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LYAPUNOV STABILITY THEORY FOR NON-SMOOTH NON-AUTONOMOUS MECHANICAL SYSTEMS APPLIED TO THE BOUNCING BALL PROBLEM

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ABSTRACT

Global attractive stability conditions for the equilibrium of the bouncing ball system have been proven in this paper using an extension of Lyapunov's direct method to non-autonomous systems. The bouncing ball system is a non-smooth non-autonomous system with a unilateral frictionless constraint and has been extensively studied in the literature as a standard example of chaotic dynamics. The system consists of a point mass in a constant gravitational field, which bounces inelastically on a flat vibrating table.

Furthermore, it is proven that the attractivity of the equilibrium is symptotic, i.e. there exists a finite time for which the solution has converged exactly to the equilibrium. For this attraction time, an upper bound is given in this paper.

NOMENCLATURE

x	scalar
$\boldsymbol{x}$	column-vector in $\mathbb{R}^n$
$\boldsymbol{x}^{\mathrm{T}}$	transpose of $\boldsymbol{x}$
x_i	i^{th} element of $\boldsymbol{x}$
$\|x\|$	absolute value of x
$\dot{x}(t)$	differentiation w. r. t. time of $x(t)$ or density of dx w. r. t. the differential measure dt
$[a,b]$	the closed interval $\{x \in \mathbb{R} \mid a \leq x \leq b\}$
(a,b)	the open interval $\{x \in \mathbb{R} \mid a < x < b\}$
$\{a,b\}$	the set comprising the elements $\{a\}$ and $\{b\}$
$f(\boldsymbol{x})$	single-valued function $\mathbb{R}^n \to \mathbb{R}$
$\boldsymbol{f}(\boldsymbol{x})$	single-valued function $\mathbb{R}^n \to \mathbb{R}^m$
$\Psi_C(\boldsymbol{x})$	indicator function of C at $\boldsymbol{x}$
$\mathrm{d}f$	differential measure of f
$\mathrm{d}t$	(differential) Lebesgue measure
$\mathrm{d}\eta$	(differential) atomic measure
$\boldsymbol{x}^*$	equilibrium point of a dynamical system
A	time-independent admissible set of a measure differential inclusion
K	time-independent set of admissible generalized coordinates
g	gravitational acceleration, $\mathrm{g} = 9.81\,\frac{\mathrm{m}}{\mathrm{s}^2}$
t_n	impact time
$e(t)$	external excitation of the table
q	absolute position of the ball
u	absolute velocity of the ball
g_N	relative distance
γ_N	relative velocity
V	Lapunov function
Γ	relative acceleration of the table

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INTRODUCTION

The stability of non-smooth dynamical systems is a novel research field which is receiving much attention in the mathematical as well as engineering community. Mechanical systems with impact phenomena and unilateral constraints form an important class of non-smooth systems as they arise in many engineering applications [3].

Non-smooth systems with impulsive effects expose discontinuities (or jumps) in the state, i.e. the state $\boldsymbol{x}(t)$ is not defined on discontinuity points t_n. Moreover, the loss of uniqueness and existence of solutions of the initial value problem greatly complicates the mathematical treatment of dynamical systems with perfect unilateral constraints, which restrict the state to an admissible domain A. Such dynamical systems, which expose discontinuities in the state, can be described by measure differential inclusions. The differential measure of the state does not only consist of a part with a density with respect to the Lebesgue measure, but is also allowed to contain an atomic part. The dynamics of the system is described by an inclusion of the differential measure of the state to a state-dependent set. Such measure differential inclusions naturally arise when considering (mechanical) systems with inequality (unilateral) constraints. Consequently, the measure differential inclusion concept describes the continuous dynamics as well as the impulsive dynamics with a single expression.

Despite the huge amount of textbooks and papers on (Lyapunov) stability in various fields of engineering science, Lyapunov stability properties of non-autonomous non-smooth systems described by measure differential inclusions has not been considered so far. Such systems emerge for example in control theory, where tracking control problems with (external) time-dependent impulsive inputs arise.

In order to study Lyapunov stability criteria of equilibria in non-smooth, explicitly time-dependent (i.e. non-autonomous) mechanical systems with unilateral constraints, we attempt to investigate the stability of the equilibrium of an apparently simple mechanical system meeting the requirements of non-smoothness and explicit time-dependence. A standard problem of chaotic dynamics, which has been extensively studied in the literature is a ball in a constant gravitational field which bounces inelastically on a flat vibrating table. The governing equations of motion are highly nonlinear due to the unilateral contact and generally do not allow for any closed form solution. Up to now, there has not been given any necessary conditions for the equilibrium of the bouncing ball system to be globally attractively stable. The vast amount of papers on the bouncing ball system rather deals with the approximate description of chaos, chaotic attractors or control of the bouncing ball system, e.g. to fix a periodic orbit at a defined height [1, 2, 4, 6, 9].

In this paper, we prove conditions for global attractive stability of the equilibrium of the bouncing ball system using an extension of the result of [7, 8] to non-autonomous systems and give a sufficient condition for global symptotic (i.e. finite time) attractive stability of the equilibrium of the bouncing ball system in terms of an inequality between the parameters of the system, the coefficient of restitution ε and the relative acceleration Γ of the harmonically vibrating table.

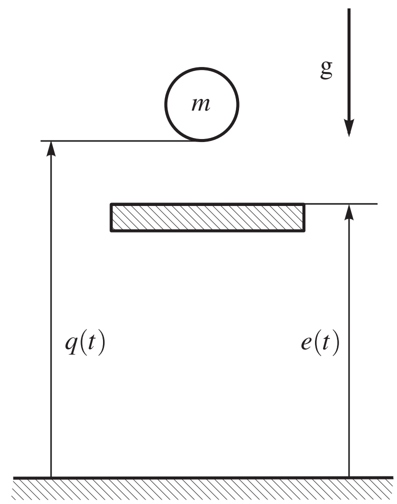

Figure 1. THE BOUNCING BALL SYSTEM.

THE BOUNCING BALL MODEL

The system consists of a rigid ball, modeled as a point mass m, bouncing in a constant gravitational field g on a rigid flat table with infinite mass M such that the movement $e(t)$ of the table is not influenced by collisions with the ball. As is shown in Figure 1, the inertial position of the ball is addressed by the absolute coordinate $q(t)$. To keep the system as simple as possible, only vertical motion is considered, leading to a system with one degree of freedom. The ball is in its equilibrium position when it is in persisting contact with the table, i.e. if it holds that

$$q(t) = e(t) \quad \forall t \geq t^*, \tag{1}$$

where t^* denotes the time when the ball has eventually come to rest on the table. The velocity $u(t)$ of the ball is given as $u(t) = \dot{q}(t)$ during flight. The state variable $u(t)$ is undefined when an impact between the ball and the table occurs, because the velocity jumps instantaneously. Subsequently, a coordinate transformation is made such that the equilibrium of the ball is located in the origin of the corresponding phase space:

$$g_N(t) = q(t) - e(t) \geq 0\,, \tag{2}$$

$$\gamma_N(t) = u(t) - \dot{e}(t)\,. \tag{3}$$

The relative distance $g_N(t)$ between the ball and table is non-negative because of the impenetrability of the rigid ball and table. The state vector $\boldsymbol{x}(t)$ is chosen as

$$\boldsymbol{x}(t) := \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} g_N(t) \\ \gamma_N(t) \end{bmatrix}. \tag{4}$$

The equations of motion are given in terms of a non-autonomous measure differential inclusion

$$\mathrm{d}\boldsymbol{x} \in \begin{bmatrix} \gamma_N \mathrm{d}t \\ -(\mathrm{g}+\ddot{e})\mathrm{d}t + \frac{1}{m}\mathrm{d}P_N \end{bmatrix} =: \mathrm{d}\boldsymbol{\Gamma}(\boldsymbol{x},t)\,, \tag{5}$$

$$A = \{\boldsymbol{x} \in \mathbb{R}^2 | x_1 \geq 0\} \Leftrightarrow K = \{g_N \in \mathbb{R} | g_N \geq 0\}\,, \tag{6}$$

$$g_N = 0: \quad -\mathrm{d}P_N \in N_{\mathbb{R}_0^+}(\xi_N)\,,\ \xi_N = \gamma_N^+ + \varepsilon\gamma_N^-\,, \tag{7}$$

where A is the time-independent set of admissible states and K is the corresponding set of admissible positions $g_N = q(t) - e(t) \geq 0$. Due to the choice of the state $\boldsymbol{x}(t)$ in Eqn. (4), both the sets A and K are time-independent. The state $\boldsymbol{x}(t)$ has to be interpreted as the result of an integration process over the differential measure $\mathrm{d}\boldsymbol{x}$,

$$\boldsymbol{x}^+(t) = \boldsymbol{x}^-(t_0) + \int_{[t_0,t]} \mathrm{d}\boldsymbol{x}\,, \quad t \geq t_0\,, \tag{8}$$

where the integration process takes the left limit $\boldsymbol{x}^-(t_0)$ of the initial value to the right limit $\boldsymbol{x}^+(t)$ of the final value over the compact time interval $[t_0,t]$. The differential measure $\mathrm{d}\boldsymbol{x}$ does therefore not only contain a density $\dot{\boldsymbol{x}}(t)$ with respect to the differential Lebesgue measure $\mathrm{d}t$ but also contains a density $\boldsymbol{x}^+ - \boldsymbol{x}^-$ with respect to the differential atomic measure $\mathrm{d}\eta$,

$$\mathrm{d}\boldsymbol{x} = \dot{\boldsymbol{x}}\,\mathrm{d}t + (\boldsymbol{x}^+ - \boldsymbol{x}^-)\mathrm{d}\eta\,. \tag{9}$$

The atomic part in Eqn. (9) is used to describe discontinuities in $\boldsymbol{x}(t)$. The function $x_2(t) = \gamma_N(t)$ is discontinuous at impact times t_n, whereas $x_1(t) = g_N(t)$ is an absolutely continuous function in time. The upper and lower limits of $\boldsymbol{x}(t)$ at collision times t_n are denoted by $\boldsymbol{x}^+(t_n) := \lim_{t\downarrow t_n} \boldsymbol{x}(t)$ and $\boldsymbol{x}^-(t_n) := \lim_{t\uparrow t_n} \boldsymbol{x}(t)$, respectively. Note that $\int_I (\cdot)\mathrm{d}\eta = 0$ if the function $\boldsymbol{x}(t)$ is absolutely continuous on I. If $\mathrm{d}\boldsymbol{x}$ is integrated over a singleton $\{t_n\}$, then $\int_{\{t_n\}} (\cdot)\mathrm{d}t = 0$ and $\int_{\{t_n\}} \mathrm{d}\boldsymbol{x} = \boldsymbol{x}^+(t_n) - \boldsymbol{x}^-(t_n)$, where the latter reduces to zero if the function $\boldsymbol{x}$ is continuous at t_n.

The total effort P_N assembles the forces that act on the ball when it is in contact with the table and its differential measure reads as $\mathrm{d}P_N = \lambda_N \mathrm{d}t + \Lambda_N \mathrm{d}\eta$, where λ_N is the non-impulsive contact force acting on the ball when the ball is in persisting contact with the table, whereas the impulsive force Λ_N causes the relative velocity γ_N to jump when an impact occurs. The force laws for λ_N and Λ_N can for a closed contact be written as inequality complementary conditions on velocity level. This yields for λ_N:

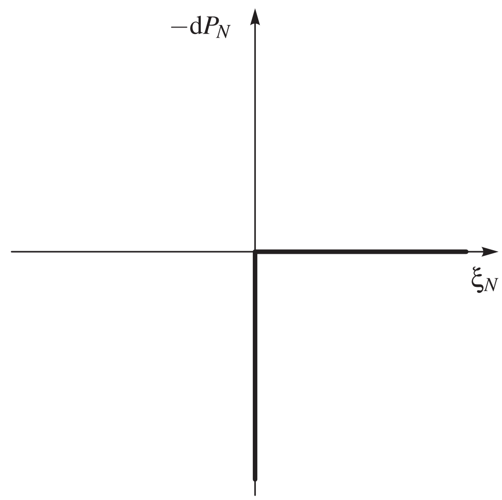

Figure 2. GRAPH OF THE SET-VALUED FORCE LAW (12).

$$g_N = 0: \qquad \gamma_N \geq 0\,, \quad \lambda_N \geq 0\,, \quad g_N\lambda_N = 0\,. \tag{10}$$

For the impulsive force Λ_N, a Newton type of restitution law is used using $\xi_N = \gamma_N^+ + \varepsilon\gamma_N^-$, where $0 \leq \varepsilon \leq 1$ is the coefficient of restitution:

$$g_N = 0: \qquad \xi_N \geq 0\,, \quad \Lambda_N \geq 0 \quad \xi_N\Lambda_N = 0\,. \tag{11}$$

Following [8], the force laws in Eqn. (10) and Eqn. (11) can be cast together in terms of a normal cone of all non-negative real numbers $\mathbb{R}_0^+$ in ξ_N as

$$-\mathrm{d}P_N \in N_{\mathbb{R}_0^+}(\xi_N) = \begin{cases} 0 & \xi_N > 0\,, \\ \mathbb{R}_0^- & \xi_N = 0\,. \end{cases} \tag{12}$$

The graph of the set-valued force law (12) for $\mathrm{d}P_N$ is depicted in Figure 2.

Consequently, the measure differential inclusion (5) evaluated on a non-impulsive interval I reduces to the ordinary differential equation

$$\dot{\gamma}_N(t) = -(\mathrm{g} + \ddot{e}(t)) + \frac{1}{m}\lambda_N$$
$$\Leftrightarrow m\ddot{q} = -m\mathrm{g} + \lambda_N\,,$$

which is the equation of motion of the ball when no impacts occur. Evaluation of the measure differential inclusion (5) over a singleton $\{t_n\}$ yields

$$\gamma_N^+ - \gamma_N^- = \frac{1}{m}\Lambda_N\,,$$

in which we recognize the impact equation.

THE EQUILIBRIUM OF THE BOUNCING BALL SYSTEM

The equilibrium of the measure differential inclusion (5) is given by

$$\boldsymbol{x}^* = \boldsymbol{0}\,, \tag{13}$$

and the ball is therefore in its equilibrium position, when $g_N(t) = 0$ and $\gamma_N(t) = 0$ for all $t \geq t^*$, where t^* is that time instant at which the ball has first come to rest on the table. In the following, we will consider a harmonic excitation $e(t)$ of the table, given by

$$e(t) = -A\sin(\Omega t)\,, \tag{14}$$

where A is the amplitude and Ω the frequency of vibration. If the ball is in persistent contact with the table, i.e. the relative distance $g_N(t)$ vanishes during an interval of time, then the contact force $\lambda_N(t)$ is given by

$$\lambda_N(t) = m(\mathrm{g} + \ddot{e}(t)) = m(\mathrm{g} + A\Omega^2\sin(\Omega t)). \tag{15}$$

The existence of the equilibrium position is guaranteed for all $t \geq t^*$, if the contact force $\lambda_N(t)$ is non-negative for all $t \geq t^*$,

$$\lambda_N(t) = m(\mathrm{g} + A\Omega^2\sin(\Omega t)) \geq 0\,, \quad \forall t \geq t^*\,.$$

Hence, the equilibrium position exists for all t if the following inequality is fulfilled

$$\begin{aligned} &\mathrm{g} - A\Omega^2 \geq 0 \\ \Leftrightarrow\ &\frac{1}{\pi} \geq \Gamma\,, \end{aligned} \tag{16}$$

where

$$\Gamma := \frac{A\Omega^2}{\pi \mathrm{g}} \tag{17}$$

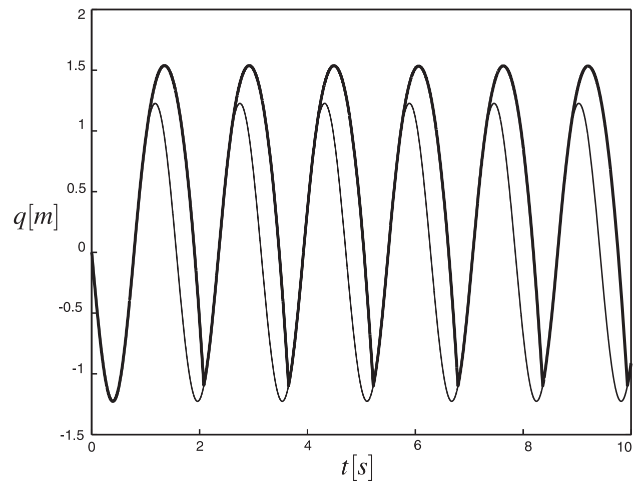

Figure 3. TRAJECTORY OF THE BOUNCING BALL FOR $\Gamma = \frac{2}{\pi} > \Gamma_{\mathrm{GAS}}$, $\varepsilon = 0$, $t_0 = 0\mathrm{s}$, $g_{N,0} = 0\mathrm{m}$, AND $\gamma_{N,0}^- = 0\frac{\mathrm{m}}{\mathrm{s}}$. THE EQUILIBRIUM $\boldsymbol{x}^* = \boldsymbol{0}$ DOES NOT EXIST.

is the relative acceleration of the table compared to gravity g. The definition of Γ with the factor π in its denominator has been chosen in accordance with [2]. If $\Gamma > \frac{1}{\pi}$, then the acceleration of the table will eventually overcome gravity such that the ball detaches from the table as is depicted in Figure 3. To guarantee the existence of the equilibrium, it must therefore hold that $\Gamma \leq \frac{1}{\pi}$.

LYAPUNOV STABILITY OF THE EQUILIBRIUM

Subsequently, the stability properties of the equilibrium of the bouncing ball system are considered using the time-autonomous positive definite Lyapunov function

$$\begin{aligned} V(\boldsymbol{x}) &= \frac{1}{2}x_2^2 + \mathrm{g}x_1 + \Psi_K(x_1) \\ &= \frac{1}{2}\gamma_N^2 + \mathrm{g}g_N + \Psi_K(g_N)\,, \end{aligned} \tag{18}$$

where

$$\Psi_K(x_1) = \begin{cases} 0 & x_1 \in K\,, \\ +\infty & x_1 \notin K\,. \end{cases} \tag{19}$$

is the indicator function on the set $K = \{g_N | g_N \geq 0\}$ of admissible positions $g_N \geq 0$.

The function $V(\boldsymbol{x}) = \frac{1}{2}x_2^2 + \mathrm{g}x_1 + \Psi_K(x_1)$ is positive definite because it is bounded from below by the positive definite function $\tilde{V}(\boldsymbol{x}) = \frac{1}{2}x_2^2 + \mathrm{g}|x_1|$. It is obvious that the indicator function is essential to make V a positive definite function. Note, that $V(\boldsymbol{x}) > 0$ if $\boldsymbol{x} \neq \mathbf{0} = \boldsymbol{x}^*$. Since the function $x_2(t)$ is discontinuous in time, the Lyapunov function V is discontinuous in time along solution curves. The differential measure $\mathrm{d}V$ of the Lyapunov function is decomposed using the differential Lebesgue measure $\mathrm{d}t$ and the differential atomic measure $\mathrm{d}\eta$

$$\mathrm{d}V = \dot{V}\mathrm{d}t + \left(V^+ - V^-\right)\mathrm{d}\eta\,. \tag{20}$$

The bouncing ball system is assumed to be consistent in the sense that $\boldsymbol{x}(t) \in A \; \forall t \geq t_0$ if $\boldsymbol{x}(t_0) \in A$, so it holds that

$$\Psi_K\left(g_N(t)\right) \equiv 0 \quad \forall t\,, \tag{21}$$

and the indicator function does therefore not contribute to the time derivative of V or to $V^+ - V^-$. The time derivative $\dot{V}$ of the Lyapunov function can be expressed, using

$$\dot{g}_N(t) = \gamma_N(t) \tag{22}$$

$$\dot{\gamma}_N(t) = -\mathrm{g} - \ddot{e}(t) = -\mathrm{g} - A\Omega^2\sin(\Omega t)\,, \tag{23}$$

as,

$$\dot{V} = \gamma_N\dot{\gamma}_N + \mathrm{g}\gamma_N = -\ddot{e}(t)\gamma_N(t) = -A\Omega^2\sin(\Omega t)\gamma_N(t)\,. \tag{24}$$

Note that the Lyapunov function (18) has been chosen to be time-independent. However, its time derivative turns out to be explicitly time-dependent due to the external excitation $e(t)$ of the table, which makes it impossible to draw any a priori conclusion with respect to the sign of $\dot{V}$. For $t \in (t_{n-1}, t_n)$, i.e. between two consecutive impacts, the Lyapunov function may therefore eventually decrease or increase, which is shown in Fig. (4).

The jump $V^+ - V^-$ of $V(t)$ at collision times t_n is given by

$$V^+(t_n) - V^-(t_n) = \frac{1}{2}\gamma_N^+(t_n)^2 - \frac{1}{2}\gamma_N^-(t_n)^2 \tag{25}$$

or, using the impact law Eqn. (11), by

$$V^+(t_n) - V^-(t_n) = -\frac{1}{2}(1-\varepsilon^2)\gamma_N^-(t_n)^2 \leq 0\,, \quad \text{for } 0 \leq \varepsilon \leq 1\,. \tag{26}$$

If the coefficient of restitution is smaller than one, then the atomic part of the differential measure $\mathrm{d}V$ is always smaller than zero, which leads to a decrease of the Lyapunov function V at impact times t_n. However, between two consecutive impacts, the Lyapunov function V may decrease or increase and the classical direct method of Lyapunov can therefore not be applied to prove the (attractive) stability of $\boldsymbol{x}^* = \mathbf{0}$ for which it is essential that V decreases in time along solution curves of the system.

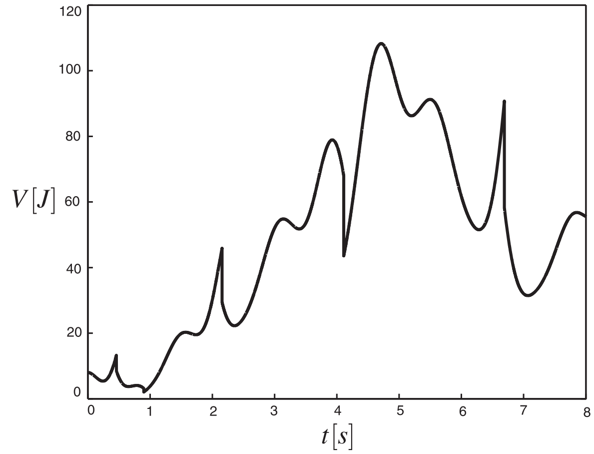

Figure 4. LYAPUNOV CANDIDATE FUNCTION V WITH $\Gamma = \frac{3}{2\pi}$, $t_0 = 0\mathrm{s}$, $\varepsilon = 0.8$, $g_{N,0} = 0\mathrm{m}$ AND $\gamma_{N,0} = -5\frac{\mathrm{m}}{\mathrm{s}}$.

The idea is now to divide the time axis into half open intervals $[t_{n-1}, t_n)$ for which the function ΔV_n is defined as

$$\Delta V_n := V^-(t_n) - V^-(t_{n-1})\,, \tag{27}$$

which can be interpreted as the cumulative change of V over one impact at the time instant t_{n-1} and the subsequent non-impulsive interval $t_{n-1} < t < t_n$. The function ΔV_n can be rewritten as

$$\Delta V_n = \int_{[t_{n-1},t_n)} \mathrm{d}V \tag{28}$$

$$= V^+(t_{n-1}) - V^-(t_{n-1}) + \int_{(t_{n-1},t_n)} \dot{V}\mathrm{d}t \tag{29}$$

$$\leq V^+(t_{n-1}) - V^-(t_{n-1}) + \int_{(t_{n-1},t_n)} |\dot{V}|\mathrm{d}t\,. \tag{30}$$

Since $\dot{V} = -A\Omega^2\sin(\Omega t)\gamma_N(t)$, an upper bound for the latter inequality is obtained as

$$\Delta V_n \leq V^+(t_{n-1}) - V^-(t_{n-1}) + A\Omega^2\int_{(t_{n-1},t_n)} |\gamma_N(t)|\mathrm{d}t\,, \tag{31}$$

where $\int_{(t_{n-1},t_n)} |\gamma_N(t)|\mathrm{d}t$ is the total variation of $g_N(t)$ on the interval (t_{n-1}, t_n).

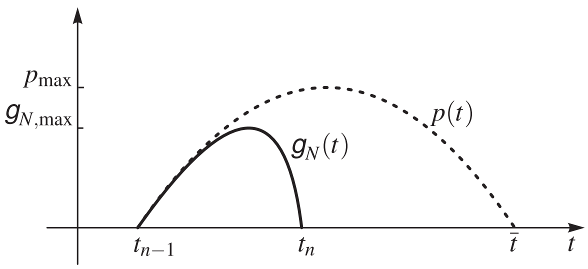

Figure 5. THE FUNCTION $p(t)$ OF CONSTANT SECOND TIME DERIVATIVE, $\ddot{p}(t) = -\mathrm{g} + A\Omega^2$, DRAWN AS DASHED LINE: $p(t_{n-1}) = 0$, $\dot{p}(t_{n-1}) = \gamma_N^+(t_{n-1})$ WITH THE RELATIVE DISTANCE $g_N(t)$: $g_N(t_{n-1}) = g_N(t_n) = 0$. NOTE THAT $p(t) \geq g_N(t)$ $\forall t \in (t_{n-1}, t_n)$.

Using the equations (14,23) and $\Gamma < \frac{1}{\pi}$, it follows that

$$\dot{\gamma}_N = -\mathrm{g} - A\Omega^2 \sin(\Omega t) < 0\,,$$

and since $\mathrm{g} > 0$, $A > 0$ and $\Omega > 0$, an inequality for $\dot{\gamma}_N$ is obtained as

$$-\mathrm{g} - A\Omega^2 \leq \dot{\gamma}_N \leq -\mathrm{g} + A\Omega^2 < 0\,, \quad \text{if } \Gamma < \frac{1}{\pi}\,. \tag{32}$$

Therefore, the relative distance $g_N(t)$ is concave for all $t \in (t_{n-1}, t_n)$, such that the variation of the relative distance in Eqn. (31) amounts to

$$\int_{(t_{n-1},t_n)} |\gamma_N(t)| \mathrm{d}t = 2 \underbrace{\max_{(t_{n-1},t_n)} g_N(t)}_{=: g_{N,\max}}\,. \tag{33}$$

With Eqn. (26) and Eqn. (33), an upper bound for ΔV_n can be given as

$$\Delta V_n \leq -\frac{1}{2}(1-\varepsilon^2)\gamma_N^-(t_{n-1})^2 + 2A\Omega^2 g_{N,\max}\,. \tag{34}$$

In the following, an upper bound for $g_{N,\max}$ has to be found. It is obvious from Eqn. (32) that, if $\Gamma < \frac{1}{\pi}$, then the second derivative of the relative distance is negative for all $t \in (t_{n-1}, t_n)$ and it holds that $-\mathrm{g} - A\Omega^2 \leq \dot{\gamma}_N \leq -\mathrm{g} + A\Omega^2 < 0$. Therefore, the function $g_N(t)$ is concave for $t \in (t_{n-1}, t_n)$ and its second time derivative $\ddot{g}_N(t)$ is smaller than or equal to $-\mathrm{g} + A\Omega^2$, which results in a unique maximum $g_{N,\max}$ of the function $g_N(t)$. Considering the function $g_N(t)$ for $t \in (t_{n-1}, t_n)$, a conservative estimate of $g_{N,\max}$ is obtained by comparison with a function with a constant second time derivative, i.e. by a parabola $p(t)$. For the estimation to be conservative, $\min_{t_{n-1}<t<t_n} |\dot{\gamma}_N(t)|$ is assigned to the second time derivative of the parabola, which is $\ddot{p}(t) = -\mathrm{g} + A\Omega^2$ if $\Gamma < \frac{1}{\pi}$. Additionally, the parabola is chosen to be tangent to the line $\gamma_N(t)$ at $t = t_{n-1}$, such that it holds that $\dot{p}(t_{n-1}) = \gamma_N^+(t_{n-1}) = -\varepsilon\gamma_N^-(t_{n-1}) > 0$ and $p(t_{n-1}) = g_N(t_{n-1}) = 0$. Therefore, the function $p(t)$ is given by

$$p(t) = -\varepsilon\gamma_N^-(t_{n-1})(t - t_{n-1}) + \frac{1}{2}(-\mathrm{g} + A\Omega^2)(t - t_{n-1})^2\,. \tag{35}$$

Since $\dot{p}(t_{n-1}) = \gamma_N^+(t_{n-1})$ and $0 > \ddot{p}(t) = -\mathrm{g} + A\Omega^2 > \ddot{g}_N(t)$, it holds that

$$p(t) \geq g_N(t)\,, \quad \forall t \in (t_{n-1}, t_n) \tag{36}$$

The maximum value of the function $p(t)$ is an upper bound for $g_{N,\max}$

$$p_{\max} = \frac{\varepsilon^2 \gamma_N^-(t_{n-1})^2}{2(\mathrm{g} - A\Omega^2)} \geq g_{N,\max}\,. \tag{37}$$

Equation (37) can now be used in Eqn. (34), which yields

$$\Delta V_n \leq \left(-\frac{1}{2}(1-\varepsilon^2) + \frac{A\Omega^2\varepsilon^2}{\mathrm{g} - A\Omega^2}\right)\gamma_N^-(t_{n-1})^2\,,$$

and can be rewritten using $\Gamma := \frac{A\Omega^2}{\pi \mathrm{g}}$ as

$$\Delta V_n \leq \left(-\frac{1}{2}(1-\varepsilon^2) + \frac{\pi\Gamma\varepsilon^2}{1 - \pi\Gamma}\right)\gamma_N^-(t_{n-1})^2\,. \tag{38}$$

We now set up a non-standard Lyapunov-type agrument. Instead of requiring that $V(t)$ decreases monotonically, we merely require that the net chance of V over the interval $[t_{n-1}, t_n)$ is negative for all $n \in \mathbb{N}$. This requirement puts an inequality condition on the coefficient of restitution ε and the relative acceleration Γ such that $\Delta V_n < 0$ for all n:

$$\Delta V_n < 0\,, \tag{39}$$

which yields

$$-\frac{1}{2}(1-\varepsilon^2) + \frac{\pi\Gamma\varepsilon^2}{1-\pi\Gamma} < 0\,,$$

$$\Leftrightarrow \Gamma < \frac{1-\varepsilon^2}{1+\varepsilon^2}\frac{1}{\pi} =: \Gamma_{\mathrm{GAS}}\,, \tag{40}$$

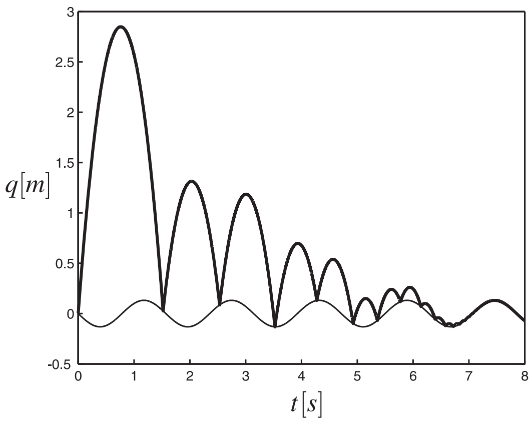

Figure 6. TRAJECTORY OF THE BOUNCING BALL FOR $t_0 = 0$s, $g_{N,0} = 0$m, $\gamma^-_{N,0} = -10\frac{\text{m}}{\text{s}}$, $\varepsilon = 0.8$ AND $\Gamma \approx 0.068 < \Gamma_{\text{GAS}}$.

where the index "GAS" refers to "globally attractively stable". Requiring that $\Delta V_n < 0$, i.e. that the value of the Lyapunov function $V^-(t_n)$ gradually decreases, is in contrast to the classical Lyapunov theory, where the Lyapunov function V is required to decrease in every point in time. Here however, the Lyapunov function (18) may indeed decrease and increase on the interval (t_{n-1}, t_n) as is depicted in Figure 7, but cumulatively, it decreases on this time interval which is guaranteed by the condition obtained in Eqn. (40). A trajectory of the bouncing ball is depicted in Figure 6 for parameters ε and Γ which fulfill Eqn. (40) such that the equilibrium is globally attractively stable.

In the next section, it will be shown that the equilibrium of the bouncing ball system is even globally symptotically attractive if $\Gamma < \Gamma_{\text{GAS}}$, which means that the ball is attracted to the equilibrium $\boldsymbol{x}^* = \boldsymbol{0}$ *within finite time* for any initial condition $\boldsymbol{x}_0$ with $g_N(t_0) \in K$.

GLOBAL SYMPTOTIC ATTRACTIVITY

Without loss of generality, the initial condition of the ball is chosen as $\boldsymbol{x}_0 = \begin{bmatrix} 0 & \gamma^-_N(t_{n-1}) \end{bmatrix}^{\text{T}}$, i.e. just prior to the $(n-1)^{\text{th}}$ impact. Using this specific initial condition $\boldsymbol{x}_0$ does not restrict the set of solutions of the measure differential inclusion (5), because for arbitrary bounded initial conditions $\tilde{\boldsymbol{x}}_0 = \begin{bmatrix} \tilde{g}_N(t_0) & \tilde{\gamma}^-_N(t_0) \end{bmatrix}^{\text{T}} \in A$, there will occur an impact such that the pre-impact state vector can be written as the aforementioned $\boldsymbol{x}_0$ for some t_{n-1} and $\gamma^-_N(t_{n-1})$.

An upper bound for the time duration between two consecutive collisions, shown in Fig. 5, in dependence of the pre-impact relative velocity $\gamma^-_N(t_{n-1})$ is obtained using Eqn. (35) by imposing that $p(\bar{t}) = 0$, which yields

$$t_n - t_{n-1} \le \bar{t} - t_{n-1} = \frac{-2\varepsilon\gamma^-_N(t_{n-1})}{\text{g} - A\Omega^2}\,. \tag{41}$$

Consequently, the time needed for the ball to be symptotically attracted to the equilibrium can be bounded from above as

$$t_\infty - t_{n-1} = \sum_{j=0}^{\infty}(t_{n+j} - t_{n-1+j}) \le \frac{2\varepsilon}{\text{g} - A\Omega^2}\sum_{j=0}^{\infty}|\gamma^-_N(t_{n-1+j})|\,. \tag{42}$$

An upper bound for the absolute value of the pre-impact velocities $|\gamma^-_N(\cdot)|$ is obtained using Eqn. (38)

$$\begin{aligned}\Delta V_n &= \frac{1}{2}\gamma^-_N(t_n)^2 - \frac{1}{2}\gamma^-_N(t_{n-1})^2 \\ &\le \left(-\frac{1}{2}(1-\varepsilon^2) + \frac{\pi\Gamma\varepsilon^2}{1-\pi\Gamma}\right)\gamma^-_N(t_{n-1})^2 < 0\,,\end{aligned}$$

if $\Gamma < \Gamma_{\text{GAS}}$. The absolute value of the pre-impact velocity at t_n is therefore bounded from above in terms of the pre-impact velocity at t_{n-1}:

$$\begin{aligned}&\gamma^-_N(t_n)^2 \le \frac{1+\pi\Gamma}{1-\pi\Gamma}\varepsilon^2\gamma^-_N(t_{n-1})^2 < \gamma^-_N(t_{n-1})^2\,, \\ \Leftrightarrow\ &|\gamma^-_N(t_n)| \le \underbrace{\sqrt{\frac{1+\pi\Gamma}{1-\pi\Gamma}}\varepsilon}_{=:\alpha}|\gamma^-_N(t_{n-1})| < |\gamma^-_N(t_{n-1})|\,.\end{aligned} \tag{43}$$

Therefore, it holds that $\alpha < 1$ if $\Gamma < \Gamma_{\text{GAS}}$ and the sum in Eqn. (42) can be estimated using a geometric series

$$\begin{aligned}t_\infty - t_{n-1} &\le \frac{2\varepsilon}{\text{g} - A\Omega^2}\sum_{j=0}^{\infty}|\gamma^-_N(t_{n-1+j})| \\ &\le \frac{2\varepsilon}{\text{g} - A\Omega^2}|\gamma^-_N(t_{n-1})|\sum_{j=0}^{\infty}\alpha^j \\ &= \frac{2\varepsilon|\gamma^-_N(t_{n-1})|}{\text{g}(1-\pi\Gamma)(1-\alpha)}\,, \quad \Gamma < \Gamma_{\text{GAS}}\,,\end{aligned} \tag{44}$$

where the latter equality holds, because $\alpha < 1$ and the geometric series therefore converges. If $\Gamma < \Gamma_{\text{GAS}}$, then the maximum time for the ball to be attracted to the equilibrium for arbitrary initial conditions $\gamma^-_N(t_{n-1})$ is bounded from above by

$$t_\infty - t_{n-1} \le \frac{2\varepsilon|\gamma^-_N(t_{n-1})|}{\text{g}(1-\pi\Gamma)(1-\varepsilon\sqrt{\frac{1+\pi\Gamma}{1-\pi\Gamma}})}\,, \quad \text{if } \Gamma < \Gamma_{\text{GAS}}\,. \tag{45}$$

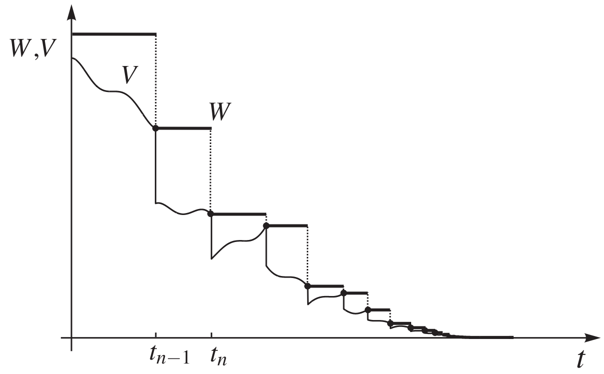

Figure 7. THE LYAPUNOV FUNCTION V FOR $\Gamma < \Gamma_{\text{GAS}}$. THE PIECEWISE CONSTANT FUNCTION $W(t) = V^-(t_{n-1})$ FOR $t \in [t_{n-1}, t_n)$ IS AN UPPER BOUND OF V.

Remarks:

1. From Eqn. (45) it follows that $t_\infty - t_{n-1} = 0$ for $\varepsilon = 0$. For a completely inelastic contact, the ball will "glue" to the table after the first impact.
2. If the table is fixed, then it can be shown that the time, which the ball needs to be attracted to the equilibrium, is exactly $\frac{2\varepsilon|\gamma_N^-(t_{n-1})|}{(1-\varepsilon)g}$. For a non-moving table, the relative acceleration Γ is zero and the upper bound for $t_\infty - t_{n-1}$ in Eqn. (45) amounts to $t_\infty - t_{n-1} \leq \frac{2\varepsilon|\gamma_N^-(t_{n-1})|}{(1-\varepsilon)g}$. Therefore, in the case of $\Gamma = 0$, the estimate given in Eqn. (45) is exact.
3. It holds that the upper bound of $t_\infty - t_{n-1}$ tends to infinity as $\Gamma \rightarrow \Gamma_{\text{GAS}}$, because α tends to 1 and the geometric series used to derive the expression for the right hand side in Eqn. (45) therefore becomes unbounded.

The estimated time for the ball to be symptotically attracted to the equilibrium is accurate for $\Gamma = 0$ and diverges for $\Gamma = \Gamma_{\text{GAS}}$.

The graph of the Lyapunov function V for $\Gamma < \Gamma_{\text{GAS}}$ is depicted in Figure 7. Additionally, a piecewise constant function W is shown, which is an upper bound of V. The function W is defined as

$$W(t) := V^-(t_{n-1}) = \frac{1}{2}\gamma_N^-(t_{n-1})^2, \quad t \in [t_{n-1}, t_n). \qquad (46)$$

The derivation for the expression $t_\infty - t_{n-1}$ in Eqn. (45) has shown that the absolute value of the pre-impact velocities $|\gamma_N^-(t_{n-1})|$ decreases as n increases for $\Gamma < \Gamma_{\text{GAS}}$. Therefore, the function W decreases. Furthermore, W is an upper bound of V for $\Gamma < \Gamma_{\text{GAS}}$, which has been guaranteed by using the total variation of the relative distance g_N on the interval (t_{n-1}, t_n) in Eqn. (30).

CONCLUSIONS

A proof for the global attractive stability for the bouncing ball system has been given which forms the stepping stone to the extension of Lyapunov's direct method to general non-autonomous non-smooth systems. Due to the choice of the state $\boldsymbol{x}(t)$, the admissible set $\mathcal{A}$ of the state and the Lyapunov function V are time-independent. It has become apparent, that the Lyapunov function V is not monotonically decreasing in time due to the harmonic excitation $e(t) = -A\sin(\Omega t)$ of the table. For the proof of global attractive stability of the system, it is thus too stringent to require that $\mathrm{d}V \leq 0$ at every time instant, but rather to think about Lyapunov's direct method in a more general form, i.e. V must not decrease at every time instant but its cumulative change between consecutive impact times must be negative. It has been shown that a monotonically decreasing step function W, converging to zero in time, exists, which equals V just prior to impacts, forming an upper bound for V.

It has been proven, that the equilibrium of the bouncing ball system is globally attractively stable if the parameter space spanned by the relative table acceleration $\Gamma = \frac{A\Omega^2}{\pi g}$ and the coefficient of restitution ε is restricted by the condition obtained in Eqn. (40), namely $\Gamma < \frac{1-\varepsilon^2}{1+\varepsilon^2}\frac{1}{\pi} =: \Gamma_{\text{GAS}}$. Under this condition, the equilibrium of the bouncing ball system is even globally symptotically attractively stable, i.e. the system converges to the equilibrium $\boldsymbol{x}^* = \mathbf{0}$ in finite time, for which an upper bound has been given in Eqn. (45).

This condition for global symptotic attractivity of the equilibrium of the bouncing ball system is sufficient but not necessary. The explicit time-dependence of $\dot{V}$ necessitates crude estimates that lead to a rather restrictive condition (Eqn. (40)) on the relative acceleration Γ. Future research will therefore focus on finding a less conservative estimation than obtained by the inequality (40). Furthermore, Lyapunov stability theorems only give sufficient conditions for stability (and sometimes attractivity), they do not state whether these conditions are also necessary. A different Lyapunov function as has been used in Eqn. (18) has to be found to obtain a less conservative estimate as in Eqn. (40). Generally, finding Lyapunov functions to prove stability (and attractivity) is a highly demanding task.

An interesting application of the stability results of the bouncing ball system derived here may be found in control theory. Certain classes of dynamical systems may be controlled such that the associated closed loop system has the form of an impact oscillator like the bouncing ball system which may open control strategies for non-smooth non-autonomous systems.

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