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Synchronization–based state observer including position jumps for impacting multibody systems

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Abstract— **We present an observer design for linear time-invariant systems with unilateral constraints using only the time information of the impacts. The approach presented in [4] is extended with switched geometric bilateral constraints. These constraints introduce position jumps which improve the synchronization rate and extend the applicability of the observer to systems with a rigid body mode.**

A master–slave synchronization setup is used for which the unidirectional coupling between the master (observed system) and the slave system (observer) consists only of the impact time information. The dynamics of the slave system is shown to be attractively incrementally stable due to the switched constraints. The decrease of the synchronization error follows directly from the property of attractive incremental stability. The slave system acts as state observer which replicates the full state for all initial conditions, also in the presence of accumulations points (Zeno behavior) or in the vicinity of grazing impacts.

The results are applied to an example of a robotic leg hopping on a moving ground.

I. INTRODUCTION

In this paper we show sufficient conditions for which a linear time-invariant system subjected to switched unilateral constraints is attractively incrementally stable. This result is used to design a state observer using master–slave synchronization which uses only the Boolean information of the impact time instants. Switched kinematic as well as switched geometric unilateral constraints are considered which extend the results presented in [4].

The present paper considers dynamical systems subjected to unilateral constraints. Such type of nonsmooth systems can conveniently be described using the framework of measure differential inclusions [17], [16], [6], [1], [10]. The states are assumed to be special functions of locally bounded variation and phenomena such as accumulation points (Zeno behavior) do not have to be excluded.

The observer is also applicable to systems with a rigid body mode, i.e. the observer system is allowed to have a free rigid body motion. For example the classical bouncing ball system has a rigid body mode since the ball is not connected to the ground with a force element such as a linear spring, which manifests itself in a singular stiffness matrix.

The switched geometric bilateral constraints are accompanied with a constraint force in the kinematic equation. This extension improves the synchronization rate and extends the applicability of the observer presented in [4], but it entails several difficulties. The generalized coordinates are no longer absolutely continuous, but discontinuous. Furthermore, it raises the question of a suitable metric for the position jumps. This problem has also been considered in the Gear-Gupta-Leimkuhler method to enforce the geometric constraints in simulations of impacting mechanical system [9], [12], [21]. This approach classically uses a non-energy consistent coordinate projection to the non-penetration constraint using the identity metric or the metric induced by the mass matrix. Energy considerations imply the use of the stiffness matrix for the metric. However, this is not applicable for systems with a rigid body mode as the stiffness matrix is only positive semi-definite.

The observer is based on a master–slave setup and the synchronization results are obtained using the property of attractive incremental stability. Incremental stability is a system property and several similar notions have been presented in the literature [23], [7], [8], [3]. Incremental stability is beneficial in many control problems such as stabilization, tracking control, output regulation problems, synchronization and observer design [18], [22], [19], [13].

In this paper an observer for impact oscillators with a rigid body mode is developed. The only measurement used is the Boolean impact time information and no additional continuous measurement is necessary. The class of mechanical systems subjected to unilateral constraints considered here are generally not strictly passive. More precisely, the transfer matrix of the linear part of the system is only positive real and not strictly positive real. Mechanical systems without any feedback do generally not fulfill the strict passivity condition. Furthermore, the concept of switched geometric unilateral constraints is used in order to cope with the rigid body mode.

This paper is organized as follows. The dynamics of the observed systems is described in Section II. The dynamics consists of an impulsive and a non-impulsive part and is accompanied by the constitutive laws of the constraint forces and impulses. The design of the state observer together with sufficient conditions for which the observer dynamics is attractively incrementally stable is presented in Section III. The results are illustrated with simulations of a 3-DOF robotic leg in Section IV and final conclusions are given in Section V.

II. DYNAMICS OF THE OBSERVED SYSTEM

In this section we describe the model of the mechanical system for which we will design a state observer. We

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consider an n-DOF linear time-invariant multibody system subjected to geometric unilateral constraints. The system has one rigid body mode, which is influenced by one of the unilateral constraints. The generalized coordinates $\boldsymbol{q}(t)$ are absolutely continuous in time and the generalized velocities $\boldsymbol{u}(t)$ are assumed to be special functions of locally bounded variation [2]. The non-impulsive dynamics is described by the kinematic equation together with the equation of motion as

$$\dot{\boldsymbol{q}} = \boldsymbol{u}, \tag{1}$$

$$\boldsymbol{M}\dot{\boldsymbol{u}} + \boldsymbol{C}\boldsymbol{u} + \boldsymbol{K}\boldsymbol{q} = \boldsymbol{W}\boldsymbol{\lambda} + \boldsymbol{f}(t). \tag{2}$$

The system matrices are assumed to be symmetric and time invariant. The mass matrix $\boldsymbol{M} = \boldsymbol{M}^{\mathsf{T}} \succ \boldsymbol{0} \in \mathbb{R}^{n\times n}$ is positive definite, whereas the stiffness matrix $\boldsymbol{K} = \boldsymbol{K}^{\mathsf{T}} \succeq \boldsymbol{0} \in \mathbb{R}^{n\times n}$ and the damping matrix $\boldsymbol{C} = \boldsymbol{C}^{\mathsf{T}} \succeq \boldsymbol{0} \in \mathbb{R}^{n\times n}$ are only positive semi-definite. Furthermore, $\boldsymbol{K}$ and $\boldsymbol{C}$ have the same nullspace and are of rank $n-1$. Therefore, the mechanical system has a rigid body mode which is described by the eigenvector corresponding to the zero eigenvalue of $\boldsymbol{K}$ and $\boldsymbol{C}$. The external forcing $\boldsymbol{f}(t)$ is independent of the state.

Equations (1)–(2) are valid almost everywhere on the time axis. The term almost everywhere captures that the state is not defined for a Lebesgue-negligible set, i.e. for the collection of the impact time instants. The impulsive dynamics is given by the impact equation

$$\boldsymbol{M}(\boldsymbol{u}^+ - \boldsymbol{u}^-) = \boldsymbol{W}\boldsymbol{\Lambda}, \tag{3}$$

where $\boldsymbol{u}^-$ and $\boldsymbol{u}^+$ denote pre- and post-impact velocities.

The mechanical system (1)–(3) is subjected to m geometric unilateral constraints. The generalized force directions are collected in the matrix $\boldsymbol{W} = [\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_m] \in \mathbb{R}^{n\times m}$ and are assumed to be linearly independent and time invariant. There is a constraint k which influences the rigid body mode and the corresponding force direction $\boldsymbol{w}_k$ will be referred to as $\boldsymbol{w}_{\text{RBM}}$. Therefore, we have

$$\boldsymbol{w}_{\text{RBM}} \notin \operatorname{range}(\boldsymbol{K}),$$
$$\boldsymbol{w}_j \in \operatorname{range}(\boldsymbol{K}) \quad \forall \boldsymbol{w}_j \neq \boldsymbol{w}_{\text{RBM}}.$$

The geometric unilateral constraints, also known as impenetrability constraints, induce constraint forces $\boldsymbol{\lambda}$ in the equation of motion (2) and constraint impulses $\boldsymbol{\Lambda}$ in the impact equation (3). The force law and impact law are constitutive laws defined using the local kinematic quantities

$$\text{constraint distance:} \quad \boldsymbol{g}(\boldsymbol{q}, t) = \boldsymbol{W}^{\mathsf{T}}\boldsymbol{q} + \boldsymbol{\eta}(t), \tag{4}$$

$$\text{constraint velocity:} \quad \boldsymbol{\gamma}(\boldsymbol{u}, t) = \boldsymbol{W}^{\mathsf{T}}\boldsymbol{u} + \dot{\boldsymbol{\eta}}(t), \tag{5}$$

where $\boldsymbol{\eta}(t)$ is an absolutely continuous function in time.

The i-th geometric unilateral constraint restricts the sign of the constraint distance $g_i \geq 0$. Its force law $0 \leq \lambda_i \perp g_i \geq 0$, also referred to as Signorini's law, can be written on velocity level (see [10]) as

$$-\lambda_i \in \begin{cases} \partial\Psi_{\mathbb{R}_0^+}(\gamma_i) & \text{if } g_i = 0, \\ 0 & \text{if } g_i > 0, \end{cases} \tag{6}$$

where $\partial\Psi_{\mathbb{R}_0^+}$ is the subdifferential of the indicator function $\Psi_{\mathbb{R}_0^+}$ on the set $\mathbb{R}_0^+$.

The impact law for the impulsive unilateral constraint forces $\boldsymbol{\Lambda}$ is given by the inclusion (see [15])

$$-\boldsymbol{\Lambda} \in \mathcal{H}_g(\bar{\boldsymbol{\gamma}}). \tag{7}$$

The set-valued map $\mathcal{H}_g$ puts a relationship between the dual variables $\bar{\boldsymbol{\gamma}} = \frac{1}{2}(\boldsymbol{\gamma}^+ + \boldsymbol{\gamma}^-)$ and $\boldsymbol{\Lambda}$. The index indicates the dependence on the index set of closed contacts (given by $\boldsymbol{g}$). Monotonicity and (inverse) strong monotonicity are useful properties of the impact map, which are defined in [20].

Definition 1: The set-valued map $\mathcal{H}_g(\bar{\boldsymbol{\gamma}})$ is called *monotone* if $\forall -\boldsymbol{\Lambda}_1 \in \mathcal{H}_g(\bar{\boldsymbol{\gamma}}_1)$, $\forall -\boldsymbol{\Lambda}_2 \in \mathcal{H}_g(\bar{\boldsymbol{\gamma}}_2)$ it fulfills

$$(\boldsymbol{\Lambda}_1 - \boldsymbol{\Lambda}_2)^{\mathsf{T}}(\bar{\boldsymbol{\gamma}}_1 - \bar{\boldsymbol{\gamma}}_2) \leq 0.$$

In addition, if

$$(\boldsymbol{\Lambda}_1 - \boldsymbol{\Lambda}_2)^{\mathsf{T}}(\bar{\boldsymbol{\gamma}}_1 - \bar{\boldsymbol{\gamma}}_2) \leq -\alpha\|\bar{\boldsymbol{\gamma}}_1 - \bar{\boldsymbol{\gamma}}_2\|^2$$

for some $\alpha > 0$, then the set-valued map is strongly monotone. Analogously, if there exists a $\beta > 0$ such that

$$(\boldsymbol{\Lambda}_1 - \boldsymbol{\Lambda}_2)^{\mathsf{T}}(\bar{\boldsymbol{\gamma}}_1 - \bar{\boldsymbol{\gamma}}_2) \leq -\beta\|\boldsymbol{\Lambda}_1 - \boldsymbol{\Lambda}_2\|^2$$

is fulfilled, then the set-valued map is inverse strongly monotone.

Commonly used impact laws such as the generalized Newton's impact law [15] or the generalized Poisson's impact law [4] are monotone under some mild assumptions. Furthermore, these impact laws are strongly monotone if none of the constraints are superfluous and inverse strongly monotone for a single constraint with a coefficient of restitution less than one. It is shown in [15] that the mapping $Z : \boldsymbol{u}^+ = Z(\boldsymbol{u}^-)$ from pre-impact to post-impact generalized velocities, defined by the impact equation (3) together with the impact law (7), is non-expansive [20] in the metric $\boldsymbol{M}$, i.e.

$$\|\boldsymbol{u}_1^+ - \boldsymbol{u}_2^+\|_{\boldsymbol{M}}^2 \leq \|\boldsymbol{u}_1^- - \boldsymbol{u}_2^-\|_{\boldsymbol{M}}^2, \tag{8}$$

if and only if the impact map is monotone. The property of non-expansivity will be used later to prove synchronization of the master–slave system.

III. Observer Design

In this section we present the design of the state observer for the class of mechanical systems presented in Section II. We assume that the model of the observed system is known, but the only available measurement is the time information when the impacts occur, i.e. which contacts are closed and which are open. We use a master–slave synchronization setup, where the observed system is the master system and the state of the slave system is the estimate.

A. Dynamics of the Observer

The slave system is a replica of the master system with two major modifications. First, the slave system is subjected to *switched kinematic unilateral constraints* instead of geometric unilateral constraints. Secondly, the kinematic equation is extended with a constraint force which influences (only) the rigid body mode. Therefore, the non-impulsive dynamics of the slave system is described by

$$\dot{\boldsymbol{q}} = \boldsymbol{u} + \boldsymbol{K}^{\perp}\boldsymbol{w}_{\text{RBM}}\,\sigma, \tag{9}$$
$$\boldsymbol{M}\dot{\boldsymbol{u}} + \boldsymbol{C}\boldsymbol{u} + \boldsymbol{K}\boldsymbol{q} = \boldsymbol{W}\boldsymbol{\lambda} + \boldsymbol{f}(t). \tag{10}$$

The matrix $\boldsymbol{K}^{\perp}$ ensures that the constraint force σ has only an influence on the nullspace of $\boldsymbol{K}$. It is defined as the orthogonal projector onto $\operatorname{null}(\boldsymbol{K})$ as

$$\boldsymbol{K}^{\perp} := \left(\boldsymbol{I} - \boldsymbol{K}^{+}\boldsymbol{K}\right),$$

where $\boldsymbol{I}$ is the identity matrix and $\boldsymbol{K}^{+}$ is the Moore–Penrose pseudoinverse of $\boldsymbol{K}$. Here, the projector $\boldsymbol{K}^{\perp}$ is given by the Cartesian product of the normalized eigenvector of the stiffness matrix $\boldsymbol{K}$ corresponding to the zero eigenvalue and it has the properties $\boldsymbol{K}^{\perp}\boldsymbol{K}^{\perp} = \boldsymbol{K}^{\perp}$ and $\boldsymbol{K}\boldsymbol{K}^{\perp} = \boldsymbol{0}$.

The constraint distances $\boldsymbol{g}$ and constraint velocities $\boldsymbol{\gamma}$ are defined as for the master system by (4)–(5). The constraint force σ implies that the generalized velocities $\boldsymbol{u}(t)$ and the constraint velocities $\boldsymbol{\gamma}(\boldsymbol{u}(t), t)$ are no longer the time derivative of the generalized coordinates $\boldsymbol{q}(t)$ and the constraint distances $\boldsymbol{g}(\boldsymbol{q}(t), t)$, respectively.

The constraint force σ might become arbitrarily large such that the impulsive dynamics has to be extended by the impact equation of the kinematic equation given by

$$\boldsymbol{q}^{+} - \boldsymbol{q}^{-} = \boldsymbol{K}^{\perp}\boldsymbol{w}_{\text{RBM}}\,\Sigma, \tag{11}$$
$$\boldsymbol{M}(\boldsymbol{u}^{+} - \boldsymbol{u}^{-}) = \boldsymbol{W}\boldsymbol{\Lambda}, \tag{12}$$

where Σ is the constraint impulse of the kinematic impact equation. The generalized coordinates $\boldsymbol{q}$ are no longer absolutely continuous, but special functions of locally bounded variation.

The constraint forces $\boldsymbol{\lambda}$ and constraint impulses $\boldsymbol{\Lambda}$ of the slave system describe switched kinematic unilateral constraints, whereas the constraint force σ and constraint impulse Σ describe a switched geometric bilateral constraint. Both types of switched constraints are switched on and off by an external Boolean switching function $\boldsymbol{\chi}(t) : \mathbb{R} \to \{0,1\}^m$.

The i-th *switched kinematic unilateral constraint* imposes a kinematic unilateral constraint $\gamma_i \geq 0$ whenever the corresponding external Boolean switching function $\chi_i(t) = 1$ and imposes no constraint if $\chi_i(t) = 0$. Its force law is described by the inequality complementarity

$$-\lambda_i \in \begin{cases} \partial\Psi_{\mathbb{R}_0^+}(\gamma_i) & \text{if } \chi_i(t) = 1, \\ 0 & \text{if } \chi_i(t) = 0. \end{cases} \tag{13}$$

The impact law for the constraint impulses $\boldsymbol{\Lambda}$ is given by the inclusion (see [15])

$$-\boldsymbol{\Lambda} \in \mathcal{H}_{\chi(t)}(\bar{\boldsymbol{\gamma}}). \tag{14}$$

The operator $\mathcal{H}_{\chi(t)}$ is chosen such that it is identical to the operator $\mathcal{H}_g$ in (7) if the same contacts are closed or switched on, respectively.

The *switched geometric bilateral constraint* ensures that the constraint corresponding to the force direction $\boldsymbol{w}_{\text{RBM}}$ is closed (i.e. g^{+}_{RBM}=0) for $\chi_{\text{RBM}}(t) = 1$ and imposes no constraint otherwise. Therefore, the constraint force σ and constraint impulse Σ are obtained as

$$\sigma \in \begin{cases} \partial\Psi_{\{0\}}(g_{\text{RBM}}) & \text{if } \chi_{\text{RBM}}(t) = 1, \\ 0 & \text{if } \chi_{\text{RBM}}(t) = 0, \end{cases} \tag{15}$$

and

$$\Sigma \in \begin{cases} \partial\Psi_{\{0\}}(g^{+}_{\text{RBM}}) & \text{if } \chi_{\text{RBM}}(t) = 1, \\ 0 & \text{if } \chi_{\text{RBM}}(t) = 0. \end{cases} \tag{16}$$

If the slave system is initialized with the same initial conditions as the master system and if the constraints of the slave system are switched on whenever the constraints of the master system are closed, then both solutions are identical. This statement is the content of the following proposition.

Proposition 1: Let the master system be described by (1)–(3) together with (6)–(7) and let the slave system be described by (9)–(16). Let $\begin{pmatrix}\boldsymbol{q}_m(t)\\ \boldsymbol{u}_m(t)\end{pmatrix}$ be the unique solution of the master system for the initial conditions $\begin{pmatrix}\boldsymbol{q}_m(t_0)\\ \boldsymbol{u}_m(t_0)\end{pmatrix} = \begin{pmatrix}\boldsymbol{q}_0\\ \boldsymbol{u}_0\end{pmatrix}$ and let $\begin{pmatrix}\boldsymbol{q}_s(t)\\ \boldsymbol{u}_s(t)\end{pmatrix}$ be the unique solution of the slave system for the same initial conditions $\begin{pmatrix}\boldsymbol{q}_s(t_0)\\ \boldsymbol{u}_s(t_0)\end{pmatrix} = \begin{pmatrix}\boldsymbol{q}_0\\ \boldsymbol{u}_0\end{pmatrix}$. Then, both solutions are identical if the Boolean switching function $\boldsymbol{\chi}(t)$ in (13)–(16) is generated by $\boldsymbol{g}_m(t)$, i.e.

$$\chi_i(t) = \begin{cases} 1 & \text{if } g_{m,i}(t) = 0, \\ 0 & \text{if } g_{m,i}(t) > 0, \end{cases} \tag{17}$$

for all $i \in \{1, 2, \ldots, m\}$, where $\boldsymbol{g}_m(t)$ are the constraint distances of the master system.

Proof: First, the relation (17) directly implies that the force laws (6) and (13) as well as the impact laws (7) and (14) are identical. Secondly, we seek an explicit expression for the constraint force σ and constraint impulse Σ in order to show that both vanish if the relation (17) holds. The impact law (16) for $\chi_{\text{RBM}}(t) = 1$ can be written[1] as $g^{+}_{\text{RBM},s} = 0$, where $\boldsymbol{g}_s$ are the constraint distances of the slave system. Multiplying (11) from the left with $\boldsymbol{w}^{\mathsf{T}}_{\text{RBM}}$ and substituting (4) and $g^{+}_{\text{RBM},s} = 0$ yields

$$\Sigma = -\left(\boldsymbol{w}^{\mathsf{T}}_{\text{RBM}}\,\boldsymbol{K}^{\perp}\boldsymbol{w}_{\text{RBM}}\right)^{-1} g^{-}_{\text{RBM},s}. \tag{18}$$

The scalar $\boldsymbol{w}^{\mathsf{T}}_{\text{RBM}}\,\boldsymbol{K}^{\perp}\boldsymbol{w}_{\text{RBM}}$ is non-zero since $\boldsymbol{w}_{\text{RBM}} \notin \operatorname{range}(\boldsymbol{K})$ and, thus, $\boldsymbol{w}_{\text{RBM}} \notin \operatorname{null}(\boldsymbol{K}^{\perp})$. The absolute continuity of the generalized coordinates $\boldsymbol{q}_m$, and therefore of the generalized constraint distances $\boldsymbol{g}_m$, implies $g^{-}_{\text{RBM},m} = 0$ for $\chi_{\text{RBM}}(t) = 1$. Together with (18), we obtain that $\boldsymbol{q}^{-}_m = \boldsymbol{q}^{-}_s$ implies $\Sigma = 0$, which yields $\boldsymbol{q}^{+}_m = \boldsymbol{q}^{+}_s$. Furthermore, the generalized coordinates $\boldsymbol{q}_s$ of the slave system are absolutely continuous.

The force law (15) for $\chi_{\text{RBM}}(t) = 1$ can be written on velocity level as $\dot{g}_{\text{RBM}} = 0$. Together with (9) and (4)–(5) and using similar steps as before, we obtain

$$\sigma = -\left(\boldsymbol{w}^{\mathsf{T}}_{\text{RBM}}\,\boldsymbol{K}^{\perp}\boldsymbol{w}_{\text{RBM}}\right)^{-1} \gamma_{\text{RBM},s}. \tag{19}$$

[1]The inverse of $\partial\Psi_{\{0\}}$ is the zero function.

The absolute continuity of $\boldsymbol{q}_m$ implies $\gamma_{\text{RBM},m} = 0$ for $\chi_{\text{RBM}}(t) = 1$ almost everywhere. Together with (19), we obtain that $\boldsymbol{u}_m = \boldsymbol{u}_s$ implies $\sigma = 0$. Therefore, we obtain $\dot{\boldsymbol{q}}_m = \dot{\boldsymbol{q}}_s$ almost everywhere, which concludes the proof. ■

In the following we will show that the slave system described by the set of equations (9)–(12) together with the force and impact laws (13)–(16) is attractively incrementally stable.

B. Attractive Incremental Stability of the Observer

The synchronization of the master and the slave system is based on the attractive incremental stability of the slave system. Attractive incremental stability (a.i.s.) is a stability property of dynamical systems which implies that all solution curves are globally uniformly attractively stable. Therefore, all solution curves approach each other and remain close in the sense of Lyapunov for all initial conditions. The information of the initial condition is lost. Here, we consider the definition of a.i.s for measure differential inclusions as presented in [4]. Other incremental stability notions have been presented in literature, see e.g. [3]. We briefly recall the definition of a.i.s. for the considered class of systems.

Definition 2: System (9)–(16) for a given switching function $\boldsymbol{\chi}(t)$ is called *attractively incrementally stable* if for any two solution curves $\boldsymbol{x}_1(t) = \begin{pmatrix}\boldsymbol{q}_1(t)\\ \boldsymbol{u}_1(t)\end{pmatrix}$, $\boldsymbol{x}_2(t) = \begin{pmatrix}\boldsymbol{q}_2(t)\\ \boldsymbol{u}_2(t)\end{pmatrix}$ it holds that $\forall\varepsilon > 0, \exists\delta = \delta(\varepsilon)$ such that $\|\boldsymbol{x}_1(t_0) - \boldsymbol{x}_2(t_0)\| < \delta$ implies $\|\boldsymbol{x}_1(t) - \boldsymbol{x}_2(t)\| < \varepsilon$ for almost all $t \geq t_0$ and additionally $\lim_{t\to\infty} \|\boldsymbol{x}_1(t) - \boldsymbol{x}_2(t)\| = 0$.

Before we state the assumptions on the system, we define the classes of functions $\mathbb{K}$ and $\mathbb{K}^m$. Hereto, we introduce the notation

$$\tilde{a}_{\Delta t}(t) = \frac{1}{\Delta t}\int_t^{t+\Delta t} a(\tau)\,\mathrm{d}\tau \text{ for a given } \Delta t > 0. \quad (20)$$

Definition 3: A Boolean switching function $\chi(t) : \mathbb{R} \to \{0, 1\}$ is of class $\mathbb{K}$ if for each t there exists a $t^*(t) > t$ and a number $\Delta t > 0$ independent of t such that $\tilde{\chi}_{\Delta t}(t^*) = 0$. Furthermore, a function $\boldsymbol{\chi}(t) : \mathbb{R} \to \{0, 1\}^m$ is of class $\mathbb{K}^m$ if each component $\chi_i(t)$ is of class $\mathbb{K}$.

We will make the following assumptions on the system:

A1 The switching function $\boldsymbol{\chi}(t)$ is of class $\mathbb{K}^m$ and is generated by an absolutely continuous function.

A2 There exists a maximal time interval for which the constraint influencing the RBM is open or superfluous, i.e. $\forall t\ \exists t^* \in (t, t + T_{\text{max}})$ such that $\Lambda_{\text{RBM}}(t^*) \neq 0$ for a given $T_{\text{max}} \in (0, \infty)$.

A3 The external forcing $\boldsymbol{f}(t)$ is bounded, i.e. $\sup_{t\in\mathbb{R}} \|\boldsymbol{f}(t)\| \leq f_{\text{max}}$ for a given bound $f_{\text{max}} < \infty$.

A4 The impact map $\mathcal{H}_{\chi(t)}(\bar{\boldsymbol{\gamma}})$ is monotone for all $\boldsymbol{\chi}(t)$. Furthermore, it is strongly monotone in $\bar{\gamma}_{\text{RBM}}$ for $\Lambda_{\text{RBM}} \neq 0$ and inverse strongly monotone in Λ_{RBM}.

Assumption A1 guarantees that the Lebesgue measure of the sum of time intervals for which each contact is switched off is infinite. Furthermore, it is used for the existence of solutions, but this is not in the scope of this paper. Assumption A2 gives an upper bound of the time intervals for which the rigid body mode is not detectable. The following theorem states the a.i.s. of the slave system under the previous assumptions and is an extension of the corresponding theorem in [4].

Theorem 1: System (9)–(16) for a given switching function $\boldsymbol{\chi}(t)$ is attractively incrementally stable if the Assumptions A1–A4 are fulfilled.

Proof: We will show that all solution curves of system (9)–(16) are globally uniformly attractively stable. Therefore, consider two solutions $\begin{pmatrix}\boldsymbol{q}_1(t)\\ \boldsymbol{u}_1(t))\end{pmatrix}$ and $\begin{pmatrix}\boldsymbol{q}_2(t)\\ \boldsymbol{u}_2(t)\end{pmatrix}$ for a given switching function $\boldsymbol{\chi}(t)$. The state vector of the error dynamics is given by $\boldsymbol{x}_e = \begin{pmatrix}\boldsymbol{e}\\ \boldsymbol{v}\end{pmatrix}$, where $\boldsymbol{e} = \boldsymbol{q}_1 - \boldsymbol{q}_2$ and $\boldsymbol{v} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ are the position and the velocity errors between these two solutions. We introduce the Lyapunov function

$$V(\boldsymbol{x}_e) = \frac{1}{2}\|\boldsymbol{v}\|^2_{\boldsymbol{M}} + \frac{1}{2}\|\boldsymbol{e}\|^2_{\boldsymbol{K}}, \quad (21)$$

which gives a notion of distance between these two solutions. It is only positive semi-definite since the stiffness matrix is singular.

The differential measure of the Lyapunov function $\mathrm{d}V$ contains a density $\dot{V}$ with respect to the Lebesgue measure $\mathrm{d}t$ and a density $V^+ - V^-$ with respect to the atomic measure $\mathrm{d}\eta$. According to (9)–(10) and (5), the density $\dot{V}$ is obtained as

$$\begin{aligned}\dot{V} &= \boldsymbol{v}^\mathsf{T}\boldsymbol{M}\dot{\boldsymbol{v}} + \boldsymbol{e}^\mathsf{T}\boldsymbol{K}\dot{\boldsymbol{e}}\\ &= \boldsymbol{v}^\mathsf{T}\left(-\boldsymbol{C}\boldsymbol{v} + \boldsymbol{W}(\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2)\right) + \boldsymbol{e}^\mathsf{T}\boldsymbol{K}\boldsymbol{K}^\perp\boldsymbol{w}_{\text{RBM}}(\sigma_1 - \sigma_2),\\ &= -\|\boldsymbol{v}\|^2_{\boldsymbol{C}} + (\boldsymbol{\gamma}_1 - \boldsymbol{\gamma}_2)^\mathsf{T}(\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2),\end{aligned} \quad (22)$$

where $\boldsymbol{K}\boldsymbol{K}^\perp = \boldsymbol{0}$ has been used. Due to the maximal monotonicity of the force law (13), we obtain $\dot{V} \leq -W(\boldsymbol{x}_e)$, where $W(\boldsymbol{x}_e) = \|\boldsymbol{v}\|^2_{\boldsymbol{C}}$ is a positive semi-definite function in $\boldsymbol{x}_e$.

The jump in the Lyapunov function at impulsive time-instants is obtained as

$$\begin{aligned}V^+ - V^- =& \frac{1}{2}(\boldsymbol{v}^+ + \boldsymbol{v}^-)^\mathsf{T}\boldsymbol{M}(\boldsymbol{v}^+ - \boldsymbol{v}^-)\\ &+ \frac{1}{2}(\boldsymbol{e}^+ + \boldsymbol{e}^-)^\mathsf{T}\boldsymbol{K}(\boldsymbol{e}^+ - \boldsymbol{e}^-)\\ =& \frac{1}{2}\|\boldsymbol{v}^+\|^2_{\boldsymbol{M}} - \frac{1}{2}\|\boldsymbol{v}^-\|^2_{\boldsymbol{M}}\\ &+ \frac{1}{2}(\boldsymbol{e}^+ + \boldsymbol{e}^-)^\mathsf{T}\boldsymbol{K}\boldsymbol{K}^\perp\boldsymbol{w}_{\text{RBM}}(\Sigma_1 - \Sigma_2)\end{aligned} \quad (23)$$

where (11)–(12) have been used. The monotonicity of the impact map $\mathcal{H}_{\chi(t)}(\bar{\boldsymbol{\gamma}})$ (Assumption A4) implies non-expansivity of the mapping Z. Substituting (8) and $\boldsymbol{K}\boldsymbol{K}^\perp = \boldsymbol{0}$ into (23) yields $V^+ - V^- \leq 0$. Consequently, the Lyapunov function V cannot increase neither during continuous nor discontinuous flow. Since the Lyapunov function V is bounded from below and non-increasing, the limit

$$V_\infty := \lim_{t\to\infty} V(\boldsymbol{x}_e(t)) = \lim_{t\to\infty}\left(\frac{1}{2}\|\boldsymbol{v}\|^2_{\boldsymbol{M}} + \frac{1}{2}\|\boldsymbol{e}\|^2_{\boldsymbol{K}}\right) \quad (24)$$

exists and lies in the interval $0 \le V_\infty \le V(\boldsymbol{x}_e(t_0))$. Note that $\|\boldsymbol{e}(t)\|^2_{\boldsymbol{K}}$ is absolutely continuous in time and $V(\boldsymbol{x}_e(t))$ tends to an absolutely continuous function (constant function) as shown in (23). From these two observations, we conclude that $\|\boldsymbol{v}(t)\|^2_{\boldsymbol{M}}$ must tend to an absolutely continuous function as well.

From (22) and (23) follows

$$V_\infty - V(\boldsymbol{x}_e^-(t_0)) \le -\lim_{t\to\infty}\int_{t_0}^{t} W(\boldsymbol{x}_e(\tau))\mathrm{d}\tau. \tag{25}$$

Since the left-hand side in (25) is finite, we deduce that

$$\lim_{t\to\infty}\int_{t_0}^{t} W(\boldsymbol{x}_e(\tau))\mathrm{d}\tau = \lim_{t\to\infty}\int_{t_0}^{t} \|\boldsymbol{v}(\tau)\|^2_{\boldsymbol{C}}\mathrm{d}\tau < \infty. \tag{26}$$

We cannot invoke Barbalat's lemma [14] since the solution $\boldsymbol{x}_e(t)$ is not continuous in time. The extension of this lemma for asymptotically absolutely continuous functions is presented in [4]. Applying the extended Barbalat's lemma to (26) yields

$$\lim_{t\to\infty}\|\boldsymbol{v}(t)\|^2_{\boldsymbol{C}} = 0. \tag{27}$$

We introduce an orthogonal decomposition of the position error as $\boldsymbol{e} = \boldsymbol{e}^{\parallel} + \boldsymbol{e}^{\perp}$, where $\boldsymbol{e}^{\parallel} = \boldsymbol{K}^{+}\boldsymbol{K}\boldsymbol{e}$ and $\boldsymbol{e}^{\perp} = \boldsymbol{K}^{\perp}\boldsymbol{e}$. The velocity error $\boldsymbol{v} = \boldsymbol{v}^{\parallel} + \boldsymbol{v}^{\perp}$ is decomposed accordingly. Using this decomposition, (27) yields $\lim_{t\to\infty}\boldsymbol{v}^{\parallel} = \boldsymbol{0}$, since $\boldsymbol{K}$ and $\boldsymbol{C}$ have the same nullspace and, thus, $\boldsymbol{C}\boldsymbol{v} = \boldsymbol{C}\boldsymbol{v}^{\parallel}$.

An impact with $\Lambda_{\mathrm{RBM}} \neq 0$ occurs at least once every time interval $[t, t+T_{\max}]$ due to Assumption A2. Using (23) and Assumption A4, the Lyapunov function is non-decreasing at these time instants only if $\bar{\gamma}_{\mathrm{RBM},1} = \bar{\gamma}_{\mathrm{RBM},2}$ and $\Lambda_{\mathrm{RBM},1} = \Lambda_{\mathrm{RBM},2}$, which imply $\gamma^-_{\mathrm{RBM},1} = \gamma^-_{\mathrm{RBM},2}$. Therefore, the lower bound of V implies $\lim_{t\to\infty}\boldsymbol{v}^{\perp} = \boldsymbol{0}$ using a contradiction argument[2].

Substituting $\lim_{t\to\infty}\boldsymbol{v} = \boldsymbol{0}$ into (24) yields $\lim_{t\to\infty}\boldsymbol{e}^{\parallel} = \boldsymbol{c}$ for some $\boldsymbol{c}$ satisfying $\frac{1}{2}\|\boldsymbol{c}\|^2_{\boldsymbol{K}} \le V(\boldsymbol{x}_e(t_0))$. In the next step, we show that $\boldsymbol{c} = \boldsymbol{0}$.

The error dynamics is governed by the equality of measures

$$\boldsymbol{M}\mathrm{d}\boldsymbol{v} + \boldsymbol{C}\boldsymbol{v}\mathrm{d}t + \boldsymbol{K}\boldsymbol{e}\mathrm{d}t = \boldsymbol{W}(\mathrm{d}\boldsymbol{P}_1 - \mathrm{d}\boldsymbol{P}_2), \tag{28}$$

where $\mathrm{d}\boldsymbol{P}_1 = \boldsymbol{\lambda}_1\mathrm{d}t + \boldsymbol{\Lambda}_1\mathrm{d}\eta$ and $\mathrm{d}\boldsymbol{P}_2 = \boldsymbol{\lambda}_2\mathrm{d}t + \boldsymbol{\Lambda}_2\mathrm{d}\eta$ are the constraint impulse measures. Integrating the equality of measures (28) over the time interval $[t, t+\Delta t]$ yields

$$\begin{aligned}&\boldsymbol{M}(\boldsymbol{v}^+(t+\Delta t) - \boldsymbol{v}^-(t)) + \boldsymbol{C}(\boldsymbol{e}^{\parallel}(t+\Delta t) - \boldsymbol{e}^{\parallel}(t))\\ &+ \boldsymbol{K}\int_t^{t+\Delta t}\boldsymbol{e}^{\parallel}(\tau)\mathrm{d}\tau = \int_{[t,t+\Delta t]}\boldsymbol{W}(\mathrm{d}\boldsymbol{P}_1 - \mathrm{d}\boldsymbol{P}_2),\end{aligned} \tag{29}$$

where $\Delta t > 0$ is arbitrary. For the second term in (29), we used $\boldsymbol{C}\dot{\boldsymbol{e}} = \boldsymbol{C}\boldsymbol{v}$, since $\boldsymbol{K}$ and $\boldsymbol{C}$ have the same nullspace.

[2]We assume the contrary, i.e. $\lim_{t\to\infty}\boldsymbol{v}^{\perp}(t) \neq \boldsymbol{0}$. Then, $\lim_{t\to\infty}\gamma^-_{\mathrm{RBM},1} - \gamma^-_{\mathrm{RBM},2} \neq \boldsymbol{0}$ implies $V^+ - V^- \le -\beta < 0$ for some constant $\beta > 0$ at least once every time interval $[t, t+T_{\max}]$. Therefore, the Lyapunov function decreases unboundedly, which is in contradiction with $V \ge 0$.

It proves useful to introduce the quantity $\tilde{\boldsymbol{\lambda}}_{\Delta t}(t) = \frac{1}{\Delta t}\int_{[t,t+\Delta t]}(\mathrm{d}\boldsymbol{P}_1 - \mathrm{d}\boldsymbol{P}_2)$, which can be regarded as the average constraint force of the error dynamics over the time lapse $[t, t+\Delta t]$. Subsequently, we take the limit $t\to\infty$ and use $\boldsymbol{v}(t) \to \boldsymbol{0}$ and $\boldsymbol{e}^{\parallel}(t) \to \boldsymbol{c}$ for $t\to\infty$. The integrated equality of measures (29), divided by Δt, yields

$$\boldsymbol{K}\boldsymbol{c} = \lim_{t\to\infty}\boldsymbol{W}\tilde{\boldsymbol{\lambda}}_{\Delta t}(t) = \lim_{t\to\infty}\sum_{i=1}^{m}\boldsymbol{w}_i\tilde{\lambda}_{\Delta t,i}(t), \tag{30}$$

where the notation (20) has bee used. Equation (30) describes, in an averaged sense, the equilibrium of forces at infinity. The columns $\boldsymbol{w}_i$ of $\boldsymbol{W}$ are linearly independent from which we deduce that each of the limits $\lim_{t\to\infty}\tilde{\lambda}_{\Delta t,i}(t)$ has to exist. Moreover, it holds that $\lambda_{1,i}(t) = \lambda_{2,i}(t) = 0$ and $\Lambda_{1,i}(t) = \Lambda_{2,i}(t) = 0$ for $\chi_i(t) = 0$. Taking a small enough Δt, we conclude that the limit of $\tilde{\lambda}_{\Delta t,i}(t)$ must vanish, since each switching function $\chi_i(t)$ is of class $\mathbb{K}$ by Assumption A1. Since $\boldsymbol{c}$ is in the range of $\boldsymbol{K}$, we obtain $\boldsymbol{c} = \boldsymbol{0}$.

The Lyapunov function is only positive semi-definite since it is independent of $\boldsymbol{e}^{\perp}$. For $\chi_{\mathrm{RBM}}(t) = 1$, we have $\boldsymbol{e}^{\perp} = \boldsymbol{0}$ according to (15). Otherwise, $\lim_{t\to\infty}\boldsymbol{u} = \boldsymbol{0}$ implies $\lim_{t\to\infty}\boldsymbol{e}^{\perp} = \boldsymbol{0}$, since Assumption A2 together with (16) implies that $\boldsymbol{e}^{\perp +} = \boldsymbol{0}$ at least once every time interval $[t, t+T_{\max}]$. Therefore, we obtain $\lim_{t\to\infty}\boldsymbol{x}_e = \boldsymbol{0}$ which shows global attractivity. Uniform stability of $\boldsymbol{x}_e = \boldsymbol{0}$ does not follow from $\mathrm{d}V \le 0$, since V is only positive semi-definite. Nevertheless, $\boldsymbol{e}^{\perp}$ is bounded by V, since the increase of $\|\boldsymbol{e}^{\perp}\|$ in any interval $[t, t+T_{\max}]$ is bounded by $\|\boldsymbol{v}\|$, which itself is bounded by V. Therefore, $\boldsymbol{x}_e(t) = \boldsymbol{0}$ is uniformly stable, which concludes the proof. ■

We have shown that the master system can be written using the equations of the slave system as shown in Proposition 1. Furthermore, any two solutions of the slave system approach each other due to the a.i.s shown in Theorem 1. Therefore, synchronization of the master–slave system is guaranteed.

IV. Example: Robotic Leg

We illustrate the synchronization-based observer proposed in Section III using the example of a 3-DOF robotic leg as depicted in Fig. 1. The hip is vertically guided and the thigh as well as the shank are connected by rotational joints with spring–damper elements. All three parts have non-zero mass and gravitation is present. The master and the slave system are subjected to a switched unilateral constraint between the foot and the harmonically moving ground. The switching function is generated by the constraint distance of the master system in both cases. Therefore, the master system is subjected to a geometric unilateral constraint according to Proposition 1.

A perfectly elastic generalized Poisson's impact law is chosen for the normal direction. We introduce friction with a sufficiently large friction coefficient in order to prevent the foot from sliding on the ground. The frictional impacts [11] are chosen to be inelastic. Therefore, the friction can be modeled as two opposing switched unilateral constraints acting in the tangential direction together with a perfectly inelastic impact law. Since these constraints are switched on whenever the constraint in the normal direction is closed, it does not change the analysis presented in Section III.

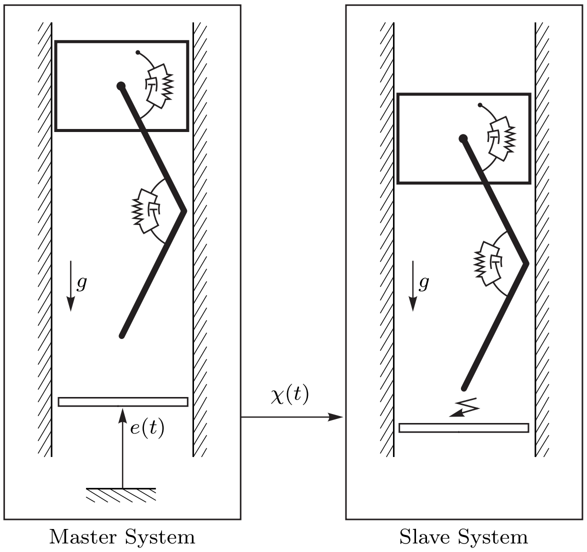

Fig. 1. Master-slave synchronization setup for the hopping leg example. The unidirectional coupling consists of the Boolean switching function $\chi(t)$

The system is nonlinear and therefore not linear time-invariant. Nevertheless, large spring constants are chosen such that the configuration of the leg remains close to the equilibrium position. Therefore, the geometric nonlinearities remain small. The parameters are chosen such that the solution of the master system is chaotic and the jumping height is bounded. Thus, the switching function $\chi(t)$ is of class $\mathbb{K}$ and there exists a maximal time interval for which the constraint is open or superfluous. Since the Assumptions A1–A4 from Section III are fulfilled[3], the system is attractively incrementally stable according to Theorem 1. Therefore, the synchronization error tends to zero and the slave system reproduces the full state of the master system using only the impact time instants.

A simulation of the master–slave system is depicted in Fig. 2. Shown are the foot positions and the foot velocities of the master system (blue) and the slave system (red). The position and velocity of the ground is shown in gray. During the time interval $4 \leq t \leq 8$, the Boolean switching function $\chi(t)$ is generated by the master system and the slave system (having switched constraints) acts as a state observer. For illustrative purposes, the switching function is generated by the slave system itself during the time intervals $0 \leq t < 4$ and $8 < t \leq 16$. Therefore, both systems are subjected to a geometric unilateral constraint during these intervals and there is no coupling. The uncoupled case shows that the master and the slave system do generally not synchronize without any coupling. Furthermore, the system has extreme sensitivity on initial conditions such that the solutions will diverge for any small initial error. Therefore, there is no local synchronization and the zero-solution of the error dynamics is unstable.

[3]The system is a.i.s. although the impact law in normal direction is not inverse strongly monotone. The inverse strong monotonicity is not a necessary condition here, since the rigid body mode is not decoupled from the remaining dynamics.

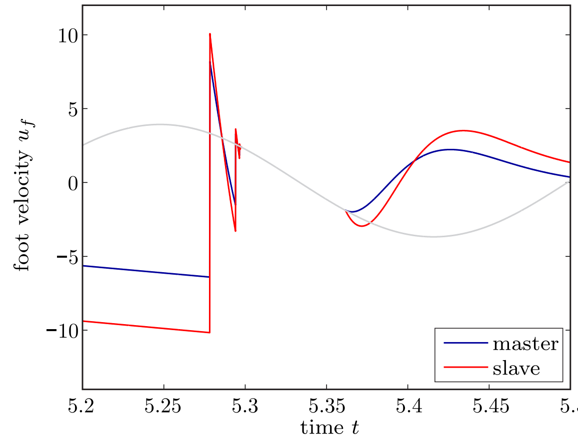

Fig. 3. Magnification of the foot velocities showing a simultaneous accumulation point followed by an interval of persistent contact

Fig. 3 shows a magnification of the foot velocities. The master system and, thus, the slave system repeatedly experience accumulation points, which correspond to infinitely many impacts in a finite time (also called Zeno behavior). This behavior is inherently part of the dynamics also in this case of a perfectly elastic impact law. The accumulation points are followed by an interval of persistent contact and a subsequent flight phase.

The time evolution of the Lyapunov function (21) for this simulation example is depicted in Fig. 4. During the time interval where the observer is active, the impact time instants of the master and slave system coincide. The Lyapunov is discontinuous but non-increasing and it tends to zero. The impact time instants do not coincide during the uncoupled time intervals and the Lyapunov function shows a 'peaking behavior' [5].

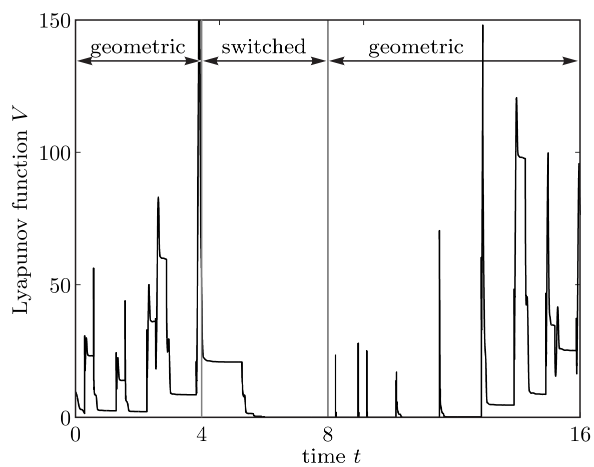

Fig. 4. Time evolution of the Lyapunov function

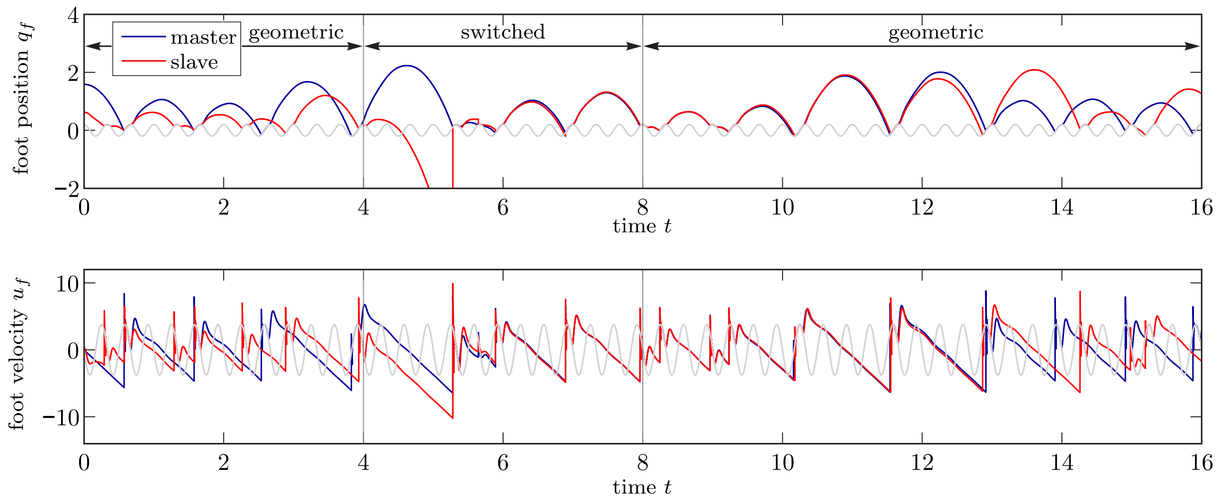

Fig. 2. Foot position and velocity of the master (blue) and slave system (red). The systems are decoupled at the beginning and the end of the simulation for which there is generally no (local) synchronization

V. CONCLUSIONS

The proposed observer reproduces the full state of the observed system using only the impact time instants. The applicability has been extended to linear time-invariant mechanical systems with a rigid body mode. The kinematic equation has been extended with a constraint force, which renders the generalized coordinates discontinuous.

The observer is based on master–slave synchronization, where the solution of the slave system (observer) tends to the solution of the master system (observed system) for every initial condition and also in the presence of accumulation points. In order to obtain the synchronization results, the property of attractive incremental stability has been shown for this type of systems.

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