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Attractivity of Equilibrium Sets of Systems with Dry Friction

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Abstract. The dynamics of mechanical systems with dry friction elements, modelled by set-valued force laws, can be described by differential inclusions. An equilibrium set of such a differential inclusion corresponds to a stationary mode for which the friction elements are sticking. The attractivity properties of the equilibrium set are of major importance for the overall dynamic behaviour of this type of systems. Conditions for the attractivity of the equilibrium set of MDOF mechanical systems with multiple friction elements are presented. These results are obtained by application of a generalisation of LaSalle's principle for differential inclusions of Filippov-type. Besides passive systems, also systems with negative viscous damping are considered. For such systems, only local attractivity of the equilibrium set can be assured under certain conditions. Moreover, an estimate for the region of attraction is given for these cases. The effectiveness of the results is illustrated by means of both 1DOF and MDOF examples.

Keywords: discontinuous systems, Coulomb friction, Lyapunov stability, differential inclusions

1. Introduction

The presence of dry friction can influence the behaviour and performance of mechanical systems as it can induce several phenomena, such as friction-induced limit-cycling, damping of vibrations and stiction. Practical examples of friction-related phenomena are torsional stick-slip vibrations in oil-well drill-strings [1], reduction of vibrations by turbine-blade dampers in airborne gas turbines [2] and stick-slip limit-cycling in controlled mechanical systems [3], such as pick-and-place machines. Stiction plays, in these applications, a crucial role and should therefore be included in the modelling of friction.

Dry friction in mechanical systems is often modelled using set-valued constitutive models [4], such as the set-valued Coulomb's law. Set-valued friction models are a simplification of reality in the sense that each frictional contact is considered to be either in a stick phase or in a slip phase. The transitions between the stick and slip phase are modelled to be instantaneous. Moreover, set-valued friction models have the advantage to properly model stiction, since the friction force is allowed to be non-zero at zero relative velocity.

The dynamics of mechanical systems with set-valued friction laws are described by differential inclusions [5]. We limit ourselves to set-valued friction laws which lead to Filippov-type systems [6], for which existence of solutions is ensured by the Filippov's solution concept. Filippov systems, describing systems with friction, can exhibit equilibrium sets, which correspond to the stiction behaviour of those systems.

The overall dynamics of mechanical systems is largely affected by the stability and attractivity properties of the equilibrium sets. For example, the loss of stability of the equilibrium set can, in certain applications, cause limit-cycling. In [7], a numerical study of the dynamics of a forced pendulum with Coulomb friction shows that the global dynamics of this system is largely influenced by the friction. Moreover, the stability and attractivity properties of the equilibrium set can also seriously affect the performance of control systems [8].

Many publications deal with stability and attractivity properties of (sets of) equilibria in differential inclusions [9–13]. For example, in [9, 10] the attractivity of the equilibrium set of a passive, one-degree-of-freedom friction oscillator with one switching boundary (i.e. one dry friction element) is discussed. Moreover, in [10–12] the Lyapunov stability of an equilibrium point in the equilibrium set is shown. Most papers are limited to either one-degree-of-freedom systems or to systems exhibiting only one switching boundary.

We will provide conditions under which the equilibrium set is attractive for multi-degree-of-freedom mechanical systems with an arbitrary number of Coulomb friction elements using Lyapunov-type stability analysis and a generalisation of LaSalle's invariance principle for non-smooth systems. Moreover, passive as well as non-passive systems will be considered. In Appendix A, the global asymptotic stability of a passive one-degree-of-freedom oscillator will be proven. The non-passive systems that will be studied are linear mechanical systems with a non-positive definite damping matrix with additional dry friction elements. The non-positive-definiteness of the damping matrix of linearised systems can be caused by fluid, aeroelastic, control and gyroscopical forces, which can cause instabilities such as flutter vibrations of airfoils [14], shimmying of wheels in vehicle systems [15] or flutter instabilities of fluid-conveying tubes [16]. It will be demonstrated in this paper that the presence of dry friction in such an unstable linear system can (conditionally) ensure the local attractivity of the equilibrium set of the resulting system with dry friction. Moreover, an estimate of the region of attraction for the equilibrium set will be given. A rigid multibody approach will be used for the description of mechanical systems with friction, which allows for a natural physical interpretation of the conditions for attractivity.

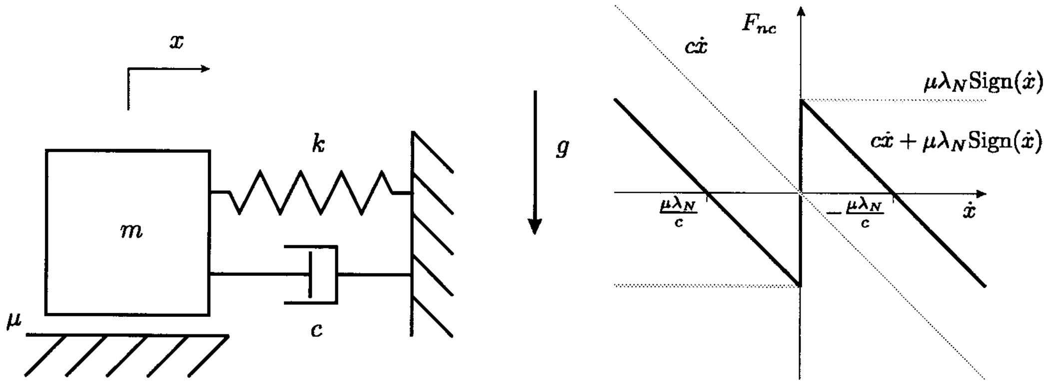
The conservativeness of the estimates of the region of attraction is evaluated by means of numerical simulation. Hereto, an event-driven integration method was used as described in [17]. The event-driven integration method is a hybrid integration technique that uses a standard ODE solver for the integration of smooth phases of the system dynamics and a LCP (Linear Complementarity Problem) formulation to determine the next hybrid mode at the switching boundaries.

In Section 2, a motivating example is presented which illuminates the basic idea of the theorems stated in this paper. In Section 3, the equations of motion for mechanical systems with frictional elements are formulated and the equilibrium set is defined. Subsequently, the attractivity properties of the equilibrium set are studied in Sections 4 and 5 by means of a generalisation of LaSalle's invariance principle. In Section 6, a number of examples are studied in order to illustrate the theoretical results and to investigate the correspondence between the estimated and actual region of attraction. Moreover, these examples demonstrate under which conditions the equilibrium set is (locally) attractive. Finally, a discussion of the obtained results and concluding remarks are given in Section 7.

2. Motivation

Consider a 1DOF mass-spring-damper system with dry friction, as depicted in Figure 1. The system is described by the differential inclusion:

$$m\ddot{x} + c\dot{x} + kx \in -\mu\lambda_N \mathrm{Sign}(\dot{x}), \tag{1}$$



(a) 1DOF system. (b) Non-conservative forces.

Figure 1. 1DOF mass spring damper system with Coulomb friction.

where $\lambda_N = mg$ is the normal contact force, μ is the friction coefficient and m, k and c are the mass, spring and damping constants, respectively. We used the set-valued sign-function

$$\mathrm{Sign}(x) = \begin{cases} \{-1\} & x < 0, \\ [-1, 1] & x = 0, \\ \{1\} & x > 0, \end{cases} \tag{2}$$

which is set-valued at $x = 0$. The global asymptotic stability of the equilibrium set of system (1) for $c \geq 0$ is proven in Appendix A. Let us consider the case that the damping constant $c < 0$. Clearly, the equilibrium point of the system without friction ($\mu = 0$) is unstable. The question rises whether the equilibrium set $\mathcal{E}$ of the system with friction ($\mu > 0$), given by

$$\mathcal{E} = \{(x, \dot{x}) \mid (\dot{x} = 0) \wedge kx \in -\mu\lambda_N \mathrm{Sign}(0)\}, \tag{3}$$

can still be (locally) attractive. In Figure 1b, the non-conservative forces F_{nc} working on the system are shown as a function of the velocity (in gray). Moreover, the net non-conservative force $c\dot{x} + \mu\lambda_N \mathrm{Sign}(\dot{x})$ is depicted in black. Clearly, in the set defined by $\dot{x} \in \{\dot{x} \mid \mu mg/c < \dot{x} < -\mu mg/c\}$, this net non-conservative force is dissipative. Consequently, one can expect that, for initial conditions sufficiently close to the equilibrium set, solutions are attracted to the equilibrium set.

In the next sections, this idea will be used to derive the attractivity properties of the equilibrium set of mechanical systems with an arbitrary number of degrees of freedom and an arbitrary number of friction elements.

3. Modelling of Mechanical Systems with Coulomb Friction

In this section, we will formulate the equations of motion for mechanical systems with frictional translational joints. These translational joints restrict the motion of the system to a manifold determined by the bilateral holonomic constraint equations imposed by these joints (sliders). Coulomb's friction law is assumed to hold in the tangential direction of the manifold.

Let us consider an autonomous mechanical system with dry friction in the following variational form:

$$\delta z^{\mathrm{T}} \left(\boldsymbol{M}(z)\, \ddot{z} - \boldsymbol{h}(z, \dot{z}) - \boldsymbol{W}_T(z) \boldsymbol{\lambda}_T \right) = 0, \tag{4}$$

for all δz which satisfy the holonomic constraints

$$\boldsymbol{W}_N^{\mathrm{T}}(z)\delta z = \mathbf{0}. \tag{5}$$

Herein, z is column of dependent generalised coordinates, $\boldsymbol{M}$ is the symmetric, positive-definite mass-matrix, $\boldsymbol{h}$ is the column with state-dependent forces, $\boldsymbol{\lambda}_T$ are the friction forces and $\boldsymbol{W}_T$ contains the related generalised force directions as columns. The dependent generalised coordinates z must satisfy bilateral, holonomic constraints, imposed by the sliders, of the form

$$\boldsymbol{g}_N(z) = \mathbf{0}. \tag{6}$$

These constraint equations can also be expressed in the variational form (5) with $\boldsymbol{W}_N^{\mathrm{T}} = \partial \boldsymbol{g}_N/\partial z$.

In (4), the friction force λ_{T_i} at the contact point in each slider i obeys the following Coulomb's set-valued friction law:

$$\lambda_{T_i} \in -\mu_i|\lambda_{N_i}|\mathrm{Sign}(\dot{g}_{T_i}), \tag{7}$$

where $\mu_i \geq 0$ is the friction coefficient, λ_{N_i} is the normal contact force and $\dot{g}_{T_i}$ is the relative velocity of the bodies interconnected by slider i. We used the set-valued sign-function as defined in (2). The tangential friction forces of all m contacts are gathered in a column $\boldsymbol{\lambda}_T = \{\lambda_{T_i}\}$ and the corresponding tangential relative velocities are gathered in a column $\dot{\boldsymbol{g}}_T = \{\dot{g}_{T_i}\}$, for $i = 1, \ldots, m$. We can therefore write the friction law as

$$\boldsymbol{\lambda}_T \in -\boldsymbol{\Lambda}\mathrm{Sign}(\dot{\boldsymbol{g}}_T), \tag{8}$$

with $\boldsymbol{\Lambda} = \mathrm{diag}([\ \mu_1|\lambda_{N_1}|\ \ \ldots\ \ \mu_m|\lambda_{N_m}|\ ])$. It should be noted that $\boldsymbol{W}_T^{\mathrm{T}} = \partial\dot{\boldsymbol{g}}_T/\partial\dot{z}$.

According to the Lagrangian multiplier theorem, we can introduce a column of Lagrangian multipliers $\boldsymbol{\lambda}_N$ such that

$$\delta z^{\mathrm{T}}\left(\boldsymbol{M}(z)\ddot{z} - \boldsymbol{h}(z,\dot{z}) - \boldsymbol{W}_N(z)\boldsymbol{\lambda}_N - \boldsymbol{W}_T(z)\boldsymbol{\lambda}_T\right) = 0, \quad \forall \delta z, \tag{9}$$

which is known as the virtual work equation. We can interpret $\boldsymbol{\lambda}_N$ as the normal constraint forces ensuring the satisfaction of the constraints (6).

Let us assume that we know an independent set of generalised coordinates $\boldsymbol{q}$ such that $z(\boldsymbol{q})$ is known and satisfies $\boldsymbol{g}_N(z(\boldsymbol{q})) = \mathbf{0}$. The variation of the dependent set of coordinates z, which satisfies the constraints (6), is denoted by the admissible variations δz_a, i.e.

$$\delta z_a = \boldsymbol{T}\delta\boldsymbol{q}, \tag{10}$$

with $\boldsymbol{T}(z) = \partial z/\partial\boldsymbol{q}$. We introduce columns $\tilde{\boldsymbol{\kappa}}$ and $\bar{\boldsymbol{\kappa}}$ such that we can express the generalised velocities and accelerations as

$$\dot{z} = \boldsymbol{T}\dot{\boldsymbol{q}} + \tilde{\boldsymbol{\kappa}}, \quad \ddot{z} = \boldsymbol{T}\ddot{\boldsymbol{q}} + \bar{\boldsymbol{\kappa}}, \tag{11}$$

respectively, where $\tilde{\boldsymbol{\kappa}}(z) = \partial z/\partial t$ and $\bar{\boldsymbol{\kappa}}(z,\dot{z}) = \dot{\boldsymbol{T}}\dot{\boldsymbol{q}} + \dot{\tilde{\boldsymbol{\kappa}}}$. The virtual work equation (9) holds for all δz implying that it also holds for admissible variations δz_a. Consequently, using (10) and (11):

$$\delta\boldsymbol{q}^{\mathrm{T}}(\boldsymbol{T}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{T}\ddot{\boldsymbol{q}} - \boldsymbol{T}^{\mathrm{T}}\boldsymbol{h} - \boldsymbol{T}^{\mathrm{T}}\boldsymbol{W}_N\boldsymbol{\lambda}_N - \boldsymbol{T}^{\mathrm{T}}\boldsymbol{W}_T\boldsymbol{\lambda}_T + \boldsymbol{T}^{\mathrm{T}}\boldsymbol{M}\bar{\boldsymbol{\kappa}}) = 0, \quad \forall\delta\boldsymbol{q}. \tag{12}$$

Note that $\boldsymbol{M} = \boldsymbol{M}(\boldsymbol{z}(\boldsymbol{q}))$, $\boldsymbol{h} = \boldsymbol{h}(\boldsymbol{z}(\boldsymbol{q}), \dot{\boldsymbol{z}}(\boldsymbol{q}, \dot{\boldsymbol{q}}))$, $\boldsymbol{T} = \boldsymbol{T}(\boldsymbol{z}(\boldsymbol{q}))$, $\boldsymbol{W}_N = \boldsymbol{W}_N(\boldsymbol{z}(\boldsymbol{q}))$ and $\boldsymbol{W}_T = \boldsymbol{W}_T(\boldsymbol{z}(\boldsymbol{q}))$. Using the fact that $\boldsymbol{T}^{\mathrm{T}}\boldsymbol{W}_N = \boldsymbol{0}$, which follows from $\boldsymbol{W}_N^{\mathrm{T}}\delta \boldsymbol{z}_a = \boldsymbol{W}_N^{\mathrm{T}}\boldsymbol{T}\delta\boldsymbol{q} = \boldsymbol{0}, \forall\delta\boldsymbol{q}$ (note that this follows from the variational constraint equations (5)), we arrive at the following equation of motion in the independent generalised coordinates $\boldsymbol{q}$:

$$\bar{\boldsymbol{M}}\ddot{\boldsymbol{q}} - \bar{\boldsymbol{h}} - \bar{\boldsymbol{W}}_T\boldsymbol{\lambda}_T = \boldsymbol{0}, \tag{13}$$

with

$$\begin{aligned}
\bar{\boldsymbol{M}} &= \boldsymbol{T}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{T},\\
\bar{\boldsymbol{h}} &= \boldsymbol{T}^{\mathrm{T}}\boldsymbol{h} - \boldsymbol{T}^{\mathrm{T}}\boldsymbol{M}\bar{\boldsymbol{\kappa}},\\
\bar{\boldsymbol{W}}_T &= \boldsymbol{T}^{\mathrm{T}}\boldsymbol{W}_T.
\end{aligned} \tag{14}$$

We will adopt the following assumptions:

1. linearity of $\boldsymbol{h}$: $\boldsymbol{h}(\boldsymbol{z}, \dot{\boldsymbol{z}}) = -\boldsymbol{C}\dot{\boldsymbol{z}} - \boldsymbol{K}\boldsymbol{z} + \boldsymbol{f}$;
2. $\boldsymbol{z}(\boldsymbol{q}) = \boldsymbol{T}\boldsymbol{q}$ where $\boldsymbol{T}$ is a constant matrix. This implies that $\tilde{\boldsymbol{\kappa}} = \boldsymbol{0}$ and $\bar{\boldsymbol{\kappa}} = \boldsymbol{0}$;
3. $\boldsymbol{W}_T$ is a constant matrix;
4. $\dot{\boldsymbol{g}}_T = \boldsymbol{W}_T^{\mathrm{T}}\dot{\boldsymbol{z}}$.

The equilibrium point of (13) without friction ($\boldsymbol{\lambda}_T = \boldsymbol{0}$) is denoted by $\boldsymbol{q}_{\mathrm{eq}}$, i.e. $\boldsymbol{K}\boldsymbol{T}\boldsymbol{q}_{\mathrm{eq}} = \boldsymbol{f}$. Consequently, we can introduce new coordinates $\bar{\boldsymbol{q}} = \boldsymbol{q} - \boldsymbol{q}_{\mathrm{eq}}$ such that

$$\bar{\boldsymbol{M}}\ddot{\bar{\boldsymbol{q}}} + \bar{\boldsymbol{C}}\dot{\bar{\boldsymbol{q}}} + \bar{\boldsymbol{K}}\bar{\boldsymbol{q}} - \bar{\boldsymbol{W}}_T\boldsymbol{\lambda}_T = \boldsymbol{0}, \tag{15}$$

in which

$$\bar{\boldsymbol{K}} = \boldsymbol{T}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{T}, \quad \bar{\boldsymbol{C}} = \boldsymbol{T}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{T}. \tag{16}$$

Note that from assumption 4 follows that $\dot{\boldsymbol{g}}_T = \boldsymbol{W}_T^{\mathrm{T}}\boldsymbol{T}\dot{\boldsymbol{q}} = \bar{\boldsymbol{W}}_T^{\mathrm{T}}\dot{\boldsymbol{q}} = \bar{\boldsymbol{W}}_T^{\mathrm{T}}\dot{\bar{\boldsymbol{q}}}$, since $\dot{\bar{\boldsymbol{q}}} = \dot{\boldsymbol{q}}$. Equation (15) together with a set-valued friction law (8) forms a differential inclusion. Differential inclusions of this type are called Filippov systems which obey Filippov's solution concept (Filippov's convex method). Consequently, the existence of solutions of system (15) is guaranteed. Moreover, due to the fact that $\mu_i \geq 0$, $i = 1, \ldots, m$, which excludes the possibility of repulsive sliding modes along the switching boundaries, also uniqueness of solutions in forward time is guaranteed [5].

If we consider Equation (13) with friction, then the equilibrium point of the system without friction is replaced by an equilibrium set. Note that $\dot{\bar{\boldsymbol{q}}} = \boldsymbol{0}$ implies $\dot{\boldsymbol{g}}_T = \boldsymbol{0}$, see assumption 4. This means that every equilibrium implies sticking in all contact points. Every equilibrium position has to obey the equilibrium inclusion:

$$\bar{\boldsymbol{K}}\bar{\boldsymbol{q}} + \bar{\boldsymbol{W}}_T\boldsymbol{\Lambda}\mathrm{Sign}(\boldsymbol{0}) \ni \boldsymbol{0}. \tag{17}$$

The equilibrium set is therefore given by

$$\mathcal{E} = \left\{(\bar{\boldsymbol{q}}, \dot{\bar{\boldsymbol{q}}}) \in \mathbb{R}^{2n} \mid (\dot{\bar{\boldsymbol{q}}} = \boldsymbol{0}) \wedge \bar{\boldsymbol{q}} \in -\bar{\boldsymbol{K}}^{-1}\bar{\boldsymbol{W}}_T\boldsymbol{\Lambda}\mathrm{Sign}(\boldsymbol{0})\right\} \tag{18}$$

and is positively invariant due to the uniqueness of the solutions in forward time. A set is invariant when for every initial condition in this set the solutions remain in this set for all time. Moreover, a set is positively invariant when it is invariant in forward time.

4. Attractivity Analysis of the Equilibrium Set

In this section, we study the attractivity properties of the equilibrium set $\mathcal{E}$ given by (18) of system (15). We will use LaSalle's principle, as described in [18], but applied to Filippov systems with uniqueness of solutions in forward time:

THEOREM 1 (LaSalle's Principle). *Let $\boldsymbol{\phi}(t, \boldsymbol{x}_0, t_0)$ denote a solution of a dynamical system with existence and uniqueness (in forward time) of solutions. Herein, t denotes time and $\boldsymbol{x}_0$ is an initial condition at time t_0. Let $V(\boldsymbol{x})$ be a continuously differentiable and positive definite function and suppose that*

$$\mathcal{I}_\rho = \{\boldsymbol{x} \mid V(\boldsymbol{x}) \leq \rho\} \tag{19}$$

is bounded and that $\dot{V} \leq 0$ for all $\boldsymbol{x} \in \mathcal{I}_\rho$. Define $\mathcal{S} \subset \mathcal{I}_\rho$ by

$$\mathcal{S} = \{\boldsymbol{x} \in \mathcal{I}_\rho \mid \dot{V}(\boldsymbol{x}) = 0\} \tag{20}$$

and let $\mathcal{M}$ be the largest positively invariant set in $\mathcal{S}$. Then, whenever $\boldsymbol{x}_0 \in \mathcal{I}_\rho$, the solution $\boldsymbol{\phi}(t, \boldsymbol{x}_0, t_0)$ approaches $\mathcal{M}$ as $t \to \infty$.

Note that the above theorem was originally stated for systems satisfying the Lipschitz condition. However, it can be naturally extended to non-smooth systems of Filippov-type with existence and uniqueness of solutions (in forward time). The Lipschitz condition is only necessary in the proof of the original LaSalle's principle to ensure existence and uniqueness of solutions. For the class of systems studied in this paper, the existence and uniqueness of solutions (in forward time) is ensured by the assumption that the systems are of Filippov-type without repulsive sliding modes. In the original theorem of LaSalle, the set $\mathcal{M}$ is an invariant set (in both positive and negative time). In the theorem stated above, merely invariance of $\mathcal{M}$ in positive time is required.

Let us first consider the stability of linear systems with friction and positive definite matrices $\bar{\boldsymbol{M}}$, $\bar{\boldsymbol{K}}$ and $\bar{\boldsymbol{C}}$. Note that this implies that the equilibrium point of the linear system without friction is globally asymptotically stable. We can prove that the equilibrium set of the system with friction is globally attractive.

THEOREM 2 (Global attractivity of an equilibrium set). *Consider system (15) with friction law (8). If matrices $\bar{\boldsymbol{M}}$, $\bar{\boldsymbol{K}}$ and $\bar{\boldsymbol{C}}$ are positive definite, then the equilibrium set (18) is globally attractive.*

Proof. We consider a positive definite function

$$V = \frac{1}{2}\dot{\bar{\boldsymbol{q}}}^{\mathrm{T}} \bar{\boldsymbol{M}} \dot{\bar{\boldsymbol{q}}} + \frac{1}{2}\bar{\boldsymbol{q}}^{\mathrm{T}} \bar{\boldsymbol{K}} \bar{\boldsymbol{q}}. \tag{21}$$

The time-derivative of V is, using friction law (8) and assumption 4,

$$\begin{aligned} \dot{V} &= \dot{\bar{\boldsymbol{q}}}^{\mathrm{T}} \left(-\bar{\boldsymbol{C}}\dot{\bar{\boldsymbol{q}}} - \bar{\boldsymbol{K}}\bar{\boldsymbol{q}} + \bar{\boldsymbol{W}}_T \boldsymbol{\lambda}_T\right) + \dot{\bar{\boldsymbol{q}}}^{\mathrm{T}} \bar{\boldsymbol{K}} \bar{\boldsymbol{q}} \\ &= -\dot{\bar{\boldsymbol{q}}}^{\mathrm{T}} \bar{\boldsymbol{C}} \dot{\bar{\boldsymbol{q}}} + \dot{\bar{\boldsymbol{q}}}^{\mathrm{T}} \bar{\boldsymbol{W}}_T \boldsymbol{\lambda}_T \\ &= -\dot{\bar{\boldsymbol{q}}}^{\mathrm{T}} \bar{\boldsymbol{C}} \dot{\bar{\boldsymbol{q}}} - \dot{\bar{\boldsymbol{q}}}^{\mathrm{T}} \bar{\boldsymbol{W}}_T \boldsymbol{\Lambda} \operatorname{Sign}(\dot{\boldsymbol{g}}_T) \\ &= -\dot{\bar{\boldsymbol{q}}}^{\mathrm{T}} \bar{\boldsymbol{C}} \dot{\bar{\boldsymbol{q}}} - \dot{\boldsymbol{g}}_T^{\mathrm{T}} \boldsymbol{\Lambda} \operatorname{Sign}(\dot{\boldsymbol{g}}_T) \\ &= -\dot{\bar{\boldsymbol{q}}}^{\mathrm{T}} \bar{\boldsymbol{C}} \dot{\bar{\boldsymbol{q}}} - \boldsymbol{p}^{\mathrm{T}} |\dot{\boldsymbol{g}}_T|, \end{aligned} \tag{22}$$

where the column $\boldsymbol{p}$ and $|\dot{\boldsymbol{g}}_T|$ are defined by

$$\boldsymbol{p} = \{\Lambda_{ii}\}, \quad |\dot{\boldsymbol{g}}_T| = \{|\dot{g}_{T_i}|\}, \quad \text{for } i = 1, \ldots, m. \tag{23}$$

It can be seen from (22) that $\dot{V}$ is a continuous single-valued function (of $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$). It holds that $\boldsymbol{p} \geq \boldsymbol{0}$ and by using assumption 4 that if $\dot{\bar{\boldsymbol{q}}} = \boldsymbol{0}$ then $\dot{\boldsymbol{g}}_T = \boldsymbol{0}$. It therefore holds that

$$\begin{aligned} &\dot{V} = 0 \quad \text{if and only if} \quad \dot{\bar{\boldsymbol{q}}} = \boldsymbol{0} \\ &\dot{V} < 0 \quad \text{for} \quad \dot{\bar{\boldsymbol{q}}} \neq \boldsymbol{0}. \end{aligned} \tag{24}$$

We now apply LaSalle's theorem (Theorem 1) for Filippov-type systems with existence and uniqueness of solutions (in forward time). Herein, $\mathcal{L}_\rho$ is defined by (19), which is a positively invariant set due to the choice of V, and ρ is chosen such that the equilibrium set $\mathcal{E}$ is contained in $\mathcal{L}_\rho$. Moreover, $\mathcal{S} = \{(\bar{\boldsymbol{q}}, \dot{\bar{\boldsymbol{q}}}) \mid \dot{\bar{\boldsymbol{q}}} = 0\}$, which follows from (24). The largest invariant set in $\mathcal{S}$ is $\mathcal{E}$. Therefore, it can be concluded that $\mathcal{E}$ is an attractive set. Since ρ can be taken arbitrarily large, global attractivity of $\mathcal{E}$ is assured. □

THEOREM 3 (Local attractivity of a subset of the equilibrium set). *Consider system (15) with friction law (8). If the matrices $\bar{\boldsymbol{M}}$, $\bar{\boldsymbol{K}}$ are positive definite and the matrix $\bar{\boldsymbol{C}}$ is not positive definite but symmetric, then a convex subset of the equilibrium set (18) is locally attractive under the following condition: $\boldsymbol{U}_{c_i} \in$ span$\{\bar{\boldsymbol{W}}_T\}$ for $i = 1, \ldots, n_q$, where $\boldsymbol{U}_c = \{\boldsymbol{U}_{c_i}\}$ is a matrix containing the n_q eigencolumns corresponding to the eigenvalues of $\bar{\boldsymbol{C}}$, which lie in the closed left-half complex plane.*

Proof. We take V as defined in (21) and we arrive at the following expression for the time-derivative of V:

$$\dot{V} = -\dot{\bar{\boldsymbol{q}}}^{\mathrm{T}}\bar{\boldsymbol{C}}\dot{\bar{\boldsymbol{q}}} - \boldsymbol{p}^{\mathrm{T}}|\bar{\boldsymbol{W}}_T^{\mathrm{T}}\dot{\bar{\boldsymbol{q}}}|. \tag{25}$$

We now apply a spectral decomposition of $\bar{\boldsymbol{C}} = \boldsymbol{U}_c^{-\mathrm{T}}\boldsymbol{\Omega}_c\boldsymbol{U}_c^{-1}$, where $\boldsymbol{U}_c$ is the matrix containing all eigencolumns and $\boldsymbol{\Omega}_c$ is the diagonal matrix containing all eigenvalues of $\bar{\boldsymbol{C}}$, which are real. Moreover, we introduce coordinates $\boldsymbol{\eta}$ such that $\bar{\boldsymbol{q}} = \boldsymbol{U}_c\boldsymbol{\eta}$. Consequently, $\dot{V}$ satisfies

$$\begin{aligned} \dot{V} &= -\dot{\bar{\boldsymbol{q}}}^{\mathrm{T}}\boldsymbol{U}_c^{-\mathrm{T}}\boldsymbol{\Omega}_c\boldsymbol{U}_c^{-1}\dot{\bar{\boldsymbol{q}}} - \boldsymbol{p}^{\mathrm{T}}|\bar{\boldsymbol{W}}_T^{\mathrm{T}}\dot{\bar{\boldsymbol{q}}}| \\ &= -\dot{\boldsymbol{\eta}}^{\mathrm{T}}\boldsymbol{\Omega}_c\dot{\boldsymbol{\eta}} - \boldsymbol{p}^{\mathrm{T}}|\bar{\boldsymbol{W}}_T^{\mathrm{T}}\boldsymbol{U}_c\dot{\boldsymbol{\eta}}|. \end{aligned} \tag{26}$$

The matrix $\bar{\boldsymbol{C}}$ has n_q eigenvalues in the closed left-half complex plane; all other eigenvalues lie in the open right-half complex plane. Consequently, $\dot{V}$ obeys the inequality

$$\dot{V} \leq -\sum_{i=1}^{n_q} \lambda_i \dot{\eta}_i^2 - \boldsymbol{p}^{\mathrm{T}}|\bar{\boldsymbol{W}}_T^{\mathrm{T}}\boldsymbol{U}_c\dot{\boldsymbol{\eta}}|, \quad \forall\, \dot{\boldsymbol{\eta}}, \tag{27}$$

where we assumed that the eigenvalues (and eigencolumns) of $\bar{\boldsymbol{C}}$ are ordered in such a manner that λ_i, $i = 1, \ldots, n_q$, correspond to the eigenvalues of $\bar{\boldsymbol{C}}$ in the closed left-half complex plane. Assume that $\exists\, \alpha > 0$ such that

$$\sum_{i=1}^{n_q} |\boldsymbol{e}_i^{\mathrm{T}}\dot{\boldsymbol{\eta}}| \leq \alpha\boldsymbol{p}^{\mathrm{T}}|\bar{\boldsymbol{W}}_T^{\mathrm{T}}\boldsymbol{U}_c\dot{\boldsymbol{\eta}}|, \quad \forall\, \dot{\boldsymbol{\eta}}. \tag{28}$$

Herein, $\boldsymbol{e}_i$ is a unit-column with a the non-zero element on the i-th position. Assuming that such an α can be found, (27) results in

$$\dot{V} \leq -\sum_{i=1}^{n_q} \lambda_i \dot{\eta}_i^2 - \beta \sum_{i=1}^{n_q} |\dot{\eta}_i| \leq 0, \quad \forall\, \dot{\boldsymbol{\eta}} \in \left\{ \dot{\boldsymbol{\eta}} \mid \frac{\beta}{\lambda_i} \leq \dot{\eta}_i \leq -\frac{\beta}{\lambda_i},\ i = 1, \ldots, n_q \right\}, \quad (29)$$

with $\beta = 1/\alpha$ and $\dot{\eta}_i = \boldsymbol{e}_i^{\mathrm{T}} \dot{\boldsymbol{\eta}}$. Note, moreover, that if

$$\boldsymbol{e}_i \in \mathrm{span}\{\boldsymbol{U}_c^{\mathrm{T}} \bar{\boldsymbol{W}}_T\}, \quad \forall\, i \in [1, \ldots, n_q],$$

then $\exists\, \boldsymbol{\gamma}^{\mathrm{T}}$ such that

$$\boldsymbol{e}_i^{\mathrm{T}} = \boldsymbol{\gamma}^{\mathrm{T}} \bar{\boldsymbol{W}}_T^{\mathrm{T}} \boldsymbol{U}_c.$$

It therefore holds that

$$|\boldsymbol{e}_i^{\mathrm{T}} \dot{\boldsymbol{\eta}}| = |\boldsymbol{\gamma}^{\mathrm{T}} \bar{\boldsymbol{W}}_T^{\mathrm{T}} \boldsymbol{U}_c \dot{\boldsymbol{\eta}}|, \quad |\boldsymbol{e}_i^{\mathrm{T}} \dot{\boldsymbol{\eta}}| \leq |\boldsymbol{\gamma}^{\mathrm{T}}||\bar{\boldsymbol{W}}_T^{\mathrm{T}} \boldsymbol{U}_c \dot{\boldsymbol{\eta}}|.$$

Choose the smallest $\tilde{\alpha}_i$ such that $|\boldsymbol{\gamma}^{\mathrm{T}}| \leq \tilde{\alpha}_i \boldsymbol{p}^{\mathrm{T}}$, where the sign $\leq$ has to be understood component-wise. Then it holds that

$$|\boldsymbol{e}_i^{\mathrm{T}} \dot{\boldsymbol{\eta}}| \leq \tilde{\alpha}_i \boldsymbol{p}^{\mathrm{T}} |\bar{\boldsymbol{W}}_T^{\mathrm{T}} \boldsymbol{U}_c \dot{\boldsymbol{\eta}}|, \quad \forall\, \dot{\boldsymbol{\eta}}, \quad \forall\, i \in \left[1, \ldots, n_q\right].$$

Note that α in (28) can be taken as $\alpha = \sum_{i=1}^{n_q} \tilde{\alpha}_i$. Finally, one should realise that if and only if

$$\boldsymbol{U}_c \boldsymbol{e}_i \in \mathrm{span}\{\boldsymbol{U}_c \boldsymbol{U}_c^{\mathrm{T}} \bar{\boldsymbol{W}}_T\}, \quad (30)$$

or, in other words, if the i-th column $\boldsymbol{U}_{c_i}$ of $\boldsymbol{U}_c$ satisfies

$$\boldsymbol{U}_{c_i} \in \mathrm{span}\{\bar{\boldsymbol{W}}_T\}$$

(note in this respect that $\boldsymbol{U}_c$ is real and symmetric), then it holds that

$$\boldsymbol{e}_i \in \mathrm{span}\{\boldsymbol{U}_c^{\mathrm{T}} \bar{\boldsymbol{W}}_T\}.$$

Therefore, a sufficient condition for the validity of (29) can be given by

$$\boldsymbol{U}_{c_i} \in \mathrm{span}\{\bar{\boldsymbol{W}}_T\}, \quad \forall\, i \in \left[1, \ldots, n_q\right]. \quad (31)$$

Now, we can once more apply Theorem 1 (LaSalle's Principle). Let us, hereto, define a set $\mathcal{C}$ by

$$\mathcal{C} = \left\{ (\bar{\boldsymbol{q}}, \dot{\bar{\boldsymbol{q}}}) \mid \left| \left(\boldsymbol{U}_c^{-1} \dot{\bar{\boldsymbol{q}}}\right)_i \right| \leq -\frac{\beta}{\lambda_i},\ i = 1, \ldots, n_q \right\}, \quad (32)$$

where $\left(\boldsymbol{U}_c^{-1} \dot{\bar{\boldsymbol{q}}}\right)_i$ denotes the i-th element of the column $\boldsymbol{U}_c^{-1} \dot{\bar{\boldsymbol{q}}}$. The constant ρ (which defines the set $\mathcal{I}_\rho$ in (19)) is chosen such that $\mathcal{I}_\rho \subset \mathcal{C}$. Moreover, the set $\mathcal{S} \subset \mathcal{I}_\rho$ is given by $\mathcal{S} = \left\{ (\bar{\boldsymbol{q}}, \dot{\bar{\boldsymbol{q}}}) \in \mathcal{I}_\rho : \dot{\bar{\boldsymbol{q}}} = \boldsymbol{0} \right\}$. Furthermore, the largest invariant set in $\mathcal{S}$ is a subset $\tilde{\mathcal{E}}$ of the equilibrium set $\mathcal{E}$, where $\tilde{\mathcal{E}} = \mathcal{E} \bigcap \mathrm{int}(\mathcal{I}_{\rho^*})$ and

$$\rho^* = \max_{\{\rho : \mathcal{I}_\rho \subset \mathcal{C}\}} \rho. \quad (33)$$

This concludes the proof of the local attractivity of $\tilde{\mathcal{E}}$ under condition (31). □

The following theorem states the conditions under which the entire equilibrium set $\mathcal{E}$ is locally attractive:

THEOREM 4 (Local attractivity of the equilibrium set). *Consider system (15) with friction law (8). If the equilibrium set $\mathcal{E}$ is contained in the interior of the set $\mathcal{I}_{\rho^*}$, then under the conditions of Theorem 3, $\mathcal{E}$ is locally attractive.*

Proof. The proof follows directly from the proof of Theorem 3, with $\mathcal{E} = \tilde{\mathcal{E}}$. □

At this point several remarks should be made:

- It should be noted that the proof of Theorem 3 provides us with a conservative estimate of the region of attraction $\mathcal{A}$ of the locally attractive equilibrium set $\mathcal{E}$. The estimate $\mathcal{B}$ can be formulated in terms of the generalised displacements and velocities:

$$\mathcal{B} = \mathcal{I}_{\rho^*}, \tag{34}$$

  where ρ^* satisfies (33), the set $\mathcal{I}_\rho = \left\{(\bar{\boldsymbol{q}}, \dot{\bar{\boldsymbol{q}}}) \mid V(\bar{\boldsymbol{q}}, \dot{\bar{\boldsymbol{q}}}) \leq \rho\right\}$, the set $\mathcal{C}$ is given by (32) and V is given by (21); In Section 5, we will give an explicit expression for ρ^*.
- The proofs of Theorems 2 and 3 also show that boundedness of solutions (starting in $\mathcal{B}$) is ensured.
- The proofs of Theorems 2 and 3 also show that the equilibrium *point* $(\boldsymbol{q}, \dot{\boldsymbol{q}}) = (\boldsymbol{0}, \boldsymbol{0})$ is Lyapunov stable;
- It can be shown that if it holds that $\boldsymbol{\Lambda}^{\mathrm{T}} \bar{\boldsymbol{W}}_T^{\mathrm{T}} \bar{\boldsymbol{K}}^{-\mathrm{T}} \bar{\boldsymbol{W}}_T \boldsymbol{\Lambda} < 2\rho^*$, then $\mathcal{E} \subset \mathcal{I}_{\rho^*}$.

5. Estimation of the Convergence Region

The value of ρ^*, introduced in the previous section, characterises the size of the estimate $\mathcal{B}$ of the region of attraction of (a subset of) the equilibrium set. Moreover, when only the attractivity of a subset $\tilde{\mathcal{E}}$ can be shown, by application of Theorem 3, then ρ^* defines $\tilde{\mathcal{E}}$ through $\tilde{\mathcal{E}} = \mathcal{E} \bigcap \mathrm{int}(\mathcal{I}_{\rho^*})$. An explicit expression for ρ^* can be found, as is shown in the sequel. Consider the positive definite function V as in (21) and a spectral decomposition of the damping matrix $\bar{\boldsymbol{C}} = \boldsymbol{U}_c^{-\mathrm{T}} \boldsymbol{\Omega}_c \boldsymbol{U}_c^{-1}$, i.e. $\bar{\boldsymbol{q}} = \boldsymbol{U}_c \boldsymbol{\eta}$. The function V can be written as

$$V = \frac{1}{2} \boldsymbol{x}^{\mathrm{T}} \boldsymbol{P} \boldsymbol{x}, \tag{35}$$

with $\boldsymbol{x}^{\mathrm{T}} = [\ \boldsymbol{\eta}^{\mathrm{T}} \ \ \dot{\boldsymbol{\eta}}^{\mathrm{T}}\ ]$ and

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{U}_c^{\mathrm{T}} \bar{\boldsymbol{K}} \boldsymbol{U}_c & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{U}_c^{\mathrm{T}} \bar{\boldsymbol{M}} \boldsymbol{U}_c \end{bmatrix}. \tag{36}$$

The value ρ^* is the lowest value of ρ for which the set

$$\mathcal{I}_\rho = \left\{ \boldsymbol{x} \mid \frac{1}{2} \boldsymbol{x}^{\mathrm{T}} \boldsymbol{P} \boldsymbol{x} \leq \rho \right\}$$

touches one of the hyperplanes of $\partial \mathcal{C}$. We define ρ_i, $i = 1, \ldots, n_q$, to be that value of ρ for which the set

$$\mathcal{I}_\rho = \left\{ \boldsymbol{x} \mid \frac{1}{2} \boldsymbol{x}^{\mathrm{T}} \boldsymbol{P} \boldsymbol{x} \leq \rho \right\}$$

touches the hyperplane $|\dot{\eta}_i| = -\beta/\lambda_i$. Accordingly, ρ^* is defined by

$$\rho^* = \min_{i=1,\dots,n_q} \rho_i. \tag{37}$$

Equating the hyperplane $|\dot{\eta}_i| = -\frac{\beta}{\lambda_i}$ with $\partial \mathcal{I}_{\rho_i}$ gives the relation

$$\sup_{(1/2)\|\boldsymbol{x}\|_{\boldsymbol{P}}^2=\rho_i} |\dot{\eta}_i| = -\frac{\beta}{\lambda_i}, \tag{38}$$

where $\|\boldsymbol{x}\|_{\boldsymbol{P}}^2 = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}$. A decomposition of $\boldsymbol{P}$

$$\boldsymbol{P} = \boldsymbol{S}^{\mathrm{T}}\boldsymbol{S}, \quad \boldsymbol{P} = \boldsymbol{U}_p^{\mathrm{T}}\boldsymbol{\Omega}_p\boldsymbol{U}_p, \quad \boldsymbol{S} = \boldsymbol{U}_p^{\mathrm{T}}\boldsymbol{\Omega}_p^{\frac{1}{2}}\boldsymbol{U}_p, \tag{39}$$

where $\boldsymbol{S}$ is the square root of $\boldsymbol{P}$ and a transformation $\boldsymbol{y} = \boldsymbol{S}\boldsymbol{x}$ gives the relationship

$$\sup_{\|\boldsymbol{y}\|=\sqrt{2\rho_i}} |\boldsymbol{e}_{n+i}^{\mathrm{T}}\boldsymbol{S}^{-1}\boldsymbol{y}| = -\frac{\beta}{\lambda_i}, \tag{40}$$

with $\|\boldsymbol{y}\| = \|\boldsymbol{x}\|_{\boldsymbol{P}}$ and $\dot{\eta}_i = x_{n+i} = \boldsymbol{e}_{n+i}^{\mathrm{T}}\boldsymbol{x}$. With a transformation $\boldsymbol{z} = \boldsymbol{y}/\sqrt{2\rho_i}$, (40) transforms into

$$\sqrt{2\rho_i}\sup_{\|\boldsymbol{z}\|=1} |\boldsymbol{e}_{n+i}^{\mathrm{T}}\boldsymbol{S}^{-1}\boldsymbol{z}| = -\frac{\beta}{\lambda_i}. \tag{41}$$

Using the definition of the norm of a matrix $\boldsymbol{A}$ as $\|\boldsymbol{A}\| = \sup_{\|\boldsymbol{x}\|=1}\|\boldsymbol{A}\boldsymbol{x}\|$, (41) yields

$$\sqrt{2\rho_i}\|\boldsymbol{e}_{n+i}^{\mathrm{T}}\boldsymbol{S}^{-1}\| = -\frac{\beta}{\lambda_i}. \tag{42}$$

Consequently, ρ_i is given by

$$\rho_i = \frac{\beta^2}{2\lambda_i^2}\frac{1}{\|\boldsymbol{e}_{n+i}^{\mathrm{T}}\boldsymbol{S}^{-1}\|^2}. \tag{43}$$

Let us now consider once more the 1DOF mass-spring-damper system, as discussed in Section 2. Using the Lyapunov function (21),

$$V = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2,$$

and (35) to (43), we can estimate the region of attraction of the equilibrium set $\mathcal{E}$ as given in (3). Evaluating (43) with

$$\boldsymbol{P} = \begin{bmatrix} k & 0 \\ 0 & m \end{bmatrix}, \quad \boldsymbol{S} = \begin{bmatrix} \sqrt{k} & 0 \\ 0 & \sqrt{m} \end{bmatrix}, \tag{44}$$

and $n_q = 1$ yields the estimate of the region of attraction $\mathcal{I}_{\rho^*}$, with

$$\rho^* = \frac{1}{2}m\left(\frac{\mu m g}{c}\right)^2. \tag{45}$$

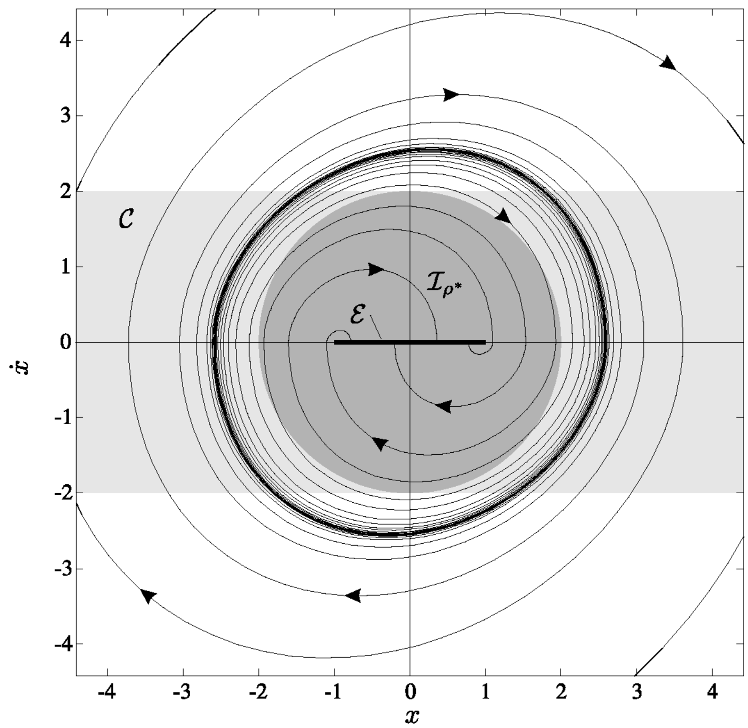


Figure 2. Phase plane of the 1DOF mass-spring-damper system with Coulomb friction.

The phase plane of system (1) is depicted in Figure 2, with $m = 1$, $k = 1$, $c = -0.5$ and $\mu g = 1$. It can be observed that the equilibrium set $\mathcal{E}$ is locally attractive, despite the negative viscous damping. Moreover, the region of attraction is bounded by an unstable limit cycle. Clearly, $\mathcal{I}_{\rho^*}$ is a conservative though fair estimate of the region of attraction. This 1DOF example illustrates the ideas used to develop the theory discussed in Sections 4 and 5.

It should be noted that when we consider the 1DOF mass-spring-damper system with a combination of a linear spring and a cubic spring ($k_1x + k_2x^3$, with $k_2 \geq 0$, instead of kx in (1)), the same strategy can be used to prove the local attractivity of the equilibrium set of this system. Note that the equilibrium set is convex. The Lyapunov candidate function used would once more consist of a combination of kinetic and potential energy:

$$V = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}k_1x^2 + \frac{1}{4}k_2x^4.$$

Despite the nonlinearity in the restoring force, the time-derivative of V is of the form (25). So, the local attractivity of (a subset of) the equilibrium set can be proven. An estimate for the corresponding region of attraction can be found by following a procedure analogous to the procedure discussed in this section. The resulting ρ^* remains the same as in the case of the 1DOF mass-spring-damper system with only a linear restoring force, but the estimate of the region of attraction changes due to the fact that V differs.

In the next section, we will apply the theory to multi-degree-of-freedom systems with multiple friction elements.

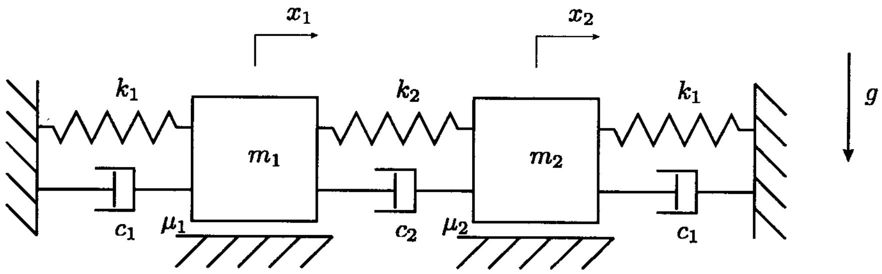


Figure 3. 2DOF mass-spring-damper system with Coulomb friction.

6. Illustrating Examples

In this section, we will illustrate the results of the previous sections by means of examples concerning two-degree-of-freedom (2DOF) mass-spring-damper systems. These simple mass-spring-damper systems characterise different situations with respect to the viscous damping properties and the generalised force directions of the dry friction elements and, thereby, illuminate the conditions for local attractivity of the equilibrium set.

6.1. Example 1: A 2DOF System with Two Friction Elements

Consider the two-degree-of-freedom (2DOF) mass-spring-damper system as depicted in Figure 3. The equation of motion of this system can be written in the form (15), with $\boldsymbol{q}^{\mathrm{T}} = [\, x_1 \;\; x_2 \,]$ and the generalised friction forces $\boldsymbol{\lambda}_T$ given by the Coulomb friction law (8). Herein the matrices $\bar{\boldsymbol{M}}$, $\bar{\boldsymbol{C}}$, $\bar{\boldsymbol{K}}$, $\bar{\boldsymbol{W}}_T$ and $\boldsymbol{\Lambda}$ are given by

$$\bar{\boldsymbol{M}} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \quad \bar{\boldsymbol{C}} = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_1 + c_2 \end{bmatrix}, \quad \bar{\boldsymbol{K}} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_1 + k_2 \end{bmatrix},$$

$$\bar{\boldsymbol{W}}_T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \boldsymbol{\Lambda} = \begin{bmatrix} \mu_1 m_1 g & 0 \\ 0 & \mu_2 m_2 g \end{bmatrix}, \tag{46}$$

with $m_1, m_2, k_1, k_2 > 0$ and $\mu_1, \mu_2 \geq 0$. Moreover, the tangential velocity $\dot{\boldsymbol{g}}_T$ in the frictional contacts is given by $\dot{\boldsymbol{g}}_T = \begin{bmatrix} \dot{x}_1 & \dot{x}_2 \end{bmatrix}^{\mathrm{T}}$. Let us first compute the spectral decomposition of the damping-matrix, $\bar{\boldsymbol{C}} = \boldsymbol{U}_c^{-\mathrm{T}} \boldsymbol{\Omega}_c \boldsymbol{U}_c^{-1}$, with (for non-singular $\bar{\boldsymbol{C}}$):

$$\boldsymbol{U}_c = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad \boldsymbol{\Omega}_c = \begin{bmatrix} c_1 & 0 \\ 0 & c_1 + 2c_2 \end{bmatrix}. \tag{47}$$

The equilibrium set $\mathcal{E}$, as defined by (18), is given by

$$\mathcal{E} = \Bigg\{ (x_1, x_2, \dot{x}_1, \dot{x}_2) \mid |x_1| \leq \frac{(k_1 + k_2)\mu_1 m_1 g + k_2 \mu_2 m_2 g}{k_1^2 + 2k_1 k_2} \wedge |x_2| \leq \frac{(k_1 + k_2)\mu_2 m_2 g + k_2 \mu_1 m_1 g}{k_1^2 + 2k_1 k_2} \wedge \dot{x}_1 = 0 \wedge \dot{x}_2 = 0 \Bigg\}. \tag{48}$$

Let us now consider a number of different cases for the damping parameters c_1 and c_2:

1. $c_1 > 0$ and $c_2 > -c_1/2$.
   Note that $\bar{\boldsymbol{C}} > 0$ if and only if $c_1 > 0$ and $c_2 > -c_1/2$. Consequently, due to Theorem 2

the global attractivity of the equilibrium set $\mathcal{E}$ is assured. It should be noted that this is also the case when one or both of the friction coefficients μ_1 and μ_2 vanish.

2. $c_1 > 0$ and $c_2 < -c_1/2$.
Clearly, the damping matrix is not positive definite in this case. As a consequence, the equilibrium point of the system without friction is unstable. Still the equilibrium set of the system with friction can be locally attractive. Therefore, Theorem 3 (or Theorem 4) can be used to investigate the attractivity properties of (a subset of) the equilibrium set. For the friction situation depicted in Figure 3, condition (31) is satisfied if $\mu_1 > 0$ and $\mu_2 > 0$. Namely, $\bar{\boldsymbol{W}}_T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ spans the two-dimensional space and, consequently, the eigencolumn of the damping matrix corresponding to the unstable eigenvalue $c_1 + 2c_2$, namely $[\,-1 \;\; 1\,]^{\mathrm{T}}$, lies in the space spanned by the columns of $\bar{\boldsymbol{W}}_T$.
Since the attractivity is only local, it is desirable to provide an estimate of the region of attraction of (a subset of) the equilibrium set and to determine whether we can prove attractivity of $\mathcal{E}$ or only of a subset $\tilde{\mathcal{E}}$. Hereto, we use the estimate $\mathcal{B}$ as defined by (34), in which we need a constant β (or $\alpha = 1/\beta$). A choice for α can be obtained from (28), which in this example is given by

$$|\dot{\eta}_2| \le \frac{\alpha}{\sqrt{2}} \begin{bmatrix} \mu_1 m_1 g & \mu_2 m_2 g \end{bmatrix} \left| \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \end{bmatrix} \right|$$
$$\Rightarrow |\dot{\eta}_2| \le \frac{\alpha}{\sqrt{2}} \left(\mu_1 m_1 g |\dot{\eta}_1 - \dot{\eta}_2| + \mu_2 m_2 g |\dot{\eta}_1 + \dot{\eta}_2| \right).$$

This inequality is satisfied for a minimal value for α,

$$\alpha = \sqrt{2} \max \left(\frac{1}{\mu_1 m_1 g}, \frac{1}{\mu_2 m_2 g} \right),$$

and arbitrary values for $\dot{\eta}_1$ and $\dot{\eta}_2$. This results in the following choice for β:

$$\beta = \frac{1}{\sqrt{2}} \min(\mu_1 m_1 g, \mu_2 m_2 g).$$

Subsequently, we can use (34) to estimate the region of attraction $\mathcal{B}$ of the equilibrium set with

$$\mathcal{B} = \mathcal{I}_{\rho^*} = \left\{ \boldsymbol{x} \in \mathbb{R}^4 \mid \frac{1}{2} \left(m_1 \dot{x}_1^2 + m_2 \dot{x}_2^2 + (k_1 + k_2)(x_1^2 + x_2^2) - 2k_2 x_1 x_2 \right) < \rho^* \right\}, \quad (49)$$

with $\boldsymbol{x}^{\mathrm{T}} = [\, \boldsymbol{q}^{\mathrm{T}} \;\; \dot{\boldsymbol{q}}^{\mathrm{T}} \,] = [\, x_1 \;\; x_2 \;\; \dot{x}_1 \;\; \dot{x}_2 \,]$. Now, ρ^* is defined by (33) with the set $\mathcal{C}$ given by

$$\mathcal{C} = \left\{ \boldsymbol{x} \in \mathbb{R}^4 \mid |-\dot{x}_1 + \dot{x}_2| \le -\frac{\min(\mu_1 m_1 g, \mu_2 m_2 g)}{c_1 + 2c_2} \right\}. \quad (50)$$

Using the fact that $\bar{\boldsymbol{K}}$ is positive-definite it is possible to prove that $\partial\mathcal{C} \bigcap \mathcal{I}_{\rho^*}$ lies within the plane $\{\boldsymbol{x} \in \mathbb{R}^4 \mid x_1 = x_2 = 0\}$. The maximal value of ρ, for which $\mathcal{I}_\rho \subset \mathcal{C}$, is therefore given by ρ^*, where ρ^* can be computed using (43)

$$\rho^* = \frac{1}{2} \frac{m_1 m_2 \gamma^2}{m_1 + m_2}, \quad \text{where } \gamma = -\frac{\min(\mu_1 m_1 g, \mu_2 m_2 g)}{c_1 + 2c_2}. \quad (51)$$

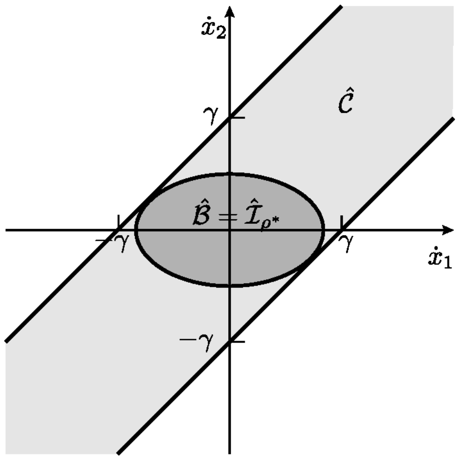


Figure 4. Cross-section of the sets $\mathcal{C}$ and $\mathcal{I}_{\rho^*}$ with the plane defined by $x_1 = 0$ and $x_2 = 0$.

The cross-section of the sets $\mathcal{C}$ and $\mathcal{I}_{\rho^*}$ with the plane defined by $x_1 = 0$ and $x_2 = 0$, denoted by $\hat{\mathcal{C}}$ and $\hat{\mathcal{I}}_{\rho^*}$, respectively, is shown in Figure 4. The set $\mathcal{B} = \mathcal{I}_{\rho^*}$ is a conservative estimate for the region of attraction $\mathcal{A}$ of the equilibrium set $\mathcal{E}$. We will present a comparison between the actual region of attraction (obtained by numerical simulation) and the estimate $\mathcal{B}$ for the following parameter set: $m_1 = m_2 = 1$ kg, $k_1 = k_2 = 1$ N/m, $c_1 = 0.5$ Ns/m, $c_2 = -0.375$ Ns/m, $\mu_1 = \mu_2 = 0.1$ and $g = 10$ m/s^2. Since, for these parameter settings, $\mathcal{E} \subset \text{int}(\mathcal{I}_{\rho^*})$, the local attractivity of the entire equilibrium set $\mathcal{E}$ is ensured. In Figure 5, we show a cross-section of $\mathcal{A}$ with the plane $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$, denoted by $\hat{\mathcal{A}}$, which was obtained numerically. Hereto, a grid of initial conditions in the plane $\dot{x}_1 = \dot{x}_2 = 0$ was defined, for which the solutions were obtained by numerically integrating the system over a given time span T. Subsequently, a check was performed to inspect whether the state of the system at time T was in the equilibrium set $\mathcal{E}$. Initial conditions corresponding to attractive solutions are depicted with a light colour (set $\hat{\mathcal{A}}$) and initial conditions corresponding to non-attractive solutions are depicted with a dark grey colour (set $\hat{\mathcal{D}}$). Moreover, $\hat{\mathcal{E}}$ and $\hat{\mathcal{B}}$ are also shown in the figure, where the $\hat{}$ indicates that we are referring to cross-sections of the sets. It should be noted that $\hat{\mathcal{E}} \subset \hat{\mathcal{B}}$. Similarly, Figure 6 was obtained for the cross-section with the plane $x_1 = x_2 = 0$. As expected the set $\mathcal{B}$ is a conservative estimate for the region of attraction $\mathcal{A}$. Figure 5 reveals that the estimate $\mathcal{B}$ is conservative in the direction $x_1 = -x_2$, on the plane $\dot{x}_1 = \dot{x}_2 = 0$. However, the estimate $\mathcal{B}$ is very conservative in the direction $x_1 = x_2$. Figure 6 reveals that the estimate $\mathcal{B}$ is conservative in the direction $\dot{x}_1 = -\dot{x}_2$, on the plane $x_1 = x_2 = 0$. However, the estimate $\mathcal{B}$ is very conservative in the direction $\dot{x}_1 = \dot{x}_2$. We have to bear in mind that the cross-sections in Figure 5 and 6 only give partial information on the region of attraction in the four-dimensional state-space. The set $\hat{\mathcal{C}}$ in Figure 6 seems to be a subset of $\hat{\mathcal{A}}$ and seems to indicate quite well the border of $\hat{\mathcal{A}}$ in the plane $x_1 = x_2 = 0$. We do not know whether $\mathcal{C} \subset \mathcal{A}$.

3. $c_1 < 0$ and $c_2 > -c_1/2$.
Again the damping-matrix is not positive definite in this case. This time the first eigendirection corresponds to the negative eigenvalue of $\bar{\boldsymbol{C}}$. However, $\bar{\boldsymbol{W}}_T$ spans the whole $\mathbb{R}^2$

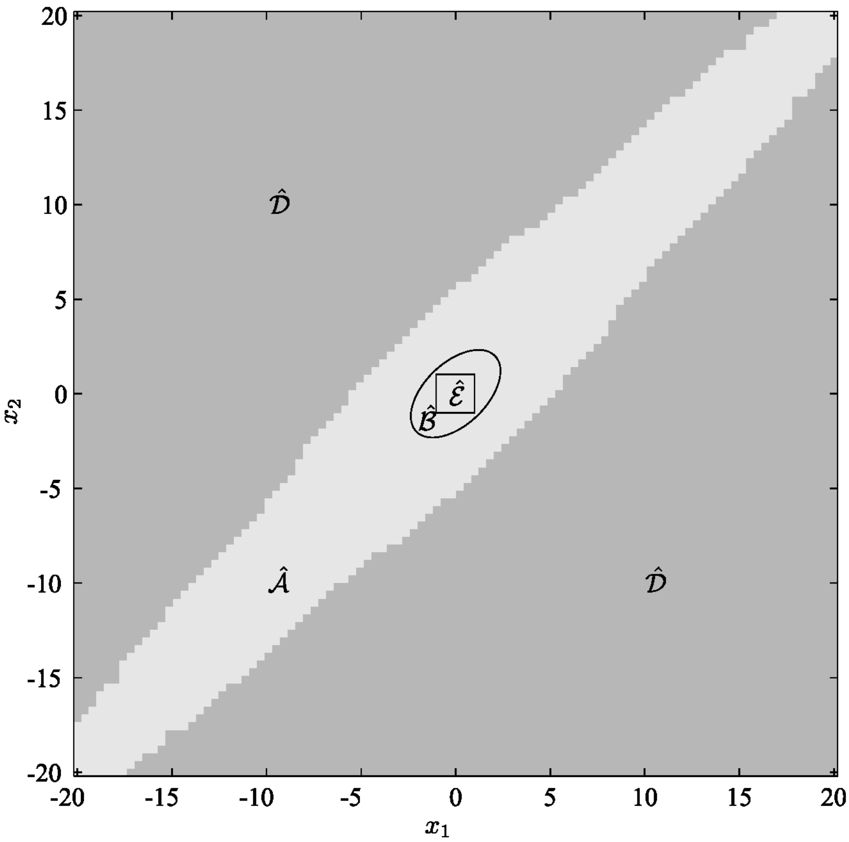


Figure 5. Cross-section of the region of attraction $\mathcal{A}$ with the plane defined by $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$.

and we can again ensure local attractivity of the set $\mathcal{E}$ for $\mu_1, \mu_2 > 0$ and find an estimate for region of attraction.

6.2. Example 2: A 2DOF System with One Friction Element

6.2.1. *Case A*

Let us consider a similar system as was discussed in the previous section, but with only one friction element between mass m_1 and mass m_2 (Figure 7). The mass-matrix, damping-matrix and stiffness-matrix are defined by (46). The normal contact force in the friction element is $m_1 g$. Moreover, the matrix $\bar{\boldsymbol{W}}_T = [\,-1 \;\; 1\,]^{\mathrm{T}}$ expresses the fact that the generalised friction force can only act in one specific direction and $\boldsymbol{\Lambda} = \mu_1 m_1 g$. The equilibrium set $\mathcal{E}$ is given by

$$\mathcal{E} = \left\{ (x_1, x_2, \dot{x}_1, \dot{x}_2) \mid |x_1| \le \frac{\mu_1 m_1 g}{k_1 + 2k_2} \wedge |x_2| \le \frac{\mu_1 m_1 g}{k_1 + 2k_2} \wedge \dot{x}_1 = 0 \wedge \dot{x}_2 = 0 \right\}. \tag{52}$$

Again we consider three different cases for the parameter values c_1 and c_2:

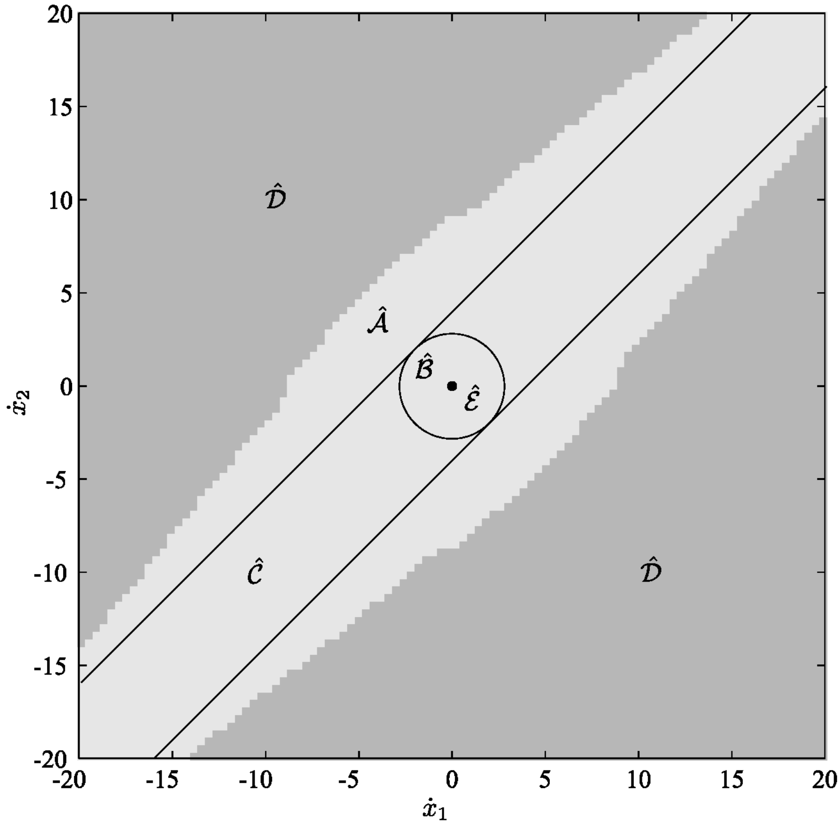


Figure 6. Cross-section of the region of attraction $\mathcal{A}$ with the plane defined by $x_1 = 0$ and $x_2 = 0$.

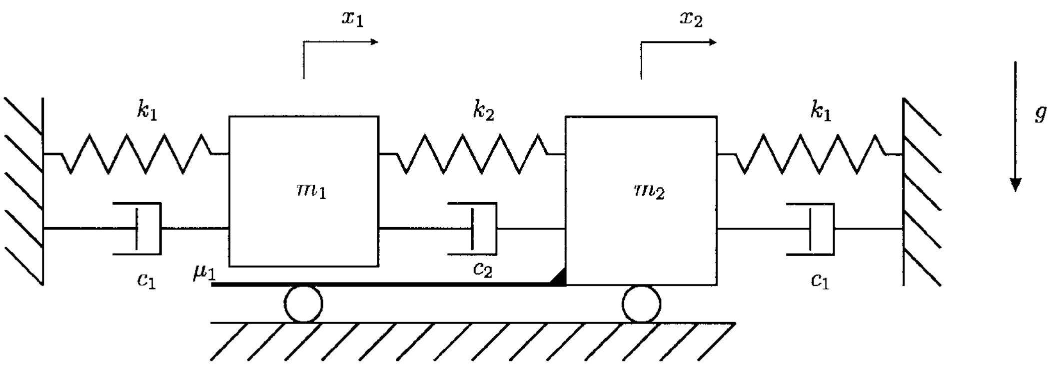


Figure 7. 2DOF mass spring damper system with Coulomb friction.

1. $c_1 > 0$ and $c_2 > -c_1/2$.
   Global attractivity of the equilibrium set is assured by Theorem 2.
2. $c_1 > 0$ and $c_2 < -c_1/2$.
   Notice that the second eigencolumn ($[\,-1 \;\; 1\,]^{\mathrm{T}}$) of $\bar{\boldsymbol{C}}$ corresponding to the open left-half plane eigenvalue $c_1 + 2c_2$ is identical to $\bar{\boldsymbol{W}}_T = [\,-1 \;\; 1\,]^{\mathrm{T}}$. So, the condition (31) of Theorem 3 (or Theorem 4) is satisfied and the local attractivity of (a subset of) the equilibrium set is assured.

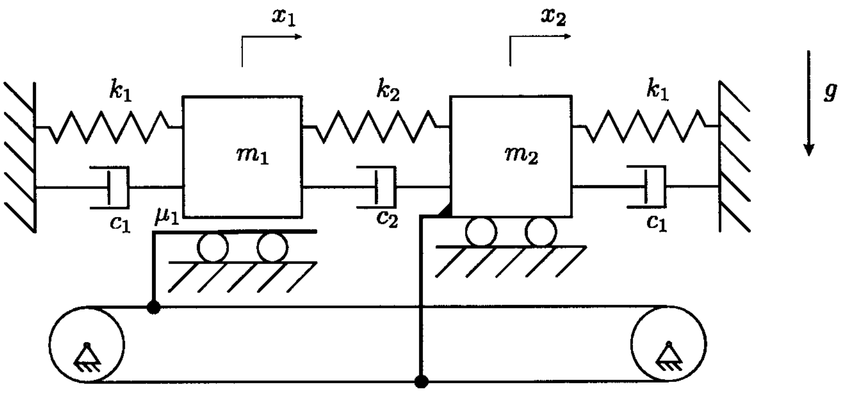


Figure 8. 2DOF mass spring damper system with Coulomb friction.

3. $c_1 < 0$ and $c_2 > -c_1/2$.
   In this case, the direction $\boldsymbol{W}_T$ of the generalised friction force does not coincide with the eigendirection of $\bar{\boldsymbol{C}}$ corresponding to its left-half plane eigenvalue. Consequently, attractivity of the equilibrium set can not be guaranteed.

6.2.2. *Case B*

Let us consider once more a system with only one friction element acting upon the sum of the velocities $\dot{x}_1 + \dot{x}_2$ (Figure 8). The second block in the system is rigidly connected by a cable to the support of block 1, such that the horizontal velocity of block 2 is opposite of the velocity of the support of block 1. The mass-matrix, damping-matrix and stiffness-matrix are defined by (46). The normal contact force in the friction element is $m_1 g$. Moreover, the matrix $\bar{\boldsymbol{W}}_T = [\,1 \;\; 1\,]^{\mathrm{T}}$ expresses the fact that the generalised friction force can only act in one specific direction and $\boldsymbol{\Lambda} = \mu_1 m_1 g$. The equilibrium set $\mathcal{E}$ is given by (52). Again we consider three different cases for the parameter values c_1 and c_2:

1. $c_1 > 0$ and $c_2 > -c_1/2$.
   Global attractivity of the equilibrium set is assured by Theorem 2.
2. $c_1 > 0$ and $c_2 < -c_1/2$.
   In this case, the direction $\boldsymbol{W}_T$ of the generalised friction force does not coincide with the eigendirection of $\bar{\boldsymbol{C}}$ corresponding to its left-half plane eigenvalue. Consequently, attractivity of the equilibrium set can not be guaranteed.
3. $c_1 < 0$ and $c_2 > -c_1/2$.
   Notice that the second eigencolumn ($[\,1 \;\; 1\,]^{\mathrm{T}}$) of $\bar{\boldsymbol{C}}$ corresponding to the open left-half plane eigenvalue c_1 is identical to $\bar{\boldsymbol{W}}_T = [\,1 \;\; 1\,]^{\mathrm{T}}$. So, the condition (31) of Theorem 3 (or Theorem 4) is satisfied and the local attractivity of (a subset of) the equilibrium set is assured.

7. Conclusions

Conditions for the (local) attractivity of (subsets of) equilibrium sets of mechanical systems with friction were derived. The systems were allowed to have multiple degrees-of-freedom and multiple switching boundaries (friction elements). It was shown that the equilibrium set $\mathcal{E}$ of a mechanical system, which without friction exhibits a stable equilibrium point E, will always be attractive when Coulomb friction elements are added provided that the conditions 1 to 4 in Section 3 are met. Moreover, it has been shown that even if the system without friction has an unstable equilibrium point E, then (a subset of) the equilibrium set $\mathcal{E}$ of the system with friction can under certain conditions be locally attractive and the equilibrium point $E \subset \mathcal{E}$ is stable. The crucial condition can be interpreted as follows: the space spanned by the eigendirections of the damping matrix, related to negative eigenvalues, lies in the space spanned by the generalised force directions of the dry friction elements.

In Appendix A, the global asymptotic stability of a passive one-degree-of-freedom oscillator is proven. Lyapunov stability of the equilibrium set of non-passive systems is not addressed, however, the combination of the attractivity property of the equilibrium set and the boundedness of solutions within $\mathcal{B}$ can be a valuable characteristic when the equilibrium set is a desired steady state of the system. The proposed analysis tools can also be effectively used to assess the performance of controllers for mechanical systems with friction aiming at the positioning of the system at the equilibrium set.

The examples studied in this paper show that the estimated region of attraction of the equilibrium is fairly accurate in some directions and very conservative in other directions, for which the actual region of attraction is stretched. The latter directions seem to be related to the eigendirection of the damping matrix corresponding to its eigenvalues in the open right-half complex plane.

The results are confined to mechanical systems with frictional sliders, i.e. bilateral constraints. Unilateral contact and spatial friction are not addressed. Furthermore, the systems were assumed to be piece-wise linear. The extension of the presented results towards nonlinear systems is complicated by the fact that a linearisation of the system (without friction) is only valid near a certain point in state space and not along the entire equilibrium set.

Appendix A. Global Asymptotic Stability for the 1DOF System with Friction

Consider the 1DOF mass-spring-damper system with friction, described by the differential inclusion (1):

$$m\ddot{x} + c\dot{x} + kx \in -\mu\lambda_N \mathrm{Sign}(\dot{x}), \tag{53}$$

with $c \geq 0$. We introduce a coordinate transformation $z = [\, z_1 \;\; z_2 \,]^{\mathrm{T}} = [\, \alpha x \;\; \beta\dot{x} \,]^{\mathrm{T}}$, $\alpha > 0$ and $\beta > 0$. The equilibrium set $\mathcal{E}$ of system (1) is given by

$$\mathcal{E} = \left\{ z \in \mathbb{R}^2 \mid |z_1| \leq \alpha\frac{\mu\lambda_N}{k},\; z_2 = 0 \right\}. \tag{54}$$

The set $\mathcal{E}$ is a convex set with a non-smooth boundary. We define the following subsets in the z-space:

$$\mathcal{I}_L = \left\{ z \in \mathbb{R}^2 \mid z_1 < -\alpha \frac{\mu\lambda_N}{k} \right\},$$

$$\mathcal{I}_M = \left\{ z \in \mathbb{R}^2 \mid |z_1| \leq \alpha \frac{\mu\lambda_N}{k} \right\},$$

$$\mathcal{I}_R = \left\{ z \in \mathbb{R}^2 \mid z_1 > \alpha \frac{\mu\lambda_N}{k} \right\}. \tag{55}$$

We will use the following properties of the distance with respect to and the proximal points on a convex set C [19]

$$\operatorname{prox}_C(\boldsymbol{x}) = \underset{\forall \boldsymbol{x}^* \in C}{\operatorname{argmin}} \|\boldsymbol{x} - \boldsymbol{x}^*\|, \quad \operatorname{dist}_C(\boldsymbol{x}) = \|\boldsymbol{x} - \operatorname{prox}_C(\boldsymbol{x})\|, \tag{56}$$

$$\nabla \frac{1}{2} \operatorname{dist}_C^2(\boldsymbol{x}) = \boldsymbol{x} - \operatorname{prox}_C(\boldsymbol{x}). \tag{57}$$

Consider the Lyapunov candidate function V, as defined by

$$V(z) = \frac{1}{2} \operatorname{dist}_C^2(z). \tag{58}$$

This Lyapunov function V has the desirable property that $V(z) = 0$ for $z \in \mathcal{E}$, which allows for the stability analysis of the equilibrium set as a whole. The time-derivative of V will be subsequently assessed in the sets $\mathcal{I}_M$, $\mathcal{I}_L$ and $\mathcal{I}_R$:

– In $\mathcal{I}_M$, $\operatorname{prox}_C(z) = [\ \alpha x \ \ 0\ ]^{\mathrm{T}}$ and, therefore,

$$\dot{V} = \left(z - \operatorname{prox}_C(z)\right)^{\mathrm{T}} \dot{z} = \frac{\beta^2}{m} \dot{x} \left(-c\dot{x} - kx - \mu\lambda_N \operatorname{Sign}(\dot{x})\right). \tag{59}$$

Choose $\beta^2 = m$, from which we obtain $\dot{V} = -c\dot{x}^2 - kx\dot{x} - \mu\lambda_N|\dot{x}| \leq 0$ for $z \in \mathcal{I}_M$.

– In $\mathcal{I}_L$, $\operatorname{prox}_C(z) = [\ -\alpha\mu\lambda_N/k \ \ 0\ ]^{\mathrm{T}}$ and, therefore,

$$\dot{V} = -c\dot{x}^2 - kx\dot{x} - \mu\lambda_N|\dot{x}| + \alpha^2 \left(x + \frac{\mu\lambda_N}{k}\right) \dot{x}. \tag{60}$$

Let us consider $\dot{V}$ in three subsets of $\mathcal{I}_L$:

- for $\dot{x} < 0$,

$$\dot{V} = -c\dot{x}^2 + \left((\mu\lambda_N + kx)\left(-1 + \frac{\alpha^2}{k}\right) + 2\mu\lambda_N\right)\dot{x}; \tag{61}$$

- for $\dot{x} = 0$, $\dot{V} = 0$;
- for $\dot{x} > 0$,

$$\dot{V} = -c\dot{x}^2 + (\mu\lambda_N + kx)\left(-1 + \frac{\alpha^2}{k}\right)\dot{x}. \tag{62}$$

Let us choose α such that $\alpha^2/k = 1 + \epsilon$, with $\epsilon > 0$. Consequently, $\dot{V} \leq 0\ \forall z \in \mathcal{I}_L$ if ϵ is taken to be arbitrarily small.

– In $\mathcal{I}_R$, a similar line of reasoning can be followed as in $\mathcal{I}_L$ due to the symmetry of the dynamics of (53) with respect to $z = \mathbf{0}$.

Resuming, we can conclude that $\dot{V} \leq 0\ \forall z \in \mathbb{R}^2$. Therefore, Lyapunov stability of the equilibrium set $\mathcal{E}$ has been proven for $c \geq 0$.

Global asymptotic stability of this equilibrium set can be proven using LaSalle's invariance principle, see Theorem 1. Hereto, realise that $\dot{V} = 0$ on the set $\mathcal{S}$ defined by

$$\mathcal{S} = \left\{z \in \mathbb{R}^2 \mid \dot{x} = 0\right\} \cup \left\{z \in \mathbb{R}^2 \mid x = -\frac{\mu\lambda_N}{k},\ \dot{x} > 0\right\} \cup \left\{z \in \mathbb{R}^2 \mid x = \frac{\mu\lambda_N}{k},\ \dot{x} < 0\right\}, \tag{63}$$

for $c = 0$, and

$$\mathcal{S} = \left\{z \in \mathbb{R}^2 \mid \dot{x} = 0\right\}, \tag{64}$$

for $c > 0$. Furthermore, $\mathcal{E}$ is the largest invariant set in $\mathcal{S}$, for $c \geq 0$. Application of LaSalle's invariance principle, combined with the Lyapunov stability of $\mathcal{E}$, proves global asymptotic stability of $\mathcal{E}$ for $c \geq 0$.

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