International Journal of Bifurcation and Chaos, Vol. 18, No. 5 (2008) 1435–1457

UNIFORM CONVERGENCE OF MONOTONE MEASURE DIFFERENTIAL INCLUSIONS: WITH APPLICATION TO THE CONTROL OF MECHANICAL SYSTEMS WITH UNILATERAL CONSTRAINTS

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Received May 22, 2007; Revised July 10, 2007

In this paper, we present theorems which give sufficient conditions for the uniform convergence of measure differential inclusions with certain maximal monotonicity properties. The framework of measure differential inclusions allows us to describe systems with state discontinuities. Moreover, we illustrate how these convergence results for measure differential inclusions can be exploited to solve tracking problems for certain classes of nonsmooth mechanical systems with friction and one-way clutches. Illustrative examples of convergent mechanical systems are discussed in detail.

Keywords: Discontinuous and impulsive systems; incremental stability; convergence.

1. Introduction

In this paper, we show that measure differential inclusions with certain maximal monotonicity conditions exhibit the convergence property. A system, which is excited by an input, is called convergent if it has a unique solution that is bounded on the whole time axis and this solution is globally asymptotically stable. Obviously, if such a solution does exist, then all other solutions converge to this solution, regardless of their initial conditions, and can be considered as a steady-state solution [Demidovich, 1967; Pavlov *et al.*, 2004]. Similar notions describing the property of solutions converging to each other are studied in literature. The notion of contraction has been introduced in [Lohmiller & Slotine, 1998] (see also references therein). An operator-based approach towards studying the property that all solutions of a system converge to each other is pursued in [Fromion *et al.*, 1996, 1999]. In [Angeli, 2002], a Lyapunov approach has been developed to study the global uniform asymptotic stability of all solutions of a system (in [Angeli, 2002], this property is called incremental stability) as well as the so-called incremental input-to-state stability property, which is compatible with the input-to-state stability approach (see e.g. [Sontag, 1995]).

The property of convergence can be beneficial from several points of view. Firstly, in many control problems it is required that controllers are designed in such a way that all solutions of the corresponding closed-loop system "forget" their initial conditions. Actually, one of the main tasks of feedback is to eliminate the dependency of solutions on initial conditions. In this case, all solutions converge to some steady-state solution that is determined only by the input of the closed-loop system. This input can be, for example, a command signal or a signal generated by a feedforward part of the controller or, as in the observer design problem, it can be the measured signal from the observed system. Such a convergence property of a system plays an important role in many nonlinear control problems including tracking, synchronization, observer design, and the output regulation problem, see e.g. [Pavlov *et al.*, 2005c, 2005d; Pogromsky, 1998; van de Wouw *et al.*, 2006] and references therein. Secondly, from a dynamics point of view, convergence is an interesting property because it excludes the possibility of different coexisting steady-state solutions: namely, a convergent system excited by a periodic input has a *unique* globally asymptotically stable periodic solution. Moreover, the notion of convergence is a powerful tool for the analysis of time-varying systems. This tool can be used, for example, for performance analysis of nonlinear control systems, see e.g. [Heertjes *et al.*, 2006].

In [Demidovich, 1967], conditions for the convergence property were formulated for smooth nonlinear systems. In [Yakubovich, 1964], Lur'e-type systems, possibly with discontinuities, were considered and convergence conditions proposed. Only recently, in [Pavlov *et al.*, 2005b] sufficient conditions for continuous (though nonsmooth) piecewise affine (PWA) systems are proposed in terms of the existence of a common quadratic Lyapunov function for all affine systems constituting the PWA system. In [Pavlov *et al.*, 2005a], it was shown that the existence of such a common quadratic Lyapunov function is by no means sufficient for convergence of discontinuous PWA systems and a necessary and sufficient condition for convergence of bimodal discontinuous PWA systems is proposed. Here, we will consider systems described by measure differential inclusions, which includes systems with discontinuities but also allows for impulsive right-hand sides.

Nonsmooth dynamical systems, with or without impulsive dynamics, are studied by various scientific communities using different mathematical frameworks [Leine & Nijmeijer, 2004]: singular perturbations, switched or hybrid systems, complementarity systems, (measure) differential inclusions. The singular perturbation approach replaces the nonsmooth system by a singularly perturbed smooth system. The resulting ordinary differential equation is extremely stiff and hardly suited for numerical integration. In the field of systems and control theory, the term hybrid system is frequently used for systems composed of continuous differential equations and discrete event parts [Brogliato, 1999; van der Schaft & Schumacher, 2000; Goebel & Teel, 2006; Lygeros *et al.*, 2003]. The switched or hybrid system concept switches between differential equations with possible state reinitializations and is not able to describe accumulation points, e.g. infinitely many switching events which occur in a finite time such as a bouncing ball coming to rest on a table, in the sense that solutions cannot proceed over the accumulation point. Systems described by differential equations with a discontinuous right-hand side, but with a time-continuous state, can be extended to differential inclusions with a set-valued right-hand side [Filippov, 1988]. The differential inclusion concept gives a simultaneous description of the dynamics in terms of a single inclusion, which avoids the need to switch between different differential equations. Moreover, this framework is able to describe accumulation points of switching events. Systems which expose discontinuities in the state and/or vector field can be described by measure differential inclusions [Brogliato, 1999; Monteiro Marques, 1993; Moreau, 1988b]. The differential measure of the state vector does not only consist of a part with a density with respect to the Lebesgue measure (i.e. the time-derivative of the state vector), but is also allowed to contain an atomic part. The dynamics of the system is described by an inclusion of the differential measure of the state to a state-dependent set (similar to the concept of differential inclusions). Consequently, the measure differential inclusion concept describes the continuous dynamics as well as the impulse dynamics with a single statement in terms of an inclusion and is able to describe accumulation phenomena. Moreover, the framework of measure differential inclusions leads directly to a numerical discretization, called the time-stepping method [Moreau, 1988b], which is a robust algorithm to simulate the dynamics of nonsmooth systems.

The framework of measure differential inclusions allows us to describe systems with state discontinuities and this framework is therefore more general than differential inclusions. However, the great advantage of this framework over other frameworks is, that physical interaction laws, such as friction and impact in mechanics or diode characteristics in electronics, can be formulated as set-valued force laws and be seamlessly incorporated in the formulation [Glocker, 2001, 2005]. We will therefore use the framework of measure differential inclusions in this work to study convergence properties of nonsmooth systems.

Stability properties of nonsmooth systems are essential both in bifurcation analysis and the control of such systems. The analysis of bifurcation phenomena in nonsmooth systems has received much attention lately in literature and conferences (see [Leine & Nijmeijer, 2004; Leine & van Campen, 2006] and references therein). Many novel bifurcation phenomena have been revealed, but the progress of the analysis of bifurcations is hampered by a lack of tools to prove the presence and loss of stability in nonsmooth systems. Currently, many research efforts are employed to develop stabilizing controllers for nonsmooth systems, aiming at the stabilization of equilibria (i.e. solving the stabilization problem), see e.g. [Arcak & Kokotović, 2001; Brogliato, 2004; Galeani *et al.*, 2004; Indri & Tornambè, 2006; Mallon *et al.*, 2006; Menini & Tornambè, 2001b; Rantzer & Johansson, 2000] and many others. In this context, previous work of the authors [Leine & van de Wouw, 2007; van de Wouw & Leine, 2004, 2006; van de Wouw *et al.*, 2005] focussed on the stability properties of equilibrium sets for nonsmooth systems. Moreover, in [Brogliato, 2004] stability properties of an equilibrium of measure differential inclusions of Lur'e-type are studied. The nonlinearities in the feedback loop are required to exhibit monotonicity properties and, if additionally passivity conditions on the linear part of the system are assured, then stability of the equilibrium can be guaranteed. Furthermore, the Lagrange–Dirichlet stability theorem is extended in [Brogliato, 2004] to measure differential inclusions describing mechanical systems with frictionless impact. Note that this work does not address the convergence property and only studies the stability of stationary solutions. However, many control problems, such as tracking control, output regulation, synchronization and observer design [Byrnes *et al.*, 1997; Isidori, 1995; Khalil, 1996; Pavlov *et al.*, 2005d] require the stability analysis of time-varying solutions. The research on the stability properties of time-varying solutions of nonsmooth systems is still in its infancy and the current paper should be placed in this context.

The paper is organized as follows. First, the reader is provided with the necessary background information on maximal monotonicity of set-valued operators in Sec. 2 and measure differential inclusions in Sec. 3. Subsequently, we define the convergence property of dynamical systems in Sec. 4 and state the associated properties of convergent systems. The essential contribution of this paper lies in Sec. 5: theorems are presented which give sufficient conditions for the uniform convergence of measure differential inclusions with certain maximal monotonicity properties. In Sec. 6, we illustrate how these convergence results for measure differential inclusions can be exploited to solve tracking problems for certain classes of nonsmooth mechanical systems with friction and one-way clutches. Illustrative examples of convergent mechanical systems are discussed in detail in Sec. 7. Finally, Sec. 8 presents concluding remarks.

2. Maximal Monotonicity

In this section we consider the main notions concerning set-valued functions and their properties. We first define what we mean by a set-valued function.

Definition 1 (Set-Valued Function). A set-valued function $\boldsymbol{\mathcal{F}} : \mathbb{R}^n \to \mathbb{R}^n$ is a map that associates with any $\mathbf{x} \in \mathbb{R}^n$ a set $\boldsymbol{\mathcal{F}}(\mathbf{x}) \subset \mathbb{R}^n$.

A set-valued function can therefore contain vertical segments on its graph denoted by Graph($\boldsymbol{\mathcal{F}}$). We use the graph to define monotonicity of a set-valued function [Aubin & Frankowska, 1990].

Definition 2 (Monotone Set-Valued Function). A set-valued function $\boldsymbol{\mathcal{F}}(\mathbf{x})$ is called monotone if its graph is monotone in the sense that for all $(\mathbf{x}, \mathbf{y}) \in$ Graph($\boldsymbol{\mathcal{F}}$) and for all $(\mathbf{x}^*, \mathbf{y}^*) \in$ Graph($\boldsymbol{\mathcal{F}}$) it holds that

$$(\mathbf{y} - \mathbf{y}^*)^{\mathrm{T}}(\mathbf{x} - \mathbf{x}^*) \geq 0.$$

In addition, if

$$(\mathbf{y} - \mathbf{y}^*)^{\mathrm{T}}(\mathbf{x} - \mathbf{x}^*) \geq \alpha \|\mathbf{x} - \mathbf{x}^*\|^2$$

for some $\alpha > 0$, then the set-valued map is strictly monotone.

For example, the scalar set-valued function

$$\mathcal{F}(x) = \begin{cases} -1 & x < 0 \\ \{-1, +1\} & x = 0 \\ +1 & x > 0 \end{cases} \tag{1}$$

is a monotone set-valued function, whereas $\mathcal{F}(x) + cx$ is strictly monotone for $c > 0$. Note that $\mathcal{F}(0)$ consists of the two elements -1 and 1. We can extend $\mathcal{F}(x)$ with additional points, such that the monotonicity of the graph is not destroyed. In particular, the set-valued function

$$\mathcal{F}(x) = \partial|x| = \begin{cases} -1 & x < 0 \\ [-1, +1] & x = 0\,, \\ +1 & x > 0 \end{cases} \tag{2}$$

being the subdifferential of the absolute value function, is the largest extension of (1) which is still monotone. Such a function is called maximal monotone.

Definition 3 (Maximal Monotone Set-Valued Function). A monotone set-valued function $\boldsymbol{\mathcal{F}}(\mathbf{x})$ is called maximal monotone if there exists no other monotone set-valued function whose graph strictly contains the graph of $\boldsymbol{\mathcal{F}}$. If $\boldsymbol{\mathcal{F}}$ is strictly monotone and maximal, then it is called strictly maximal monotone.

It follows from this definition that if $\boldsymbol{\mathcal{F}}$ is maximal monotone, then the image of $\mathbf{x}$ under $\boldsymbol{\mathcal{F}}$ is closed and convex for each $\mathbf{x} \in \mathbb{R}^n$. It can be shown, that the subdifferential ∂f of a lower semi-continuous convex function $f(\mathbf{x})$ is maximal monotone (see Corollary 31.5.1 in [Rockafellar, 1970]).

3. Measure Differential Inclusions

In this section, we introduce the measure differential inclusion

$$\mathrm{d}\mathbf{x} \in \mathrm{d}\boldsymbol{\Gamma}(t, \mathbf{x}(t)) \tag{3}$$

as has been proposed by Moreau [1988a].

We usually describe a time-evolution of the state $\mathbf{x}(t)$ of a system by a differential equation of the form $\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x})$. Differential equations can only describe a smooth evolution of the state $\mathbf{x}(t)$. A larger class of systems is formed by differential inclusions of the form $\dot{\mathbf{x}} \in \boldsymbol{\mathcal{F}}(t, \mathbf{x})$ (almost everywhere), where $\boldsymbol{\mathcal{F}}(t, \mathbf{x})$ is a set-valued function. With a differential inclusion we are able to describe an absolutely continuous time-evolution $\mathbf{x}(t)$ which is nondifferentiable on an at most countable number of time-instants. Roughly speaking, we can say that the function $\mathbf{x}(t)$ is allowed to have a "kink". Discontinuities in the time-evolution can however not be described by a differential inclusion. In the following, we will extend the concept of differential inclusions to measure differential inclusions in order to allow for discontinuities in $\mathbf{x}(t)$. The assumption of absolute continuity of $\mathbf{x}(t)$ will be relaxed to locally bounded variation in time.

Consider $\mathbf{x} \in \mathrm{lbv}(\mathcal{I}, \mathcal{X})$, i.e. $\mathbf{x}(t)$ is a function of locally bounded variation in time. The function $\mathbf{x}(t)$ is therefore admitted to undergo jumps at discontinuity points $t_i \in \mathcal{I}$, $i = 1, 2, \ldots$, but the discontinuity points are either

(1) separated in time such that $t_1 < t_2 < t_3, \ldots$, leading to a finite number of discontinuities on a compact time-interval, or,
(2) accumulate to an accumulation point $\lim_{i\to\infty} t_i = t^*$, leading to infinitely many discontinuities on a compact time-interval. The jump $\mathbf{x}^+(t_i) - \mathbf{x}^-(t_i)$ is assumed to decrease to zero for $i \to \infty$.

In the first case, each discontinuity point t_i is followed by a nonzero time-interval of absolutely continuous evolution. In the case of an accumulation point, the time-interval $t_{i+1} - t_i$ tends to zero as well as the discontinuity in $\mathbf{x}(t)$, which guarantees that $\mathbf{x}(t)$ is of locally bounded variation. At a time-instant t, including the discontinuity points $t = t_i$ and accumulation points, we can therefore define a right limit $\mathbf{x}^+(t)$ and a left limit $\mathbf{x}^-(t)$ of $\mathbf{x}$ as one of the properties of functions of bounded variation:

$$\mathbf{x}^+(t) = \lim_{\tau\downarrow 0} \mathbf{x}(t+\tau), \quad \mathbf{x}^-(t) = \lim_{\tau\uparrow 0} \mathbf{x}(t+\tau). \tag{4}$$

If $\mathbf{x}(t)$ is locally continuous at time-instant t, then it holds that $\mathbf{x}(t) = \mathbf{x}^-(t) = \mathbf{x}^+(t)$. Moreover, we define the right derivative $\dot{\mathbf{x}}^+(t)$ and left derivative $\dot{\mathbf{x}}^-(t)$ of $\mathbf{x}$ at t as

$$\dot{\mathbf{x}}^+(t) = \lim_{\tau\downarrow 0} \frac{\mathbf{x}^+(t+\tau) - \mathbf{x}^+(t)}{\tau}, \tag{5}$$

and

$$\dot{\mathbf{x}}^-(t) = \lim_{\tau\uparrow 0} \frac{\mathbf{x}^-(t+\tau) - \mathbf{x}^-(t)}{\tau}, \tag{6}$$

whenever these limits exist. If $\mathbf{x}$ is locally continuous at t and $\dot{\mathbf{x}}^+(t) = \dot{\mathbf{x}}^-(t)$, then $\mathbf{x}$ is locally differentiable at t. A function $\mathbf{x} : \mathcal{I} \to \mathbb{R}^n$ is said to be smooth if it is locally smooth for all $t \in \mathcal{I}$. A function is said to be *almost everywhere* continuous, if the set $\mathcal{D} \subset \mathcal{I}$ of discontinuity points $t_i \in \mathcal{D}$,

$k = 1, 2, \ldots$ is of measure zero with respect to the Lebesgue measure. Similarly, a continuous function can be differentiable almost everywhere.

We want to describe with $\mathbf{x}(t)$ an evolution in time and therefore consider $\mathbf{x}(t)$ to be the result of an integration process

$$\mathbf{x}(t) = \mathbf{x}(t_0) + \int_{t_0}^{t} \mathrm{d}\mathbf{x}, \quad t \geq t_0, \tag{7}$$

where we call $\mathrm{d}\mathbf{x}$ the differential measure of $\mathbf{x}$ [Moreau, 1988a]. If $\mathbf{x}(t)$ is an absolutely continuous and differentiable function, then $\mathrm{d}\mathbf{x}$ admits a density function, say $\mathbf{x}'_t(t)$, with respect to the Lebesgue measure $\mathrm{d}t$, i.e. $\mathrm{d}\mathbf{x} = \mathbf{x}'_t(t)\mathrm{d}t$. We usually immediately associate the density function $\mathbf{x}'_t(t)$ with the derivative $\dot{\mathbf{x}}(t)$.

Subsequently, we consider $\mathbf{x}(t)$ to be an absolutely continuous function, which is nondifferentiable at the set $\mathcal{D} \subset \mathcal{I}$ of points $t_i \in \mathcal{D}$. The derivative $\dot{\mathbf{x}}(t)$ does therefore not exist for $t = t_i$. Lebesgue integration over a singleton $\{t_i\}$, i.e. an interval with zero length, results in zero

$$\int_{\{t_i\}} \mathrm{d}\mathbf{x} = 0, \quad \text{with } \mathrm{d}\mathbf{x} = \mathbf{x}'_t(t)\mathrm{d}t, \tag{8}$$

even if $\dot{\mathbf{x}}(t)$ does not exist for $t = t_i$. The derivative $\dot{\mathbf{x}}(t)$ exists almost everywhere because $\mathbf{x}(t)$ is absolutely continuous. We say that the set $\mathcal{D}$ of points t_i for which $\dot{\mathbf{x}}(t)$ does not exist is Lebesgue negligible. Lebesgue integration over a Lebesgue negligible set results in zero. Consequently, (7) also holds for absolutely continuous functions, which are nondifferentiable at a Lebesgue negligible set D of time-instants t_i.

Finally, we consider $\mathbf{x}(t)$ to be a function of bounded variation on the interval $\mathcal{I}$, which is discontinuous at the set $\mathcal{D} \subset \mathcal{I}$ of points $t_i \in \mathcal{D}$. Moreover, we assume that $\mathbf{x}(t)$ does not contain any singular terms, i.e. fractal-like functions such as the Cantor function. Although the function $\mathbf{x}(t)$ does not exist at the discontinuity points $t = t_i$, it admits a right limit $\mathbf{x}^+(t)$ and left limit $\mathbf{x}^-(t)$ at every time-instant t. Just as before, we consider $\mathbf{x}(t)$ to be the result of an integration process

$$\mathbf{x}^+(t) = \mathbf{x}^-(t_0) + \int_{[t_0,t]} \mathrm{d}\mathbf{x}, \quad t \geq t_0, \tag{9}$$

where the integration process takes the left limit $\mathbf{x}^-(t_0)$ of the initial value to the right limit $\mathbf{x}^+(t)$ of the final value over the closed time-interval $[t_0, t] = \{\tau \in \mathcal{I} | t_0 \leq \tau \leq t\}$. The differential measure $\mathrm{d}\mathbf{x}$ does therefore not only contain a density $\mathbf{x}'_t$ with respect to the Lebesgue measure $\mathrm{d}t$ but also contains a density $\mathbf{x}'_\eta$ with respect to the atomic measure $\mathrm{d}\eta$, which gives a nonzero result when integrated over a singleton, such that

$$\mathrm{d}\mathbf{x} = \mathbf{x}'_t(t)\mathrm{d}t + \mathbf{x}'_\eta(t)\mathrm{d}\eta, \tag{10}$$

with $\int_{\{t_i\}} \mathrm{d}\eta = 1$, $t_i \in \mathcal{D}$. The atomic measure $\mathrm{d}\eta$ can be interpreted as the sum of Dirac point measures $\mathrm{d}\delta_i$,

$$\mathrm{d}\eta = \sum_i \mathrm{d}\delta_i, \tag{11}$$

where

$$\int_{[t_l,t_k]} \mathrm{d}\delta_i = \begin{cases} 1 & t_i \in [t_l, t_k] \\ 0 & t_i \notin [t_l, t_k] \end{cases}, \tag{12}$$

for any interval $[t_l, t_k] \subset \mathcal{I}$. Measure theory with atomic measures is therefore related to distribution theory. Integration of the differential measure $\mathrm{d}\mathbf{x}$ over a singleton $\{t_k\}$ yields

$$\begin{aligned} \mathbf{x}^+(t_k) - \mathbf{x}^-(t_k) &= \int_{\{t_k\}} \mathrm{d}\mathbf{x} \\ &= \int_{\{t_k\}} \mathbf{x}'_\eta(t)\mathrm{d}\eta \\ &= \mathbf{x}'_\eta(t_k) \int_{\{t_k\}} \mathrm{d}\eta. \end{aligned} \tag{13}$$

It follows that $\mathbf{x}^+(t_k) = \mathbf{x}^-(t_k)$ when $t_k \notin \mathcal{D}$, which obviously must hold for a locally continuous function at $t = t_k$. Moreover, if we choose $t_k = t_i \in \mathcal{D}$, then we can immediately associate the density $\mathbf{x}'_\eta(t)$ with respect to the atomic measure $\mathrm{d}\eta$ as the jump in $\mathbf{x}(t)$ at the discontinuity point t_i, i.e. $\mathbf{x}'_\eta(t_i) = \mathbf{x}^+(t_i) - \mathbf{x}^-(t_i)$, $t_i \in \mathcal{D}$. We therefore usually write the differential measure (10) as

$$\mathrm{d}\mathbf{x} = \dot{\mathbf{x}}(t)\mathrm{d}t + (\mathbf{x}^+(t) - \mathbf{x}^-(t))\mathrm{d}\eta. \tag{14}$$

Consequently, using the differential measure (14) we are able to describe a locally absolutely continuously varying time-evolution (using the Lebesgue measurable part of $\mathrm{d}\mathbf{x}$) together with discontinuities at time-instants $t_i \in \mathcal{D}$ (using the atomic part). Integration of the differential measure $\mathrm{d}\mathbf{x}$ therefore gives the total increment over the interval $[t_l, t_k]$

$$\mathbf{x}^+(t_k) - \mathbf{x}^-(t_l) = \int_{[t_l,t_k]} \mathrm{d}\mathbf{x}, \tag{15}$$

or singleton $\{t_k\}$

$$\mathbf{x}^+(t_k) - \mathbf{x}^-(t_k) = \int_{\{t_k\}} \mathrm{d}\mathbf{x}. \tag{16}$$

With the differential inclusion $\dot{\mathbf{x}}(t) \in \boldsymbol{\mathcal{F}}(t, \mathbf{x}(t))$, in which $\boldsymbol{\mathcal{F}}(t, \mathbf{x}(t))$ is a set-valued mapping, we are able to describe a nonsmooth absolutely continuous time-evolution $\mathbf{x}(t)$. The solution $\mathbf{x}(t)$ fulfills the differential inclusion almost everywhere, because $\dot{\mathbf{x}}(t)$ does not exist on a Lebesgue negligible set $\mathcal{D}$ of time-instants $t_i \in \mathcal{D}$ related to nonsmooth state evolution. Instead of using the density $\dot{\mathbf{x}}(t)$, we can also write the differential inclusion using the differential measure

$$\mathrm{d}\mathbf{x} \in \boldsymbol{\mathcal{F}}(t, \mathbf{x}(t))\mathrm{d}t, \tag{17}$$

which yields a measure differential inclusion. The solution $\mathbf{x}(t)$ fulfills the measure differential inclusion (17) *for all* $t \in I$ because of the underlying integration process being associated with measures. Moreover, writing the dynamics in terms of a measure differential inclusion allows us to study a larger class of functions $\mathbf{x}(t)$, as we can let $\mathrm{d}\mathbf{x}$ contain parts other than the Lebesgue integrable part. In order to describe a time-evolution of bounded variation which is discontinuous at isolated time-instants, we let the differential measure $\mathrm{d}\mathbf{x}$ also have an atomic part such as in (14) and therefore extend the measure differential inclusion (17) with an atomic part as well:

$$\mathrm{d}\mathbf{x} \in \boldsymbol{\mathcal{F}}(t, \mathbf{x}(t))\mathrm{d}t + \boldsymbol{\mathcal{G}}(t, \mathbf{x}(t))\mathrm{d}\eta. \tag{18}$$

Here, $\boldsymbol{\mathcal{G}}(t, \mathbf{x}(t))$ is a set-valued mapping, which is in general dependent on t, $\mathbf{x}^-(t)$ and $\mathbf{x}^+(t)$. Following [Moreau, 1988a], we simply write $\boldsymbol{\mathcal{G}}(t, \mathbf{x}(t))$. More conveniently, we write the measure differential inclusion as

$$\mathrm{d}\mathbf{x} \in \mathrm{d}\boldsymbol{\Gamma}(t, \mathbf{x}(t)), \tag{19}$$

where $\mathrm{d}\boldsymbol{\Gamma}(t, \mathbf{x}(t))$ is a set-valued measure function defined as

$$\mathrm{d}\boldsymbol{\Gamma}(t, \mathbf{x}(t)) = \boldsymbol{\mathcal{F}}(t, \mathbf{x}(t))\mathrm{d}t + \boldsymbol{\mathcal{G}}(t, \mathbf{x}(t))\mathrm{d}\eta. \tag{20}$$

The measure differential inclusion (19) has to be understood in the sense of integration. A solution $\mathbf{x}(t)$ of (19) is a function of locally bounded variation which fulfills

$$\mathbf{x}^+(t) = \mathbf{x}^-(t_0) + \int_I \mathbf{f}(t, \mathbf{x})\mathrm{d}t + \mathbf{g}(t, \mathbf{x})\mathrm{d}\eta, \tag{21}$$

for every compact interval $I = [t_0, t]$, where the functions $\mathbf{f}(t, \mathbf{x})$ and $\mathbf{g}(t, \mathbf{x})$ have to obey

$$\mathbf{f}(t, \mathbf{x}) \in \boldsymbol{\mathcal{F}}(t, \mathbf{x}), \quad \mathbf{g}(t, \mathbf{x}) \in \boldsymbol{\mathcal{G}}(t, \mathbf{x}). \tag{22}$$

Substitution of (14) in the measure differential inclusion (19) gives

$$\begin{aligned}\dot{\mathbf{x}}(t)\mathrm{d}t + (\mathbf{x}^+(t) - \mathbf{x}^-(t))\mathrm{d}\eta \\ \in \boldsymbol{\mathcal{F}}(t, \mathbf{x}(t))\mathrm{d}t + \boldsymbol{\mathcal{G}}(t, \mathbf{x}(t))\mathrm{d}\eta,\end{aligned} \tag{23}$$

which we can separate in the Lebesgue integrable part

$$\dot{\mathbf{x}}(t)\mathrm{d}t \in \boldsymbol{\mathcal{F}}(t, \mathbf{x}(t))\mathrm{d}t, \tag{24}$$

and atomic part

$$(\mathbf{x}^+(t) - \mathbf{x}^-(t))\mathrm{d}\eta \in \boldsymbol{\mathcal{G}}(t, \mathbf{x}(t))\mathrm{d}\eta \tag{25}$$

from which we can retrieve $\dot{\mathbf{x}}(t) \in \boldsymbol{\mathcal{F}}(t, \mathbf{x}(t))$ and the jump condition $\mathbf{x}^+(t) - \mathbf{x}^-(t) \in \boldsymbol{\mathcal{G}}(t, \mathbf{x}(t))$. Moreover, by considering the limits $t \downarrow t_i$ and $t \uparrow t_i$ we obtain the differential inclusions for post- and pre-jump times

$$\dot{\mathbf{x}}^+(t) \in \boldsymbol{\mathcal{F}}(t, \mathbf{x}^+(t)), \quad \dot{\mathbf{x}}^-(t) \in \boldsymbol{\mathcal{F}}(t, \mathbf{x}^-(t)), \tag{26}$$

which we call the directional differential inclusions.

Note that the jump condition $\mathbf{x}^+(t) - \mathbf{x}^-(t) \in \boldsymbol{\mathcal{G}}(t, \mathbf{x}(t))$ is an implicit inclusion for the post-jump state $\mathbf{x}^+(t)$, because $\boldsymbol{\mathcal{G}}$ is in general dependent on $\mathbf{x}^+(t)$. Such an implicit description of the post-jump state makes this formalism especially useful for the description of physical processes with set-valued reset laws, such as mechanical systems with unilateral constraints and electrical systems with set-valued elements (spark plugs, diodes and the like) [Glocker, 2005]. The solution of the post-jump state constitutes a combinatorial problem which is inherent to the physical nature of unilateral constraints. The implicit description of the post-jump state is the key difference between the measure differential inclusion formalism and the hybrid system formalism, which pre-supposes an explicit jump map. Moreover, a description in terms of differential measures allows to describe accumulation points as an intrinsic part of the dynamics and also opens the way to the numerical treatment of systems with accumulation points.

A special class of systems is described by set-valued measure functions $\mathrm{d}\boldsymbol{\Gamma}(t, \mathbf{x}(t))$ for which each density function is a conic subset[1] of $\mathbb{R}^n$. In particular, the set-valued functions $\boldsymbol{\mathcal{F}}(t, \mathbf{x}(t))$ and $\boldsymbol{\mathcal{G}}(t, \mathbf{x}(t))$ are often equal to the same cone

[1] If $\mathcal{K} \subset \mathbb{R}^n$ is a cone, then it holds that $\lambda\mathbf{a} \in \mathcal{K}$ for each $\mathbf{a} \in \mathcal{K}$ and $\lambda \geq 0$.

$\mathcal{K}(t, \mathbf{x}(t))$, i.e. $\boldsymbol{\mathcal{F}}(t, \mathbf{x}(t)) = \boldsymbol{\mathcal{G}}(t, \mathbf{x}(t)) = \mathcal{K}(t, \mathbf{x}(t))$. Following Moreau [1988a], we write in this case

$$\mathrm{d}\mathbf{x} \in \mathcal{K}(t, \mathbf{x}(t)), \tag{27}$$

and refrain from prescribing a measure in the right-hand side in advance. It is to be understood from (27) that, if $\mathrm{d}\mathbf{x}$ possesses a density function $\mathbf{f}'_\mu$ with respect to the non-negative differential measure $\mathrm{d}\mu$, then this density function belongs to the cone $\mathcal{K}$, i.e. $\mathbf{f}'_\mu(t, \mathbf{x}) \in \mathcal{K}(t, \mathbf{x})$. In particular, this applies for the Lebesgue measure as well as for the atomic measure.

In Lagrangian dynamics, the measure differential inclusion (19) typically describes the time-evolution of absolutely continuous generalized coordinates $\mathbf{q}(t)$ and generalized velocities $\mathbf{u}(t)$, which are of locally bounded variation. The set-valued measure function $\mathrm{d}\boldsymbol{\Gamma}(t, \mathbf{x}(t))$ typically contains indicator functions, which impose constraints on the system. Unilateral constraints $g(\mathbf{q}) \geq 0$ and bilateral constraints $g(\mathbf{q}) = 0$ restrict the generalized coordinates $\mathbf{q}(t)$ to an admissible set. In a first-order description (19), we denote the admissible set of the state $\mathbf{x}(t)$ as $\mathcal{X} \subset \mathbb{R}^n$.

Differential inclusions and measure differential inclusions do not in general possess existence and uniqueness of solutions. However, if the (measure) differential inclusion is a model of a physical system, then $\mathrm{d}\boldsymbol{\Gamma}$ stems from a set-valued constitutive law, which is usually associated with a nonsmooth (pseudo)-potential, which often (but not always) leads to the existence of solutions. Consider for instant the Painlevé example [Brogliato, 1999; Leine *et al.*, 2002], which is a famous mechanical system with frictional unilateral contact showing existence and uniqueness problems. When existence problems occur, then we have to rethink the adopted solution concept. If the system does not admit solutions within the chosen solution concept, then it may have existence of solutions for a larger solution concept. For instant, in the Painlevé example we can extend the solution concept by allowing for impacts without collisions. Nonuniqueness of solutions is abundant in models of reality, and is simply a fact with which we have to live. For the special class of differential inclusions of the form $\dot{\mathbf{x}} \in -\boldsymbol{\mathcal{A}}(\mathbf{x})$, where $\boldsymbol{\mathcal{A}}$ is a maximal monotone set-valued function, uniqueness of solutions is guaranteed (see [Brézis, 1973]). In the remainder of the paper, we will consider measure differential inclusions with certain maximal monotonicity properties exhibiting existence and uniqueness of solutions as well as a continuous dependence on initial conditions.

4. Convergent Systems

In this section, we will briefly discuss the definition of convergence and certain properties of convergent systems. In the definition of convergence, the Lyapunov stability of solutions of (28) plays a central role. Definitions of (uniform) stability and attractivity of measure differential inclusions are given in the Appendix.

The definitions of convergence properties presented here extend the definition given in [Demidovich, 1967] (see also [Pavlov *et al.*, 2005c]). Consider a system described by the measure differential inclusion

$$\mathrm{d}\mathbf{x} \in \mathrm{d}\boldsymbol{\Gamma}(\mathbf{x}, t), \tag{28}$$

where $\mathbf{x} \in \mathbb{R}^n$, $t \in \mathbb{R}$. Let us formally define the property of convergence.

Definition 4. System (28) is said to be

- *convergent* if there exists a solution $\bar{\mathbf{x}}(t)$ satisfying the following conditions:
  - (i) $\bar{\mathbf{x}}(t)$ is defined and bounded for all $t \in \mathbb{R}$,
  - (ii) $\bar{\mathbf{x}}(t)$ is globally attractively stable.
- *uniformly convergent* if it is convergent and $\bar{\mathbf{x}}(t)$ is globally uniformly attractively stable.
- *exponentially convergent* if it is convergent and $\bar{\mathbf{x}}(t)$ is globally exponentially stable.

The wording "attractively stable" has been used instead of the usual term "asymptotically stable", because attractivity of solutions in (measure) differential inclusions can be asymptotic or symptotic (finite-time attractivity).

The solution $\bar{\mathbf{x}}(t)$ is called a *steady-state solution*. As follows from the definition of convergence, any solution of a convergent system "forgets" its initial condition and converges to some steady-state solution. In general, the steady-state solution $\bar{\mathbf{x}}(t)$ may be nonunique. But for any two steady-state solutions $\bar{\mathbf{x}}_1(t)$ and $\bar{\mathbf{x}}_2(t)$ it holds that $\|\bar{\mathbf{x}}_1(t) - \bar{\mathbf{x}}_2(t)\| \to 0$ as $t \to +\infty$. At the same time, for *uniformly* convergent systems the steady-state solution is unique, as formulated below.

Property 1 [Pavlov *et al.*, 2005c, 2005d]. *If system (28) is uniformly convergent, then the steady-state solution $\bar{\mathbf{x}}(t)$ is the only solution defined and bounded for all $t \in \mathbb{R}$.*

In many engineering problems, dynamical systems excited by time-varying perturbations are encountered. Therefore, we will consider convergence properties for systems with time-varying inputs. So, instead of systems of the form (28), we consider systems of the form

$$\mathrm{d}\mathbf{x} \in \mathrm{d}\mathbf{\Gamma}(\mathbf{x}, \mathbf{w}(t)), \tag{29}$$

with state $\mathbf{x} \in \mathbb{R}^n$ and input $\mathbf{w} \in \mathbb{R}^d$. The right-hand side of (29) is assumed to be continuous in $\mathbf{w}$. In the following, we will consider the class $\overline{\mathbb{PC}}_d$ of piecewise continuous inputs $\mathbf{w}(t) : \mathbb{R} \to \mathbb{R}^d$ which are bounded on $\mathbb{R}$. Below we define the convergence property for systems with inputs.

Definition 5. System (29) is said to be (*uniformly, exponentially*) *convergent* if it is (uniformly, exponentially) convergent for every input $\mathbf{w} \in \overline{\mathbb{PC}}_d$. In order to emphasize the dependency on the input $\mathbf{w}(t)$, the steady-state solution is denoted by $\bar{\mathbf{x}}_w(t)$.

Uniformly convergent systems excited by periodic or constant inputs exhibit the following property, that is particularly useful in, for example, bifurcation analyses of periodically perturbed systems.

Property 2 [Demidovich, 1967; Pavlov *et al.*, 2005d]. *Suppose system (29) with a given input $\mathbf{w}(t)$ is uniformly convergent. If the input $\mathbf{w}(t)$ is constant, the corresponding steady-state solution $\bar{\mathbf{x}}_w(t)$ is also constant; if the input $\mathbf{w}(t)$ is periodic with period T, then the corresponding steady-state solution $\bar{\mathbf{x}}_w(t)$ is also periodic with the same period T.*

5. Convergence of Maximal Monotone Systems

In this section we will consider the dynamics of measure differential inclusions (29) with certain maximal monotonicity conditions on $\mathbf{\Gamma}(\mathbf{x}, \mathbf{w}(t))$. In particular, we study systems for which $\mathbf{\Gamma}(\mathbf{x}, \mathbf{w}(t))$ can be split in a state-dependent part and an input-dependent part. The state-dependent part is, with the help of a maximal monotonicity requirement, assumed to be strictly passive with respect to the Lebesgue measure and passive with respect to the atomic measure. Such kind of systems will be simply referred to as "maximal monotone systems" in the following.

We first formalize maximal monotone systems in Sec. 5.1, subsequently give sufficient conditions for the existence of a compact positively invariant set in Sec. 5.2 and finally give sufficient conditions for convergence in Sec. 5.3.

5.1. *Maximal monotone systems*

Let $\mathbf{x} \in \mathbb{R}^n$ be the state-vector of the system and $\mathbf{w} \in \mathbb{R}^m$ be the input vector. Consider the time-evolution of $\mathbf{x}$ to be governed by a measure differential equation of the form

$$\mathrm{d}\mathbf{x} = -\mathrm{d}\mathbf{a} - \mathbf{c}(\mathbf{x})\mathrm{d}t + \mathrm{d}\mathbf{b}(\mathbf{w}), \tag{30}$$

where $\mathbf{c} : \mathbb{R}^n \to \mathbb{R}^n$ is a single-valued function and $\mathrm{d}\mathbf{a}$ and $\mathrm{d}\mathbf{b}(\mathbf{w})$ are differential measures with densities with respect to $\mathrm{d}t$ and $\mathrm{d}\eta$, i.e.

$$\mathrm{d}\mathbf{a} = \mathbf{a}'_t\mathrm{d}t + \mathbf{a}'_\eta\mathrm{d}\eta, \tag{31}$$

and

$$\mathrm{d}\mathbf{b}(\mathbf{w}) = \mathbf{b}'_t(\mathbf{w})\mathrm{d}t + \mathbf{b}'_\eta(\mathbf{w})\mathrm{d}\eta. \tag{32}$$

In the following, we will assume $\mathbf{x}^{\mathrm{T}}\mathbf{b}'_\eta(\mathbf{w})$ to be bounded from above by a constant β. Basically, this gives an upper-bound on the energy input of the impulsive inputs. Such an assumption makes sense from the physical point of view, see the example in Sec. 7.1. The quantities $\mathbf{a}'_t$ and $\mathbf{a}'_\eta$, which are functions of time, obey the set-valued laws

$$\mathbf{a}'_t \in \boldsymbol{\mathcal{A}}(\mathbf{x}), \tag{33}$$

$$\mathbf{a}'_\eta \in \boldsymbol{\mathcal{A}}(\mathbf{x}^+), \tag{34}$$

where $\boldsymbol{\mathcal{A}}$ is a set-valued mapping. The dynamics can be decomposed in a Lebesgue measurable part and an atomic part. The Lebesgue measurable part gives the differential equation

$$\dot{\mathbf{x}}(t) := \mathbf{x}'_t = -\mathbf{a}'_t(\mathbf{x}(t)) - \mathbf{c}(\mathbf{x}(t)) + \mathbf{b}'_t(\mathbf{w}(t)), \tag{35}$$

which forms with the set-valued law (33) a differential inclusion

$$\dot{\mathbf{x}} \in -\boldsymbol{\mathcal{A}}(\mathbf{x}) - \mathbf{c}(\mathbf{x}) + \mathbf{b}'_t(\mathbf{w}) \quad \text{a.e.} \tag{36}$$

The atomic part gives the state-reset rule

$$\mathbf{x}^+ - \mathbf{x}^- := \mathbf{x}'_\eta = -\mathbf{a}'_\eta + \mathbf{b}'_\eta(\mathbf{w}). \tag{37}$$

In mechanics, the state-reset rule is called the impact equation. The above impact law (34), for which $\boldsymbol{\mathcal{A}}$ is only a function of $\mathbf{x}^+$, corresponds to a completely inelastic impact equation. Because of the similarity between the laws (33) and (34), we can combine these laws into the measure law

$$\mathrm{d}\mathbf{a} \in \mathrm{d}\boldsymbol{\mathcal{A}}(\mathbf{x}^+) = \boldsymbol{\mathcal{A}}(\mathbf{x}^+)(\mathrm{d}t + \mathrm{d}\eta). \tag{38}$$

The equality of measures (30) together with the measure law (38) constitutes a measure differential inclusion

$$\mathrm{d}\mathbf{x} \in -\mathrm{d}\boldsymbol{\mathcal{A}}(\mathbf{x}^+) - \mathbf{c}(\mathbf{x})\mathrm{d}t + \mathrm{d}\mathbf{b}(\mathbf{w}) := \mathrm{d}\boldsymbol{\Gamma}(\mathbf{x}, \mathbf{w}). \tag{39}$$

The set-valued operator $\boldsymbol{\mathcal{A}}(\mathbf{x})$ models the non-smooth dissipative elements in the system. We assume that $\boldsymbol{\mathcal{A}}(\mathbf{x})$ is a maximal monotone set-valued mapping, i.e. $\boldsymbol{\mathcal{A}}(\mathbf{x})$ satisfies

$$(\mathbf{x}_2 - \mathbf{x}_1)^{\mathrm{T}}(\boldsymbol{\mathcal{A}}(\mathbf{x}_2) - \boldsymbol{\mathcal{A}}(\mathbf{x}_1)) \geq 0, \tag{40}$$

for any two states $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{X}$. Moreover, we assume that $\mathbf{0} \in \boldsymbol{\mathcal{A}}(\mathbf{0})$. This last assumption together with the monotonicity assumption implies the condition

$$\mathbf{x}^{\mathrm{T}}\boldsymbol{\mathcal{A}}(\mathbf{x}) \geq \mathbf{0} \tag{41}$$

for any $\mathbf{x} \in \mathcal{X}$, i.e. the action of $\boldsymbol{\mathcal{A}}$ is passive. Furthermore, we assume that $\boldsymbol{\mathcal{A}}(\mathbf{x}) + \mathbf{c}(\mathbf{x})$ is a strictly maximal monotone set-valued mapping, i.e. there exists an $\alpha > 0$ such that

$$\begin{aligned}(\mathbf{x}_2 - \mathbf{x}_1)^{\mathrm{T}}&(\boldsymbol{\mathcal{A}}(\mathbf{x}_2) + \mathbf{c}(\mathbf{x}_2) - \boldsymbol{\mathcal{A}}(\mathbf{x}_1) - \mathbf{c}(\mathbf{x}_1)) \\ &\geq \alpha\|\mathbf{x}_2 - \mathbf{x}_1\|^2,\end{aligned} \tag{42}$$

for any two states $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{X}$.

5.2. *Existence of a compact positively invariant set*

The existence of a compact positively invariant set is useful in the proof of convergence as will become clear in Sec. 5.3. Clearly, if the impulsive inputs are passive in the sense that $(\mathbf{x}^+)^{\mathrm{T}}\mathbf{b}'_\eta(\mathbf{w}(t)) \leq 0$ for all t, then the system is dissipative for large $\|\mathbf{x}\|$ and all solutions must be bounded. In the following theorem, we give a less stringent sufficient condition for the existence of a compact positively invariant set of (39) based on a dwell-time condition [Hespanha *et al.*, 2005; Hespanha & Morse, 1999].

Theorem 1. *A measure differential inclusions of the form (39) has a compact positively invariant set if*

1. $\boldsymbol{\mathcal{A}}(\mathbf{x})$ *is a maximal monotone set-valued mapping with* $\mathbf{0} \in \boldsymbol{\mathcal{A}}(\mathbf{0})$,
2. $\boldsymbol{\mathcal{A}}(\mathbf{x}) + \mathbf{c}(\mathbf{x})$ *is a strictly maximal monotone set-valued mapping, i.e. there exists an* $\alpha > 0$ *such that (42) is satisfied,*
3. *there exists a* $\beta \in \mathbb{R}$ *such that* $(\mathbf{x}^+)^{\mathrm{T}}\mathbf{b}'_\eta(\mathbf{w}) \leq \beta$ *for all* $\mathbf{x} \in \mathcal{X}$, *i.e. the energy input of the impulsive inputs is bounded from above,*
4. *the time-instants* t_i *for which the input is impulsive are separated by the dwell-time* $\tau \leq t_{i+1} - t_i$, *with*

$$\tau = \frac{\delta}{2(\delta - 1)\alpha}\ln\left(1 + \frac{2\beta}{\delta^2\gamma^2}\right)$$

and $\gamma := (1/\alpha)\sup_{t\in\mathbb{R}, \mathbf{a}'_t(\mathbf{0})\in\boldsymbol{\mathcal{A}}(\mathbf{0})}\{-\mathbf{a}'_t(\mathbf{0}) - \mathbf{c}(\mathbf{0}) + \mathbf{b}'_t(\mathbf{w}(t))\}$ *for some* $\delta > 1$.

Proof. Consider the Lyapunov candidate function $W = (1/2)\mathbf{x}^{\mathrm{T}}\mathbf{x}$. The differential measure of W has a density $\dot{W}$ with respect to the Lebesgue measure $\mathrm{d}t$ and a density $W^+ - W^-$ with respect to the atomic measure $\mathrm{d}\eta$, i.e. $\mathrm{d}W = \dot{W}\mathrm{d}t + (W^+ - W^-)\mathrm{d}\eta$. We first evaluate the density $\dot{W}$:

$$\begin{aligned}\dot{W} &= \mathbf{x}^{\mathrm{T}}(-\mathbf{a}'_t - \mathbf{c}(\mathbf{x}) + \mathbf{b}'_t(\mathbf{w})) \\ &= \mathbf{x}^{\mathrm{T}}(-\mathbf{a}'_t - \mathbf{c}(\mathbf{x}) + \mathbf{a}'_t(\mathbf{0}) + \mathbf{c}(\mathbf{0})) \\ &\quad + \mathbf{x}^{\mathrm{T}}(-\mathbf{a}'_t(\mathbf{0}) - \mathbf{c}(\mathbf{0}) + \mathbf{b}'_t(\mathbf{w})),\end{aligned} \tag{43}$$

with $\mathbf{a}'_t \in \boldsymbol{\mathcal{A}}(\mathbf{x})$ and $\mathbf{a}'_t(\mathbf{0}) \in \boldsymbol{\mathcal{A}}(\mathbf{0})$. Due to strict monotonicity of $\boldsymbol{\mathcal{A}}(\mathbf{x}) + \mathbf{c}(\mathbf{x})$, there exists a constant $\alpha > 0$ such that

$$\begin{aligned}\dot{W} &\leq -\alpha\|\mathbf{x}\|^2 + \mathbf{x}^{\mathrm{T}}(-\mathbf{a}'_t(\mathbf{0}) - \mathbf{c}(\mathbf{0}) + \mathbf{b}'_t(\mathbf{w})), \\ &\leq -\|\mathbf{x}\|\left(\alpha\|\mathbf{x}\| - \sup_{t\in\mathbb{R}, \mathbf{a}'_t(\mathbf{0})\in\boldsymbol{\mathcal{A}}(\mathbf{0})}\{-\mathbf{a}'_t(\mathbf{0}) \right. \\ &\qquad \left. - \mathbf{c}(\mathbf{0}) + \mathbf{b}'_t(\mathbf{w}(t))\}\right).\end{aligned} \tag{44}$$

Note that $\dot{W} < 0$ for $\mathbf{x}$ satisfying

$$\begin{aligned}\|\mathbf{x}\| > \frac{1}{\alpha}&\sup_{t\in\mathbb{R}, \mathbf{a}'_t(\mathbf{0})\in\boldsymbol{\mathcal{A}}(\mathbf{0})}\{-\mathbf{a}'_t(\mathbf{0}) \\ &- \mathbf{c}(\mathbf{0}) + \mathbf{b}'_t(\mathbf{w}(t))\}.\end{aligned} \tag{45}$$

Let γ be

$$\begin{aligned}\gamma = \max&\left(0, \frac{1}{\alpha}\sup_{t\in\mathbb{R}, \mathbf{a}'_t(\mathbf{0})\in\boldsymbol{\mathcal{A}}(\mathbf{0})}\{-\mathbf{a}'_t(\mathbf{0}) \right. \\ &\left. - \mathbf{c}(\mathbf{0}) + \mathbf{b}'_t(\mathbf{w}(t))\}\right).\end{aligned} \tag{46}$$

For $\|\mathbf{x}\| > \gamma$ we can prove an exponential decay of W (in between state jumps at $t = t_i$). The function $f(x) = -(1 - (1/\delta))\alpha x^2$ is greater than $g(x) = -\alpha x^2 + \gamma\alpha x$ for $x > \delta\gamma$, where $\delta > 1$ is an arbitrary constant and $\gamma > 0$. It therefore holds that $\dot{W} \leq -(1 - (1/\delta))\alpha\|\mathbf{x}\|^2$ for $\|\mathbf{x}\| \geq \delta\gamma$, i.e.

$$\dot{W} \leq -2\left(1 - \frac{1}{\delta}\right)\alpha W \quad \text{for } \|\mathbf{x}\| \geq \delta\gamma. \tag{47}$$

Subsequently, we consider the jump $W^+ - W^-$ of W:

$$W^+ - W^- = \frac{1}{2}(\mathbf{x}^+ + \mathbf{x}^-)^{\mathrm{T}}(\mathbf{x}^+ - \mathbf{x}^-) \tag{48}$$

with $\mathbf{x}^+ - \mathbf{x}^- = -\mathbf{a}'_\eta + \mathbf{b}'_\eta(\mathbf{w})$ and $\mathbf{a}'_\eta \in \boldsymbol{\mathcal{A}}(\mathbf{x}^+)$. Elimination of $\mathbf{x}^-$ and exploiting the monotonicity of $\boldsymbol{\mathcal{A}}(\mathbf{x})$ gives

$$\begin{aligned} W^+ - W^- &= \frac{1}{2}(2\mathbf{x}^+ + \mathbf{a}'_\eta - \mathbf{b}'_\eta(\mathbf{w}))^{\mathrm{T}}(-\mathbf{a}'_\eta + \mathbf{b}'_\eta(\mathbf{w})) \\ &= (\mathbf{x}^+)^{\mathrm{T}}(-\mathbf{a}'_\eta + \mathbf{b}'_\eta(\mathbf{w})) \\ &\quad - \frac{1}{2}\|\mathbf{a}'_\eta - \mathbf{b}'_\eta(\mathbf{w})\|^2 \\ &\leq \beta, \end{aligned} \tag{49}$$

in which we used the assumption that the energy input of the impulsive inputs $\mathbf{b}'_\eta(\mathbf{w})$ is bounded from above by β (see condition 3 in the theorem) and the monotonicity and passivity of $\boldsymbol{\mathcal{A}}$. Then, due to (44) and (45), for the nonimpulsive part of the motion it holds that if $\|\mathbf{x}(t_0)\| \leq \gamma$ then $\|\mathbf{x}(t)\| \leq \gamma$ for all $t \in [t_0, t^*]$ (if no state resets occur in this time interval). Moreover, as far as the state resets are concerned, (49) shows that a state reset from a state $\mathbf{x}^-(t_i) \in \mathcal{V}$ with $\mathcal{V} = \{\mathbf{x} \in \mathcal{X} \,|\, \|\mathbf{x}\| \leq \delta\gamma\}$ can only occur to $\mathbf{x}^+(t_i)$ such that $W(\mathbf{x}^+(t_i)) := (1/2)\|\mathbf{x}^+(t_i)\|^2 \leq W(\mathbf{x}^-(t_i)) + \beta \leq (1/2)\delta^2\gamma^2 + \beta$ (note hereto the specific form of $W = (1/2)\mathbf{x}^{\mathrm{T}}\mathbf{x}$). During the following open time-interval (t_i, t_{i+1}) for which $\mathbf{b}'_\eta(\mathbf{w}(t)) = \mathbf{0}$, the function W evolves as

$$W(\mathbf{x}^-(t_{i+1})) = W(\mathbf{x}^+(t_i)) + \int_{(t_i, t_{i+1})} \mathrm{d}W, \tag{50}$$

which may involve impulsive motion due to dissipative impulses $\mathbf{a}'_\eta$. Let $t_\mathcal{V} \in (t_i, t_{i+1})$ be the time-instant for which $\|\mathbf{x}^-(t_\mathcal{V})\| = \delta\gamma$. The function W will necessarily decrease during the time-interval $(t_i, t_\mathcal{V})$ due to (47) and $W^+ - W^- = -(\mathbf{x}^+)^{\mathrm{T}}\mathbf{a}'_\eta - (1/2)\|\mathbf{a}'_\eta\|^2 \leq 0$ (the state-dependent impulses are passive). It therefore holds that

$$W(\mathbf{x}^-(t_\mathcal{V})) \leq e^{-2(1-\frac{1}{\delta})\alpha(t_\mathcal{V} - t_i)} W(\mathbf{x}^+(t_i)), \tag{51}$$

because $\mathrm{d}W \leq -2(1 - (1/\delta))\alpha W \mathrm{d}t + (W^+ - W^-)\mathrm{d}\eta \leq -2(1-(1/\delta))\alpha W \mathrm{d}t$ for positive measures. Using $W(\mathbf{x}^-(t_\mathcal{V})) = (1/2)\delta^2\gamma^2$ and $W(\mathbf{x}^+(t_i)) \leq (1/2)\delta^2\gamma^2 + \beta$ in the exponential decrease (51) gives

$$\frac{1}{2}\delta^2\gamma^2 \leq e^{-2(1-\frac{1}{\delta})\alpha(t_\mathcal{V} - t_i)} \left(\frac{1}{2}\delta^2\gamma^2 + \beta\right) \tag{52}$$

or

$$t_\mathcal{V} - t_i \leq \frac{\delta}{2(\delta - 1)\alpha} \ln\left(1 + \frac{2\beta}{\delta^2\gamma^2}\right). \tag{53}$$

Consequently, if the next impulsive time-instant t_{i+1} of the input is after $t_\mathcal{V}$, then the solution $\mathbf{x}(t)$ has enough time to reach $\mathcal{V}$. Hence, if the impulsive time-instant of the input are separated by the dwell-time τ, i.e. $t_{i+1} - t_i \geq \tau$, with

$$\tau = \frac{\delta}{2(\delta - 1)\alpha} \ln\left(1 + \frac{2\beta}{\delta^2\gamma^2}\right), \tag{54}$$

then the set

$$\mathcal{W} = \left\{\mathbf{x} \in \mathcal{X} \,\middle|\, \frac{1}{2}\|\mathbf{x}\|^2 \leq \frac{1}{2}\delta^2\gamma^2 + \beta\right\} \tag{55}$$

is a compact positively invariant set. ■

Typically, we would like the invariant set $\mathcal{W}$ to be as small as possible, as it gives an upper-bound for the trajectories of the system. On the other hand, we also want the dwell-time to be as small as possible. The constant $\delta > 1$ plays an interesting role in the above theorem. By increasing δ, we allow the invariant set $\mathcal{W}$ to be larger, thereby decreasing the dwell-time τ. So, there is a kind of pay-off between the size of the invariant set and the dwell-time. Any *finite* value of δ is sufficient to prove the existence of a *compact* positively invariant set. We therefore can take the dwell-time τ to be an arbitrary small value, but not infinitely small. This brings us to the following corollary:

Corollary 1. *If the size of the compact positively invariant set is not of interest, then Condition 4 in Theorem 1 can be replaced by an arbitrary small dwell-time* $\tau > 0$.

Proof. Taking the limit of $\delta \to \infty$ gives the condition $\tau > 0$ for arbitrary γ and β. ■

It therefore suffices to assume that the impulsive inputs are separated in time (which is not a strange assumption from a physical point of view) and simply put τ equal to the (unknown) minimal time-lapse between the impulsive inputs. Then, we calculate the corresponding value of δ and obtain the size of the compact positively invariant set.

In this section, we presented a sufficient condition for the existence of a compact positively invariant set, but the attractivity of solutions outside $\mathcal{W}$ to $\mathcal{W}$ is not guaranteed. If in addition the system is incrementally attractively stable, for which we will give a sufficient condition in Sec. 5.3, then it is also assured that all solutions outside $\mathcal{W}$ converge to $\mathcal{W}$.

5.3. *Conditions for convergence*

In the following theorem, it is stated that strictly maximal monotone measure differential inclusions are exponentially convergent.

Theorem 2. *A measure differential inclusion of the form (39) is exponentially convergent if*

(1) $\boldsymbol{\mathcal{A}}(\mathbf{x})$ *is a maximal monotone set-valued mapping, with* $\mathbf{0} \in \boldsymbol{\mathcal{A}}(\mathbf{0})$,
(2) $\boldsymbol{\mathcal{A}}(\mathbf{x})+\mathbf{c}(\mathbf{x})$ *is a strictly maximal monotone set-valued mapping,*
(3) *system (39) exhibits a compact positively invariant set.*

Proof. Let us first show that system (39) is incrementally attractively stable, i.e. all solutions of (39) converge to each other for positive time. Consider hereto two solutions $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ of (39) and a Lyapunov candidate function $V = (1/2)\|\mathbf{x}_2 - \mathbf{x}_1\|^2$. Consequently, the differential measure of V satisfies:

$$\mathrm{d}V = \frac{1}{2}(\mathbf{x}_2^+ + \mathbf{x}_2^- - \mathbf{x}_1^+ - \mathbf{x}_1^-)^{\mathrm{T}}(\mathrm{d}\mathbf{x}_2 - \mathrm{d}\mathbf{x}_1), \quad (56)$$

with

$$\begin{aligned} \mathrm{d}\mathbf{x}_1 &= -\mathrm{d}\mathbf{a}_1 - \mathbf{c}(\mathbf{x}_1)\mathrm{d}t + \mathrm{d}\mathbf{b}(\mathbf{w}), \\ \mathrm{d}\mathbf{x}_2 &= -\mathrm{d}\mathbf{a}_2 - \mathbf{c}(\mathbf{x}_2)\mathrm{d}t + \mathrm{d}\mathbf{b}(\mathbf{w}), \end{aligned} \quad (57)$$

where $\mathrm{d}\mathbf{a}_1 \in \boldsymbol{\mathcal{A}}(\mathbf{x}_1^+)$ and $\mathrm{d}\mathbf{a}_2 \in \boldsymbol{\mathcal{A}}(\mathbf{x}_2^+)$. The differential measure of V has a density $\dot{V}$ with respect to the Lebesgue measure $\mathrm{d}t$ and a density $V^+ - V^-$ with respect to the atomic measure $\mathrm{d}\eta$, i.e. $\mathrm{d}V = \dot{V}\mathrm{d}t + (V^+ - V^-)\mathrm{d}\eta$. We first evaluate the density $\dot{V}$:

$$\begin{aligned} \dot{V} &= -(\mathbf{x}_2 - \mathbf{x}_1)^{\mathrm{T}}(\mathbf{a}_t'(\mathbf{x}_2) + \mathbf{c}(\mathbf{x}_2) - \mathbf{b}_t'(\mathbf{w}) \\ &\quad - \mathbf{a}_t'(\mathbf{x}_1) - \mathbf{c}(\mathbf{x}_1) + \mathbf{b}_t'(\mathbf{w})) \\ &= -(\mathbf{x}_2 - \mathbf{x}_1)^{\mathrm{T}}(\mathbf{a}_t'(\mathbf{x}_2) + \mathbf{c}(\mathbf{x}_2) \\ &\quad - \mathbf{a}_t'(\mathbf{x}_1) - \mathbf{c}(\mathbf{x}_1)), \end{aligned} \quad (58)$$

where $\mathbf{a}_t'(\mathbf{x}_1) \in \boldsymbol{\mathcal{A}}(\mathbf{x}_1)$ and $\mathbf{a}_t'(\mathbf{x}_2) \in \boldsymbol{\mathcal{A}}(\mathbf{x}_2)$, since both solutions $\mathbf{x}_1$ and $\mathbf{x}_2$ correspond to the same perturbation $\mathbf{w}$. Due to strict monotonicity of $\boldsymbol{\mathcal{A}}(\mathbf{x}) + \mathbf{c}(\mathbf{x})$, there exists a constant $\alpha > 0$ such that

$$\dot{V} \le -\alpha\|\mathbf{x}_2 - \mathbf{x}_1\|^2. \quad (59)$$

Subsequently, we consider the jump $V^+ - V^-$ of V:

$$V^+ - V^- = \frac{1}{2}(\mathbf{x}_2^+ + \mathbf{x}_2^- - \mathbf{x}_1^+ - \mathbf{x}_1^-)^{\mathrm{T}} (\mathbf{x}_2^+ - \mathbf{x}_2^- - \mathbf{x}_1^+ + \mathbf{x}_1^-), \quad (60)$$

with

$$\begin{aligned} \mathbf{x}_1^+ - \mathbf{x}_1^- &= -\mathbf{a}_\eta'(\mathbf{x}_1) + \mathbf{b}_\eta'(\mathbf{w}), \quad \mathbf{a}_\eta'(\mathbf{x}_1) \in \boldsymbol{\mathcal{A}}(\mathbf{x}_1^+), \\ \mathbf{x}_2^+ - \mathbf{x}_2^- &= -\mathbf{a}_\eta'(\mathbf{x}_2) + \mathbf{b}_\eta'(\mathbf{w}), \quad \mathbf{a}_\eta'(\mathbf{x}_2) \in \boldsymbol{\mathcal{A}}(\mathbf{x}_2^+). \end{aligned} \quad (61)$$

Elimination of $\mathbf{x}_1^-$ and $\mathbf{x}_2^-$ and exploiting the monotonicity of $\boldsymbol{\mathcal{A}}(\mathbf{x})$ gives

$$\begin{aligned} V^+ - V^- &= \frac{1}{2}(2\mathbf{x}_2^+ + \mathbf{a}_\eta'(\mathbf{x}_2) - 2\mathbf{x}_1^+ - \mathbf{a}_\eta'(\mathbf{x}_1))^{\mathrm{T}} \\ &\quad (-\mathbf{a}_\eta'(\mathbf{x}_2) + \mathbf{a}_\eta'(\mathbf{x}_1)) \\ &= -(\mathbf{x}_2^+ - \mathbf{x}_1^+)^{\mathrm{T}}(\mathbf{a}_\eta'(\mathbf{x}_2) - \mathbf{a}_\eta'(\mathbf{x}_1)) \\ &\quad - \frac{1}{2}\|\mathbf{a}_\eta'(\mathbf{x}_2) - \mathbf{a}_\eta'(\mathbf{x}_1)\|^2 \\ &\le 0. \end{aligned} \quad (62)$$

It therefore holds that V strictly decreases over every nonempty compact time-interval as long as $\mathbf{x}_2 \neq \mathbf{x}_1$. In turn, this implies that all solutions of (39) converge to each other exponentially (and therefore uniformly).

Finally we use Lemma 2 in [Yakubovich, 1964], which formulates that if a system exhibits a compact positively invariant set, then the existence of a solution that is bounded for $t \in \mathbb{R}$ is guaranteed. We will denote this "steady-state" solution by $\bar{\mathbf{x}}_w(t)$. The original lemma is formulated for differential equations (possibly with discontinuities, therewith including differential inclusions, with bounded right-hand sides). Here, we use this lemma for measure differential inclusions and would like to note that the proof of the lemma allows for such extensions if we only require continuous dependence on initial conditions. The latter is guaranteed for monotone measure differential inclusions, because incremental stability implies a continuous dependence on initial conditions.

Since all solutions of (39) are globally exponentially stable, also $\bar{\mathbf{x}}_w(t)$ is a globally exponentially stable solution. This concludes the proof that the measure differential inclusion (39) is exponentially convergent. ■

6. Tracking Control for Measure Differential Inclusions of Lur'e Type

An important application of convergence theory is the tracking control of dynamical systems, i.e. the design of a controller, such that a desired trajectory $\mathbf{x}_d(t)$ of the system exists and is globally attractively stable. Tracking control of measure differential inclusions has received very little attention in literature [Bourgeot & Brogliato, 2005; Brogliato *et al.*, 1997; Menini & Tornambè, 2001a].

In this section, we consider the tracking control problem of a nonlinear measure differential inclusion, which can be decomposed into a linear measure differential inclusion with a nonlinear maximal monotone operator in the feedback path. We allow the desired trajectory $\mathbf{x}_d(t)$ to have discontinuities in time (but assume it to be of locally bounded variation). The open-loop dynamics is described by an equality of measures:

$$\begin{aligned} \mathrm{d}\mathbf{x} &= \mathbf{A}\mathbf{x}\mathrm{d}t + \mathbf{B}\mathrm{d}\mathbf{p} + \mathbf{D}\mathrm{d}\mathbf{s}, \\ \mathbf{y} &= \mathbf{C}\mathbf{x} \\ -\mathrm{d}\mathbf{s} &\in \boldsymbol{\mathcal{H}}(\mathbf{y}), \end{aligned} \tag{63}$$

with $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times n_p}$, $\mathbf{C} \in \mathbb{R}^{m\times n}$ and $\mathbf{D} \in \mathbb{R}^{n\times m}$. Herein, $\mathbf{x} \in \mathbb{R}^n$ is the system state (of locally bounded variation), $\mathrm{d}\mathbf{p} = \mathbf{w}\mathrm{d}t + \mathbf{W}\mathrm{d}\eta$ is the differential measure of the control action and $\mathrm{d}\mathbf{s} = \boldsymbol{\lambda}\mathrm{d}t + \boldsymbol{\Lambda}\mathrm{d}\eta$ is the differential measure of the nonlinearity in the feedback loop that is characterized by the set-valued maximal monotone mapping $\boldsymbol{\mathcal{H}}(\mathbf{y})$. The problem that we consider here is the design of a control law $\mathrm{d}\mathbf{p}$ such that the attractive tracking of a desired trajectory $\mathbf{x}_d(t)$ is assured. We propose to tackle the tracking problem by means of a combination of Lebesgue measurable linear error-feedback and a possibly impulsive feedforward control:

$$\begin{aligned} \mathrm{d}\mathbf{p} &= \mathbf{w}_{fb}\mathrm{d}t + \mathrm{d}\mathbf{p}_{ff}(t) \\ &= \mathbf{K}(\mathbf{x} - \mathbf{x}_d(t))\mathrm{d}t + \mathbf{w}_{ff}(t)\mathrm{d}t + \mathbf{W}_{ff}(t)\mathrm{d}\eta, \end{aligned} \tag{64}$$

with

$$\begin{aligned} \mathbf{w}_{fb} &= \mathbf{K}(\mathbf{x} - \mathbf{x}_d(t)), \\ \mathrm{d}\mathbf{p}_{ff}(t) &= \mathbf{w}_{ff}(t)\mathrm{d}t + \mathbf{W}_{ff}(t)\mathrm{d}\eta, \end{aligned} \tag{65}$$

where $\mathbf{K} \in \mathbb{R}^{n_p\times n}$ is the feedback gain matrix and $\mathbf{x}_d(t)$ the desired state trajectory. We restrict the energy input of the impulsive control action $\mathbf{W}_{ff}(t)$ to be bounded from above

$$(\mathbf{x}^+)^{\mathrm{T}}\mathbf{B}\mathbf{W}_{ff} \leq \beta. \tag{66}$$

Note that this condition puts a bound on the jumps in the desired trajectory $\mathbf{x}_d(t)$ which can be realized. Combining the control law (64) with the system dynamics (63) yields the closed-loop dynamics:

$$\begin{aligned} \mathrm{d}\mathbf{x} &= \mathbf{A}_{cl}\mathbf{x}\mathrm{d}t + \mathbf{D}\mathrm{d}\mathbf{s} + \mathbf{B}(-\mathbf{K}\mathbf{x}_d(t)\mathrm{d}t + \mathrm{d}\mathbf{p}_{ff}(t)), \\ \mathbf{y} &= \mathbf{C}\mathbf{x} \\ -\mathrm{d}\mathbf{s} &\in \boldsymbol{\mathcal{H}}(\mathbf{y}), \end{aligned} \tag{67}$$

with

$$\mathbf{A}_{cl} = \mathbf{A} + \mathbf{B}\mathbf{K}. \tag{68}$$

We now propose a convergence-based control design. The main idea of this convergence-based control design is to find a controller of the form (64) that guarantees two properties:

(a) the closed-loop system has a trajectory which is bounded for all t and along which the tracking error $\mathbf{x} - \mathbf{x}_d(t)$ is identically zero. In other words, the feedforward $\mathbf{w}_{ff}(t)$ and $\mathbf{W}_{ff}(t)$ has to be designed such that it induces the desired solution $\mathbf{x}_d(t)$;
(b) the closed-loop system is uniformly convergent. Hereto, the control gain matrix $\mathbf{K}$ should be designed appropriately.

Condition (b) guarantees that the closed-loop system has a unique bounded UGAS steady-state solution, while condition (a) guarantees that, by Property 1, this steady-state solution equals the bounded solution of the closed-loop system with zero tracking error.

For the design of the feedback gain (to ensure that condition (b) is met), we employ the following strategy. First, we design $\mathbf{K}$ such that the linear part of system (67), (68) is strictly passive. Subsequently, using the fact that $\boldsymbol{\mathcal{H}}(\mathbf{y})$ is maximal monotone, we show that this implies that the measure differential inclusion (67), (68) is (after a coordinate transformation) maximal monotone. Hence, exponential convergence for measure differential inclusions of the form (67) can be proven using Theorem 2. A similar result was found for a class of differential inclusions by Yakubovich [1964]. In [Yakubovich, 1964] it is shown that strict passivity of the linear part of the system is sufficient for exponential convergence for Lur'e-type systems with monotone set-valued nonlinearities and absolutely continuous state (i.e. for a class of differential inclusions).

Here, we will show that for *measure* differential inclusions (67), (68) that, if the triple $(\mathbf{A}_{cl}, \mathbf{D}, \mathbf{C})$ is strictly positive real (i.e. the linear part of the system (67) is strictly passive) and the nonlinearity $\boldsymbol{\mathcal{H}}(\mathbf{y})$ is a monotone nonlinearity, then the system is uniformly convergent. Therefore, the feedback gain matrix $\mathbf{K}$ should be designed such that the triple $(\mathbf{A}_{cl}, \mathbf{D}, \mathbf{C})$ is strictly positive real.

Note that the triple $(\mathbf{A}_{cl}, \mathbf{D}, \mathbf{C})$ is rendered strictly passive by means of the feedback design. In other words we design $\mathbf{K}$ such that there exists a positive definite matrix $\mathbf{P} = \mathbf{P}^{\mathrm{T}} > 0$ for which the following conditions are satisfied:

$$\begin{aligned} \mathbf{A}_{cl}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{A}_{cl} &< 0, \\ \mathbf{D}^{\mathrm{T}}\mathbf{P} &= \mathbf{C}. \end{aligned} \tag{69}$$

Let us now introduce a linear coordinate transformation $\tilde{\mathbf{x}} = \mathbf{S}\mathbf{x}$, where $\mathbf{P} = \mathbf{S}^{\mathrm{T}}\mathbf{S}$, i.e. $\mathbf{S}$ is the square root of $\mathbf{P}$. Using these transformed coordinates, the closed-loop dynamics can then be formulated in the form (39):

$$\mathrm{d}\tilde{\mathbf{x}} \in -\mathrm{d}\boldsymbol{\mathcal{A}}(\tilde{\mathbf{x}}^+) - \mathbf{c}(\tilde{\mathbf{x}})\mathrm{d}t + \mathrm{d}\mathbf{b}(\mathbf{w}) \tag{70}$$

with

$$\mathrm{d}\boldsymbol{\mathcal{A}}(\tilde{\mathbf{x}}^+) = \mathbf{S}\mathbf{D}\boldsymbol{\mathcal{H}}(\mathbf{C}\mathbf{S}^{-1}\tilde{\mathbf{x}}^+)(\mathrm{d}t + \mathrm{d}\eta), \tag{71}$$

$$\mathbf{c}(\tilde{\mathbf{x}}) = -\mathbf{S}\mathbf{A}_{cl}\mathbf{S}^{-1}\tilde{\mathbf{x}}, \tag{72}$$

$$\mathrm{d}\mathbf{b}(\mathbf{w}) = \mathbf{S}\mathbf{B}(-\mathbf{K}\mathbf{x}_d(t)\mathrm{d}t + \mathrm{d}\mathbf{p}_{ff}(t)). \tag{73}$$

We will now show that condition (69) together with the monotonicity of the set-valued mapping $\boldsymbol{\mathcal{H}}(\mathbf{y})$ implies strict monotonicity of the differential inclusion (70). Hereto, we prove the strict monotonicity of the set-valued operator $-\mathbf{S}\mathbf{A}_{cl}\mathbf{S}^{-1}\tilde{\mathbf{x}} + \mathbf{S}\mathbf{D}\boldsymbol{\mathcal{H}}(\mathbf{C}\mathbf{S}^{-1}\tilde{\mathbf{x}})$. Using $\boldsymbol{\lambda}_i \in -\boldsymbol{\mathcal{H}}(\mathbf{C}\mathbf{S}^{-1}\tilde{\mathbf{x}}_i)$, $i = 1, 2$, we can verify that it holds

$$\begin{aligned}
&(\tilde{\mathbf{x}}_1 - \tilde{\mathbf{x}}_2)^{\mathrm{T}}(-\mathbf{S}\mathbf{A}_{cl}\mathbf{S}^{-1}\tilde{\mathbf{x}}_1 - \mathbf{S}\mathbf{D}\boldsymbol{\lambda}_1 + \mathbf{S}\mathbf{A}_{cl}\mathbf{S}^{-1}\tilde{\mathbf{x}}_2 + \mathbf{S}\mathbf{D}\boldsymbol{\lambda}_2)\\
&= (\tilde{\mathbf{x}}_1 - \tilde{\mathbf{x}}_2)^{\mathrm{T}}(-\mathbf{S}\mathbf{A}_{cl}\mathbf{S}^{-1})(\tilde{\mathbf{x}}_1 - \tilde{\mathbf{x}}_2) + (\tilde{\mathbf{x}}_1 - \tilde{\mathbf{x}}_2)^{\mathrm{T}}\mathbf{S}\mathbf{D}(\boldsymbol{\lambda}_2 - \boldsymbol{\lambda}_1)\\
&= -(\mathbf{x}_1 - \mathbf{x}_2)^{\mathrm{T}}(\mathbf{S}^{\mathrm{T}}\mathbf{S}\mathbf{A}_{cl})(\mathbf{x}_1 - \mathbf{x}_2) + (\mathbf{x}_1 - \mathbf{x}_2)^{\mathrm{T}}\mathbf{S}^{\mathrm{T}}\mathbf{S}\mathbf{D}(\boldsymbol{\lambda}_2 - \boldsymbol{\lambda}_1)\\
&= -\frac{1}{2}(\mathbf{x}_1 - \mathbf{x}_2)^{\mathrm{T}}(\mathbf{P}\mathbf{A}_{cl} + \mathbf{A}_{cl}^{\mathrm{T}}\mathbf{P})(\mathbf{x}_1 - \mathbf{x}_2)\\
&\quad + (\mathbf{x}_1 - \mathbf{x}_2)^{\mathrm{T}}\mathbf{P}\mathbf{D}(\boldsymbol{\lambda}_2 - \boldsymbol{\lambda}_1).
\end{aligned} \tag{74}$$

Using the conditions (69), we can write $-\mathbf{x}^{\mathrm{T}}(\mathbf{A}_{cl}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{A}_{cl})\mathbf{x} \geq \alpha\|\mathbf{x}\|^2$ for some $\alpha > 0$ and $\mathbf{x}^{\mathrm{T}}\mathbf{P}\mathbf{D} = \mathbf{x}^{\mathrm{T}}\mathbf{C}^{\mathrm{T}} = \mathbf{y}^{\mathrm{T}}$ and Eq. (74) becomes

$$\begin{aligned}
&(\tilde{\mathbf{x}}_1 - \tilde{\mathbf{x}}_2)^{\mathrm{T}}(-\mathbf{S}\mathbf{A}_{cl}\mathbf{S}^{-1}\tilde{\mathbf{x}}_1 - \mathbf{S}\mathbf{D}\boldsymbol{\lambda}_1 + \mathbf{S}\mathbf{A}_{cl}\mathbf{S}^{-1}\tilde{\mathbf{x}}_2 + \mathbf{S}\mathbf{D}\boldsymbol{\lambda}_2)\\
&\geq \frac{\alpha}{2}\|\mathbf{x}_1 - \mathbf{x}_2\|^2 + (\mathbf{y}_1 - \mathbf{y}_2)^{\mathrm{T}}(\boldsymbol{\lambda}_2 - \boldsymbol{\lambda}_1).
\end{aligned} \tag{75}$$

Finally, we use the fact that $\boldsymbol{\lambda}_i \in -\boldsymbol{\mathcal{H}}(\mathbf{y}_i)$, $i = 1, 2$, and the monotonicity of the set-valued nonlinearity $\boldsymbol{\mathcal{H}}(\mathbf{y})$ to conclude that

$$\begin{aligned}
&(\tilde{\mathbf{x}}_1 - \tilde{\mathbf{x}}_2)^{\mathrm{T}}(-\mathbf{S}\mathbf{A}_{cl}\mathbf{S}^{-1}\tilde{\mathbf{x}}_1 - \mathbf{S}\mathbf{D}\boldsymbol{\lambda}_1\\
&\quad + \mathbf{S}\mathbf{A}_{cl}\mathbf{S}^{-1}\tilde{\mathbf{x}}_2 + \mathbf{S}\mathbf{D}\boldsymbol{\lambda}_2) \geq \frac{\alpha}{2}\|\mathbf{x}_1 - \mathbf{x}_2\|^2.
\end{aligned} \tag{76}$$

In other words, strict monotonicity of the $\tilde{\mathbf{x}}$-dynamics is guaranteed. Earlier in the paper we have shown that strict monotonicity implies uniform convergence. Moreover, the convergence property is conserved under smooth coordinate transformations (see [Pavlov *et al.*, 2005d]). Consequently, if the $\tilde{\mathbf{x}}$-dynamics is uniformly convergent, then also the $\mathbf{x}$-dynamics is uniformly convergent.

7. Illustrative Examples

In the next sections, examples concerning models for the control of mechanical systems with set-valued friction and one-way clutches illustrate the power of the result in Theorem 2. Moreover, the results of Sec. 6 on tracking control are applied to mechanical systems with friction and a one-way clutch in Secs. 7.2 and 7.3.

7.1. *One-way clutch*

The time-evolution of the velocity of a mass m (see Fig. 1) subjected to a one-way clutch, a dashpot $b > 0$ and an external input (considering both bounded and impulsive contributions) can be described by the equality of measures

$$m\mathrm{d}u = \mathrm{d}p + \mathrm{d}s - bu\mathrm{d}t. \tag{77}$$

We can decompose the differential measure $\mathrm{d}s$ of the one-way clutch in

$$\mathrm{d}s = \lambda\mathrm{d}t + \Lambda\mathrm{d}\eta, \tag{78}$$

where $\lambda := s'_t$ is the contact force and $\Lambda = s'_\eta$ is the contact impulse. The differential impulse measure $\mathrm{d}s$ of the one-way clutch obeys the set-valued force law

$$-\mathrm{d}s \in \mathrm{Upr}(u^+). \tag{79}$$

The set-valued function $\mathrm{Upr}(x)$ is the unilateral primitive [Glocker, 2001]:

$$\begin{aligned}
-y \in \mathrm{Upr}(x) &\Leftrightarrow 0 \leq x \perp y \geq 0\\
&\Leftrightarrow x \geq 0,\ y \geq 0,\ xy = 0,
\end{aligned} \tag{80}$$

being a maximal monotone operator (Fig. 2).

The input consists of a bounded force f and an impulse F

$$\mathrm{d}p = f\mathrm{d}t + F\mathrm{d}\eta. \tag{81}$$

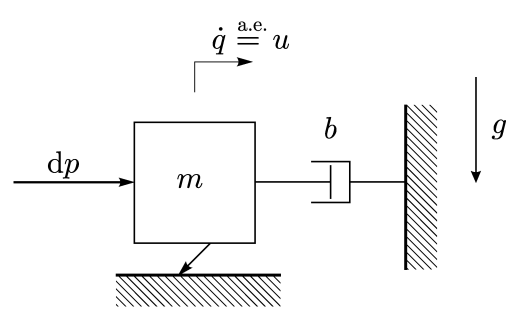

Fig. 1. Mass with one-way clutch and impulsive actuation.

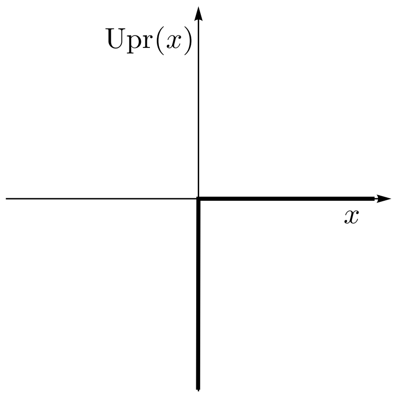

Fig. 2. Unilateral primitive.

We assume that an impulsive input $F > 0$ is transmitted by firing bullets with mass m_0 and constant speed $v \le v_{\max}$ on the left side of the mass m. We assume a completely inelastic impact. If $u \ge v$, then the bullet is not able to hit the mass m and then the impulse F equals zero. If $u^+ < v$, then the impulse F equals the mass of the bullet multiplied with its velocity jump:

$$F = m_0(v - u^+). \tag{82}$$

Similarly, we assume that an impulsive input $F < 0$ is transmitted by firing on the right side of the mass m with a speed $v < 0$, bounded by $|v| \le v_{\max}$. The energy input $u^+F = m_0u^+(v - u^+)$ of the impulse F is maximal when $u^+ = (1/2)v$ and is therefore bounded from above by $\beta := (1/4)m_0v_{\max}^2 \ge |u^+F|$.

We first prove the existence of a compact positively invariant set with Theorem 1 which uses the Lyapunov function $W(u) = (1/2)u^2$, that we recognize to be the kinetic energy divided by the mass m. The time-derivative $\dot{W}$ gives, using $u\lambda = 0$,

$$\dot{W} \le -\frac{b}{m}u^2 + u\sup_{t\in\mathbb{R}}(f(t)), \tag{83}$$

and it therefore holds that $\alpha = b/m$ and $\gamma = (1/b)\sup_{t\in\mathbb{R}}(f(t))$ with α and γ defined in Theorem 1. Theorem 1 states that if the time-instants t_i of the impulses F are separated by the dwell-time

$$\tau = \frac{\delta}{2(\delta-1)\alpha}\ln\left(1 + \frac{2\beta}{\delta^2\gamma^2}\right), \tag{84}$$

then the set

$$\mathcal{W} = \left\{u \in \mathbb{R}^+ \,\middle|\, \frac{1}{2}u^2 \le \frac{1}{2}\delta^2\gamma^2 + \beta\right\} \tag{85}$$

is a compact positively invariant set for arbitrary $\delta > 1$. Following Corollory 1, we conclude that the dwell-time can be made arbitrary small by increasing δ. We therefore take τ to be smaller than the minimal time-lapse between two succeeding impulsive time-instants, which gives a lower bound for δ.

Just as in the proof of Theorem 2, we prove incremental stability using the Lyapunov function

$$V = \frac{1}{2}(u_2 - u_1)^2. \tag{86}$$

First, we consider the time-derivative $\dot{V}$:

$$\begin{aligned}\dot{V} &= (u_2 - u_1)(\dot{u}_2 - \dot{u}_1)\\ &= (u_2 - u_1)\frac{1}{m}(\lambda_2 - bu_2 - \lambda_1 + bu_1)\\ &= (u_2 - u_1)\frac{1}{m}(\lambda_2 - \lambda_1) - \frac{b}{m}(u_2 - u_1)^2,\\ &\le -\frac{b}{m}(u_2 - u_1)^2,\end{aligned} \tag{87}$$

with $-\lambda_1 \in \mathrm{Upr}(u_1)$, $-\lambda_2 \in \mathrm{Upr}(u_2)$.

Subsequently, we consider a jump in V:

$$\begin{aligned}V^+ - V^- &= V(u_1^+, u_2^+) - V(u_1^-, u_2^-)\\ &= \frac{1}{2}(u_2^+ - u_1^+)^2 - \frac{1}{2}(u_2^- - u_1^-)^2\\ &= \frac{1}{2}(u_2^+ + u_2^- - u_1^+ - u_1^-)\\ &\quad (u_2^+ - u_2^- - u_1^+ + u_1^-).\end{aligned} \tag{88}$$

Following the proof of Theorem 2, we eliminate u_1^- and u_2^- by substituting the impact equation $m(u_j^+ - u_j^-) = \Lambda_j + F$, $j = 1, 2$:

$$\begin{aligned}V^+ - V^- &= \frac{1}{2}\left(2u_2^+ - \frac{1}{m}\Lambda_2 - 2u_1^+ + \frac{1}{m}\Lambda_1\right)\frac{1}{m}(\Lambda_2 - \Lambda_1)\\ &= (u_2^+ - u_1^+)\frac{1}{m}(\Lambda_2 - \Lambda_1) - \frac{1}{2m^2}(\Lambda_2 - \Lambda_1)^2\\ &\le 0.\end{aligned} \tag{89}$$

Hence, it holds for the differential measure $\mathrm{d}V$ that

$$\begin{aligned}\mathrm{d}V &= \dot{V}\mathrm{d}t + (V^+ - V^-)\mathrm{d}\eta\\ &\le -\alpha(u_2 - u_1)^2\mathrm{d}t,\end{aligned} \tag{90}$$

with

$$\alpha = \frac{b}{m}.$$

Integration of $\mathrm{d}V$ over a nonempty time-interval therefore leads to a strict decrease of the function V as long as $u_2 \ne u_1$. This proves incremental stability. Consequently, the system is exponentially convergent (see Theorem 2).

7.2. *Tracking control for mechanical systems with set-valued friction*

In this section, we consider the tracking control problem for mechanical systems with set-valued friction. Hereto, we study a common motor-load configuration as depicted in Fig. 3. The essential problem here is the fact that the friction and the actuation are noncollocated (i.e. the motor, mass m_1, is actuated and the load, mass m_2, is subject to friction). Note that the spring-damper combination, with stiffness c and viscous damping constant b, reflects a finite-stiffness coupling between the motor and load as is usual in many motion systems. A common approach in tackling control problems for systems with friction is that of friction compensation. This angle of attack is clearly not feasible here since the actuation cannot compensate directly for the friction. Another common approach in compensating for nonlinearities can be recognized in the backstepping control schemes [Khalil, 1996]. However, these generally require differentiability of the nonlinearity, which is not the case here due to the set-valued nature of the friction law.

In many applications, mainly the velocity of the load is of interest. In this context, one can think of controlling a printhead in a printer, where the printhead is to achieve a constant velocity when moving across the paper or drilling systems where the bottom-hole-assemble (including the drill bit) should achieve a constant cutting speed. From this perspective, the following third-order differential inclusion describes the dynamics of the system under study:

$$\begin{aligned} \dot{\mathbf{x}} &= \overline{\mathbf{A}}\mathbf{x} + \mathbf{B}w + \mathbf{D}\overline{\lambda}, \\ y &= \mathbf{C}\mathbf{x} \\ \overline{\lambda} &\in -\overline{\mathcal{H}}(y), \end{aligned} \tag{91}$$

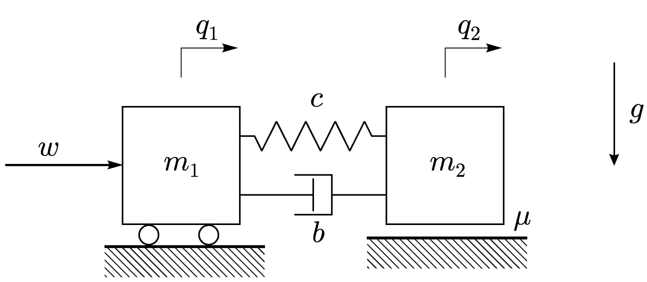

Fig. 3. Typical motor-load configuration with noncollocated friction and actuation.

with

$$\overline{\mathbf{A}} = \begin{bmatrix} 0 & -1 & 1 \\ \dfrac{c}{m_1} & -\dfrac{b}{m_1} & \dfrac{b}{m_1} \\ -\dfrac{c}{m_2} & \dfrac{b}{m_2} & -\dfrac{b}{m_2} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ \dfrac{1}{m_1} \\ 0 \end{bmatrix}, \tag{92}$$

$$\mathbf{D} = \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{m_2} \end{bmatrix}, \quad \mathbf{C}^{\mathrm{T}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Herein, $\mathbf{x} = [q_2 - q_1 \quad \dot{q}_1 \quad \dot{q}_2]^{\mathrm{T}}$ is the absolutely continuous system state, $w \in \mathbb{R}$ is the control action and $\overline{\lambda} \in \mathbb{R}$ is the friction force that is characterized by the set-valued mapping $\overline{\mathcal{H}}(\cdot) : \mathbb{R} \to \mathbb{R}$. The set-valued friction law adopted here includes a combination of Coulomb friction, viscous friction and the Stribeck effect:

$$\overline{\mathcal{H}}(y) = m_2 g \left(\mu_0 \operatorname{Sign}(y) + \mu_1 y - \frac{\mu_2 y}{1 + \mu_2 |y|} \right),$$

$$\text{with } \operatorname{Sign}(y) = \begin{cases} \{-1\} & y < 0 \\ [-1, 1] & y = 0\,, \\ \{1\} & y > 0 \end{cases} \tag{93}$$

where g is the gravitational acceleration, $\mu_0 > 0$ is the Coulomb friction coefficient, $\mu_1 > 0$ is the viscous friction coefficient and μ_2 is an additional coefficient characterizing the modeling of the Stribeck effect. It is well known that exactly such a Stribeck effect can induce instabilities, complicating the design of stabilizing controllers, see e.g. [Armstrong–Hélouvry *et al.*, 1994]. In Fig. 4, such type of set-valued static friction is depicted schematically. At this point, we will transform the friction law $\overline{\mathcal{H}}(y)$ to a strictly maximal monotone operator $\mathcal{H}(y)$

$$\mathcal{H}(y) = \overline{\mathcal{H}}(y) + \kappa y, \quad \text{with } \kappa = m_2 g(\mu_2 - \mu_1), \tag{94}$$

where the choice of κ ensures that the set-valued mapping $\mathcal{H}(y)$ is a maximal monotone mapping. System (91) can therefore be transformed into the form (63)

$$\begin{aligned} \mathrm{d}\mathbf{x} &= \mathbf{A}\mathbf{x}\mathrm{d}t + \mathbf{B}\mathrm{d}p + \mathbf{D}\mathrm{d}s, \\ y &= \mathbf{C}\mathbf{x} \\ -\mathrm{d}s &\in \mathcal{H}(y), \end{aligned} \tag{95}$$

where $\mathbf{A} = \overline{\mathbf{A}} + \kappa\mathbf{D}\mathbf{C}$, $\mathrm{d}p = w\mathrm{d}t$, $\mathrm{d}s = \lambda\mathrm{d}t$ and $\mathcal{H}(y)$ is a maximal monotone mapping. We now

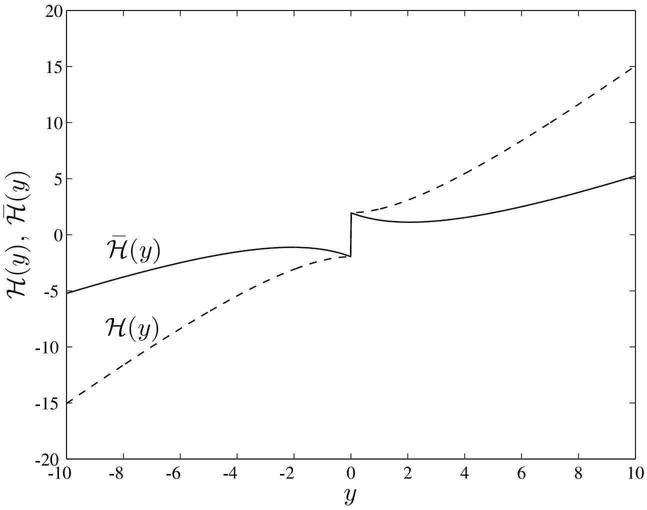

Fig. 4. Friction law $\mathcal{H}(y)$ and transformed friction law $\overline{\mathcal{H}}(y)$.

use the convergence-based tracking control strategy proposed in Sec. 6 to solve the tracking problem of this mechanical system with friction. Hereto, we use a combination of linear error-feedback and feedforward control as in (64), where $\mathbf{K} = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$ is the feedback gain matrix which has to be chosen such that the triple $(\mathbf{A}_{cl}, \mathbf{D}, \mathbf{C})$ is strictly positive real.

We adopt the following system parameters: $g = 9.81$, $m_1 = 1$, $m_2 = 1$, $c = 100$, $b = 1$, $\mu_0 = 0.2$, $\mu_1 = 0.1$ and $\mu_2 = 0.2$. The resulting friction map and the transformed (monotone) friction map are shown in Fig. 4. Moreover, we aim at tracking a constant velocity solution (with desired velocity v_d) for both the motor and the load; i.e. the desired state-trajectory is given by

$$\mathbf{x}_d(t) = \left[-\frac{m_2 g}{c} \left(\mu_0 + \mu_1 v_d - \frac{\mu_2 v_d}{1 + \mu_2 |v_d|} \right) v_d \ v_d \right]^{\mathrm{T}},$$

where $v_d = 1$. Note that this velocity lies in the range in which the friction law exhibits a pronounced Stribeck effect. Let us design the controller in the form (64). Firstly, the feedforward which induces the desired solution is given by

$$w_{ff} = m_2 g \left(\mu_0 + \mu_1 v_d - \frac{\mu_2 v_d}{1 + \mu_2 |v_d|} \right). \qquad (96)$$

Secondly, by checking appropriate LMI conditions or frequency-domain conditions (see e.g. [Khalil, 1996]) for the strictly positive realness of the triple $(\mathbf{A}_{cl}, \mathbf{D}, \mathbf{C})$, suitable controller gains can be selected: $k_1 = -30$, $k_2 = -150$ and $k_3 = -150$. The strict positive realness of the triple $(\mathbf{A}_{cl}, \mathbf{D}, \mathbf{C})$ can be proven using the following symmetric, positive definite matrix $\mathbf{P}$

$$\mathbf{P} = \begin{bmatrix} 50.98 & -0.33 & 0 \\ -0.33 & 0.005 & 0 \\ 0 & 0 & 1 \end{bmatrix} > 0. \qquad (97)$$

satisfying the LMIs (69).

Next, we implement control law (64) on system (91), with these control gains and feedforward (96) and use numerical time-stepping schemes [Acary *et al.*, 2008; Leine & Nijmeijer, 2004; Moreau, 1988b] to numerically compute the solution of the closed-loop system. In Fig. 5, the velocities of both masses are depicted when the controller is active and asymptotic tracking of the constant velocity solution is achieved. Note that when only the feedforward is applied, the desired solution is still a solution of the system; however, no asymptotic tracking was achieved, see Fig. 6. In this figure, it is shown that both masses ultimately come to a standstill. Clearly, the system now exhibits at least two steady-state solutions; the desired solution and the solution on which $x_1 = -u_{ff}/c$, $x_2 = 0$ and $x_3 = 0$, as depicted in Fig. 6. Consequently, the system without feedback is not convergent. For both cases the initial condition $\mathbf{x}(0) = \begin{bmatrix} 0 & 0.8 & 0.8 \end{bmatrix}^{\mathrm{T}}$ was used.

Note that we solve a stabilization problem in this example. However, using the strategy discussed here, we can make any bounded feasible time-varying desired solution $\mathbf{x}_d(t)$ attractively stable using the same feedback gain matrix $\mathbf{K}$.

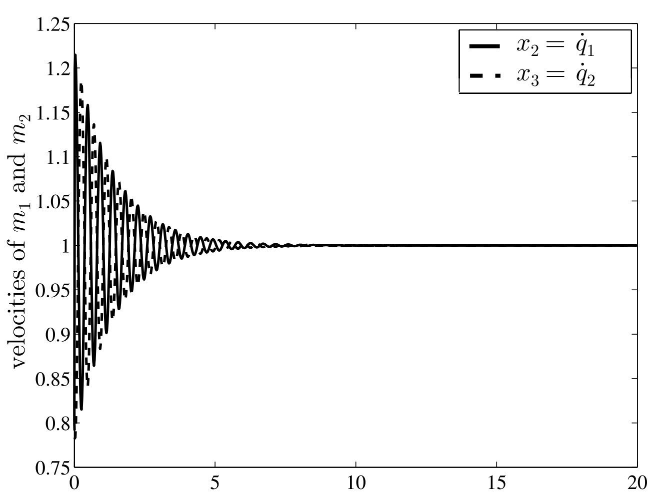

Fig. 5. Feedback and feedforward control.

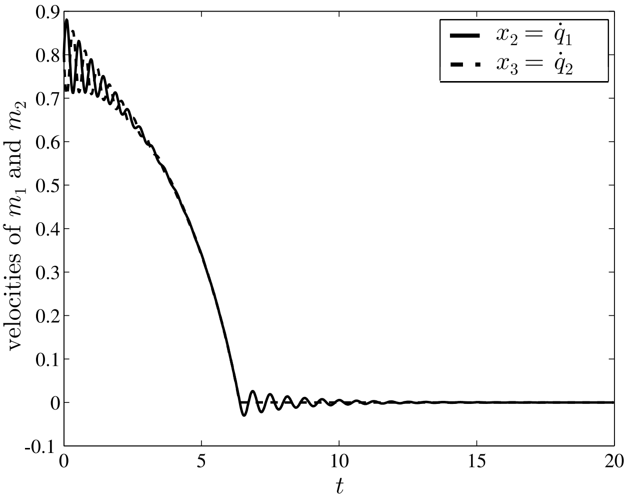

Fig. 6. Only feedforward control.

7.3. *Tracking control for a mechanical system using an impulsive input*

In the example of Sec. 7.2 we solved the tracking problem mechanical motion system with set-valued friction. In the current example, we consider a similar system; however, the set-valued friction is replaced by a one-way clutch and impulsive control action is needed to achieve tracking of a periodic trajectory.

More specifically, we study a variant of the previous problem and replace the friction element by a one-way clutch and add an additional damper b_2, see Fig. 7. Moreover, we allow for impulsive inputs on the first mass. The open-loop dynamics is now described by an equality of measures:

$$\begin{aligned} \mathrm{d}\mathbf{x} &= \mathbf{A}\mathbf{x}\mathrm{d}t + \mathbf{B}\mathrm{d}p + \mathbf{D}\mathrm{d}s, \\ y &= \mathbf{C}\mathbf{x} \\ -\mathrm{d}s &\in \mathcal{H}(y), \end{aligned} \tag{98}$$

with

$$\mathbf{A} = \begin{bmatrix} 0 & -1 & 1 \\ \dfrac{c}{m_1} & -\dfrac{b_1+b_2}{m_1} & \dfrac{b_1}{m_1} \\ -\dfrac{c}{m_2} & \dfrac{b_1}{m_2} & -\dfrac{b_1}{m_2} \end{bmatrix},$$

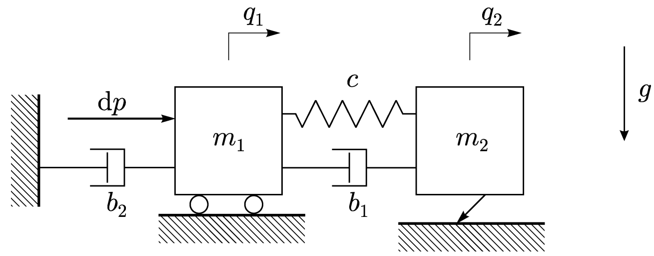

Fig. 7. Motor-load configuration with one-way clutch and impulsive actuation.

$$\mathbf{B} = \begin{bmatrix} 0 \\ \dfrac{1}{m_1} \\ 0 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 \\ 0 \\ \dfrac{1}{m_2} \end{bmatrix}, \quad \mathbf{C}^{\mathrm{T}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{99}$$

The evolution $\mathbf{x}(t)$ of the state vector $\mathbf{x} = \begin{bmatrix} q_2 - q_1 & u_1 & u_2 \end{bmatrix}^{\mathrm{T}}$ is of locally bounded variation. The differential measure of the control action $\mathrm{d}p = w\mathrm{d}t + W\mathrm{d}\eta$ now also contains an impulsive part W. The differential measure $\mathrm{d}s$ of the force in the one-way clutch is characterized by the scalar set-valued maximal monotone mapping $\mathcal{H}(x) = \mathrm{Upr}(x)$.

In this example, we try to let the velocity $x_3(t) = \dot{q}_2(t)$ approach the desired trajectory $x_{d3}(t)$. Hereto, we design trajectories $x_{d1}(t)$ and $x_{d2}(t)$ which generate the desired $x_{d3}(t)$. Subsequently, we aim at tracking of the desired state trajectory $\mathbf{x}_d(t)$. The state-tracking problem is solved by making the system uniformly convergent with a feedback $\mathbf{K}(\mathbf{x} - \mathbf{x}_d(t))$. As in Sec. 7.2, we can design $\mathbf{K}$ such that the triple $(\mathbf{A}_{cl}, \mathbf{D}, \mathbf{C})$ is rendered strictly passive, which, given the monotonicity of $\mathcal{H}(y)$, makes the system uniformly convergent.

We adopt the following system parameters: $g > 0$, $m_1 = 1$, $m_2 = 1$, $c = 10$, $b_1 = 1$ and $b_2 = -1.4$. The negative damping $b_2 < 0$ causes the system matrix $\mathbf{A}$ to have a positive real eigenvalue. The desired velocity of the second mass is characterized by a periodic sawtooth wave with period time T:

$$x_{d3}(t) = \begin{cases} \mathrm{mod}\,(t,T) & \text{for } 0 \le \mathrm{mod}\,(t,T) \le \dfrac{T}{4} \quad \text{(ramp-up)} \\ -\mathrm{mod}\,(t,T) + \dfrac{T}{2} & \text{for } \dfrac{T}{4} \le \mathrm{mod}\,(t,T) \le \dfrac{T}{2} \quad \text{(ramp-down)} \\ 0 & \text{for } \dfrac{T}{2} \le \mathrm{mod}\,(t,T) \le T \quad \text{(deadband)} \end{cases}.$$

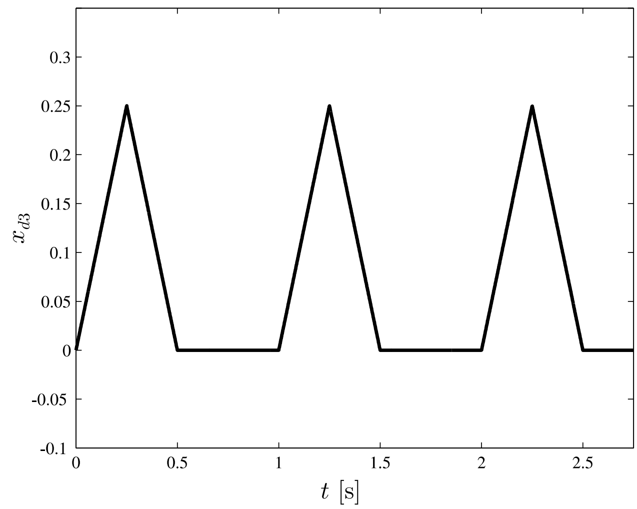

Fig. 8. Desired trajectory $x_{d3}(t)$.

The signal $x_{d3}(t)$ for $T = 1$ s is shown in Fig. 8. The desired trajectory $x_{d3}(t)$ is a periodic signal which is time-continuous but has three kinks in each period. Kinks in $x_{d3}(t)$ can be achieved by applying an impulsive force on the first mass which causes an instantaneous change in the velocity $x_2 = \dot{q}_1$ and therefore a discontinuous force in the damper b_1. The one-way clutch on the second mass prevents negative values of x_{d3} and no impulsive force on the first mass is therefore necessary for the change from ramp-down to deadband. In a first step, the signals $x_{d1}(t)$, $x_{d2}(t)$ and $\mathrm{d}s(t)$ are designed such that

$$\begin{aligned}
&\dot{x}_{d1}(t) = -x_{d2}(t) + x_{d3}(t)\\
&\mathrm{d}x_{d3}(t) = \left(-\frac{c}{m_2}x_{d1}(t) - \frac{b_1}{m_2}(-x_{d2}(t) + x_{d3}(t))\right)\mathrm{d}t\\
&\qquad\qquad + \frac{1}{m_2}\mathrm{d}s(t)\\
&-\mathrm{d}s(t) \in \mathrm{Upr}(x_{d3}(t)),
\end{aligned} \tag{100}$$

for the given periodic trajectory $x_{d3}(t)$. The solution of this problem is not unique as we are free to choose $\mathrm{d}s(t) \geq 0$ for $x_{d3}(t) = 0$. By fixing $ds(t) = \dot{s}_0\mathrm{d}t$ to a constant value for $x_{d3}(t) = 0$ (i.e. $\dot{s}_0$ is a constant), we obtain the following discontinuous differential equation for $x_{d1}(t)$:

$$\dot{x}_{d1} = \begin{cases} \dfrac{m_2}{b_1}\left(-\dot{x}_{d3}(t) - \dfrac{c}{m_2}x_{d1}\right) & x_{d3}(t) > 0, \\ \dfrac{m_2}{b_1}\left(-\dfrac{c}{m_2}x_{d1} + \dfrac{1}{m_2}\dot{s}_0\right) & x_{d3}(t) = 0 \end{cases} \tag{101}$$

The numerical solution of $x_{d1}(t)$ gives (after a transient) a periodic signal $x_{d1}(t)$ and $x_{d2}(t) = -\dot{x}_{d1}(t) + x_{d3}(t)$ (see the dotted lines in Figs. 12 and 13 which are mostly below the solid lines). We have taken $\dot{s}_0 = 1$. Subsequently, the feedforward input $\mathrm{d}p_{ff} = w_{ff}\mathrm{d}t + W_{ff}\mathrm{d}\eta$ is designed such that

$$\begin{aligned}\mathrm{d}p_{ff} = m_1\mathrm{d}x_{d2} - (cx_{d1} + b_1(-x_{d2} + x_{d3})\\ - b_2x_{d2})\mathrm{d}t\end{aligned} \tag{102}$$

and it therefore holds that $\mathbf{x}(t) = \mathbf{x}_d(t)$ for $t \geq 0$ if $\mathbf{x}(0) = \mathbf{x}_d(0)$, where $\mathbf{x}(t)$ is a solution of (98), (99), with $\mathrm{d}p = \mathrm{d}p_{ff}$. The feedforward input $\mathrm{d}p_{ff}/\mathrm{d}t$ is shown in Fig. 9 and is equal to $w_{ff}(t)$ almost everywhere. Two impulsive inputs $W_{ff}(t)$ per period can be seen at the time-instants for which there is a "change ramp-up to ramp-down" and "ramp-down to deadband". Next, we implement the control law (64) on system (98) with the feedforward $\mathrm{d}p_{ff}$ as in (102). We choose $\mathbf{K} = \begin{bmatrix}0 & -4 & 0\end{bmatrix}$ which renders the triple $(\mathbf{A}_{cl}, \mathbf{D}, \mathbf{C})$ strictly positive real and, therefore, the closed-loop system (98), (99), (64), (102) has convergent dynamics. The strict positive realness of the triple $(\mathbf{A}_{cl}, \mathbf{D}, \mathbf{C})$ can be proven using the following matrix $\mathbf{P}$

$$\begin{aligned}
&\mathbf{P} = \begin{bmatrix} 34 & -10.5 & 0 \\ -10.5 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix} > 0,\\
&\mathbf{Q} = -(\mathbf{A}_{cl}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{A}_{cl}) > 0,\\
&\mathbf{D}^{\mathrm{T}}\mathbf{P} = \mathbf{C}, \quad \mathbf{A}_{cl} = \mathbf{A} + \mathbf{BK}.
\end{aligned} \tag{103}$$

Figure 10 shows the closed-loop dynamics for which the desired periodic solution $\mathbf{x}_d(t)$ is globally attractively stable. The attraction to the periodic solution from an arbitrary initial condition occurs in finite time (symptotic attraction). Figure 11 shows the open-loop dynamics for which there is no state-feedback. The desired periodic solution $\mathbf{x}_d(t)$ is not globally attractive, not even locally, and the solution from the choosen initial condition is attracted to a stable period-2 solution. Clearly, the system without feedback is not convergent. For both cases the initial condition $\mathbf{x}(0) = [0.16 \quad 2.17 \quad 0]^{\mathrm{T}}$ was used. Figures 12 and 13 show the time-histories of $x_1(t)$ and $x_{d1}(t)$, respectively $x_2(t)$ and $x_{d2}(t)$, in solid and dotted lines. Jumps in the state $x_2(t)$ and desired state $x_{d2}(t)$ can be seen on time-instants for which the input is impulsive.

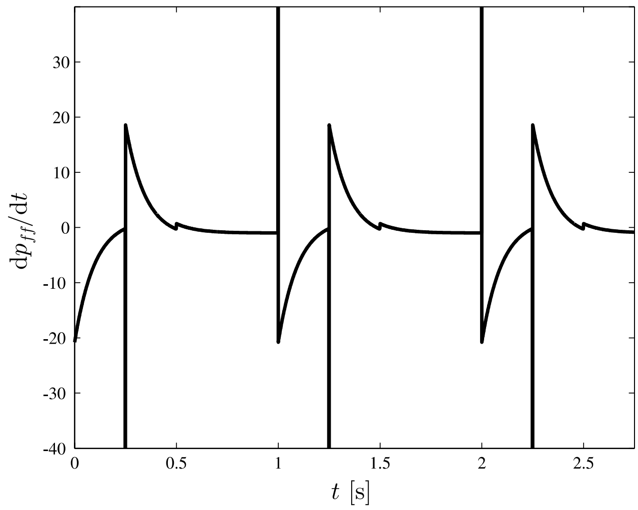

Fig. 9. Feedforward $\mathrm{d}p_{ff}/\mathrm{d}t$.

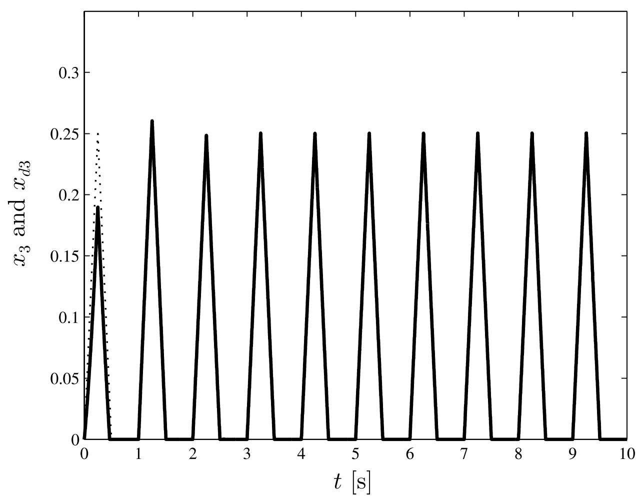

Fig. 10. Trajectories $x_3(t)$ (solid) and $x_{d3}(t)$ (dotted) for the case of feedback and feedforward control.

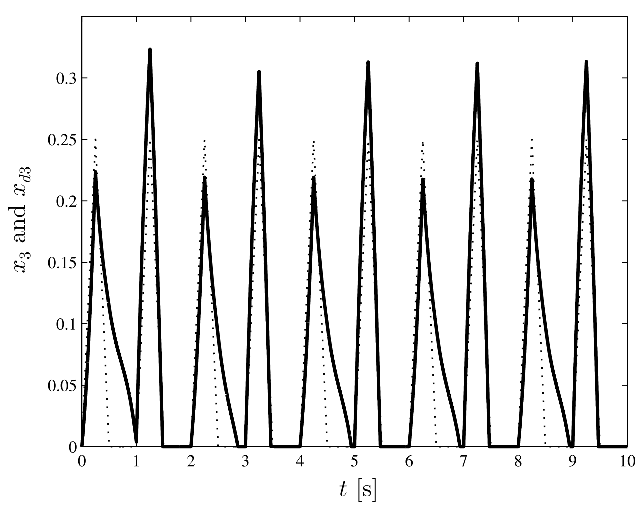

Fig. 11. Trajectories $x_3(t)$ (solid) and $x_{d3}(t)$ (dotted) for the case of only feedforward control.

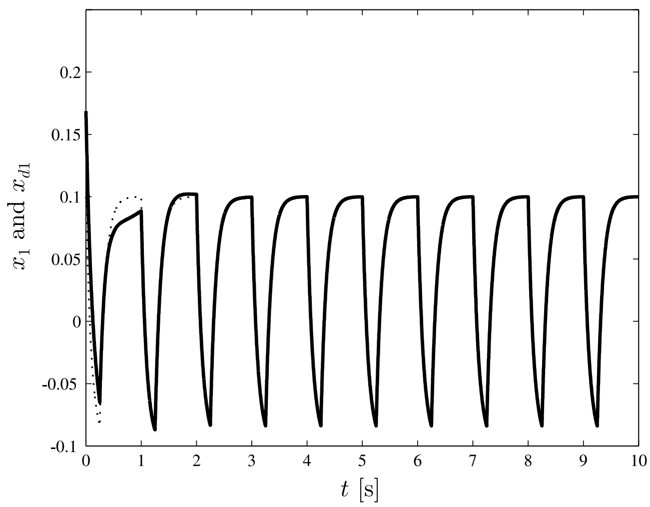

Fig. 12. Trajectories $x_1(t)$ (solid) and $x_{d1}(t)$ (dotted) for the case of feedback and feedforward control.

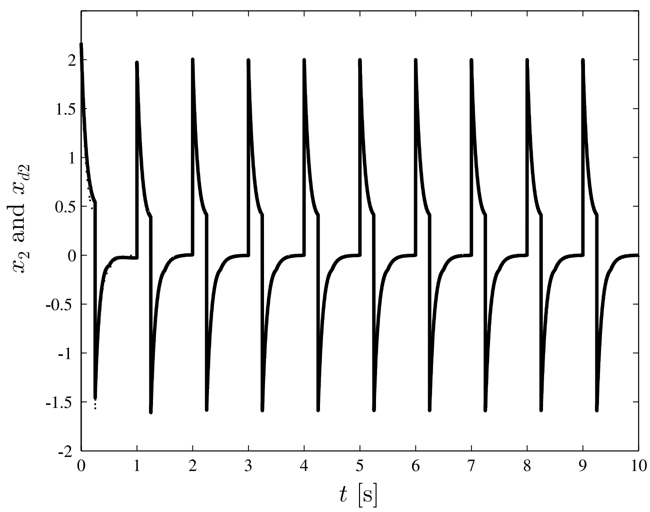

Fig. 13. Trajectories $x_2(t)$ (solid) and $x_{d2}(t)$ (dotted) for the case of feedback and feedforward control.

8. Conclusions

In the previous sections, sufficient conditions have been derived for the uniform convergence of a class of measure differential inclusions with certain maximal monotonicity properties. We will summarize the main ideas of the paper.

First, sufficient conditions have been presented in Theorem 1 for the existence of a compact positively invariant set. Theorem 1 relies on a Lyapunov argument with the squared magnitude of the state as Lyapunov function, which is a kind of energy function. The assumption of strict monotonicity of the Lebesgue part of the state dependent right-hand side equals a strict passivity requirement with a quadratic dissipation. The quadratic dissipation can always outperform the linear energy input of bounded nonimpulsive forces. Hence, during nonimpulsive motion, the system dissipates energy for large enough magnitudes of the state. The assumption of monotonicity of the atomic part of the state dependent right-hand side equals a passivity requirement. Moreover, the energy input of the impulsive inputs is assumed to be bounded. This means that, for a given size of the compact positively invariant set of which we like to prove existence, we can find a dwell-time for the impulsive inputs. If the time-lapse between subsequent impulsive inputs is larger than this dwell-time, then the Lebesgue measurable dissipative forces have enough time to "eat" the energy input of the impulsive input. This reasoning works also in the opposite direction. Given a certain dwell-time, there exists a certain compact positively invariant set of the system. This means that the dwell-time is not really a condition for the existence of a compact positively invariant set, but is merely a constant which relates to the size of such a set. The existence of such a set guarantees the existence of a solution that is bounded for all times.

Subsequently, sufficient conditions for incremental attractive stability have been derived in Theorem 2 using again a Lyapunov-based approach. The decrease of the Lyapunov function, which measures the distance between two arbitrary solutions, follows from a monotonicity condition. Incremental attractive stability implies that all solutions converge to one another. The aforementioned bounded (steady-state) solution must therefore be globally asymptotically stable for all bounded inputs, which rigourously proves uniform convergence of the system (Theorem 2).

The above theorems hinge on a few important assumptions, for which we can give the following interpretations in the context of mechanical systems with impulsive right-hand sides:

(1) *Separation of state-dependent forces and inputs.* In other words: no cross-talk between state-dependent forces and inputs. This excludes mixed terms in state and input, which for instant arise if the generalized force directions of the input forces are state-dependent.
(2) *A strict monotonicity condition on the Lebesgue measurable right-hand side.* This implies that the state-dependent forces in the system are strictly passive.
(3) *A monotonicity condition on the atomic (impulsive) right-hand side.* This implies that the state-dependent impulses in the system are passive.
(4) *Bounded energy input of the impulsive inputs.* The physical meaning of this assumption has been elucidated in Sec. 7.1.
(5) *A dwell-time condition.* The dwell-time can be choosen to be arbitrary small. In practice, there always exist a minimal time between two impulsive inputs which can be exerted on the system.

Condition 2 is the condition which may limit most of all the use of Theorem 2, simply because many systems are not dissipative. However, systems can be *made* dissipative using an appropriate control. In other words, the presented theorems give us the knowledge how to design controllers, such that the closed loop system is uniformly convergent. The uniform convergence can then be used for tracking control purposes, synchronization, etc. In Sec. 6 we presented such a convergence-based tracking control design for a class of measure differential inclusions in Lur'e form. Finally, we presented examples of mechanical systems with set-valued force laws. In these examples, it has been demonstrated that the tracking problem for a class of systems with noncollocated actuation and set-valued friction can be solved using the results presented in this paper.

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Appendix

Definitions of Stability Properties

A proper definition of convergence properties requires an exact definition of (uniform) stability and attractivity. These definitions have been well-defined for differential equations [Coddington & Levinson, 1955; Willems, 1970]. Here, we generalize these definitions to differential inclusions and measure differential inclusions. See also [Leine, 2006] for stability properties of time-autonomous measure differential inclusions.

Differential inclusions

Consider the differential inclusion

$$\dot{\mathbf{x}} \in \boldsymbol{\mathcal{F}}(t, \mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^n, \quad t \in \mathbb{R}, \tag{A.1}$$

where the function $\boldsymbol{\mathcal{F}}(t, \mathbf{x})$ is a set-valued function. A solution $\mathbf{x}(t)$ of (A.1) is an absolutely continuous function, defined for all t (at least locally), which fulfills (A.1) for almost all t. The set of forward solutions of (A.1) with $\mathbf{x}(t_0) = \mathbf{x}_0$ is denoted by $\mathcal{S}(\boldsymbol{\mathcal{F}}, t_0, \mathbf{x}_0)$.

Definition 6 (Stability). A solution $\bar{\mathbf{x}}(t)$ of system (A.1), with $\bar{\mathbf{x}}(t_0) = \bar{\mathbf{x}}_0$ and which is defined on $t \in (t_*, +\infty)$, is said to be

- *stable* if for any $t_0 \in (t_*, +\infty)$ and $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon, t_0) > 0$ such that $\|\mathbf{x}_0 - \bar{\mathbf{x}}(t_0)\| < \delta$ implies that each forward solution $\mathbf{x}(t) \in \mathcal{S}(\boldsymbol{\mathcal{F}}, t_0, \mathbf{x}_0)$ satisfies $\|\mathbf{x}(t) - \bar{\mathbf{x}}(t)\| < \varepsilon$ for all $t \geq t_0$.
- *uniformly stable* if it is stable and the number δ in the definition of stability is independent of t_0.
- *attractively stable* if it is stable and for any $t_0 \in (t_*, +\infty)$ there exists $\bar{\delta} = \bar{\delta}(t_0) > 0$ such that $\|\mathbf{x}_0 - \bar{\mathbf{x}}_0\| < \bar{\delta}$ implies that each forward solution $\mathbf{x}(t) \in \mathcal{S}(\boldsymbol{\mathcal{F}}, t_0, \mathbf{x}_0)$ satisfies $\lim_{t\to+\infty} \|\mathbf{x}(t) - \bar{\mathbf{x}}(t)\| = 0$ for all $t \geq t_0$.
- *uniformly attractively stable* if it is uniformly stable and there exists $\bar{\delta} > 0$ (independent of t_0) such that for any $\varepsilon > 0$ there exists $T = T(\varepsilon) > 0$ such that $\|\mathbf{x}_0 - \bar{\mathbf{x}}_0\| < \bar{\delta}$ for $t_0 \in (t_*, +\infty)$ implies that each forward solution $\mathbf{x}(t) \in \mathcal{S}(\boldsymbol{\mathcal{F}}, t_0, \mathbf{x}_0)$ satisfies $\|\mathbf{x}(t) - \bar{\mathbf{x}}(t)\| < \varepsilon$ for all $t \geq t_0 + T$.

Note that if $\bar{\mathbf{x}}(t)$ is a stable solution, then it must be the unique forward solution from $(t_0, \bar{\mathbf{x}}_0)$, i.e.

$\mathcal{S}(\boldsymbol{\mathcal{F}}, t_0, \bar{\mathbf{x}}_0) = \{\bar{\mathbf{x}}(\cdot)\}$. Definitions of global attractive stability and global uniform attractive stability of a solution can be given in a similar way.

Measure differential inclusions

Consider the measure differential inclusion (3), (19)

$$\mathrm{d}\mathbf{x} \in \mathrm{d}\boldsymbol{\Gamma}(t, \mathbf{x}(t)) \tag{A.2}$$

which has been introduced in Sec. 3. As has been discussed in Sec. 3, a solution of (3) is a function of locally bounded variation that fulfills (3) in a measure sense *for all* t. A solution $\mathbf{x}(t)$ of (3) is defined for *almost all* t, i.e. not for a Lebesgue negligible set of time-instant for which the solution $\mathbf{x}(t)$ jumps. A measure differential inclusion usually describes a physical process. Set-valued force-laws restrict the state $\mathbf{x}$ in (3) to some admissible set $\mathcal{A}(t)$. For instant, contact laws and restitution laws prohibit penetration of a unilateral contact in a mechanical system and therefore restrict the position to some admissible set.

Definition 7 (Stability). A solution $\bar{\mathbf{x}}(t)$ of system (3), with $\bar{\mathbf{x}}(t_0) = \bar{\mathbf{x}}_0 \in \mathcal{A}(t_0)$ and which is defined on $t \in (t_*, +\infty)$, is said to be

- *stable* if for any $t_0 \in (t_*, +\infty)$ and $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon, t_0) > 0$ such that $\|\mathbf{x}_0 - \bar{\mathbf{x}}(t_0)\| < \delta$ with $\mathbf{x}_0 \in \mathcal{A}(t_0)$ implies that each forward solution $\mathbf{x}(t) \in \mathcal{S}(\boldsymbol{\mathcal{F}}, t_0, \mathbf{x}_0)$ satisfies $\|\mathbf{x}(t) - \bar{\mathbf{x}}(t)\| < \varepsilon$ for almost all $t \geq t_0$.
- *uniformly stable* if it is stable and the number δ in the definition of stability is independent of t_0.
- *attractively stable* if it is stable and for any $t_0 \in (t_*, +\infty)$ there exists $\bar{\delta} = \bar{\delta}(t_0) > 0$ such that $\|\mathbf{x}_0 - \bar{\mathbf{x}}_0\| < \bar{\delta}$ with $\mathbf{x}_0 \in \mathcal{A}(t_0)$ implies that each forward solution $\mathbf{x}(t) \in \mathcal{S}(\boldsymbol{\mathcal{F}}, t_0, \mathbf{x}_0)$ satisfies $\lim_{t \to +\infty} \|\mathbf{x}(t) - \bar{\mathbf{x}}(t)\| = 0$ for almost all $t \geq t_0$.
- *uniformly attractively stable* if it is uniformly stable and there exists $\bar{\delta} > 0$ (independent of t_0) such that for any $\varepsilon > 0$ there exists $T = T(\varepsilon) > 0$ such that $\|\mathbf{x}_0 - \bar{\mathbf{x}}_0\| < \bar{\delta}$ with $\mathbf{x}_0 \in \mathcal{A}(t_0)$ for $t_0 \in (t_*, +\infty)$ implies that each forward solution $\mathbf{x}(t) \in \mathcal{S}(\boldsymbol{\mathcal{F}}, t_0, \mathbf{x}_0)$ satisfies $\|\mathbf{x}(t) - \bar{\mathbf{x}}(t)\| < \varepsilon$ for almost all $t \geq t_0 + T$.