol{W}_{\mathrm{N}}^{\mathrm{T}}\boldsymbol{M}^{-1}\boldsymbol{W}_{\mathrm{T}} = \boldsymbol{O}$,*
5. *$\ker \boldsymbol{C}(\boldsymbol{q}) \cap \ker \boldsymbol{W}_{\mathrm{T}}^{\mathrm{T}}(\boldsymbol{q}) \cap \ker \boldsymbol{W}_{N}^{\mathrm{T}}(\boldsymbol{q}) = \{\boldsymbol{0}\}\, \forall \boldsymbol{q}$, and $\boldsymbol{0} \in \mathrm{int}\, C_{\mathrm{T}i}$, i.e. there exist no one-way clutches,*
6. *$0 \leq e_{\mathrm{N}i} < 1$, $|e_{\mathrm{T}i}| < 1\ \forall i \in I_G$,*
7. *one of the following conditions holds*
   a. *the restitution coefficients are small in the sense that $\frac{2e_{\max}}{1+e_{\max}} < \frac{1}{cond(\boldsymbol{G}(\boldsymbol{q}))}\ \forall \boldsymbol{q} \in \mathcal{C}$ where $\boldsymbol{G}(\boldsymbol{q}) := \boldsymbol{W}(\boldsymbol{q})^{\mathrm{T}}\boldsymbol{M}(\boldsymbol{q})^{-1}\boldsymbol{W}(\boldsymbol{q})$ and $e_{\max}$ is the largest restitution coefficient, i.e. $e_{\max} \geq \max(e_{\mathrm{N}i}, e_{\mathrm{T}i})\ \forall i \in I_G$,*
   b. *all restitution coefficients are equal, i.e. $e = e_{\mathrm{N}i} = e_{\mathrm{T}i} \forall i \in I_G$,*
   c. *friction is absent, i.e. $\mu_i = 0\, \forall i \in I_G$,*
8. *$\mathcal{E} \subset \mathcal{I}_{\rho^*}$ in which the set $\mathcal{I}_{\rho^*}$, with $\mathcal{I}_\rho = \{(\boldsymbol{q},\boldsymbol{u}) \in \mathbb{R}^n \times \mathbb{R}^n \mid V(\boldsymbol{q},\boldsymbol{u}) < \rho\}$, is the largest level set of V (60) that is contained in $\mathcal{V}$ and $\mathcal{Q} = \{(\boldsymbol{q},\boldsymbol{u}) \in \mathbb{R}^n \times \mathbb{R}^n \mid \boldsymbol{q} \in \mathcal{U}\}$, i.e.*

   $$\rho^* = \max_{\{\rho : \mathcal{I}_\rho \subset (\mathcal{V}\cap\mathcal{Q})\}} \rho,$$

9. *each limit set in $\mathcal{I}_{\rho^*}$ is positively invariant,*

then the equilibrium set $\mathcal{E}$ is locally attractive and $\mathcal{I}_{\rho^*}$ is a conservative estimate for the region of attraction.

Condition 4 of Corollary 2 replaces condition 4 of Theorem 1 due to Proposition 1. Condition 5 of Corollary 2 and Propositions 2 and 3 replace condition 5 of Theorem 1. Moreover, note that Conditions 3, 5 and 6 of Corollary 2 together imply that for all $(\boldsymbol{q},\boldsymbol{u})$, for which $\boldsymbol{u} \neq \boldsymbol{0}$ and $\boldsymbol{g}_{\mathrm{N}} = \boldsymbol{0}$, the sum of the (smooth) non-conservative forces and the dry friction forces are dissipating energy, which ensures $\dot{V}$ (with V as in (60) being positive definite) to satisfy $\dot{V} < 0$. Consequently, no oscillations can sustain in any sub-space of the generalised coordinate space. Note furthermore, that condition 5 of Corollary 2 implies a friction law (not a one-way clutch) with $\mu_i > 0$, $i \in I_G$, and that the normal forces $\lambda_{\mathrm{N}i}$, $i \in I_G$, do not equal zero. A careful inspection of the proof of Theorem 1 learns that this condition with respect to the normal forces can be relaxed even further. Namely, when the normal forces only equal zero on the set $\{(\boldsymbol{q}, \boldsymbol{u}) \mid \boldsymbol{u} = \boldsymbol{0}\}$ attractivity of the equilibrium set $\mathcal{E}$ can still be guaranteed.

Corollary 2 includes the case of a system for which all smooth forces are conservative, i.e. $\boldsymbol{f}^{\mathrm{nc}} = \boldsymbol{0}$. Dissipation is then only due to impact and friction. Note that in this case the conditions of the corollary imply that $\dot{V} = -\sum_i \Psi^*_{C_{\mathrm{T}i}}(\gamma_{\mathrm{T}i})$ (with V as in (60) being positive definite). Then, condition 5 implies that the columns of $\boldsymbol{W}_{\mathrm{T}}$ span the space $\ker \boldsymbol{W}_{\mathrm{N}}^{\mathrm{T}}$, i.e. that $\boldsymbol{\gamma}_{\mathrm{T}} = \boldsymbol{0}$ if and only if $\boldsymbol{u} = \boldsymbol{0}$ (for $\boldsymbol{u} \in \ker \boldsymbol{W}_{\mathrm{N}}^{\mathrm{T}}$). In combination with condition 6, this ensures that $\mathrm{d}V$ obeys (84). When all smooth forces are conservative, then condition 5 expresses the fact that the dry friction forces should always be dissipative and that the related generalised force directions span the tangent space of the unilateral constraints at every point in the $(\boldsymbol{q}, \boldsymbol{u})$-space.

6 Systems with bilateral constraints and dry friction

In this section, we focus on systems with bilateral constraints with dry friction (frictional sliders). The restriction to bilateral constraints excludes unilateral contact phenomena such as impact and detachment. These kind of systems are very common in engineering practice; think for example of industrial robots with play-free joints. We assume that a set of independent generalised coordinates is known (denoted by $\boldsymbol{q} \in \mathbb{R}^n$ in this section), for which these bilateral constraints are eliminated from the formulation of the dynamics of the system. We formulate the dynamics of the system using a Lagrangian approach, resulting in

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}(T_{,\boldsymbol{u}}) - T_{,\boldsymbol{q}} + U_{,\boldsymbol{q}}\right)^{\mathrm{T}} = \boldsymbol{f}^{\mathrm{nc}} + \boldsymbol{W}_{\mathrm{T}}(\boldsymbol{q})\boldsymbol{\lambda}_{\mathrm{T}}, \tag{90}$$

or, alternatively,

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} - \boldsymbol{h}(\boldsymbol{q}, \boldsymbol{u}) = \boldsymbol{W}_{\mathrm{T}}(\boldsymbol{q})\boldsymbol{\lambda}_{\mathrm{T}}. \tag{91}$$

Herein, $\boldsymbol{M}(\boldsymbol{q}) = \boldsymbol{M}^{\mathrm{T}}(\boldsymbol{q}) > 0$ is the mass-matrix and $T = \frac{1}{2}\boldsymbol{u}^{\mathrm{T}}\boldsymbol{M}(\boldsymbol{q})\boldsymbol{u}$ represents kinetic energy. Moreover, the friction forces are assumed to obey Coulomb's set-valued force law (11). Note that no unilateral contact forces are present in this formulation. Since (normal and tangential) impact is excluded, there is no need to formulate the dynamics on momentum level, since no impulsive forces occur. Consequently, the Equation (90) or (91) together with the set-valued force law (11) represent a differential inclusion on force level. An equilibrium set of Equation (91), being a simply connected set of equilibria, obeys

$$\mathcal{E} \subset \left\{ (\boldsymbol{q}, \boldsymbol{u}) \in \mathbb{R}^n \times \mathbb{R}^n \,|\, (\boldsymbol{u} = \boldsymbol{0}) \wedge \boldsymbol{h}(\boldsymbol{q}, \boldsymbol{0}) - \sum_{i \in I_G} \boldsymbol{W}_{\mathrm{T}i}(\boldsymbol{q}) C_{\mathrm{T}i} \ni \boldsymbol{0} \right\}, \tag{92}$$

where I_G is the set of all frictional bilateral contact points (frictional sliders). An equilibrium set is positively invariant if we assume uniqueness of solutions in forward time.

In Section 6.1, sufficient conditions for the attractivity of equilibrium sets of systems defined by Equations (91) and (11) are stated, based on the results for systems with unilateral contact and impact, proposed in the previous section. In Section 6.2, the instability of an equilibrium set is investigated. Hereto, first a theorem is proposed which states sufficient conditions for the instability of an equilibrium set of a differential inclusion. Subsequently, this result is used to derive sufficient conditions under which an equilibrium set of a linear mechanical system with dry friction is unstable. The latter result in combination with the results on the attractivity of equilibrium sets of a linear mechanical system with dry friction, as proposed in [45], provides a rather complete picture of the stability-related properties of equilibrium sets of such systems.

6.1 Attractivity of equilibrium sets of systems with frictional bilateral constraints

The following result is a corollary of Theorem 1.

Corollary 3 (Attractivity of the equilibrium set). *Consider an equilibrium set $\mathcal{E}$ of system (91) with friction law (11). If*

1. $T = \frac{1}{2}\boldsymbol{u}^{\mathrm{T}}\boldsymbol{M}(\boldsymbol{q})\boldsymbol{u}$, with $\boldsymbol{M}(\boldsymbol{q}) = \boldsymbol{M}^{\mathrm{T}}(\boldsymbol{q}) > 0$,
2. the equilibrium position $\boldsymbol{q}_{\mathrm{e}}$ is a local minimum of the potential energy $U(\boldsymbol{q})$ and $U(\boldsymbol{q})$ has a non-vanishing generalised gradient for all $\boldsymbol{q} \in \mathcal{U}\backslash\{\boldsymbol{q}_{\mathrm{e}}\}$, i.e. $\mathbf{0} \notin \partial U(\boldsymbol{q})\, \forall \boldsymbol{q} \in \mathcal{U}\backslash\{\boldsymbol{q}_{\mathrm{e}}\}$, and the equilibrium set $\mathcal{E}_{\boldsymbol{q}}$ is contained in $\mathcal{U}$, i.e. $\mathcal{E}_{\boldsymbol{q}} \subset \mathcal{U}$,
3. $D_{\boldsymbol{q}}^{\mathrm{nc}}(\boldsymbol{u}) = -\boldsymbol{u}^{\mathrm{T}}\boldsymbol{f}^{\mathrm{nc}} \geq 0$, i.e. the smooth non-conservative forces are dissipative, and $\boldsymbol{f}^{\mathrm{nc}} = \mathbf{0}$ for $\boldsymbol{u} = \mathbf{0}$,
4. $D_{\boldsymbol{q}}^{\mathrm{nc}\,-1}(0) \bigcap D_{\boldsymbol{q}}^{\lambda_{\mathrm{T}}\,-1}(0) = \{\mathbf{0}\}\ \forall \boldsymbol{q}$ with $D_{\lambda_{\mathrm{T}}}$ given by (52) for $I_{\mathrm{N}} = I_G$,

then the equilibrium set $\mathcal{E}$ is attractive.

Since we now consider systems without unilateral contact, the proof of Corollary 3 follows the proof of Theorem 1 with the Lyapunov candidate function $V = T(\boldsymbol{q}, \boldsymbol{u}) + U(\boldsymbol{q})$. It should be noted that condition 4 on the dissipation rate functions of the smooth non-conservative forces and the dry friction forces implies that, firstly, the joint generalised force directions of the smooth non-conservative forces $\boldsymbol{f}^{\mathrm{nc}}$ and the dry friction forces $\boldsymbol{\lambda}_{\mathrm{T}}$ should span the n-dimensional generalised coordinate space for all $(\boldsymbol{q}, \boldsymbol{u})$ with $\boldsymbol{u} \neq \mathbf{0}$, and, secondly, the normal forces of those friction forces do not equal zero or do not change sign. In this context, we would like to refer to Proposition 3, which relates the zero set of the dissipation rate function of the dry friction forces to the kernel of the matrix $\boldsymbol{W}_{\mathrm{T}}^{\mathrm{T}}$ related to the generalised force direction of the dry friction forces. In this proposition the condition $\mathbf{0} \in \operatorname{int} C_T$ implies that the normal force can not be zero; in other words, if the normal force is zero, then the friction force is zero and thus not dissipative.

In [45], the attractivity of equilibrium sets of linear mechanical systems with dry friction was investigated. In that paper, it was also shown that the equilibrium set of a linear mechanical system with dry friction can be (locally) attractive even when the linear mechanical system without dry friction is unstable due to negative damping (i.e. the smooth non-conservative forces are non-dissipative in certain generalised force directions). The fact that the presence of dry friction can have such a 'stabilising' effect can be explained by pointing out that the dry friction forces are of zero-th order (in terms of generalised velocities) whereas the 'destabilising' linear damping forces are only of first order. Consequently, the 'stabilising' effect of the dry friction forces can locally dominate the 'destabilising' smooth damping forces leading to the local attractivity of the equilibrium set. In [45], these facts have been proved rigorously. Here, we want to refrain from such mathematically rigourous formulations, while still motivating that attractivity properties of equilibrium sets in non-linear mechanical system may still persist in the presence of non-dissipative smooth non-conservative forces. The conditions under which such attractivity can still be preserved is that, firstly, the generalised force directions of the dry friction forces span, at all times, the generalised force directions of $\boldsymbol{f}^{\mathrm{nc}}$ in which it is non-dissipative (a simple, though rather strict condition guaranteeing this demand is that $\boldsymbol{W}_{\mathrm{T}}(\boldsymbol{q})$ spans $\mathbb{R}^n$ for all $\boldsymbol{q}$). Secondly, the non-dissipative smooth forces should be of first (or higher) order in terms of the generalised velocities. The latter condition is needed to ensure that locally the dry friction forces (of zero-th order nature) dominate these non-dissipative forces.

Resuming, we can conclude that, in this section, we have formulated sufficient conditions for the (local) attractivity of equilibrium sets of a rather wide class of non-linear mechanical systems with bilateral frictional sliders. The non-linearities may involve: nonlinearities in the mass-matrix, both non-linear conservative forces and non-conservative forces (possibly even non-dissipative). Moreover, the generalised force directions of the dry friction forces may depend on the generalised coordinates and the normal forces in the friction sliders may depend on both the generalised coordinates and the generalised velocities.

6.2 Instability of equilibrium sets of systems with frictional bilateral constraints

We aim at proving the instability of equilibrium sets of mechanical systems with dry friction, under certain conditions, by proving that these equilibrium sets are not stable (in the sense of Lyapunov), i.e. by showing that we can not find for *every* ε-environment of the equilibrium set a δ-neighbourhood of the equilibrium set such that for every initial condition in the δ-neighbourhood the solution will stay in the ε-environment. We aim to do so by generalising the instability theorem for equilibrium points of smooth vectorfields (see [25]) to an instability theorem for equilibrium sets of differential inclusions[2] (see also [20, 21]):

[2] Note that Equations (91) and (11) together constitute a differential inclusion of the form (93).

Theorem 2. (Instability Theorem for Equilibrium Sets). *Let $\mathcal{E}$ be an equilibrium set of the differential inclusion*

$$\dot{\boldsymbol{x}} \in \boldsymbol{F}(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^n, \quad \boldsymbol{F} : \mathbb{R}^n \to \mathbb{R}^n, \quad \textit{almost everywhere}, \tag{93}$$

where $\boldsymbol{F}(\boldsymbol{x})$ is bounded and upper semi-continuous with a closed and (minimal) convex image. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function such that $V(\boldsymbol{x}_0) > V_{\mathcal{E}} > 0$ for some $\boldsymbol{x}_0$, for which dist$(\boldsymbol{x}_0, \mathcal{E})$ is arbitrarily small, and where $V_{\mathcal{E}} = \max_{\boldsymbol{x}\in\mathcal{E}} V(\boldsymbol{x})$. Define a set $\mathcal{U}$ by

$$\mathcal{U} = \{\boldsymbol{x} \in \mathcal{D}_r \mid V(\boldsymbol{x}) \geq 0\},$$

where $\mathcal{D}_r = \{\boldsymbol{x} \in \mathbb{R}^n \mid$ dist$(\boldsymbol{x}, \mathcal{E}) \leq r\}$ and choose $r > 0$ such that $\mathcal{E} \subset \mathcal{D}_r$ is the largest stationary set in $\mathcal{D}_r$. Now, three statements can be made:

1. *If $\dot{V}(\boldsymbol{x}) > 0$ in $\mathcal{U}\backslash\mathcal{E}$, then $\mathcal{E}$ is unstable;*
2. *If $\dot{V}(\boldsymbol{x}) \geq 0$ in $\mathcal{U}\backslash\mathcal{E}$ and $\mathcal{E} \subset$ int$\mathcal{U}$, then $\mathcal{E}$ is not attractive;*
3. *If $\dot{V}(\boldsymbol{x}) \geq 0$ in $\mathcal{U}\backslash\mathcal{E}$ and in a bounded environment of $\mathcal{E}$ solutions of (93) cannot stay in $\mathcal{S}\backslash\mathcal{E}$ with $\mathcal{S} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \dot{V} = 0\}$, then $\mathcal{E}$ is unstable.*

Proof: The point $\boldsymbol{x}_0$ is in the interior of $\mathcal{U}$ and $V(\boldsymbol{x}_0) = V_{\mathcal{E}} + \delta V$ with $\delta V > 0$.

Let us first prove statement 1 using that $\dot{V}(\boldsymbol{x}) > 0$ in $\mathcal{U}\backslash\mathcal{E}$: The trajectory $\boldsymbol{x}(t)$ starting in $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ must leave the set $\mathcal{U}$. To prove this, notice that as long as $\boldsymbol{x}(t)$ is inside $\mathcal{U}$, $V(\boldsymbol{x}(t)) > V_{\mathcal{E}} + \delta V \ \forall t > t_0$ since $\dot{V} > 0$ in $\mathcal{U}\backslash\mathcal{E}$. Note that $\dot{V} = 0$ in $\mathcal{E}$ since it is an equilibrium set. Define

$$\gamma = \min_{\boldsymbol{x}\in\mathcal{U}, V(\boldsymbol{x})\geq V_{\mathcal{E}}+\delta V} \dot{V}(\boldsymbol{x}).$$

Note that the function $\dot{V}(\boldsymbol{x}) = \frac{\partial V}{\partial \boldsymbol{x}}\dot{\boldsymbol{x}}$ has a minimum on the compact set $\{\boldsymbol{x} \in \mathbb{R}^n | (\boldsymbol{x} \in \mathcal{U}) \wedge (V(\boldsymbol{x}) \geq V_{\mathcal{E}} + \delta V)\} = \{\boldsymbol{x} \in \mathbb{R}^n | (\boldsymbol{x} \in \mathcal{D}_r) \wedge (V(\boldsymbol{x}) \geq V_{\mathcal{E}} + \delta V)\}$. Then, $\gamma > 0$ since $\dot{V}(\boldsymbol{x}) > 0$ in $\mathcal{U}\backslash\mathcal{E}$ and

$$\begin{aligned} V(\boldsymbol{x}(t)) = {} & V(\boldsymbol{x}_0) + \int_{t_0}^{t} \dot{V}(\boldsymbol{x}(s))\,\mathrm{d}s \geq V_{\mathcal{E}} + \delta V \\ & + \int_{t_0}^{t} \gamma\,\mathrm{d}s \quad \forall\, t > t_0, \\ \Rightarrow {} & V(\boldsymbol{x}(t)) \geq V_{\mathcal{E}} + \delta V + \gamma(t - t_0) \quad \forall\, t > t_0, \end{aligned} \tag{94}$$

because the set of time-instances for which $\dot{V}(t)$ is not defined is of Lebesgue measure zero. This inequality shows that $\boldsymbol{x}(t)$ cannot stay forever in $\mathcal{U}$ because $V(\boldsymbol{x})$ is bounded on $\mathcal{U}$. Now, $\boldsymbol{x}(t)$ must leave $\mathcal{U}$ through the surface $\{\boldsymbol{x} \in \mathbb{R}^n |$ dist$(\boldsymbol{x}, \mathcal{E}) = r\}$. Note, hereto that $\boldsymbol{x}(t)$ cannot leave $\mathcal{U}$ through the surface $V(\boldsymbol{x}) = 0$, since $V(\boldsymbol{x}(t)) > V_{\mathcal{E}} + \delta V > 0, \ \forall\, t > t_0$. Since this can happen for $\boldsymbol{x}_0$ such that dist$(\boldsymbol{x}_0, \mathcal{E})$ is arbitrarily small, the equilibrium set $\mathcal{E}$ is unstable.

Let us now prove statement 2 (exclusion of attractivity) using the fact that $\dot{V}(\boldsymbol{x}) \geq 0$ in $\mathcal{U}\backslash\mathcal{E}$: repeat the above reasoning and realise that now $\gamma \geq 0$ and thus $V(\boldsymbol{x}(t)) \geq V_{\mathcal{E}} + \delta V \ \forall t > t_0$. This excludes the possibility of $\boldsymbol{x}(t)$ ultimately converging to $\mathcal{E}$ since, firstly, $V < V_{\mathcal{E}} \ \forall \boldsymbol{x} \in \mathcal{E}$ and, secondly, the fact that $\mathcal{E}$ is enclosed in the interior of $\mathcal{U}$. Since this is true for $\boldsymbol{x}_0$ arbitrarily close to $\mathcal{E}$, no neighbourhood of $\mathcal{E}$ exists such that for any initial condition in this neighbourhood the solution will ultimately converge to $\mathcal{E}$ as $t \to \infty$, i.e. $\mathcal{E}$ is not attractive.

Finally, let us prove statement 3. Since solutions cannot stay on $\mathcal{S}\backslash\mathcal{E}$, $\exists t > t_0$ such that $\boldsymbol{x}(t) \notin \mathcal{S}$. Moreover, every solution $\boldsymbol{x}(t)$ of (93) is absolutely continuous in time and $\boldsymbol{x}(t) \notin \mathcal{S}$ for some small open time domain (t_0, t_1). Therefore, it holds that $\dot{V} > 0$ for $t \in (t_0, t_1)$. Consequently, $\int_{t_0}^{t_1} \dot{V}(s)\,\mathrm{d}s > 0$. This implies that $V(t)$ is strictly increasing for (t_0, t_1). As $t \to \infty$, the positive contributions to $V(t)$ will ensure that the solution will be bounded away from the equilibrium set for an initial condition arbitrarily close to the equilibrium set. As a consequence, $\mathcal{E}$ is unstable. □

In Section 7, this result will be illustrated by studying a non-linear mechanical system with dry friction. In the remainder of this section, we will apply Theorem 2 to a class of linear mechanical systems with dry friction. The attractivity of equilibrium sets of linear mechanical systems, which have an equilibrium point that is (in the absence of dry friction) unstable due to negative linear damping has been studied in [45]. Here, we will show that the equilibrium set of a linear mechanical system with dry friction, where the underlying equilibrium point is unstable due to negative stiffness, is unstable under some mild additional assumptions. Let us introduce the class of systems described by:

$$\boldsymbol{M}\dot{\boldsymbol{u}} + \boldsymbol{C}\boldsymbol{u} + \boldsymbol{K}\boldsymbol{q} = \boldsymbol{W}_{\mathrm{T}}\boldsymbol{\lambda}_{\mathrm{T}}, \tag{95}$$

with mass-matrix $\boldsymbol{M} = \boldsymbol{M}^{\mathrm{T}} > 0$, stiffness matrix $\boldsymbol{K} = \boldsymbol{K}^{\mathrm{T}}$, damping matrix $\boldsymbol{C} \geq 0$ and $\boldsymbol{\lambda}_{\mathrm{T}}$ given by (11). Note that the equilibrium set $\mathcal{E}$ of Equation (95) is given (for nonsingular $\boldsymbol{K}$) by

$$\mathcal{E} = \left\{ (\boldsymbol{q}, \boldsymbol{u}) \in \mathbb{R}^n \times \mathbb{R}^n \,|\, (\boldsymbol{u} = \boldsymbol{0}) \right. \\ \left. \wedge \boldsymbol{q} \in -\boldsymbol{K}^{-1} \sum_{i \in I_G} \boldsymbol{W}_{\mathrm{T}i} C_{\mathrm{T}i} \right\}. \tag{96}$$

The following theorem states the conditions under which the equilibrium set (96) of Equation (95) is unstable.

Theorem 3. (Instability of Equilibrium Sets of Linear Mechanical Systems) *Consider system (95), (11). Suppose $\boldsymbol{M} = \boldsymbol{M}^{\mathrm{T}} > 0$, $\boldsymbol{K} = \boldsymbol{K}^{\mathrm{T}} \not\geq 0$, $\boldsymbol{C} \geq 0$. The admissible set of friction forces is assumed to fulfil $\boldsymbol{0} \in \operatorname{int} C_{\mathrm{T}i}$ for all $i \in I_G$. If, moreover, the following condition is satisfied: $\boldsymbol{U}_{c_i} \in \operatorname{span}\{\boldsymbol{W}_{\mathrm{T}}\}$, for $i = 1, \ldots, n_q$, where $\boldsymbol{U}_c = \{\boldsymbol{U}_{c_i}\}$ is a matrix containing the n_q eigencolumns corresponding to the purely imaginary eigenvalues of $\boldsymbol{C}$, then the equilibrium set (96) is unstable.*

Proof: Consider a function V given by

$$V = -\frac{1}{2}\boldsymbol{u}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{u} - \frac{1}{2}\boldsymbol{q}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{q}. \tag{97}$$

The time-derivative of V is given by

$$\begin{aligned} \dot{V} &= -\boldsymbol{u}^{\mathrm{T}}(-\boldsymbol{C}\boldsymbol{u} - \boldsymbol{K}\boldsymbol{q} + \boldsymbol{W}_{\mathrm{T}}\boldsymbol{\lambda}_{\mathrm{T}}) - \boldsymbol{u}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{q} \\ &= \boldsymbol{u}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{u} - \boldsymbol{u}^{\mathrm{T}}\boldsymbol{W}_{\mathrm{T}}\boldsymbol{\lambda}_{\mathrm{T}} \\ &= \boldsymbol{u}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{u} - \boldsymbol{\gamma}_{\mathrm{T}}^{\mathrm{T}}\boldsymbol{\lambda}_{\mathrm{T}} \\ &= \boldsymbol{u}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{u} + \sum_{i \in I_G} \Psi^*_{C_{\mathrm{T}i}}(\gamma_{\mathrm{T}i}). \end{aligned} \tag{98}$$

Consequently, it holds that $\dot{V} \geq 0$. We define a set $\mathcal{S}$ by $\mathcal{S} = \{(\boldsymbol{q}, \boldsymbol{u}) \mid \dot{V} = 0\}$. Under the conditions stated in the theorem this set is given by: $\mathcal{S} = \{(\boldsymbol{q}, \boldsymbol{u}) \mid \boldsymbol{u} = \boldsymbol{0}\}$. It therefore holds that

$$\begin{aligned} &\dot{V} = 0 \quad \text{if and only if } \ \boldsymbol{u} = \boldsymbol{0}, \\ &\dot{V} > 0 \quad \text{for } \ \boldsymbol{u} \neq \boldsymbol{0}. \end{aligned} \tag{99}$$

Let us define a point $(\boldsymbol{q}_{\mathcal{E}}, \boldsymbol{u}_{\mathcal{E}}) = (c\,\boldsymbol{u}_{k_i}, \boldsymbol{0})$, $i \in \{1, \ldots, n_k\}$, with a positive constant $c > 0$ and $\boldsymbol{u}_{k_i}$ an eigencolumn corresponding to an eigenvalue λ_{k_i} of $\boldsymbol{K}$, which lies in the open left-half complex plane. Since $\boldsymbol{K}$ is symmetric, λ_{k_i} is real and $\lambda_{k_i} < 0$. We choose c such that $\boldsymbol{q}_{\mathcal{E}} \in \operatorname{bdry}(\mathcal{E})$. Moreover, we define a point $(\boldsymbol{q}_0, \boldsymbol{u}_0) = (\boldsymbol{q}_{\mathcal{E}}, \boldsymbol{u}_{\mathcal{E}}) + (\delta\,\boldsymbol{u}_{k_i}, \boldsymbol{0}) = ((c + \delta)\boldsymbol{u}_{k_i}, \boldsymbol{0})$ with $\delta > 0$ an arbitrarily small positive constant. We consider $(\boldsymbol{q}_0, \boldsymbol{u}_0)$ to be an initial condition which can be chosen arbitrarily close to the boundary point $(\boldsymbol{q}_{\mathcal{E}}, \boldsymbol{u}_{\mathcal{E}})$ of the equilibrium set by choosing δ arbitrarily small. Moreover, note that $V(\boldsymbol{q}_0, \boldsymbol{u}_0) > V(\boldsymbol{q}_{\mathcal{E}}, \boldsymbol{u}_{\mathcal{E}}) > 0$, since $V(\boldsymbol{q}_0, \boldsymbol{u}_0) = -\frac{1}{2}(c + \delta)^2\lambda_{k_i} > 0$ and $V(\boldsymbol{q}_{\mathcal{E}}, \boldsymbol{u}_{\mathcal{E}}) = -\frac{1}{2}c^2\lambda_{k_i} > 0$.

Regarding the equations of motion (95), with the set-valued friction law (11), on $\mathcal{S}$, it can be concluded that the accelerations $\dot{\boldsymbol{u}}$ are always non-zero for $(\boldsymbol{q}, \boldsymbol{u}) \notin \mathcal{E}$. Consequently, the solutions of the system cannot stay in $\mathcal{S} \backslash \mathcal{E}$.

Now, all conditions of Theorem 2, with statement 3, are satisfied and we conclude that the equilibrium set $\mathcal{E}$ is unstable. □

Theorem 3, together with the results in [45], provide a rather complete picture of the stability-related properties of the equilibrium set of a linear mechanical systems with Coulomb friction:

- For linear mechanical systems (without Coulomb friction) with an asymptotically stable equilibrium point, the equilibrium set of the system with Coulomb friction is globally attractive,
- For linear mechanical systems (without Coulomb friction) with an unstable equilibrium point due to ‘negative damping’ effects, the equilibrium set of the system with Coulomb friction can still, under conditions stated in [45], be shown to be locally attractive,
- For linear mechanical systems (without Coulomb friction) with an unstable equilibrium point due to ‘negative stiffness’ effects, the equilibrium set of the system with Coulomb friction is unstable.

7 Examples

In this section, we show how the above theorems can be used to prove the attractivity (or instability) of an equilibrium set of a number of mechanical systems. Sections 7.1–7.3 involve examples of mechanical systems with unilateral contact, impact and friction and are of increasing complexity. Section 7.4 treats an example of a mechanical system with bilateral frictional constraints to illustrate the results of Section 6.

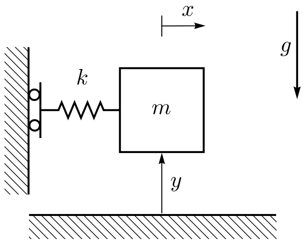


Fig. 5 Falling block

7.1 Falling block

Consider a planar rigid block (see Fig. 5) with mass m under the action of gravity (gravitational acceleration g), which is attached to a vertical wall with a spring. The block can freely move in the vertical direction but is not able to undergo a rotation. The coordinates x and y describe the position of the block. The spring is unstressed for $x = 0$. The block comes into contact with a horizontal floor when the contact distance $g_{\mathrm{N}} = y$ becomes zero. The constitutive properties of the contact are the friction coefficient μ and the restitution coefficients $0 \leq e_{\mathrm{N}} < 1$ and $e_{\mathrm{T}} = 0$. The equations of motion for impact free motion read as

$$\begin{aligned} m\ddot{x} + kx &= \lambda_{\mathrm{T}}, \\ m\ddot{y} &= -mg + \lambda_{\mathrm{N}}. \end{aligned} \tag{100}$$

Using the generalised coordinates $\boldsymbol{q} = [x \quad y]^{\mathrm{T}}$, we can describe the system in the form (33) with

$$\begin{aligned} \boldsymbol{M} &= \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \quad \boldsymbol{h} = \begin{bmatrix} -kx \\ -mg \end{bmatrix}, \\ \boldsymbol{W}_{\mathrm{N}} &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{W}_{\mathrm{T}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \end{aligned} \tag{101}$$

The system for $\mu = 0$ admits a unique equilibrium position $\boldsymbol{q}_{\mathrm{e}} = \boldsymbol{0}$. For $\mu > 0$ there exists an equilibrium set $\mathcal{E} = \{(x, y, \dot{x}, \dot{y}) \mid k|x| \leq \mu mg,\ y = 0,\ \dot{x} = \dot{y} = 0\}$ and it holds that $(\boldsymbol{q}_{\mathrm{e}}, \boldsymbol{0}) \in \mathcal{E}$.

The total potential energy function used in condition 2 of Theorem 1 reads as

$$\begin{aligned} Q(\boldsymbol{q}) &= U(\boldsymbol{q}) + \Psi^*_{C_{\mathrm{N}}}(g_{\mathrm{N}}(\boldsymbol{q})) \\ &= \frac{1}{2}kx^2 + mgy + \Psi^*_{\mathbb{R}^-}(y) \\ &= \frac{1}{2}kx^2 + mgy + \Psi_{\mathbb{R}^+}(y). \end{aligned} \tag{102}$$

Notice that the term $mgy + \Psi_{\mathbb{R}^+}(y)$ is a positive definite term in y. It holds that Q is a positive definite function in $\boldsymbol{q}$, because it is above or equal to another positive definite function $Q(\boldsymbol{q}) \geq \frac{1}{2}kx^2 + mg|y|$. Moreover, the minimum of Q is located at the equilibrium point $\boldsymbol{q}_{\mathrm{e}} = \boldsymbol{0}$, because $\partial Q(\boldsymbol{q}_{\mathrm{e}}) \ni \boldsymbol{0}$ and is unique because of the convexity of Q. Condition 2 of Theorem 1 is therefore fulfilled for all $\boldsymbol{q} \in \mathbb{R}^n$. The system does not contain smooth non-conservative forces, i.e. $\boldsymbol{f}^{\mathrm{nc}} = \boldsymbol{0}$, which fulfills condition 3 of Theorem 1. Denote the contact between block and floor as contact 1 and take $I_C = I_G = \{1\}$. It holds that $\dot{\gamma}_{\mathrm{N}} = -g$ for $g_{\mathrm{N}} = y > 0$, which guarantees the satisfaction of condition 4 of Theorem 1. Furthermore, it holds that $D_{\boldsymbol{q}}^{\mathrm{nc}-1}(0) = \mathbb{R}^n$ and $D_{\boldsymbol{q}}^{\lambda_{TC}-1}(0) = \ker \boldsymbol{W}_{\mathrm{T}}^{\mathrm{T}}$. Because the vectors $\boldsymbol{W}_{\mathrm{N}}$ and $\boldsymbol{W}_{\mathrm{T}}$ are linearly independent it holds that $\ker \boldsymbol{W}_{\mathrm{T}}^{\mathrm{T}} \cap \ker \boldsymbol{W}_{\mathrm{N}}^{\mathrm{T}} = \{\boldsymbol{0}\}$ and condition 5 of Theorem 1 is therefore fulfilled. Consequently, Theorem 1 proves that the equilibrium set $\mathcal{E}$ is globally attractive.

7.2 Rocking bar

Consider a planar rigid bar with mass m and inertia J_S around the centre of mass S, which is attached to a vertical wall with a spring (Fig. 6). The gravitational acceleration is denoted by g. The position and orientation of the bar are described by the generalised coordinates

$$\boldsymbol{q} = [x \quad y \quad \varphi]^{\mathrm{T}}, \tag{103}$$

where x and y are the displacements of the centre of mass S with respect to the coordinate frame $(\mathbf{e}_x^I, \mathbf{e}_y^I)$ and φ is the inclination angle. The spring is unstressed for $x = 0$. The bar has length $2a$ and two endpoints which can come into contact with the floor. The contact be-

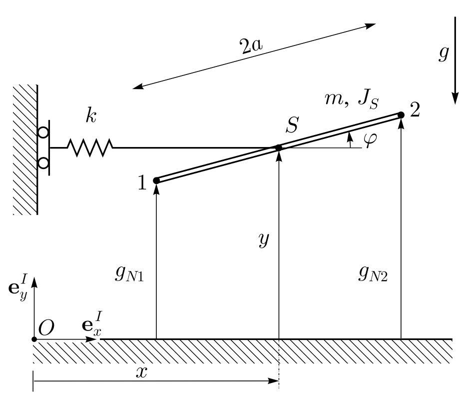


Fig. 6 Rocking bar

tween bar and floor is described by a friction coefficient $\mu > 0$ and a normal restitution coefficient $0 \le e_{\mathrm{N}} < 1$ that is equal to the tangential restitution $e_{\mathrm{T}} = e_{\mathrm{N}}$. The contact distances, indicated in Fig. 6, are

$$\begin{aligned} g_{N1} &= y - a\sin\varphi, \\ g_{N2} &= y + a\sin\varphi. \end{aligned} \tag{104}$$

The relative velocities of contact points 1 and 2 with respect to the floor read as

$$\begin{aligned} \gamma_{T1} &= \dot{x} + a\dot{\varphi}\sin\varphi, \\ \gamma_{T2} &= \dot{x} - a\dot{\varphi}\sin\varphi. \end{aligned} \tag{105}$$

We can describe the system in the form (33) with

$$\boldsymbol{M} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & J_S \end{bmatrix}, \quad \boldsymbol{h} = \begin{bmatrix} -kx \\ -mg \\ 0 \end{bmatrix}, \tag{106}$$

$$\begin{aligned} \boldsymbol{W}_{\mathrm{N}}^{\mathrm{T}} &= \begin{bmatrix} 0 & 1 & -a\cos\varphi \\ 0 & 1 & a\cos\varphi \end{bmatrix}, \\ \boldsymbol{W}_{\mathrm{T}}^{\mathrm{T}} &= \begin{bmatrix} 1 & 0 & a\sin\varphi \\ 1 & 0 & -a\sin\varphi \end{bmatrix}. \end{aligned} \tag{107}$$

The system contains a number of equilibrium sets. We will consider the equilibrium set

$$\begin{aligned} \mathcal{E} = \{&(x, y, \varphi, \dot{x}, \dot{y}, \dot{\varphi}) \mid k|x| \le \mu mg, \\ &y = 0,\ \varphi = 0,\ \dot{x} = \dot{y} = \dot{\varphi} = 0\}, \end{aligned} \tag{108}$$

for which $g_{N1} = g_{N2} = 0$. The total potential energy function

$$\begin{aligned} Q(\boldsymbol{q}) &= U(\boldsymbol{q}) + \Psi^*_{C_{\mathrm{N}}}(g_{N1}(\boldsymbol{q})) + \Psi^*_{C_{\mathrm{N}}}(g_{N2}(\boldsymbol{q})) \\ &= \frac{1}{2}kx^2 + mgy + \Psi^*_{\mathbb{R}^-}(g_{N1}) + \Psi^*_{\mathbb{R}^-}(g_{N2}) \\ &= \frac{1}{2}kx^2 + mgy + \Psi_{\mathbb{R}^+}(g_{N1}) + \Psi_{\mathbb{R}^+}(g_{N2}) \end{aligned} \tag{109}$$

contains a quadratic term in x, a linear term in y and two indicator functions on the contact distances. Notice that $Q(\boldsymbol{q}) = 0$ for $\boldsymbol{q} = 0$. Moreover, it holds that if $g_{N1} \ge 0$ and $g_{N2} \ge 0$ then $y \ge 0$ and $a|\sin\varphi| \le y$. We therefore deduce that

$$\begin{aligned} g_{N1} \ge 0 \wedge g_{N2} \ge 0 \Longrightarrow Q(\boldsymbol{q}) &= \frac{1}{2}kx^2 + mgy \\ Q(\boldsymbol{q}) &= \frac{1}{2}kx^2 + \frac{mg}{2}(|y| + y) \\ Q(\boldsymbol{q}) &\ge \frac{1}{2}kx^2 + \frac{mg}{2}(|y| + a|\sin\varphi|) \end{aligned} \tag{110}$$

and

$$\begin{aligned} g_{N1} < 0 \vee g_{N2} < 0 &\Longrightarrow Q(\boldsymbol{q}) = +\infty \\ Q(\boldsymbol{q}) &> \frac{1}{2}kx^2 + \frac{mg}{2}(|y| + a|\sin\varphi|). \end{aligned} \tag{111}$$

The function $f(\boldsymbol{q}) = \frac{1}{2}kx^2 + \frac{mg}{2}(|y| + a|\sin\varphi|)$ is locally positive definite in the set $\mathcal{U} = \{\boldsymbol{q} \in \mathbb{R}^n \mid |\varphi| < \frac{\pi}{2}\}$. Consequently, the total potential energy function $Q(\boldsymbol{q}) \ge f(\boldsymbol{q})$ is locally positive definite in the set $\mathcal{U}$ as well. It can be easily checked that the generalised gradient

$$\begin{aligned} &\partial Q(\boldsymbol{q}) \\ &= \begin{bmatrix} kx \\ mg + \partial\Psi_{\mathbb{R}^+}(g_{N1}) + \partial\Psi_{\mathbb{R}^+}(g_{N2}) \\ -\partial\Psi_{\mathbb{R}^+}(g_{N1})a\cos\varphi + \partial\Psi_{\mathbb{R}^+}(g_{N2})a\cos\varphi \end{bmatrix} \end{aligned} \tag{112}$$

can only vanish in the set $\mathcal{U}$ for $\boldsymbol{q} = \boldsymbol{q}_{\mathrm{e}}$, i.e. $\boldsymbol{0} \notin \partial Q(\boldsymbol{q}) \forall \boldsymbol{q} \in \mathcal{U}\backslash\{\boldsymbol{q}_{\mathrm{e}}\}$ and $\boldsymbol{0} \in \partial Q(\boldsymbol{q}_{\mathrm{e}})$.

Smooth non-conservative forces are absent in this system, i.e. $\boldsymbol{f}^{\mathrm{nc}} = \boldsymbol{0}$ and $D^{\mathrm{nc}}_{\boldsymbol{q}}(\boldsymbol{u}) = 0$. We now want to prove that condition 4 of Theorem 1 holds with $I_C = \{1, 2\}$. Consider the open sub-set $\mathcal{V} = \{(\boldsymbol{q}, \boldsymbol{u}) \in \mathbb{R}^n \times \mathbb{R}^n \mid \mu|\tan\varphi| < 1, a\dot{\varphi}^2 < g\}$ which contains the equilibrium set, i.e. $\mathcal{E} \subset \mathcal{V}$. We consider the following cases with $(\boldsymbol{q}, \boldsymbol{u}) \in \mathcal{V}$:

- $I_{\mathrm{N}} = \emptyset$: both contacts are open, i.e. $g_{N1} > 0$ and $g_{N2} > 0$. It holds for $(\boldsymbol{q}, \boldsymbol{u}) \in \mathcal{V}$ that

$$\begin{aligned} \dot{\gamma}_{N1} &= \ddot{y} - a\ddot{\varphi}\cos\varphi + a\dot{\varphi}^2\sin\varphi \\ &= -g + a\dot{\varphi}^2\sin\varphi \\ &< 0 \\ \dot{\gamma}_{N2} &= \ddot{y} + a\ddot{\varphi}\cos\varphi - a\dot{\varphi}^2\sin\varphi \\ &= -g - a\dot{\varphi}^2\sin\varphi \\ &< 0. \end{aligned} \tag{113}$$

- $I_N = \{1\}$: contact 1 is closed and contact 2 is open, i.e. $g_{N1} = 0$ and $g_{N2} > 0$. We consider contact 1 to be closed for a nonzero time-interval. The normal contact acceleration of the closed contact 1 must vanish:

$$\begin{aligned}
\dot{\gamma}_{N1} &= \ddot{y} - a\ddot{\varphi}\cos\varphi + a\dot{\varphi}^2\sin\varphi \\
0 &= -g + \frac{1}{m}\lambda_{N1} + \frac{a^2}{J_S}\cos^2\varphi\,\lambda_{N1} \\
&\quad - \frac{a^2}{J_S}\cos\varphi\sin\varphi\,\lambda_{T1} + a\dot{\varphi}^2\sin\varphi \\
0 &= -g + \left(\frac{1}{m} + \frac{a^2}{J_S}\cos\varphi(\cos\varphi - \bar{\mu}\sin\varphi)\right)\lambda_{N1} \\
&\quad + a\dot{\varphi}^2\sin\varphi,
\end{aligned} \tag{114}$$

with $\lambda_{T1} = \bar{\mu}\lambda_{N1}$, i.e. $\bar{\mu} \in -\mu\,\mathrm{Sign}(\gamma_{T1})$. It follows from (114) that the normal contact force λ_{N1} is a function of φ and $\dot{\varphi}$. The contact acceleration of contact 2 therefore becomes

$$\begin{aligned}
\dot{\gamma}_{N2} &= \ddot{y} + a\ddot{\varphi}\cos\varphi - a\dot{\varphi}^2\sin\varphi \\
&= -g + \frac{1}{m}\lambda_{N1} - \frac{a^2}{J_S}\cos^2\varphi\,\lambda_{N1} \\
&\quad + \frac{a^2}{J_S}\cos\varphi\sin\varphi\,\lambda_{T1} - a\dot{\varphi}^2\sin\varphi \\
&= -g + \left(\frac{1}{m} - \frac{a^2}{J_S}\cos\varphi(\cos\varphi - \bar{\mu}\sin\varphi)\right)\lambda_{N1} \\
&\quad - a\dot{\varphi}^2\sin\varphi \\
&= \frac{\frac{1}{m} - \frac{a^2}{J_S}\cos\varphi(\cos\varphi - \bar{\mu}\sin\varphi)}{\frac{1}{m} + \frac{a^2}{J_S}\cos\varphi(\cos\varphi - \bar{\mu}\sin\varphi)}(g - a\dot{\varphi}^2\sin\varphi) \\
&\quad - g - a\dot{\varphi}^2\sin\varphi \\
&= -2g\frac{a^2\frac{m}{J_S}\cos\varphi(\cos\varphi - \bar{\mu}\sin\varphi)}{1 + a^2\frac{m}{J_S}\cos\varphi(\cos\varphi - \bar{\mu}\sin\varphi)} \\
&\quad - \frac{2a\dot{\varphi}^2\sin\varphi}{1 + a^2\frac{m}{J_S}\cos\varphi(\cos\varphi - \bar{\mu}\sin\varphi)}.
\end{aligned} \tag{115}$$

  Using $|\bar{\mu}| \le \mu$ and $(\boldsymbol{q}, \boldsymbol{u}) \in \mathcal{V}$ it follows that $\dot{\gamma}_{N2} < 0$.
- $I_N = \{2\}$: contact 1 is open and contact 2 is closed, i.e. $g_{N1} > 0$ and $g_{N2} = 0$. Similar to the previous case we can prove that $\dot{\gamma}_{N1} < 0$.

Hence, there exists a non-empty set $I_C = \{1, 2\}$, such that $\dot{\gamma}_{Ni}(\boldsymbol{q}, \boldsymbol{u}) < 0$ (a.e.) for $\forall i \in I_C \backslash I_N$ and $\forall(\boldsymbol{q}, \boldsymbol{u}) \in \mathcal{V}$. Condition 4 of Theorem 1 is therefore fulfilled.

It holds that $D_{\boldsymbol{q}}^{\mathrm{nc}-1}(0) = \mathbb{R}^n$ and using Proposition 3 it follows that $D_{\boldsymbol{q}}^{\lambda_T-1}(0) = \ker \boldsymbol{W}_T^T(\boldsymbol{q})$. Furthermore, for $\boldsymbol{q} \in \mathcal{C} = \{\boldsymbol{q} \in \mathbb{R}^n \mid g_{N1} = g_{N2} = 0\}$ follows the implication $\boldsymbol{W}_N^T(\boldsymbol{q})\boldsymbol{u} = 0 \Longrightarrow \dot{y} = 0 \wedge \dot{\varphi} = 0$ and similarly $\boldsymbol{W}_T^T(\boldsymbol{q})\boldsymbol{u} = 0 \Longrightarrow \dot{x} = 0$. We conclude that there is always dissipation when both contacts are closed and $\boldsymbol{u} \neq \boldsymbol{0}$ because

$$\ker \boldsymbol{W}_T^T(\boldsymbol{q}) \cap \ker \boldsymbol{W}_N^T(\boldsymbol{q}) = \{\boldsymbol{0}\} \quad \forall \boldsymbol{q} \in \mathcal{C}, \tag{116}$$

and condition 5 of Theorem 1 is therefore fulfilled. The largest level set of $V = T(\boldsymbol{q}, \boldsymbol{u}) + Q(\boldsymbol{q})$ which lies entirely in $\mathcal{Q} = \{(\boldsymbol{q}, \boldsymbol{u}) \in \mathbb{R}^n \times \mathbb{R}^n \mid \boldsymbol{q} \in \mathcal{U}\}$ is given by $V(\boldsymbol{q}, \boldsymbol{u}) < mga$. The largest level set of V which lies entirely in $\mathcal{V}$ is determined by $V(\boldsymbol{q}, \boldsymbol{u}) < \frac{1}{2}J_S\frac{g}{a}$ and $V(\boldsymbol{q}, \boldsymbol{u}) < \frac{mga}{\sqrt{1+\mu^2}}$. We therefore choose the set $\mathcal{I}_{\rho^*}$ as

$$\begin{aligned}
&\mathcal{I}_{\rho^*} = \{(\boldsymbol{q}, \boldsymbol{u}) \in \mathbb{R}^n \times \mathbb{R}^n \mid V(\boldsymbol{q}, \boldsymbol{u}) < \rho^*\}, \text{ with} \\
&\rho^* = \min\left(\frac{1}{2}J_S\frac{g}{a}, \frac{mga}{\sqrt{1+\mu^2}}\right).
\end{aligned} \tag{117}$$

If additionally

$$\frac{1}{2}\frac{(\mu m g)^2}{k} < \rho^*, \tag{118}$$

then it holds that $\mathcal{E} \subset \mathcal{I}_{\rho^*}$. We conclude that Theorem 1 proves conditionally the local attractivity of the equilibrium set $\mathcal{E}$ and that $\mathcal{I}_{\rho^*}$ is a conservative estimate of the region of attraction. Naturally, the attractivity is only local, because the system has also other attractive equilibrium sets for $\varphi = n\pi$ with $n \in \mathbb{Z}$ and unstable equilibrium sets around $\varphi = \frac{\pi}{2} + n\pi$.

7.3 Rocking block

The theory presented in this paper has been applied in the Section 7.2 to a simple rocking bar system. In this Section we study a rocking block on a rigid floor, which seems like a slight modification of the previous example, but which shows that the analysis can already become very elaborate for a relatively simple system.

Consider a planar rigid block with mass m and inertia J_S around the centre of mass S, which is attached to a vertical wall with a spring (Fig. 7). The block has a hight $2b$, a width $2a$ and the gravitational acceleration is denoted by g. The position and orientation of the

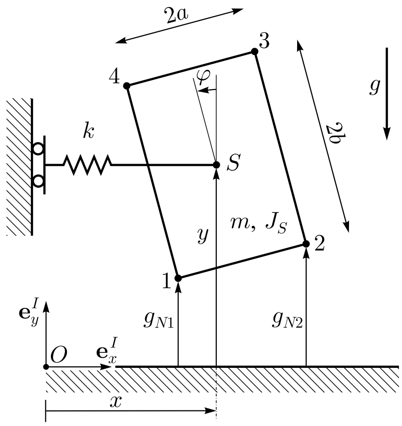


Fig. 7 Rocking block

block are described by the generalised coordinates

$$\boldsymbol{q} = (x \quad y \quad \varphi^{\mathrm{T}}), \tag{119}$$

where x and y are the displacements of the centre of mass S with respect to the coordinate frame $(\mathbf{e}_x^I, \mathbf{e}_y^I)$ and φ is the inclination angle. The spring is unstressed for $x = 0$. The block has four corner points which can come into contact with the floor (friction coefficient $0 < \mu < \frac{a}{b}$ and a normal restitution coefficient $0 \le e_{\mathrm{N}} < 1$ that is equal to the tangential restitution $e_{\mathrm{T}} = e_{\mathrm{N}}$). We will only be interested in contact points 1 and 2 of which the contact distances are

$$\begin{aligned} g_{N1} &= y - a\sin\varphi - b\cos\varphi, \\ g_{N2} &= y + a\sin\varphi - b\cos\varphi. \end{aligned} \tag{120}$$

The relative velocities of contact points 1 and 2 with respect to the floor read as

$$\begin{aligned} \gamma_{T1} &= \dot{x} + (a\sin\varphi + b\cos\varphi)\dot{\varphi}, \\ \gamma_{T2} &= \dot{x} + (-a\sin\varphi + b\cos\varphi)\dot{\varphi}. \end{aligned} \tag{121}$$

We can describe the system in the form (33) with

$$\boldsymbol{M} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & J_S \end{bmatrix}, \quad \boldsymbol{h} = \begin{bmatrix} -kx \\ -mg \\ 0 \end{bmatrix}, \tag{122}$$

$$\boldsymbol{W}_{\mathrm{N}}^{\mathrm{T}} = \begin{bmatrix} 0 & 1 & -a\cos\varphi + b\sin\varphi \\ 0 & 1 & a\cos\varphi + b\sin\varphi \end{bmatrix},$$

$$\boldsymbol{W}_{\mathrm{T}}^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & a\sin\varphi + b\cos\varphi \\ 1 & 0 & -a\sin\varphi + b\cos\varphi \end{bmatrix}. \tag{123}$$

The system contains a number of equilibrium sets. We will consider the equilibrium set

$$\begin{aligned} \mathcal{E} = \{(\boldsymbol{q}, \boldsymbol{u}) \in \mathbb{R}^n \times \mathbb{R}^n \mid k|x| \le \mu mg,\ y = b, \\ \varphi = 0,\ \dot{x} = \dot{y} = \dot{\varphi} = 0\}, \end{aligned} \tag{124}$$

for which $g_{N1} = g_{N2} = 0$. We study the system for $\mu < \frac{a}{b}$, i.e. the equilibrium set $\mathcal{E}$ is isolated. The functions g_{N1} and g_{N2} are locally monotonous functions in φ for $|\varphi| < \arctan\frac{a}{b}$. The total potential energy function

$$\begin{aligned} Q(\boldsymbol{q}) &= U(\boldsymbol{q}) + \Psi^*_{C_{\mathrm{N}}}(g_{N1}(\boldsymbol{q})) + \Psi^*_{C_{\mathrm{N}}}(g_{N2}(\boldsymbol{q})) \\ &= \frac{1}{2}kx^2 + mg(y-b) + \Psi^*_{\mathbb{R}^-}(g_{N1}) + \Psi^*_{\mathbb{R}^-}(g_{N2}) \\ &= \frac{1}{2}kx^2 + mg(y-b) + \Psi_{\mathbb{R}^+}(g_{N1}) + \Psi_{\mathbb{R}^+}(g_{N2}) \end{aligned} \tag{125}$$

contains a quadratic term in x, a linear term in y and two indicator functions on the contact distances. Notice that $Q(\boldsymbol{q}) = 0$ for $\boldsymbol{q} = \boldsymbol{q}_{\mathrm{e}} = [0 \quad b \quad 0]^{\mathrm{T}}$. Moreover, if $g_{N1} \ge 0$ and $g_{N2} \ge 0$ then it holds that $y \ge a|\sin\varphi| + b\cos\varphi$. We therefore deduce that

$$\begin{aligned} & g_{N1} \ge 0 \wedge g_{N2} \ge 0 \Longrightarrow \\ & \quad Q(\boldsymbol{q}) = \frac{1}{2}kx^2 + mg(y-b) \\ & \quad Q(\boldsymbol{q}) = \frac{1}{2}kx^2 + \frac{mg}{2}(|y| + y - b) \\ & \quad Q(\boldsymbol{q}) \ge \frac{1}{2}kx^2 + \frac{mg}{2}(|y| + a|\sin\varphi| \\ & \qquad + b\cos\varphi - b) \end{aligned} \tag{126}$$

in which $y \ge 0$ has been used, and it follows that

$$\begin{aligned} & g_{N1} < 0 \vee g_{N2} < 0 \Longrightarrow \\ & \quad Q(\boldsymbol{q}) = +\infty \\ & \quad Q(\boldsymbol{q}) > \frac{1}{2}kx^2 + \frac{mg}{2} \\ & \qquad (|y| + a|\sin\varphi| + b\cos\varphi - b). \end{aligned} \tag{127}$$

The function $f(\boldsymbol{q}) = \frac{1}{2}kx^2 + \frac{mg}{2}(|y| + a|\sin\varphi| + b\cos\varphi - b)$ is locally positive definite in the set $\mathcal{U} = \{\boldsymbol{q} \in \mathbb{R}^n \mid |\varphi| < \arctan\frac{a}{b}\}$. Consequently, the total potential energy function $Q(\boldsymbol{q}) \ge f(\boldsymbol{q})$ is locally positive definite in the set $\mathcal{U}$ as well. It can be easily checked that the generalised gradient

$$\partial Q(\boldsymbol{q}) = \begin{bmatrix} kx \\ mg + \partial\Psi_{\mathbb{R}^+}(g_{N1}) + \partial\Psi_{\mathbb{R}^+}(g_{N2}) \\ -\partial\Psi_{\mathbb{R}^+}(g_{N1})(a\cos\varphi - b\sin\varphi) \\ \quad + \partial\Psi_{\mathbb{R}^+}(g_{N2})(a\cos\varphi + b\sin\varphi) \end{bmatrix} \tag{128}$$

can only vanish in the set $\mathcal{U}$ for $\boldsymbol{q} = \boldsymbol{q}_{\mathrm{e}}$, i.e. $\mathbf{0} \notin \partial Q(\boldsymbol{q})\,\forall \boldsymbol{q} \in \mathcal{U}\backslash\{\boldsymbol{q}_{\mathrm{e}}\}$ and $\mathbf{0} \in \partial Q(\boldsymbol{q}_{\mathrm{e}})$.

Smooth non-conservative forces are absent in this system, i.e. $\boldsymbol{f}^{\mathrm{nc}} = \mathbf{0}$ and $D^{\mathrm{nc}}_{\boldsymbol{q}}(\boldsymbol{u}) = 0$. We now want to prove that condition 4 of Theorem 1 holds with $I_C = \{1, 2\}$. Consider the open sub-set $\mathcal{V} = \{(\boldsymbol{q}, \boldsymbol{u}) \in \mathbb{R}^n \times \mathbb{R}^n \mid d(\varphi) > 0, \sqrt{a^2+b^2}\dot{\varphi}^2 < g\}$ with $d(\varphi) = a\cos\varphi - b\sin\varphi - \mu(a\sin\varphi + b\cos\varphi)$. The set $\mathcal{V}$ is a neighbourhood of the equilibrium set, i.e. $\mathcal{E} \subset \mathcal{V}$. We consider the following cases with $(\boldsymbol{q}, \boldsymbol{u}) \in \mathcal{V}$:

- $I_{\mathrm{N}} = \emptyset$: both contacts are open, i.e. $g_{N1} > 0$ and $g_{N2} > 0$. It holds for $(\boldsymbol{q}, \boldsymbol{u}) \in \mathcal{V}$ that

$$\begin{aligned} \dot{\gamma}_{N1} &= \ddot{y} + (-a\cos\varphi + b\sin\varphi)\ddot{\varphi} \\ &\quad + (a\sin\varphi + b\cos\varphi)\dot{\varphi}^2 \\ &= -g + (a\sin\varphi + b\cos\varphi)\dot{\varphi}^2 \\ &< 0 \\ \dot{\gamma}_{N2} &= \ddot{y} + (a\cos\varphi + b\sin\varphi)\ddot{\varphi} \\ &\quad + (-a\sin\varphi + b\cos\varphi)\dot{\varphi}^2 \\ &= -g + (-a\sin\varphi + b\cos\varphi)\dot{\varphi}^2 \\ &< 0. \end{aligned} \tag{129}$$

- $I_{\mathrm{N}} = \{1\}$: contact 1 is closed and contact 2 is open, i.e. $g_{N1} = 0$ and $g_{N2} > 0$. We consider contact 1 to be closed for a nonzero time-interval. The normal contact acceleration of the closed contact 1 must vanish:

$$\begin{aligned} \dot{\gamma}_{N1} &= \ddot{y} + (-a\cos\varphi + b\sin\varphi)\ddot{\varphi} \\ &\quad + (a\sin\varphi + b\cos\varphi)\dot{\varphi}^2 \\ 0 &= -g + \frac{1}{m}\lambda_{N1} + \frac{1}{J_S}(-a\cos\varphi + b\sin\varphi)^2\lambda_{N1} \\ &\quad + \frac{1}{J_S}(-a\cos\varphi + b\sin\varphi)(a\sin\varphi \\ &\quad + b\cos\varphi)\lambda_{T1} + (a\sin\varphi + b\cos\varphi)\dot{\varphi}^2 \\ 0 &= -g + \left(\frac{1}{m} + \frac{1}{J_S}(-a\cos\varphi + b\sin\varphi)(-a\cos\varphi\right. \\ &\quad \left. + b\sin\varphi + \bar{\mu}(a\sin\varphi + b\cos\varphi))\right)\lambda_{N1} \\ &\quad + (a\sin\varphi + b\cos\varphi)\dot{\varphi}^2 \\ 0 &= -g + \left(\frac{1}{m} + \frac{1}{J_S}(a\cos\varphi - b\sin\varphi)\bar{d}(\varphi)\right)\lambda_{N1} \\ &\quad + (a\sin\varphi + b\cos\varphi)\dot{\varphi}^2, \end{aligned} \tag{130}$$

with $\bar{d}(\varphi) = a\cos\varphi - b\sin\varphi - \bar{\mu}(a\sin\varphi + b\cos\varphi)$ and $\lambda_{T1} = \bar{\mu}\lambda_{N1}$, i.e. $\bar{\mu} \in -\mu\,\mathrm{Sign}(\gamma_{T1})$, from which follows the normal contact force λ_{N1} as a function of φ and $\dot{\varphi}$. The contact acceleration of contact 2 therefore becomes

$$\begin{aligned} \dot{\gamma}_{N2} &= \ddot{y} + (a\cos\varphi + b\sin\varphi)\ddot{\varphi} + (-a\sin\varphi \\ &\quad + b\cos\varphi)\dot{\varphi}^2 \\ &= -g + \frac{1}{m}\lambda_{N1} + \frac{1}{J_S}(a\cos\varphi + b\sin\varphi) \\ &\quad \times (-a\cos\varphi + b\sin\varphi)\lambda_{N1} \\ &\quad + \frac{1}{J_S}(a\cos\varphi + b\sin\varphi)(a\sin\varphi \\ &\quad + b\cos\varphi)\lambda_{T1} + (-a\sin\varphi + b\cos\varphi)\dot{\varphi}^2 \\ &= -g + \left(\frac{1}{m} - \frac{1}{J_S}(a\cos\varphi + b\sin\varphi)\bar{d}(\varphi)\right)\lambda_{N1} \\ &\quad + (-a\sin\varphi + b\cos\varphi)\dot{\varphi}^2 \\ &= \frac{1 - \frac{m}{J_S}(a\cos\varphi + b\sin\varphi)\bar{d}(\varphi)}{1 + \frac{m}{J_S}(a\cos\varphi - b\sin\varphi)\bar{d}(\varphi)}(g - (a\sin\varphi \\ &\quad + b\cos\varphi)\dot{\varphi}^2) - g + (-a\sin\varphi + b\cos\varphi)\dot{\varphi}^2 \\ &= -\frac{2\frac{m}{J_S}a\bar{d}(\varphi)(g\cos\varphi - b\dot{\varphi}^2) + 2a\dot{\varphi}^2\sin\varphi}{1 + \frac{m}{J_S}(a\cos\varphi - b\sin\varphi)\bar{d}(\varphi)}. \end{aligned} \tag{131}$$

Using $g_{N1} = 0$ and $g_{N2} > 0$ we deduce that $\sin\varphi > 0$. Moreover, using $(\boldsymbol{q}, \boldsymbol{u}) \in \mathcal{V}$ it follows that $|\tan\varphi| < \frac{a-b\mu}{b+a\mu}$ and $\dot{\varphi}^2 < g/\sqrt{a^2+b^2}$. We study the system for $\mu < \frac{a}{b}$, i.e. the equilibrium set $\mathcal{E}$ is isolated. Hence, it must hold that $\cos\varphi > \frac{b}{\sqrt{a^2+b^2}}$. From $d(\varphi) > 0$ and $|\bar{\mu}| \le \mu$ it follows that $\bar{d}(\varphi) > 0$. Substitution of $\dot{\varphi}^2 < g/\sqrt{a^2+b^2}$ and $\cos\varphi > \frac{b}{\sqrt{a^2+b^2}}$ gives $g\cos\varphi - b\dot{\varphi}^2 > 0$. Because $\bar{d}(\varphi) > 0$, $\cos\varphi > \frac{b}{\sqrt{a^2+b^2}}$, $\sin\varphi > 0$ and $a\cos\varphi - b\sin\varphi > 0$ it follows that $\dot{\gamma}_{N2} < 0$.

- $I_{\mathrm{N}} = \{2\}$: contact 1 is open and contact 2 is closed, i.e. $g_{N1} > 0$ and $g_{N2} = 0$. Similar to the previous case we can prove that $\dot{\gamma}_{N1} < 0$.

Hence, there exists a non-empty set $I_C = \{1, 2\}$, such that $\dot{\gamma}_{Ni}(\boldsymbol{q}, \boldsymbol{u}) < 0$ (a.e.) for $\forall i \in I_C \backslash I_N$ and $\forall(\boldsymbol{q}, \boldsymbol{u}) \in \mathcal{V}$. Condition 4 of Theorem 1 is therefore fulfilled.

Now we will show that condition 5 of Corollary 2 holds which implies through Proposition 3 that condition 5 of Theorem 1 holds. Using $D_{\boldsymbol{q}}^{\mathrm{nc}^{-1}}(0) = \mathbb{R}^n$ and Proposition 3 it follows that $D_{\boldsymbol{q}}^{\lambda_{\mathrm{T}}^{-1}}(0) = \ker \boldsymbol{W}_{\mathrm{T}}^{\mathrm{T}}(\boldsymbol{q})$. Note that for $\boldsymbol{q} \in \mathcal{C} = \{\boldsymbol{q} \in \mathbb{R}^n \mid g_{N1} = g_{N2} = 0\}$ the implication $\boldsymbol{W}_{\mathrm{N}}^{\mathrm{T}}(\boldsymbol{q})\boldsymbol{u} = 0 \Longrightarrow \dot{y} = 0 \wedge \dot{\varphi} = 0$ holds and similarly $\boldsymbol{W}_{\mathrm{T}}^{\mathrm{T}}(\boldsymbol{q})\boldsymbol{u} = 0 \Longrightarrow \dot{x} = 0$. We conclude that there is always dissipation when both contacts are closed and $\boldsymbol{u} \neq \boldsymbol{0}$ because

$$\ker \boldsymbol{W}_{\mathrm{T}}^{\mathrm{T}}(\boldsymbol{q}) \cap \ker \boldsymbol{W}_{N}^{\mathrm{T}}(\boldsymbol{q}) = \{\boldsymbol{0}\} \quad \forall \boldsymbol{q} \in \mathcal{C}, \tag{132}$$

and condition 5 of Corollary 2 is therefore fulfilled.

The largest level set of $V = T(\boldsymbol{q}, \boldsymbol{u}) + Q(\boldsymbol{q})$ which lies entirely in $\mathcal{Q} = \{(\boldsymbol{q}, \boldsymbol{u}) \in \mathbb{R}^n \times \mathbb{R}^n \mid \boldsymbol{q} \in \mathcal{U}\}$ is given by $V(\boldsymbol{q}, \boldsymbol{u}) < mg(\sqrt{a^2 + b^2} - b)$. The largest level set of V which lies entirely in $\mathcal{V}$ is determined by $V(\boldsymbol{q}, \boldsymbol{u}) < \frac{1}{2}\frac{J_S g}{\sqrt{a^2+b^2}}$ and $V(\boldsymbol{q}, \boldsymbol{u}) < mg(\frac{\sqrt{a^2+b^2}}{\sqrt{1+\mu^2}} - b)$. We therefore choose the set $\mathcal{I}_{\rho^*}$ as

$$\mathcal{I}_{\rho^*} = \{(\boldsymbol{q}, \boldsymbol{u}) \in \mathbb{R}^n \times \mathbb{R}^n \mid V(\boldsymbol{q}, \boldsymbol{u}) < \rho^*\}, \tag{133}$$

with

$$\rho^* = \min\left(\frac{1}{2}\frac{J_S g}{\sqrt{a^2 + b^2}}, mg\left(\frac{\sqrt{a^2 + b^2}}{\sqrt{1 + \mu^2}} - b\right)\right). \tag{134}$$

If additionally

$$\frac{1}{2}\frac{(\mu m g)^2}{k} < \rho^*, \tag{135}$$

then it holds that $\mathcal{E} \subset \mathcal{I}_{\rho^*}$. We conclude that Theorem 1 proves conditionally the local attractivity of the equilibrium set $\mathcal{E}$ and that $\mathcal{I}_{\rho^*}$ is a conservative estimate of the region of attraction.

7.4 Constrained beam

We now study an example with bilateral constraints. Consider a beam with mass m, length $2l$ and moment of inertia J_S around its centre of mass S, see Fig. 8. The gravitational acceleration is denoted by g. The beam is subject to two holonomic constraints: Point 1 of the beam is constrained to the vertical slider and Point 2 of the beam is constrained to the horizontal slider. Coulomb friction is present in the contact between these endpoints of the beam and the grooves (friction coefficient μ_1 in the vertical slider and friction coefficient μ_2 in the horizontal slider). It should be noted that the realised friction forces depend on the constraint forces in the grooves (i.e. the friction is described by the non-associated Coulomb's law (18)). The dynamics of the system will be described in terms of the (independent) coordinate θ, see Fig. 8. The corresponding equation of motion is given by

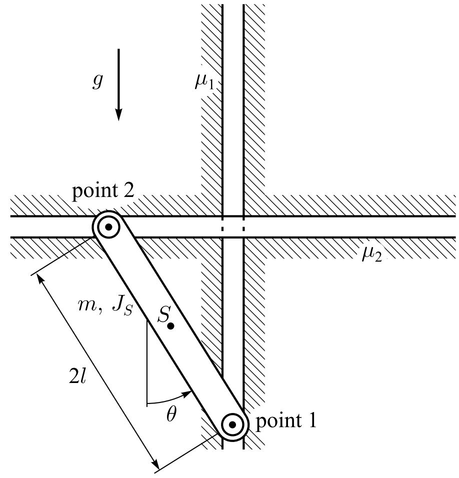


Fig. 8 Constrained beam

$$(ml^2 + J_S)\ddot{\theta} + mgl\sin\theta = 2l\sin\theta\lambda_{T_1} - 2l\cos\theta\lambda_{T_2}, \tag{136}$$

where λ_{T_1} and λ_{T_2} are the friction forces in the vertical and horizontal sliders, respectively. Equation (136) can be written in the form (91), with

$$\boldsymbol{M}(\boldsymbol{q}) = ml^2 + J_S, \quad \boldsymbol{h}(\boldsymbol{q}, \boldsymbol{u}) = -mgl\sin\theta,$$
$$\boldsymbol{W}_{\mathrm{T}}(\boldsymbol{q}) = [2l\sin\theta \quad -2l\cos\theta]. \tag{137}$$

The equilibrium set of (136) is given by Equation (92), with $C_{T_i} = \{-\lambda_{T_i} \mid -\mu_i|\lambda_{N_i}| \leq \lambda_{T_i} \leq +\mu_i|\lambda_{N_i}|\}$, $i = 1, 2$. Note that C_{T_i} depends on the normal force λ_{N_i}, which in turn may depend on the friction forces. The static equilibrium equations of the beam

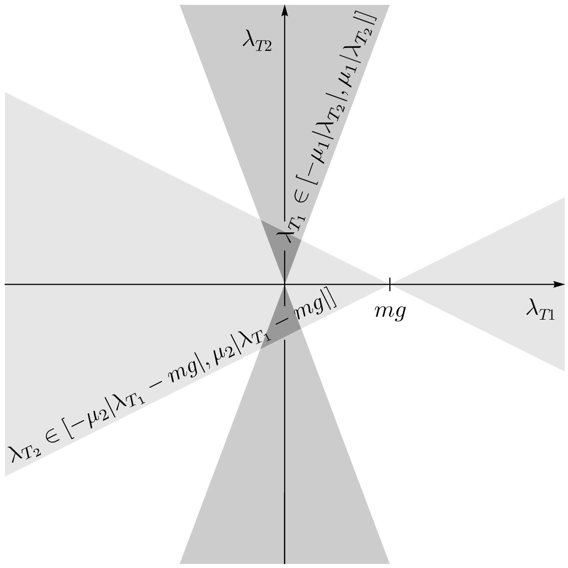


Fig. 9 Attainable friction forces in equilibrium

yield:

$$\begin{aligned}
&\lambda_{N_1} + \lambda_{T_2} = 0,\\
&\lambda_{N_2} + \lambda_{T_1} - mg = 0, \qquad (138)\\
&l\cos\theta\lambda_{N_1} - l\sin\theta\lambda_{N_2} + l\sin\theta\lambda_{T_1} - l\cos\theta\lambda_{T_2} = 0.
\end{aligned}$$

Based on the first two equations in Equation (138) and the non-associated Coulomb's law (18) the following algebraic inclusions for the friction forces in equilibrium can be derived:

$$\begin{aligned}
&\lambda_{T_1} \in [-\mu_1|\lambda_{T_2}|, \mu_1|\lambda_{T_2}|],\\
&\lambda_{T_2} \in [-\mu_2|\lambda_{T_1} - mg|, \mu_2|\lambda_{T_1} - mg|].
\end{aligned} \qquad (139)$$

The resulting set of friction forces in equilibrium is depicted schematically in Fig. 9. The equilibrium set $\mathcal{E}$ in terms of the independent generalised coordinate θ now follows from the equation of motion (136):

$$mgl\sin\theta = 2l\sin\theta\lambda_{T_1} - 2l\cos\theta\lambda_{T_2}. \qquad (140)$$

For values of θ such that $\cos\theta \neq 0$ (we assume that, for given values for m, g and l, the friction coefficients μ_1 and μ_2 are small enough to guarantee that this assumption is satisfied) we obtain:

$$\theta = \arctan\left(\frac{\lambda_{T_2}}{-\frac{mg}{2} + \lambda_{T_1}}\right) + k\pi, \quad k = 0, 1, \qquad (141)$$

for values of λ_{T_1} and λ_{T_2} taken from (139). Equation (141) describes the fact that there exist two isolated equilibrium sets (an equilibrium set $\mathcal{E}_1$ around $\theta = 0$ and $\mathcal{E}_2$ around $\theta = \pi$) for small values of the friction coefficients. The equilibrium sets are given by

$$\begin{aligned}
\mathcal{E}_k = \Bigg\{(\theta, \dot\theta) \mid \dot\theta = 0, -\arctan\left(\frac{2\mu_2}{1-\mu_1\mu_2}\right)\\
\leq \theta - (k-1)\pi \leq \arctan\left(\frac{2\mu_2}{1-\mu_1\mu_2}\right)\Bigg\},
\end{aligned} \qquad (142)$$

for $k = 1, 2$ and $\mu_1\mu_2 < 1$. Note that for $\mu_1\mu_2 \geq 1$ these isolated equilibrium sets merge into one large equilibrium set, such that any value of θ can be attained in this equilibrium set. We will consider the case of two isolated equilibrium sets here.

First, we will study the stability properties of the equilibrium set $\mathcal{E}_1$ around $\theta = 0$. Let us hereto apply Corollary 3 and check the conditions stated therein. Condition 1 of this corollary is clearly satisfied since the kinetic energy is given by: $T = \frac{1}{2}\left(ml^2 + J_S\right)\dot\theta^2$. Condition 2 is also satisfied. Namely, take the set $\mathcal{U} = \{\theta \mid |\theta| < \pi\}$ and realise that indeed the potential energy $U = mgl(1 - \cos\theta)$ is positive definite in $\mathcal{U}$ and $\partial U/\partial\theta = mgl\sin\theta$ satisfies the demand $\partial U/\partial\theta \neq 0, \forall\theta \in \mathcal{U}\backslash\{0\}$. Since there are no smooth non-conservative forces $D_{\boldsymbol{q}}^{\mathrm{nc}}(\boldsymbol{u}) = 0$, condition 3 is satisfied. Finally, we note that $D_{\boldsymbol{q}}^{\mathrm{nc}\,-1}(0) = \mathbb{R}$ and $D_{\boldsymbol{q}}^{\lambda_{\mathrm{T}}\,-1}(0) = \boldsymbol{0}$, which implies that condition 4 of Corollary 3 is satisfied. The set $\mathcal{U}$ contains the equilibrium set $\mathcal{E}_1$ and part of the equilibrium set $\mathcal{E}_2$ (see Fig. 11). We now consider the largest level set $V < c^*$ for which the set $\mathcal{E}_1$ is the only equilibrium set within the level set of $V = T + U$. This level set is an open set and the value

$$c^* = mgl\left(1 - \frac{1-\mu_1\mu_2}{\sqrt{4\mu_2^2 + (1-\mu_1\mu_2)^2}}\right) \qquad (143)$$

is chosen such that its closure touches the equilibrium set $\mathcal{E}_2$. Consequently, we can conclude that the equilibrium set $\mathcal{E}_1$ is locally attractive. The phase plane of the constrained beam system is depicted in Fig. 10 for the parameter values $m = 1$ kg, $J_S = \frac{1}{3}$ kg m^2, $l = 1$ m, $\varepsilon_{\mathrm{N}} = \varepsilon_{\mathrm{T}} = 0$, $\mu_1 = \mu_2 = 0.3$, $g = 10$ m/s^2. The trajectories in Fig. 10 have been obtained numerically using the time-stepping method (see [31] and references therein). The equilibrium sets $\mathcal{E}_1$ and $\mathcal{E}_2$

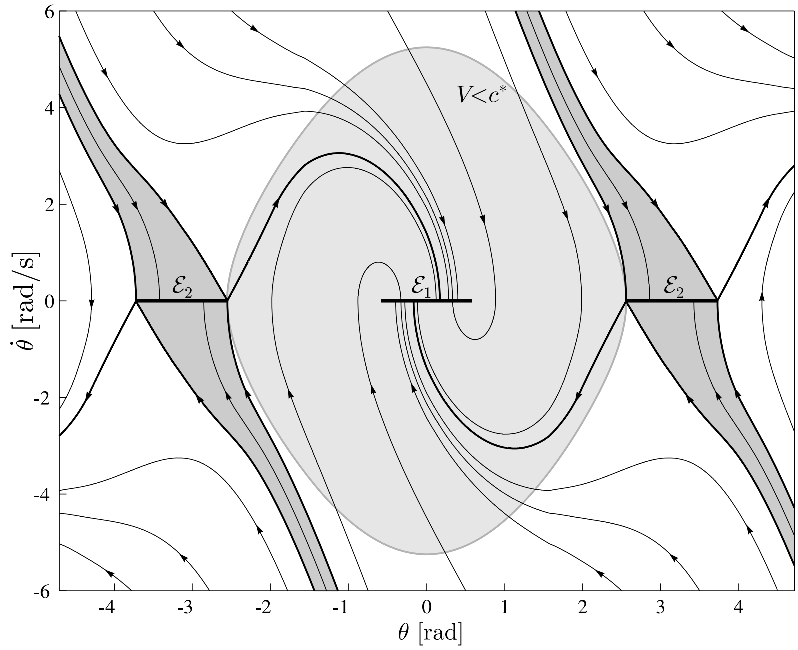


Fig. 10 Phase plane and the set in which $V = T + U < c^*$

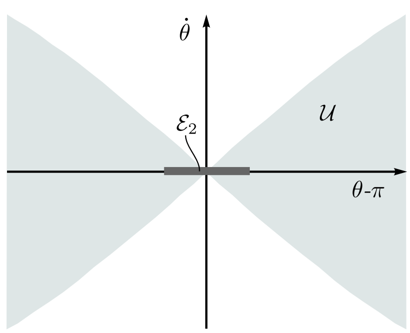


Fig. 11 Schematic representation of the set $\mathcal{U}$ in which $V \geq 0$ (V as in Equation (144))

are indicated by thick lines on the axis $\dot\theta = 0$. It can be seen in Fig. 10 that the level set $V < c^*$ is a fairly good (though conservative) estimate for the region of attraction of the equilibrium set $\mathcal{E}_1$.

Secondly, we will study the stability properties of the equilibrium set $\mathcal{E}_2$ around $\theta = \pi$. We apply Theorem 2 and check the conditions stated therein. The function V in this theorem is chosen as follows:

$$V = -\frac{1}{2}(J_S + ml^2)\dot\theta^2 + mgl\,(1 + \cos\theta)\,, \qquad (144)$$

where $V \geq 0 \in \mathcal{U}$ with the set $\mathcal{U}$ depicted schematically in Fig. 11. The time-derivative of V obeys

$$\dot V = -\dot\theta \boldsymbol{W}_\mathrm{T}\boldsymbol{\lambda}_\mathrm{T} = -\boldsymbol{\gamma}_\mathrm{T}^\mathrm{T}\boldsymbol{\lambda}_\mathrm{T},$$

with $\boldsymbol{W}_\mathrm{T} = [2l\sin\theta \quad -2l\cos\theta]$, $\boldsymbol{\lambda}_\mathrm{T}^\mathrm{T} = [\lambda_{T_1} \quad \lambda_{T_2}]$ and $\boldsymbol{\gamma}_\mathrm{T} = \boldsymbol{W}_\mathrm{T}^\mathrm{T}\dot\theta = [2l\dot\theta\sin\theta \quad -2l\dot\theta\cos\theta]^\mathrm{T}$ are the sliding velocities in the two frictional sliders. Note that $\dot V \geq 0$ for all $(\theta, \dot\theta) \in \mathcal{U}$ and $\dot V = 0$ if and only if $\dot\theta = 0$. We can easily show that solutions cannot stay in $\mathcal{S}\backslash\mathcal{E}_2$, with $\mathcal{S} = \{(\theta, \dot\theta) \mid \dot\theta = 0\}$, using the equation of motion (136). The conditions of statement 3 of Theorem 2 are satisfied and it can be concluded that the equilibrium set $\mathcal{E}_2$ is unstable.

The equilibrium set $\mathcal{E}_2$ becomes a saddle point for $\mu_{1,2} = 0$. This saddle structure in the phase plane (see Fig. 10) remains for $\mu_{1,2} > 0$, but $\mathcal{E}_2$ is a set instead of a point. Interestingly, the stable manifold of $\mathcal{E}_2$ is 'thick', i.e. there exists a bundle of solutions (depicted in dark grey) which are attracted to the unstable equilibrium set $\mathcal{E}_2$. Put differently: the equilibrium set $\mathcal{E}_2$ has a region of attraction, where the region is a set with a non-empty interior. The unstable half-manifolds of $\mathcal{E}_2$ originate at the tips of the set $\mathcal{E}_2$ and are heteroclinic orbits with the stable equilibrium set $\mathcal{E}_1$.

8 Conclusions

In this paper, conditions are given under which the equilibrium set of multi-degree-of-freedom non-linear mechanical systems with an arbitrary number of frictional unilateral constraints is attractive. The theorems for attractivity are proved by using the framework of measure differential inclusions together with a Lyapunov-type stability analysis and a generalisation of LaSalle's invariance principle for non-smooth systems. The total mechanical energy of the system, including the support function of the normal contact forces, is chosen as Lyapunov function. It has been proved that, under some conditions, the differential measure of the Lyapunov function is non-positive, which is basically a dissipativity argument. Sufficient conditions for the dissipativity of frictional unilateral constraints are given. If we do not consider dependent constraints, then the restitution coefficients must either be small enough, or, be all equal to each other. The latter condition has also been stated in [19, 35]. Attractivity of the equilibrium set is proved in Theorem 1 under a number of conditions. Condition 4 is a condition, which is difficult to satisfy and check. It guarantees that there exists no invariant set when one or more contacts are open. Still, we are able to use Theorem 1 to prove the attractivity of equilibrium sets in a number of example systems in Section 7. Moreover, we provide conservative estimates for the region of attraction of the equilibrium set.

Non-linear mechanical systems with frictional bilateral constraints form a sub-class of the class of systems that can be studied with Theorem 1. For this sub-class of systems the (local) attractivity of the equilibrium set can be proven with Corollary 3. Moreover, a result on the instability of equilibrium sets of differential inclusions is proposed in Section 6.2. This result allows us to investigate the instability of equilibrium sets of non-linear mechanical systems with frictional bilateral constraints. An example system with two frictional bilateral constraints is studied in Section 7.4 and the attractivity and instability of its equilibrium sets are discussed. Figure 10 showed that the stable manifold of a 'saddle-type' equilibrium set consists of a bundle of solutions, which are attracted to the unstable equilibrium set in finite time.

The theorems presented in this paper have been proved for dissipative systems and form the stepping stone to the analysis of non-dissipative systems for which the equilibrium set might still be attractive due to the dissipation of the frictional impacts (see also [45]). The results of this paper will be used in further research to develop control methods for systems with unilateral constraints.

Appendix A: Positive definite matrices

Proposition 4. *Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix and $\boldsymbol{B} \in \mathbb{R}^{n\times n}$ be a diagonal positive definite matrix with the diagonal elements b_{ii} which fulfil $1 \geq b_{ii} \geq b_{\min} > 0$, $i = 1, \dots, n$. If*

$$1 - b_{\min} < \frac{1}{cond(A)}$$

then it holds that the matrix $\boldsymbol{AB}$ is positive definite.

Proof: The matrix $\boldsymbol{A} = \boldsymbol{A}^{\mathrm{T}} > 0$ has real positive eigenvalues and it therefore holds that

$$\boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x} \geq \lambda_{\min}\|\boldsymbol{x}\|^2 , \tag{145}$$

where $\lambda_{\min}$ is the smallest eigenvalue of $\boldsymbol{A}$. Moreover, it holds that

$$\begin{aligned} \boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}(\boldsymbol{I}-\boldsymbol{B})\boldsymbol{x} &\leq |\boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}(\boldsymbol{I}-\boldsymbol{B})\boldsymbol{x}| \\ &\leq |\boldsymbol{A}|\,|\boldsymbol{I}-\boldsymbol{B}|\,\|\boldsymbol{x}\|^2 \\ &\leq \lambda_{\max}(1-b_{\min})\|\boldsymbol{x}\|^2 , \end{aligned} \tag{146}$$

where $\lambda_{\max}$ is the largest eigenvalue of $\boldsymbol{A}$ and $b_{\min}$ is the smallest diagonal element of $\boldsymbol{B}$. Using the above inequalities, we deduce that

$$\begin{aligned} \boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{B}\boldsymbol{x} &= \boldsymbol{x}^{\mathrm{T}}(\boldsymbol{A}-\boldsymbol{A}(\boldsymbol{I}-\boldsymbol{B}))\boldsymbol{x} \\ &\geq (\lambda_{\min} - \lambda_{\max}(1-b_{\min}))\,\|\boldsymbol{x}\|^2 . \end{aligned} \tag{147}$$

Hence, if it holds that

$$1 - b_{\min} < \frac{\lambda_{\min}}{\lambda_{\max}} =: \frac{1}{\mathrm{cond}(\boldsymbol{A})} , \tag{148}$$

then it follows that $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{B}\boldsymbol{x} > 0$ holds for all $\boldsymbol{x} \neq \boldsymbol{0}$. □

Appendix B: Convex analysis

The generalised differential of a scalar convex function, defined by Equation (149), is called the *subdifferential*

$$\begin{aligned} \partial f(\boldsymbol{x}) = \{\boldsymbol{y} \mid f(\boldsymbol{x}^*) &\geq f(\boldsymbol{x}) \\ &+ \boldsymbol{y}^{\mathrm{T}}(\boldsymbol{x}^* - \boldsymbol{x}); \forall \boldsymbol{x}^*\} \quad \subset \mathbb{R}^n . \end{aligned} \tag{149}$$

Let C be a convex set and $\boldsymbol{x} \in C$. The set of vectors $\boldsymbol{y}$ that are normal to $\boldsymbol{x} \in C$ form the *normal cone* of C in $\boldsymbol{x}$

$$N_C(\boldsymbol{x}) = \{\boldsymbol{y} \mid \boldsymbol{y}^{\mathrm{T}}(\boldsymbol{x}^* - \boldsymbol{x}) \leq 0, \quad \boldsymbol{x} \in C, \forall \boldsymbol{x}^* \in C\}. \tag{150}$$

If $\boldsymbol{x}$ is in the interior of C then $N_C(\boldsymbol{x}) = 0$. If $\boldsymbol{x} \notin C$ then $N_C(\boldsymbol{x}) = \emptyset$. The *indicator function* of C is defined as

$$\Psi_C(\boldsymbol{x}) = \begin{cases} 0, & \boldsymbol{x} \in C, \\ +\infty, & \boldsymbol{x} \notin C. \end{cases} \tag{151}$$

The indicator function is a convex function. With the definition of the subdifferential (149) and the indicator function it follows that

$$\begin{aligned} \partial\Psi_C(\boldsymbol{x}) &= \{\boldsymbol{y} \mid \Psi_C(\boldsymbol{x}^*) \geq \Psi_C(\boldsymbol{x}) + \boldsymbol{y}^{\mathrm{T}}(\boldsymbol{x}^* - \boldsymbol{x}), \\ &\qquad \boldsymbol{x} \in C, \forall \boldsymbol{x}^* \in C\} \\ &= \{\boldsymbol{y} \mid 0 \geq \boldsymbol{y}^{\mathrm{T}}(\boldsymbol{x}^* - \boldsymbol{x}), \quad \boldsymbol{x} \in C, \forall \boldsymbol{x}^* \in C\}. \end{aligned} \tag{152}$$

This is exactly the definition of the normal cone at C. The subdifferential of the indicator function at $\boldsymbol{x} \in C$ is therefore the normal cone of C at $\boldsymbol{x}$,

$$\partial\Psi_C(\boldsymbol{x}) = N_C(\boldsymbol{x}). \tag{153}$$

Let f be a convex function. The function f^* is called the *conjugate function* of f and is defined as

$$f^*(\boldsymbol{x}^*) = \sup_{\boldsymbol{x}} \{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}^* - f(\boldsymbol{x})\}. \tag{154}$$

From Fenchel's inequality [42] follows the equality

$$\begin{aligned} \boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}^* = f(\boldsymbol{x}) + f^*(\boldsymbol{x}^*) &\iff \boldsymbol{x}^* \\ \in \partial f(\boldsymbol{x}) &\iff \boldsymbol{x} \in \partial f^*(\boldsymbol{x}^*). \end{aligned} \tag{155}$$

The conjugate function of the indicator function Ψ_C on a convex set C is called *support function*,

$$\begin{aligned} \Psi_C^*(\boldsymbol{x}^*) &= \sup_{\boldsymbol{x}} \{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}^* - \Psi_C(\boldsymbol{x})\} \\ &= \sup_{\boldsymbol{x}} \{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}^* \mid \boldsymbol{x} \in C\}. \end{aligned} \tag{156}$$

The support function is positively homogeneous in the sense that

$$\Psi_C^*(a\boldsymbol{x}^*) = a\Psi_C^*(\boldsymbol{x}^*) \quad \forall a > 0. \tag{157}$$

If $\boldsymbol{x} \in \partial\Psi_C^*(\boldsymbol{x}^*)$ then it holds that $\boldsymbol{x} \in C$ and

$$\begin{aligned} \boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}^* &= \underbrace{\Psi_C(\boldsymbol{x})}_{=0} + \Psi_C^*(\boldsymbol{x}^*) \\ &= \Psi_C^*(\boldsymbol{x}^*). \end{aligned} \tag{158}$$

It follows that $\partial\Psi_C^*(\boldsymbol{0}) = C$. The support function $\Psi_C^*(\boldsymbol{x}^*)$ is a convex function with $\Psi_C^*(\boldsymbol{0}) = 0$. Hence, if $\boldsymbol{0} \in C$, then $\boldsymbol{0} \in \partial\Psi_C^*(\boldsymbol{0})$ from which follows that $\Psi_C^*(\boldsymbol{x}^*)$ attains a minimum at $\boldsymbol{x}^* = \boldsymbol{0}$, i.e.

$$\boldsymbol{0} \in C \Longrightarrow \Psi_C^*(\boldsymbol{x}^*) \geq 0. \tag{159}$$

Moreover, if $\boldsymbol{0} \in \operatorname{int} C$ then it follows that $\Psi_C^*(\boldsymbol{x}^*)$ attains a global minimum at $\boldsymbol{x}^* = \boldsymbol{0}$, i.e.

$$\boldsymbol{0} \in \operatorname{int} C \Longrightarrow \Psi_C^*(\boldsymbol{x}^*) > 0 \quad \forall \boldsymbol{x}^* \neq \boldsymbol{0}. \tag{160}$$

Appendix C: Subderivative

The differentiablility of a function $f\colon \mathbb{R}^n \to \mathbb{R}$ at a point $\boldsymbol{x}$ is connected with the existence of a tangent hyperplane to the graph of f at the point $(\boldsymbol{x}, f(\boldsymbol{x}))$ [41]. The concept of differentiablity can be generalised by considering the contingent cone to the epigraph of f instead. In this section, we consider a lower semi-continuous extended function $f\colon \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ whose domain $\mathrm{dom}(f) = \{\boldsymbol{x} \in \mathbb{R}^n \mid f(\boldsymbol{x}) < \infty\}$ is non-empty (i.e. the function is not trivial). The epigraph of the function f is closed, because f is lower semi-continuous. Various generalised notions of gradients exist, but the subderivative is the most natural object to focus on and is often called the contingent epiderivative [5] or epicontingent derivative [4].

We define the function

$$\mathrm{d}f(\boldsymbol{x})(\boldsymbol{v}) = \liminf_{t \downarrow 0,\, \boldsymbol{v}' \to \boldsymbol{v}} \frac{f(\boldsymbol{x} + t\boldsymbol{v}') - f(\boldsymbol{x})}{t}$$

as the *subderivative* of f at $\boldsymbol{x}$ in the direction $\boldsymbol{v}$ [42]. The epigraph of $\mathrm{d}f(\boldsymbol{x})(\cdot)$ is the contingent cone at the

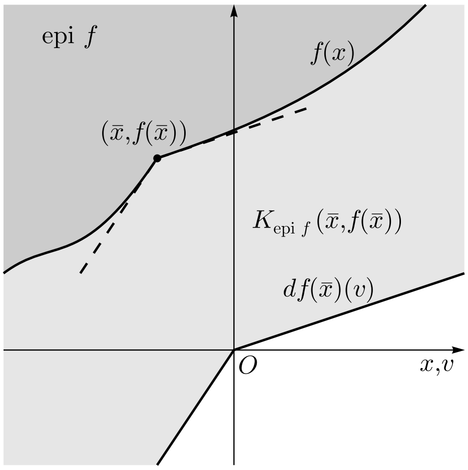


Fig. 12 Subderivative of f and its relation to the contingent cone

epigraph of f at $(\boldsymbol{x}, f(\boldsymbol{x}))$

$$\operatorname{epi} \mathrm{d} f(\boldsymbol{x})(\cdot) = K_{\operatorname{epi} f}((\boldsymbol{x}, f(\boldsymbol{x})). \tag{161}$$

We observe that $\mathrm{d} f(\boldsymbol{x})(\mathbf{0}) = 0$. If f is differentiable at $\boldsymbol{x}$, then it holds that $\mathrm{d} f(\boldsymbol{x})(\boldsymbol{v}) = -\mathrm{d} f(\boldsymbol{x})(-\boldsymbol{v}) = (\nabla f(\boldsymbol{x}))^{\mathrm{T}}\boldsymbol{v}$. The epigraph of the subderivative is a cone (the contingent cone) and the subderivative $\mathrm{d} f(\boldsymbol{x})(\cdot)$ is therefore positively homogeneous

$$\mathrm{d} f(\boldsymbol{x})(a\boldsymbol{v}) = a\, \mathrm{d} f(\boldsymbol{x})(\boldsymbol{v}), \quad \forall a \geq 0. \tag{162}$$

If the function f is convex, then we can express the subdifferential as

$$\partial f(\boldsymbol{x}) = \{\boldsymbol{y} \mid \mathrm{d} f(\boldsymbol{x})(\boldsymbol{v}) \geq \boldsymbol{v}^{\mathrm{T}}\boldsymbol{y}\}. \tag{163}$$

Of special interest is the subderivative (see Fig. 12) of an indicator function $\Psi_C(\boldsymbol{x})$,

$$\mathrm{d}\Psi_C(\boldsymbol{x})(\boldsymbol{v}) = \Psi_{K_C(\boldsymbol{x})}(\boldsymbol{v}), \tag{164}$$

where $K_C(\boldsymbol{x})$ is the contingent cone to C at the point $\boldsymbol{x}$.

Appendix D: Functions of bounded variation

Let I be a real interval and X be a Euclidean space. The function $\boldsymbol{f}: I \to X$ is said to be of locally bounded variation, $\boldsymbol{f} \in \operatorname{lbv}(I, X)$, if and only if

$$\operatorname{var}(\boldsymbol{f}, [a, b]) = \sup \sum_{i=1}^{n} \|\boldsymbol{f}(t_i) - \boldsymbol{f}(t_{i-1})\| < \infty \tag{165}$$

for every compact sub-interval $[a, b]$ of I, where the supremum is taken over all strictly increasing finite sequences $t_1 < t_2 < \cdots < t_{\mathrm{N}}$ of points on $[a, b]$.

Let $V: X \to \mathbb{R}$ be a function which is Lipschitz continuous on the closed domain $D \subset X$ with Lipschitz constant K, i.e.

$$\exists K < \infty,\ \|V(\boldsymbol{x}) - V(\boldsymbol{y})\| \leq K\|\boldsymbol{x} - \boldsymbol{y}\| \quad \forall \boldsymbol{x}, \boldsymbol{y} \in D. \tag{166}$$

If it holds that $\boldsymbol{x} \in \operatorname{lbv}(I, D)$, and consequently $\boldsymbol{x}(t) \in D\ \forall t \in I$, then it holds that the function $\tilde{V}(t) = V \circ \boldsymbol{x} = V(\boldsymbol{x}(t))$ is of locally bounded variation on I. Indeed, the variation of $\tilde{V}$ on a compact interval $[a, b] \subset I$ gives

$$\begin{aligned} \operatorname{var}(\tilde{V}, [a, b]) &= \sup \sum_{i=1}^{n} \|\tilde{V}(t_i) - \tilde{V}(t_{i-1})\| \\ &= \sup \sum_{i=1}^{n} \|V(\boldsymbol{x}(t_i)) - V(\boldsymbol{x}(t_{i-1}))\| \\ &\leq K \sup \sum_{i=1}^{n} \|\boldsymbol{x}(t_i) - \boldsymbol{x}(t_{i-1})\| \\ &\leq K \operatorname{var}(\boldsymbol{x}, [a, b]) \\ &< \infty. \end{aligned} \tag{167}$$

In particular, if $V(\boldsymbol{x}) = v(\boldsymbol{x}) + \Psi_D(\boldsymbol{x})$, where $v: X \to \mathbb{R}$ is a Lipschitz continuous function on X and $\boldsymbol{x} \in \operatorname{lbv}(I, D)$, then it follows that $\tilde{V} = V \circ \boldsymbol{x} \in \operatorname{lbv}(I, \mathbb{R})$.

Appendix E: Differential measure of a bilinear form

The following is based on [33]. Consider $\boldsymbol{x} \in \operatorname{lbv}(I, X)$ and $\boldsymbol{y} \in \operatorname{lbv}(I, Y)$ and the function $t \to F(\boldsymbol{x}(t), \boldsymbol{y}(t))$, being a continuous bilinear form $F: X \times Y \to \mathbb{R}$, denoted by $F(\boldsymbol{x}, \boldsymbol{y})$ in short. First assume that $\boldsymbol{x}$ and $\boldsymbol{y}$ are local step functions, each having their own set of discontinuity points. The set of discontinuity points of $F(\boldsymbol{x}, \boldsymbol{y})$ is the union of the discontinuity points of $\boldsymbol{x}$ and of $\boldsymbol{y}$. Construct a sequence of nodes $t_1 < t_t < \cdots < t_n$ on the discontinuity points of $F(\boldsymbol{x}, \boldsymbol{y})$ on a sub-interval $[a, b]$ of I. The functions $\boldsymbol{x}(t)$, $\boldsymbol{y}(t)$ and $F(\boldsymbol{x}, \boldsymbol{y})$ are therefore constant on each open sub-interval (t_i, t_{i+1}). The differential measure $\mathrm{d}F(\boldsymbol{x}, \boldsymbol{y})$ equals the sum of a locally finite collection of point measures placed at the discontinuity points of F

$$\int_{[a,b]} \mathrm{d}F(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{n} (F(\boldsymbol{x}^+(t_i), \boldsymbol{y}^+(t_i)) - F(\boldsymbol{x}^-(t_i), \boldsymbol{y}^-(t_i))). \tag{168}$$

Similarly, $\mathrm{d}\boldsymbol{x}$ equals the sum of point measures placed at the nodes t_i with values $\boldsymbol{x}^+(t_i) - \boldsymbol{x}^-(t_i)$ (and the same applies for $\mathrm{d}\boldsymbol{y}$). It therefore holds that

$$\begin{aligned} \int_{[a,b]} F(\mathrm{d}\boldsymbol{x}, \boldsymbol{y}^-) &= \sum_{i=1}^{n} F(\boldsymbol{x}^+(t_i) - \boldsymbol{x}^-(t_i), \boldsymbol{y}^-(t_i)), \\ \int_{[a,b]} F(\boldsymbol{x}^+, \mathrm{d}\boldsymbol{y}) &= \sum_{i=1}^{n} F(\boldsymbol{x}^+(t_i), \boldsymbol{y}^+(t_i) - \boldsymbol{y}^-(t_i)). \end{aligned} \tag{169}$$

Exploiting the bilinearity of F yields

$$\int_{[a,b]} \mathrm{d}F(\boldsymbol{x}, \boldsymbol{y}) = \int_{[a,b]} (F(\mathrm{d}\boldsymbol{x}, \boldsymbol{y}^-) + F(\boldsymbol{x}^+, \mathrm{d}\boldsymbol{y})). \tag{170}$$

Every locally bounded function can be approximated by a local step function and their difference can be made arbitrarily small by refining the partition of the local step function. Equation (170) does therefore not only hold for local step functions, but holds for arbitrary locally bounded functions $\boldsymbol{x}$ and $\boldsymbol{y}$, as has been proved rigourously in [33].

Consider now a symmetric quadratic form $G(\boldsymbol{x}) = F(\boldsymbol{x}, \boldsymbol{x}) = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x}$, with $\boldsymbol{A} = \boldsymbol{A}^{\mathrm{T}}$. We deduce from Equation (170) that

$$\begin{aligned} \int_{[a,b]} \mathrm{d}G(\boldsymbol{x}) &= \int_{[a,b]} (F(\mathrm{d}\boldsymbol{x}, \boldsymbol{x}^-) + F(\boldsymbol{x}^+, \mathrm{d}\boldsymbol{x})) \\ &= \int_{[a,b]} F(\boldsymbol{x}^+ + \boldsymbol{x}^-, \mathrm{d}\boldsymbol{x}) \\ &= \int_{[a,b]} (\boldsymbol{x}^+ + \boldsymbol{x}^-)^{\mathrm{T}}\boldsymbol{A}\,\mathrm{d}\boldsymbol{x} \end{aligned} \tag{171}$$

or simply $\mathrm{d}G = (\boldsymbol{x}^+ + \boldsymbol{x}^-)^{\mathrm{T}}\boldsymbol{A}\,\mathrm{d}\boldsymbol{x}$.

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