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# Parametric excitation of non-smooth systems: the unilaterally constrained Hill's equation

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## Abstract

The aim of this paper is to give a Lyapunov stability analysis of a parametrically excited impact oscillator, i.e. a vertically driven pendulum which can collide with a support. The impact oscillator with parametric excitation is described by Hill's equation with a unilateral constraint, being an archetype of a parametrically excited non-smooth dynamical system with state jumps. The exact stability criteria of the unilaterally constrained Hill's equation are rigorously derived using Lyapunov techniques and are expressed in the properties of the fundamental solutions of the unconstrained Hill's equation. Furthermore, an asymptotic approximation method for the critical restitution coefficient is presented based on Hill's infinite determinant and this approximation can be made arbitrarily accurate. A comparison of numerical and theoretical results is presented for the unilaterally constrained Mathieu equation.

## 1 Introduction

The objective of the paper is to give more insight in the stability properties of non-smooth dynamical systems with parametric excitation. The impact oscillator with parametric excitation is studied which is described by Hill's equation with a unilateral constraint [2].

The theory of parametrically excited systems has applications in a wide range of disciplines, e.g. parametric amplifiers, rotor dynamical instabilities, parametric resonance in power transmission belts and celestial mechanics. The vertically driven pendulum (or Kapitza pendulum), of which the suspension point is driven periodically up and down (see Figure 1a), is one of the simplest mechanical systems with parametric excitation. The dynamics of the angle $\vartheta$ is expressed by

$$\ddot{\vartheta}(t) + \left(\frac{g}{l} - \frac{a(t)}{l}\right)\sin\vartheta(t) = 0, \tag{1}$$

where $a(t) = \ddot{u}(t)$ is the vertical acceleration of the suspension point. The linearization in the vicinity of its equilibrium positions $\vartheta = 0$ and $\vartheta = \pi$

$$\ddot{y}(t) + g(t)y(t) = 0, \tag{2}$$

where $g(t)$ is a periodic function, is known as Hill's equation.

Systems with some degree of non-smoothness are often referred to as non-smooth dynamical systems of which mechanical systems with impact and/or friction form an important subclass. The stability properties of non-smooth (mechanical) systems is currently receiving much attention [3]. However, the stability of equilibria of parametrically excited non-smooth systems has hardly been addressed.

In this paper, a detailed Lyapunov stability analysis is presented of the unilaterally constrained Hill's equation with restitution coefficient $\varepsilon$,

$$\begin{aligned} &\ddot{x}(t) + g(t)x(t) = 0, \\ &x(t_i) = 0: \quad \dot{x}^+(t_i) = -\varepsilon\dot{x}^-(t_i), \end{aligned} \tag{3}$$

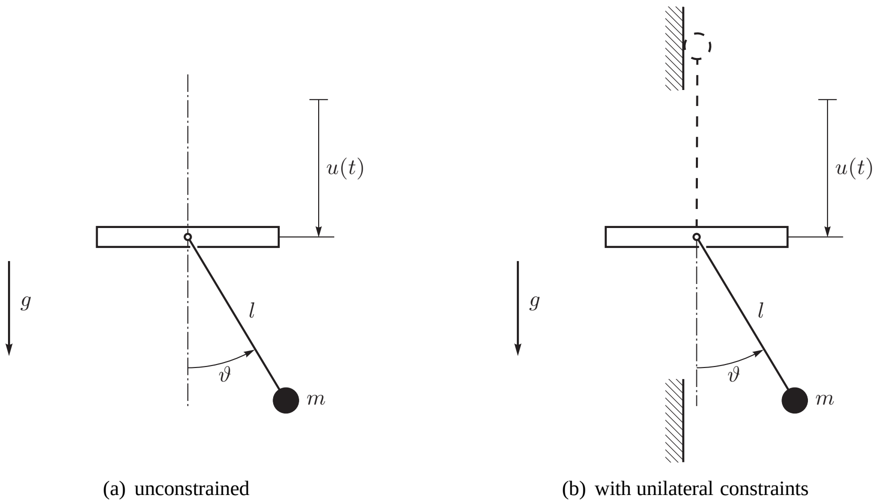



**Figure 1.** Vertically driven pendulum.

being an archetype of a parametrically excited non-smooth dynamical system with state jumps. The unilateral constraint limits the motion to $x(t) \geq 0$ and imposes a Newtonian impact law $\dot{x}^+ = -\varepsilon \dot{x}^-$. The unilaterally constrained Hill's equation (3) describes the dynamics in the vicinity of the equilibrium positions of a vertically driven pendulum with a vertical wall, limiting the angle $\vartheta$ to $0 \leq \vartheta(t) \leq \pi$ (Figure 1b).

## 2 Properties of Hill's equation

Hill's equation (2) has two continuously differentiable solutions $y_1(t)$ and $y_2(t)$ with the initial conditions

$$y_1(0) = 1, \quad \dot{y}_1(0) = 0, \quad y_2(0) = 0, \quad \dot{y}_2(0) = 1,$$

which are usually referred to as normalized solutions or fundamental solutions. The fundamental solutions of (2) can be expressed in polar coordinates (see [4]) by

$$y_1(t) = \varrho(t) \cos \psi(t), \qquad y_2(t) = \varrho(t) \sin \psi(t), \tag{4}$$

with $\varrho(t) > 0$ for all $t$ and the differential equations

$$\ddot{\varrho}(t) - \frac{1}{\varrho(t)^3} + g(t)\varrho(t) = 0, \qquad \psi(t) = \int_0^t \frac{\mathrm{d}t}{\varrho(t)^2}, \tag{5}$$

with initial conditions $\varrho(0) = 1$, $\dot{\varrho}(0) = 0$, $\psi(0) = 0$ and $\dot{\psi}(0) = 1$.

Hill's equation can be put in first-order form as

$$\dot{\boldsymbol{y}}(t) = \boldsymbol{A}(t)\boldsymbol{y}(t), \qquad \boldsymbol{A}(t) = \begin{pmatrix} 0 & 1 \\ -g(t) & 0 \end{pmatrix}, \tag{6}$$

with the state vector $\boldsymbol{y}(t) = \begin{pmatrix} y(t) & \dot{y}(t) \end{pmatrix}^{\mathrm{T}}$ and the time-dependent system matrix $\boldsymbol{A}(t)$. The fundamental solutions constitute the fundamental solution matrix

$$\boldsymbol{\Phi}(t,0) = \begin{pmatrix} y_1(t) & y_2(t) \\ \dot{y}_1(t) & \dot{y}_2(t) \end{pmatrix},$$

which is therefore the solution of the matrix differential equation $\dot{\boldsymbol{\Phi}}(t,0) = \boldsymbol{A}(t)\,\boldsymbol{\Phi}(t,0)$ with initial condition $\boldsymbol{\Phi}(0,0) = \boldsymbol{I}$. The system (6) has a unit Wronskian $\det(\boldsymbol{\Phi}(t,0)) = e^{\int_0^t \mathrm{trace}(\boldsymbol{A}(t))\mathrm{d}t} = 1$, because $\mathrm{trace}(\boldsymbol{A}(t)) = 0$. The fundamental solution matrix $\boldsymbol{\Phi}(t,0)$ maps the initial condition $\boldsymbol{y}(0)$ to the state $\boldsymbol{y}(t)$

$$\boldsymbol{y}(t) = \boldsymbol{\Phi}(t,0)\boldsymbol{y}(0). \tag{7}$$

More generally, the fundamental solution matrix $\boldsymbol{\Phi}(t_1, t_0)$ is defined as the mapping $\boldsymbol{y}(t_1) = \boldsymbol{\Phi}(t_1, t_0)\boldsymbol{y}(t_0)$ which fulfills the matrix differential equation $\dot{\boldsymbol{\Phi}}(t, t_0) = \boldsymbol{A}(t)\,\boldsymbol{\Phi}(t, t_0)$ with $\boldsymbol{\Phi}(t_0, t_0) = \boldsymbol{I}$. Furthermore, the transition property $\boldsymbol{\Phi}(t_2, t_0) = \boldsymbol{\Phi}(t_2, t_1)\,\boldsymbol{\Phi}(t_1, t_0)$ holds from which one can deduce the inverse of the fundamental solution matrix to be $\boldsymbol{\Phi}(t_1, t_0) = \boldsymbol{\Phi}(t_0, t_1)^{-1}$. The fundamental solution matrix $\boldsymbol{\Phi}_T = \boldsymbol{\Phi}(\pi, 0)$ is referred to as the monodromy matrix. The trace of the monodromy matrix

$$\Delta = \mathrm{trace}(\boldsymbol{\Phi}_T) = y_1(\pi) + \dot{y}_2(\pi) \tag{8}$$

is known as the discriminant of Hill's equation [4]. The monodromy matrix $\boldsymbol{\Phi}_T$ has the characteristic equation $\lambda^2 - \Delta\lambda + 1 = 0$, because $\det(\boldsymbol{\Phi}_T) = 1$, and the eigenvalues

$$\lambda_{1,2} = \frac{1}{2}\Delta \pm \frac{1}{2}\sqrt{\Delta^2 - 4}, \tag{9}$$

which are called characteristic multipliers (or Floquet multipliers). The characteristic multipliers are reciprocal in the sense that $\lambda_1 = 1/\lambda_2$. The discriminant $\Delta = \lambda_1 + \lambda_2$ plays an essential role in the analysis of Hill's equation and it is useful to distinguish between three cases. If it holds that $|\Delta| < 2$, then the characteristic multipliers $\lambda_1 = \overline{\lambda}_2$ are complex conjugated and located on the unit circle because $|\lambda_{1,2}| = 1$. If it holds that $|\Delta| > 2$, then the characteristic multipliers $\lambda_1$ and $\lambda_2$ are real and distinct and we define the order by $|\lambda_1| \geq |\lambda_2|$. There exist two linearly independent real eigenvectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. If $y_2(\pi) \neq 0$, then the eigenvectors are given by

$$\boldsymbol{v}_1 = \frac{\mathrm{sign}(y_2(\pi))}{\sqrt{y_2(\pi)^2 + (\lambda_1 - y_1(\pi))^2}} \begin{pmatrix} y_2(\pi) \\ \lambda_1 - y_1(\pi) \end{pmatrix}, \quad \boldsymbol{v}_2 = \frac{\mathrm{sign}(y_2(\pi))}{\sqrt{y_2(\pi)^2 + (\lambda_2 - y_1(\pi))^2}} \begin{pmatrix} y_2(\pi) \\ \lambda_2 - y_1(\pi) \end{pmatrix}. \tag{10}$$

The normalization is chosen such that $\boldsymbol{v}_{1,2}$ lie in the first or fourth quadrant and $\|\boldsymbol{v}_{1,2}\| = 1$. If $y_2(\pi) = 0$ then either $\lambda_1 - y_1(\pi) = 0$ or $\lambda_2 - y_1(\pi) = 0$ and the expression (10) for $\boldsymbol{v}_1$ or $\boldsymbol{v}_2$ degenerates. In this case there still exist two linearly independent real eigenvectors of which either $\boldsymbol{v}_1$ or $\boldsymbol{v}_2$ is equal to $\begin{pmatrix} 0 & 1 \end{pmatrix}^{\mathrm{T}}$. Lastly, if it holds that $|\Delta| = 2$, then the characteristic multipliers are equal $\lambda_1 = \lambda_2 = \Delta/2 = \pm 1$.

Hill's equation has two (complex) eigensolutions

$$f_1(t) = e^{\sigma t} p_1(t), \quad f_2(t) = e^{-\sigma t} p_2(t), \tag{11}$$

where $p_i(t) = p_i(t + \pi)$, $i = 1, 2$, are complex periodic functions and $\sigma$ is the characteristic exponent defined by $e^{\pm\pi\sigma} = \lambda_{1,2}$, i.e. $\Delta = \lambda_1 + \lambda_2 = 2\cosh(\pi\sigma)$.

In Section 3 on the stability analysis of the unilaterally constrained Hill's equation, it will be of interest to know the number of zeros of the fundamental solution $y_2(t)$ on the half-open interval $(0, \pi]$. The floor function $\lfloor \cdot \rfloor$ and fractional part $\{\cdot\}$ will be used to count the zeros.

**Proposition 1**
*Let $n$ denote the number of zeros of $y_2(t)$ on the interval $(0, \pi]$. It holds that*

$$n = \left\lfloor \frac{1}{\pi} \int_0^{\pi} \frac{\mathrm{d}t}{y_1(t)^2 + y_2(t)^2} \right\rfloor. \tag{12}$$

**Proof:** Using polar coordintes one may write $y_2(t) = \varrho(t) \sin\psi(t)$ with $\varrho(t) > 0$ for all $t$. The zeros of $y_2(t)$ are therefore given by the condition $\sin\psi(t) = 0$, i.e. $\psi(t) = \pi, 2\pi, 3\pi, \ldots$ and (12) therefore holds with $\psi(\pi) = \int_0^{\pi} \varrho(t)^{-2} \mathrm{d}t$. □

Consider a solution $y(t)$ of (2) with initial condition $y(0) = r_0 \cos\theta_0 \geq 0$ and $\dot{y}(0) = r_0 \sin\theta_0 > 0$, where $r_0 > 0$ and $\theta_0 \in (-\frac{\pi}{2}, \frac{\pi}{2}]$. With $m(t) \in \mathbb{N}_0$ we will denote the right-continuous monotonically increasing step function which counts the number of reflections in the interval $(0, t]$ such that $(-1)^{m(t)} y(t) = |y(t)|$. It can easily be verified that $m(0) = 0$ and that $m(t)$ equals the number of zeros of $y(t)$ on the interval $(0, t]$. In the following, we will be interested in the number $m(\pi)$ and use the short-hand notation $m = m(\pi)$. By definition, it holds that $m = n$ for $\theta_0 = \frac{\pi}{2}$, because $n$ is defined as the number of zeros of $y_2(t)$ on the interval $(0, \pi]$ and $y(t) = r_0 y_2(t)$ for $\theta_0 = \frac{\pi}{2}$. The calculation of $m$ is given by the following proposition.

**Proposition 2**
*Consider a solution $y(t)$ of (2) with initial condition $y(0) = r_0 \cos\theta_0 \geq 0$ and $\dot{y}(0) = r_0 \sin\theta_0$, where $r_0 > 0$ and $-\frac{\pi}{2} < \theta_0 \leq \frac{\pi}{2}$. Let $m$ denote the number of zeros of $y(t)$ on the interval $(0, \pi]$. It holds that*

$$m = \begin{cases} n & \text{if } \theta_0 > \theta_c, \\ n+1 & \text{if } \theta_0 \leq \theta_c, \end{cases} \qquad \text{with} \quad \theta_c = -\arctan\left(\frac{y_1(\pi)}{y_2(\pi)}\right). \tag{13}$$

**Proof:** From (7) follows that the solution $y(t)$ is given by the linear combination $y(t) = y_1(t)y(0) + y_2(t)\dot{y}(0)$ which we write in polar coordinates as

$$y(t) = \varrho(t) r_0 \left(\cos\theta_0 \cos\psi(t) + \sin\theta_0 \sin\psi(t)\right) = \varrho(t) r_0 \cos(\psi(t) - \theta_0)$$

with $\varrho(t) > 0$. The zeros of $y(t)$ are therefore determined by the condition $\cos(\psi(t) - \theta_0) = 0$ or equivalently $\sin(\psi(t) - \theta_0 + \frac{\pi}{2}) = 0$. The number of zeros of $y(t)$ therefore amounts to

$$m = \left\lfloor \frac{\psi(\pi) - \theta_0 + \frac{\pi}{2}}{\pi} \right\rfloor - \left\lfloor \frac{-\theta_0 + \frac{\pi}{2}}{\pi} \right\rfloor, \qquad -\frac{\pi}{2} < \theta_0 \leq \frac{\pi}{2}. \tag{14}$$

which gives together with (12) the inequality $n \leq m \leq n+1$. It follows that $m = n$ if and only if $\cot(\psi(\pi)) + \cot(-\theta_0 + \frac{\pi}{2}) > 0$, i.e. $\theta_0 > \theta_c = -\arctan\left(\cot(\psi(\pi))\right)$ where $\cot(\psi(\pi)) = y_1(\pi)/y_2(\pi)$. If the inequality does not hold, then $m \neq n$ and $m$ must therefore be equal to $n+1$. □

**Remark:** Some care needs to be taken for the case when $y_2(\pi) = 0$, i.e. $\psi(\pi) = n\pi$, $n \in \mathbb{N}_0$. In this case it holds that $\cot(\psi(\pi)) = +\infty$ which implies that $m = n$. Moreover, if $\psi(\pi) = n\pi$, then it can immediately be verified from (14) that $m = n$.

## 3 The unilaterally constrained Hill's equation

In this section the unilaterally constrained Hill's equation is analyzed which consists of Hill's differential equation

$$\ddot{x}(t) + g(t)x(t) = 0, \tag{15}$$

which holds for almost all $t$ and $x(t) \geq 0$, and the Newtonian impact law

$$\dot{x}^+(t_i) = -\varepsilon \dot{x}^-(t_i) \tag{16}$$

at time-instants for which $x(t_i) = 0$. With the notation $\dot{x}^-(t_i) = \lim_{t\uparrow t_i} \dot{x}(t)$ and $\dot{x}^+(t_i) = \lim_{t\downarrow t_i} \dot{x}(t)$ we denote the pre- and post-impact velocity and with $\varepsilon \in [0, 1]$ the restitution coefficient. The velocity $\dot{x}(t)$ of the unilaterally constrained Hill's equation is considered to be right-continuous by convention, i.e. $\dot{x}(t) = \dot{x}^+(t)$, and the initial condition $(x(0), \dot{x}(0))$ is likewise considered to be a post-impact state. Hence, if the initial condition is written in polar coordinates

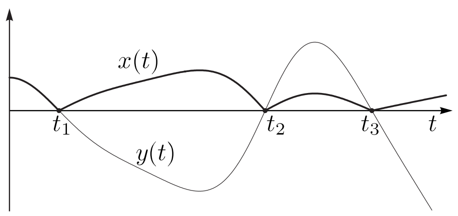
$$x(0) = r_0 \cos\theta_0, \qquad \dot{x}(0) = r_0 \sin\theta_0,$$

then it necessarily holds that $r_0 \geq 0$ and $\theta_0 \in (-\frac{\pi}{2}, \frac{\pi}{2}]$. The solution $x(t)$ of the unilaterally constrained Hill's equation is therefore confined to the domain

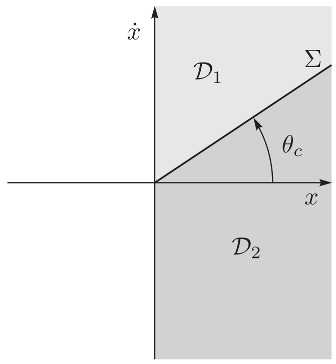
$$\mathcal{D} = \left\{ \boldsymbol{x} = \begin{pmatrix} x & \dot{x} \end{pmatrix}^{\mathrm{T}} \in \mathbb{R}^2 \,\middle|\, x = r\cos\theta,\ \dot{x} = r\sin\theta,\ r \geq 0,\ -\frac{\pi}{2} < \theta \leq \frac{\pi}{2} \right\}. \tag{17}$$

The impact law (16) becomes active when $x(t_i) = 0$ and induces a jump $\dot{x}^+(t_i) = -\varepsilon \dot{x}^-(t_i)$ in the velocity whereas the position $x(t)$ remains continuous at the collision time-instant, i.e. $x^+(t_i) = x^-(t_i) = x(t_i) = 0$. The impact law can therefore formally be written for position and velocity as a homogeneous map

$$x^+(t_i) = -\varepsilon x^-(t_i), \qquad \dot{x}^+(t_i) = -\varepsilon \dot{x}^-(t_i), \tag{18}$$




(a)




(b)

**Figure 2.** Solutions $x(t)$ and $y(t)$ of the unilaterally constrained/unconstrained Hill's equation and cones $\mathcal{D}_1$ and $\mathcal{D}_2$.

because $x^+(t_i) = x^-(t_i) = 0$. The homogeneity of the impact conditions (18) is due to the fact that the unilateral constraint is located at $x = 0$. A nonzero location of the unilateral constraint would give a completely different type of dynamics and is not studied in this paper.

The homogeneity of the linear differential equation (15) and the homogeneity of the impact conditions (18) have important consequences for the solution $x(t)$ of the unilaterally constrained Hill's equation. Consider a solution curve $x(t)$ of the unilaterally constrained Hill's equation as well as the solution curve $y(t)$ of the unconstrained Hill's equation, see Figure 2(a), both with the same initial condition $x(0) = y(0)$ and $\dot{x}(0) = \dot{y}(0)$ such that $\begin{pmatrix} x(0) & \dot{x}(0) \end{pmatrix}^{\mathrm{T}} \in \mathcal{D}$. It holds that $x(t) = y(t)$ during the interval $t \in [0, t_1]$, where $t_1$ is the time-instant for which $x(t_1) = y(t_1) = 0$. At $t = t_1$ an impact occurs in the unilaterally constrained Hill's equation. The solution $x(t)$ after $t = t_1$ is reflected and scaled with $\varepsilon$, i.e. $x(t) = -\varepsilon y(t)$ for $t \in [t_1, t_2]$ due to the linearity and homogeneity of (15). The second impact occurs when $x(t)$ becomes again zero, which is the next zero of $y(t)$. The number of zeros of $y(t)$ in the interval $(0, t]$ is given by $m(t)$, see Proposition 2. The solution $x(t)$ is therefore directly related to $y(t)$ and $m(t)$ through $x(t) = (-\varepsilon)^{m(t)} y(t)$ and, using $m = m(\pi)$, we obtain the relationship

$$x(\pi) = (-\varepsilon)^m y(\pi). \tag{19}$$

The direct relationship between the solutions of the unilaterally constrained and unconstrained Hill's equation can be expressed in first-order form and be related to the fundamental solution matrix. Using $\boldsymbol{x}(t) = \begin{pmatrix} x(t) & \dot{x}(t) \end{pmatrix}^{\mathrm{T}}$, the system is written in first-order form as

$$\begin{aligned} &\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(t)\boldsymbol{x}(t) \\ &\boldsymbol{n}^{\mathrm{T}}\boldsymbol{x}^-(t_i) = 0: \quad \boldsymbol{x}^+(t_i) = \boldsymbol{G}\boldsymbol{x}^-(t_i) \end{aligned} \tag{20}$$

with the system matrix $\boldsymbol{A}(t)$, the vector $\boldsymbol{n} = \begin{pmatrix} 1 & 0 \end{pmatrix}^{\mathrm{T}}$ and impact map $\boldsymbol{G} = -\varepsilon \boldsymbol{I}$. The homogeneity of the impact conditions (18) allows us to write the impact map $\boldsymbol{G}$ as $-\varepsilon\boldsymbol{I}$ and the matrix $\boldsymbol{G}$ therefore commutes with any arbitrary matrix. The solution of the unilaterally constrained Hill's equation can be obtained by concatenation of non-impulsive motion given by Hill's differential equation and the impact equations. The non-impulsive motion between two consecutive collision time-instants $t_i$ and $t_{i+1}$ is described by the linear homogeneous differential equation $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(t)\boldsymbol{x}(t)$. The fundamental solution matrix $\boldsymbol{\Phi}(t_{i+1}, t_i)$ therefore maps the post-impact state $\boldsymbol{x}^+(t_i)$ to the pre-impact state $\boldsymbol{x}^-(t_{i+1})$,

$$\boldsymbol{x}^-(t_{i+1}) = \boldsymbol{\Phi}(t_{i+1}, t_i)\, \boldsymbol{x}^+(t_i), \tag{21}$$

and it necessarily holds that $\boldsymbol{n}^{\mathrm{T}}\boldsymbol{x}^{\pm}(t_i) = \boldsymbol{n}^{\mathrm{T}}\boldsymbol{x}^{\pm}(t_{i+1}) = 0$. The impulsive motion at each collision time-instant $t_i$ is described by the impact map $\boldsymbol{G}$. Concatenation of non-impulsive and impulsive motion gives

$$\boldsymbol{x}(\pi) = \boldsymbol{\Phi}(\pi, t_m)\, \boldsymbol{G}\, \boldsymbol{\Phi}(t_m, t_{m-1})\, \boldsymbol{G} \ldots \boldsymbol{\Phi}(t_2, t_1)\, \boldsymbol{G}\, \boldsymbol{\Phi}(t_1, 0)\boldsymbol{x}(0). \tag{22}$$

The impact map $\boldsymbol{G} = -\varepsilon \boldsymbol{I}$ commutes with all the fundamental solution matrices $\boldsymbol{\Phi}(\cdot,\cdot)$ in (22) and equation (22) therefore simplifies to

$$\begin{aligned} \boldsymbol{x}(\pi) &= \boldsymbol{G}^m \boldsymbol{\Phi}(\pi, t_m)\, \boldsymbol{\Phi}(t_m, t_{m-1}) \ldots \boldsymbol{\Phi}(t_2, t_1)\, \boldsymbol{\Phi}(t_1, 0)\, \boldsymbol{x}(0) \\ &= \boldsymbol{G}^m \boldsymbol{\Phi}(\pi, 0)\, \boldsymbol{x}(0) \\ &= (-\varepsilon)^m \boldsymbol{\Phi}_T\, \boldsymbol{x}(0) \end{aligned} \tag{23}$$

in which the transition property of the fundamental solution matrix and the notation $\boldsymbol{\Phi}_T = \boldsymbol{\Phi}(\pi, 0)$ for the monodromy matrix has been used. Let $\boldsymbol{y}(t)$ denote the solution of Hill's equation with $\boldsymbol{y}(0) = \boldsymbol{x}(0)$. From (23) is becomes apparent that one can relate the solution $\boldsymbol{x}(\pi)$ of the unilaterally constrained Hill's equation to $\boldsymbol{y}(\pi)$ and $m$

$$\boldsymbol{x}(\pi) = (-\varepsilon)^m \boldsymbol{\Phi}_T\, \boldsymbol{y}(0) = (-\varepsilon)^m \boldsymbol{y}(\pi), \tag{24}$$

which is the same result as (19) but now in first-order form. The number $m$ depends on the angle $\theta_0$ of the initial condition $\boldsymbol{x}(0)$ and $m = n$ if $\theta_0 > \theta_c$ and $m = n + 1$ else, see Proposition 2. The domain $\mathcal{D}$, defined in (17), is therefore split into two cones $\mathcal{D}_1$ and $\mathcal{D}_2$

$$\begin{aligned} \mathcal{D}_1 &= \{\boldsymbol{x} \in \mathcal{D} \mid \theta > \theta_c\}, \\ \mathcal{D}_2 &= \{\boldsymbol{x} \in \mathcal{D} \mid \theta \leq \theta_c\}, \end{aligned} \tag{25}$$

by a half-hyperplane $\Sigma = \{\boldsymbol{x} \in \mathcal{D} \mid \theta = \theta_c\}$, where $\theta_c$ is defined by (13), such that $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$ and $\mathcal{D}_1 \cap \mathcal{D}_2 = \{\boldsymbol{0}\}$, see Figure 2(b). If $y_2(\pi) = 0$ then it holds that $\theta_c = -\frac{\pi}{2}$ and $\mathcal{D}_1 = \mathcal{D}$. The switching of the number $m$ on the half-hyperplane $\Sigma$ leads together with (24) to a piece-wise linear, or, more precisely, a cone-wise linear Poincaré map

$$\boldsymbol{x}_{j+1} = \begin{cases} \boldsymbol{A}_1\, \boldsymbol{x}_j & \text{if } \boldsymbol{x}_j \in \mathcal{D}_1, \\ \boldsymbol{A}_2\, \boldsymbol{x}_j & \text{if } \boldsymbol{x}_j \in \mathcal{D}_2, \end{cases} \tag{26}$$

where $\boldsymbol{A}_1 = (-\varepsilon)^n \boldsymbol{\Phi}_T$ and $\boldsymbol{A}_2 = -\varepsilon \boldsymbol{A}_1 = (-\varepsilon)^{n+1} \boldsymbol{\Phi}_T$. The cone-wise linearity of the Poincaré map suggests to analyze the map by using polar coordinates $\boldsymbol{x}_j = \begin{pmatrix} r_j \cos\theta_j & r_j \sin\theta_j \end{pmatrix}^{\mathrm{T}}$, i.e.

$$\tan\theta_j = \frac{\dot{x}(\pi j)}{x(\pi j)}, \qquad r_j = \sqrt{x(\pi j)^2 + \dot{x}(\pi j)^2}.$$

Evaluation of the Poincaré map for $\varepsilon > 0$ yields

$$\tan\theta_{j+1} = \frac{\dot{x}(\pi(j+1))}{x(\pi(j+1))} = \frac{(-\varepsilon)^m \left(\dot{y}_1(\pi)x(\pi j) + \dot{y}_2(\pi)\dot{x}(\pi j)\right)}{(-\varepsilon)^m \left(y_1(\pi)x(\pi j) + y_2(\pi)\dot{x}(\pi j)\right)} = \frac{\dot{y}_1(\pi) + \dot{y}_2(\pi)\tan\theta_j}{y_1(\pi) + y_2(\pi)\tan\theta_j}, \tag{27}$$

whereas $\theta_{j+1}$ is not defined if $r_{j+1} = 0$ for $\varepsilon = 0$. The map $\theta_j \mapsto \theta_{j+1}$ is therefore autonomous as it does not depend on $r_j$ which expresses the cone-wise linearity of the Poincaré map. The map $\theta_j \mapsto \theta_{j+1}$ can be simplified even further by using the nonlinear transformation $q_j = y_1(\pi) + y_2(\pi)\tan\theta_j$ and $y_2(\pi) \neq 0$. It holds that

$$q_{j+1} = Q(q_j) = \Delta - \frac{1}{q_j}, \tag{28}$$

where $\Delta = \mathrm{trace}(\boldsymbol{\Phi}_T) = y_1(\pi) + \dot{y}_2(\pi)$. The fixed points of the map $q_{j+1} = Q(q_j)$ are those values of $q$ which are mapped to themselves, i.e. $q^* = Q(q^*)$, and the fixed points therefore satisfy ${q^*}^2 - \Delta q^* + 1 = 0$ in which we recognize the characteristic equation of the monodromy matrix $\boldsymbol{\Phi}_T$. Hence, the fixed points $q^*$ agree with the real characteristic multipliers $\lambda_{1,2}$ of the monodromy matrix, see (9). The map $q_{j+1} = Q(q_j)$ is known as the Riccati difference equation (or first-order rational difference equation) and has been studied in detail in [1] for three different cases:

- $|\Delta| > 2$: The map $q_{j+1} = Q(q_j)$ has two distinct fixed points $q^*_{1,2} = \lambda_{1,2} = \frac{1}{2}\Delta \pm \frac{1}{2}\sqrt{\Delta^2 - 4}$ and we define $|q^*_1| > |q^*_2|$ and equivalently $|\lambda_1| > |\lambda_2|$. Because $\lambda_1\lambda_2 = 1$, it holds that $|q^*_1| > 1 > |q^*_2|$. The stability of the fixed points $q^*_{1,2}$ is determined by the derivative of the map $Q'(q) = 1/q^2$ from which we see that $q^*_1$ is asymptotically stable whereas $q^*_2$ is unstable.

- $|\Delta| = 2$: The map has a single fixed point $q^* = \frac{1}{2}\Delta = \pm 1$ being unstable but globally attractive.

- $|\Delta| < 2$: The map has no fixed points and the solution is quasi-periodic, wandering between $\mathbb{R}_0^-$ and $\mathbb{R}^+$ (see [1]). The number of iterations that the solution $q_j$ remains in $\mathbb{R}_0^-$ or $\mathbb{R}^+$ is bounded.

The above results on the map $q_{j+1} = Q(q_j)$ for $|\Delta| > 2$ can be understood by noting that the values of $q_{1,2}^*$ define for $y_2(\pi) \neq 0$ the angles $\theta_1^*$ and $\theta_2^*$ of the eigenvectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ (see (10)) of the monodromy matrix $\boldsymbol{\Phi}_T$:

$$\tan\theta_1^* = \frac{q_1^* - y_1(\pi)}{y_2(\pi)} = \tan\theta_c + \frac{\lambda_1}{y_2(\pi)}, \quad \tan\theta_2^* = \frac{q_2^* - y_1(\pi)}{y_2(\pi)} = \tan\theta_c + \frac{\lambda_2}{y_2(\pi)}. \tag{29}$$

If $\boldsymbol{x}_j$ is chosen in the direction of an eigenvector, say $\boldsymbol{x}_j = r_j\boldsymbol{v}_1$, then the next iteration point $\boldsymbol{x}_{j+1}$ will be again in the direction of that eigenvector, i.e. $\boldsymbol{x}_{j+1} = r_{j+1}\boldsymbol{v}_1$. In other words, if $\theta_j = \theta_1^*$ then also $\theta_{j+1} = \theta_1^*$ which implies that $\theta_1^*$ is a fixed point of the map $\theta_j \mapsto \theta_{j+1}$, or, equivalently, that $q_1^*$ is a fixed point of the map $q_{j+1} = Q(q_j)$.

The fixed points $q_{1,2}^* = \lambda_{1,2}$ are either both positive if $\Delta > 2$ or both negative if $\Delta < 2$ and the sign of $y_2(\pi)$ is given by $(-1)^n$. Hence, if $|\Delta| > 2$ and $y_2(\pi) \neq 0$, then it holds that $\boldsymbol{v}_{1,2} \in \mathcal{D}_1$ for $(-1)^n\Delta > 2$ and $\boldsymbol{v}_{1,2} \in \mathcal{D}_2$ for $(-1)^n\Delta < 2$. If $|\Delta| > 2$ and $y_2(\pi) = 0$ then it holds that $y_1(\pi)\dot{y}_2(\pi) = 1$ and $|y_1(\pi) + \dot{y}_2(\pi)| > 2$ from which we deduce that $\dot{y}_2(\pi) \neq 0$ and $\operatorname{sign}(\dot{y}_2(\pi)) = \operatorname{sign}(\Delta)$. If $y_2(\pi) = 0$ then it holds that $n > 0$ and $n$ is even for $\dot{y}_2(\pi) > 0$ and $n$ is odd for $\dot{y}_2(\pi) < 0$. Consequently, the condition $|\Delta| > 2$ and $y_2(\pi) = 0$ implies that $(-1)^n\Delta > 2$. Moreover, it holds that $\mathcal{D}_1 = \mathcal{D}$ for $y_2(\pi) = 0$. We conclude that, if $|\Delta| > 2$, then the location of the eigenvectors is determined by the condition

$$\boldsymbol{v}_{1,2} \in \begin{cases} \mathcal{D}_1 & (-1)^n\Delta > 2, \\ \mathcal{D}_2 & (-1)^n\Delta < -2. \end{cases} \tag{30}$$

for all values of $y_2(\pi)$.

The attractivity of the fixed point $q_1^*$ implies that the solution $\boldsymbol{x}_j$ will be drawn towards the eigenvector $\boldsymbol{v}_1$ when $j \to \infty$, because $\boldsymbol{v}_1$ belongs to the characteristic multiplier which is largest in magnitude. The long-term behaviour of the dynamics is therefore determined by the matrix $\boldsymbol{A}_1 = (-\varepsilon)^n\boldsymbol{\Phi}_T$ if $\boldsymbol{v}_1 \in \mathcal{D}_1$ or by the matrix $\boldsymbol{A}_2 = (-\varepsilon)^{n+1}\boldsymbol{\Phi}_T$ if $\boldsymbol{v}_1 \in \mathcal{D}_2$. This insight leads directly to a stability condition.

**Theorem 1**
*The equilibrium $\boldsymbol{x}^* = \boldsymbol{0}$ of the unilaterally constrained Hill's equation (3) is globally uniformly asymptotically stable if $0 \leq \varepsilon < \varepsilon_c$, where the critical restitution coefficient is given by*

$$\varepsilon_c = \begin{cases} 0 & \text{if } n = 0 \text{ and } \Delta \geq 2 \\ |\lambda_1|^{-\frac{1}{n}} & \text{if } n > 0 \text{ and } (-1)^n\Delta > 2, \\ |\lambda_1|^{-\frac{1}{n+1}} & \text{if } n \geq 0 \text{ and } (-1)^n\Delta < -2, \\ 1 & \text{if } n > 0 \text{ and } -2 \leq \Delta \leq 2 \text{ or if } n = 0 \text{ and } -2 \leq \Delta < 2. \end{cases} \tag{31}$$

*If $|\Delta| > 2$ and $\varepsilon > \varepsilon_c$, then the equilibrium $\boldsymbol{x}^* = \boldsymbol{0}$ is unstable.*

**Proof:** An extensive proof is given in [2] using discrete quadratic Lyapunov functions. □

For $|\Delta| > 2$ the results of Theorem 1 can be put in a more tangible form by introducing the number $m_1$ as

$$m_1 = \begin{cases} n & \text{if } (-1)^n\Delta > 2, \\ n+1 & \text{if } (-1)^n\Delta < -2. \end{cases} \tag{32}$$

The number $m_1$ is the value of $m$ of a solution curve $\boldsymbol{y}(t)$ which starts in the direction of the first eigenvector, i.e. $\boldsymbol{y}(0) = \boldsymbol{v}_1$. Hence, if $|\Delta| > 2$, then it holds that $\varepsilon_c = |\lambda_1|^{-\frac{1}{m_1}}$.

## 4 Approximation of $\varepsilon_c$ using Hill's infinite determinant

Theorem 1 shows that the stability properties of the unilaterally constrained Hill's equation are completely determined by the properties of the fundamental solutions of the unconstrained Hill's equation. More precisely, the critical restitution coefficient only depends on the value of the discriminant $\Delta$ and the number $n$. Standard approximation techniques for the unconstrained Hill's equation can be therefore used to approximate the critical restitution coefficient of the unilaterally constrained Hill's equation. In this section the method of Hill's infinite determinant is explored.

The function $g(t)$ in Hill's equation can be represented by a complex Fourier series as

$$g(t) = \sum_{k=-\infty}^{\infty} g_k e^{2ikt}, \tag{33}$$

where $g_{-k} = \overline{g_k}$. In this section we will assume that $\sqrt{g_0} \neq 0, 2, 4, 8, \ldots$, i.e. the even parametric resonances are avoided. Similarly, the first eigensolution $f_1(t) = e^{\sigma t} p_1(t)$, see (11), can be written as a complex Fourier series

$$f_1(t) = e^{\sigma t} \sum_{k=-\infty}^{\infty} c_k e^{2ikt}, \tag{34}$$

where the characteristic exponent $\sigma$ is related to the discriminant through $\Delta = 2\cosh(\pi\sigma)$. Substitution of the Fourier representations (33) and (34) in Hill's equation (2) yields the condition

$$e^{\sigma t}\left[\sum_{k=-\infty}^{\infty}\left((\sigma + 2ik)^2 + \sum_{s=-\infty}^{\infty} g_s e^{2ist}\right) c_k e^{2ikt}\right] = 0. \tag{35}$$

Reordering terms and requiring that (35) has to be fulfilled for all $t$ leads to an infinite set of linear homogeneous equations for the Fourier coefficients $c_k$

$$(\sigma + 2ik)^2 c_k + \sum_{s=-\infty}^{\infty} g_s c_{k-s} = 0 \tag{36}$$

which has only a non-trivial solution if the infinite determinant

$$D(\sigma) = |H_{kl}|_{k,l=-\infty\cdots+\infty}, \quad H_{kk} = \frac{g_0 + (\sigma + 2ik)^2}{g_0 - 4k^2}, \quad H_{kl} = \frac{g_{l-k}}{g_0 - 4k^2} \quad (l \neq k). \tag{37}$$

vanishes. Each row in (36) has been divided by $g_0 - 4k^2$ to ensure convergence [4]. In [4] it is shown that the infinite determinant $D(\sigma)$ can be expressed as

$$D(\sigma) = D(0) - \frac{\sin^2(\frac{1}{2}i\pi\sigma)}{\sin^2(\frac{1}{2}\pi\sqrt{g_0})} = D(0) - \frac{2-\Delta}{4\sin^2(\frac{1}{2}\pi\sqrt{g_0})},$$

in which the identity $\sin^2(\frac{1}{2}i\pi\sigma) = \frac{1}{4}(2-\Delta)$ has been used. The determinant condition $D(\sigma) = 0$ yields an equation for the discriminant $\Delta$ which can be expressed as

$$\Delta = 2 + 4D(0)\sin^2(\frac{1}{2}\pi\sqrt{g_0}). \tag{38}$$

If $|\Delta| > 2$, then the critical restitution coefficient is given by $\varepsilon_c = |\lambda_1|^{-\frac{1}{m_1}}$, with $|\lambda_1| = \frac{1}{2}|\Delta| + \frac{1}{2}\sqrt{\Delta^2 - 4}$, and the critical restitution coefficient can therefore be calculated from

$$\varepsilon_c = \left(\left|1 + 2D(0)\sin^2(\frac{1}{2}\pi\sqrt{g_0})\right| + \sqrt{\left(1 + 2D(0)\sin^2(\frac{1}{2}\pi\sqrt{g_0})\right)^2 - 1}\right)^{-\frac{1}{m_1}} \tag{39}$$

under the assumption that $\sqrt{g_0} \neq 0, 2, 4, 8, \ldots$. The value of $D(0)$ can be approximated by the determinant of a central $k \times k$ block.

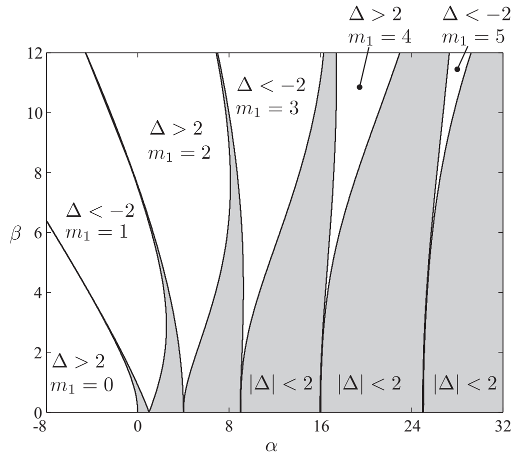



**Figure 3.** Ince-Strutt diagram and discriminant $\Delta$ of the unconstrained Mathieu equation (40). In the grey stability domains holds $|\Delta| < 2$. The instability domains are white.

## 5 The unilaterally constrained Mathieu equation

In order to illustrate the previous results, the stability properties of the unilaterally constrained Mathieu equation are studied in this section. A stability diagram for the unilaterally constrained Mathieu equation is obtained by using direct numerical integration to calculate the discriminant $\Delta$ and $n$.

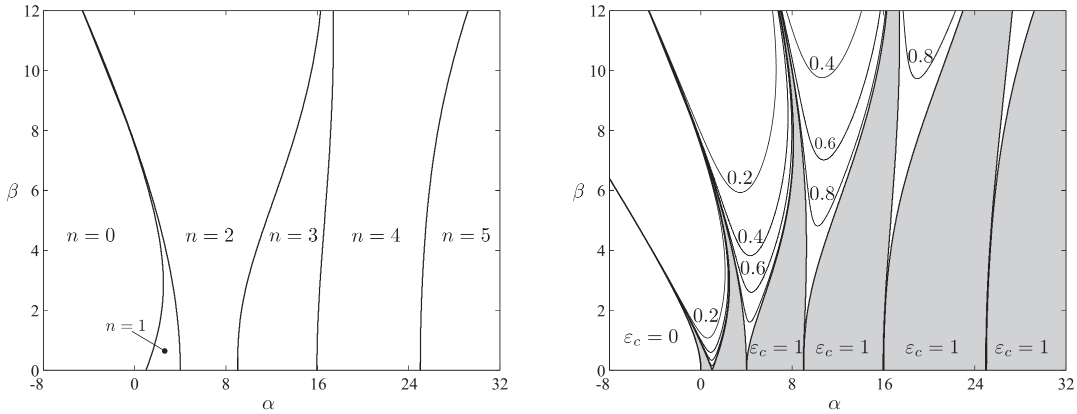
The Mathieu equation

$$\ddot{y}(t) + (\alpha + 2\beta \cos 2t)\, y(t) = 0 \tag{40}$$

is a Hill's equation with symmetry of the function $g(t) = \alpha + 2\beta \cos 2t$, i.e. $g(t) = g(-t)$ and $g(t) = g(t + \pi)$. This implies that, if $y_1(t)$ and $y_2(t)$ are solutions of (40), then also $y_1(-t)$ and $y_2(-t)$ are also solutions of (40). The function $y_1(t)$ is therefore even and $y_2(t)$ is odd. For the same reason it holds that $\dot{y}_1(t)$ is odd and $\dot{y}_2(t)$ is even. Using the transition property we deduce that $\boldsymbol{\Phi}(-\pi, 0) = \boldsymbol{\Phi}(0, \pi) = \boldsymbol{\Phi}(\pi, 0)^{-1}$. Evaluation of $\boldsymbol{\Phi}(-\pi, 0) = \boldsymbol{\Phi}(\pi, 0)^{-1}$ together with the evenness of $y_1(t)$ and $\dot{y}_2(t)$ and the oddness of $y_2(t)$ and $\dot{y}_1(t)$ yields the identity $y_1(\pi) = \dot{y}_2(\pi)$ (see [4]). Hence, it holds that $\Delta = \mathrm{trace}(\boldsymbol{\Phi}_T) = 2y_1(\pi)$.

The stability of the unconstrained Mathieu equation (40) depends on the value of $\Delta$, being a function of the parameters $\alpha$ and $\beta$. The stability boundaries in the parameter plane $(\alpha, \beta)$ are given by $\Delta(\alpha, \beta) = \pm 2$, i.e. $|y_1(\pi)| = 1$. The unity of the determinant, $\det(\boldsymbol{\Phi}_T) = y_1(\pi)^2 - y_2(\pi)\dot{y}_1(\pi) = 1$ implies that either $y_2(\pi) = 0$ and/or $\dot{y}_2(\pi) = 0$ at a stability boundary. One can therefore distinguish between stability boundaries for which $y_2(\pi) = 0$ and those for which $\dot{y}_2(\pi) = 0$. The value of the discriminant $\Delta(\alpha, \beta)$ and the number $n(\alpha, \beta)$ have been computed using direct numerical integration on a grid of $1000 \times 1000$ points for the intervals $\alpha = -8 \ldots 32$ and $\beta = 0 \ldots 12$. The stability boundaries $\Delta(\alpha, \beta) = \pm 2$ of the unconstrained Mathieu equation are depicted in Figure 3, which is often called the Ince-Strutt diagram. The number $n$ changes its value in the parameter plane $(\alpha, \beta)$ if $y_2(\pi)$ changes sign. The boundary of the domains where $n$ is constant therefore agrees with those stability boundaries of the unconstrained Mathieu equation for which $y_2(\pi) = 0$, see Figure 4(a). Figure 3 indicates the number $m_1$ in the instability domains of the unconstrained Mathieu equation. Apparently, $m_1 = k$ in the $k$-th instability domain.

The stability of the equilibrium of the unilaterally constrained Mathieu equation is dependent on the number $n(\alpha, \beta)$ and the discriminant $\Delta(\alpha, \beta)$ and the restitution coefficient $\varepsilon$. The numerical results for the critical restitution coefficient are depicted in Figure 4(b), being the Ince-Strutt diagram for the unilaterally constrained Mathieu equation. The level curves for $\varepsilon_c = 0, 0.2, 0.4, 0.6, 0.8$ and $1$ are shown in Figure 4(b). The grey areas, being the stability domains of the unconstrained Hill's equation, have a critical restitution coefficient $\varepsilon_c = 1$. It can be seen that a decrease of the restitution coefficient enlarges the stability domain in those regions of the parameter space for which $n > 0$, especially when $n$ is large. The value of $n$ is zero in the so-called zeroth instability domain (the lower left part of Figure 4(b) labeled with $\varepsilon_c = 0$) and




(a) value of $n$ (b) Ince-Strutt diagram

**Figure 4.** Diagrams of the unilaterally constrained Mathieu equation with critical value $\varepsilon_c$.

$\Delta > 2$. As follows from Theorem 1, the value of the restitution coefficient has no influence in the zeroth instability domain as the long-term behaviour is governed by non-impacting motion.

## 6 Conclusions and discussion

In this paper the stability conditions of the unilaterally constrained Hill's equation have been addressed in detail using Floquet theory and Lyapunov techniques. It has been shown that the stability of the equilibrium of the unilaterally constrained Hill's equation depends on the discriminant $\Delta$ and the number $n$ (i.e. the number of zeros of the second fundamental solution within one period) of the unconstrained Hill's equation and on the restitution coefficient $\varepsilon$. The remarkable simplicity of the unilaterally constrained Hill's equation stems from the fact that, although the system can be considered to be strongly nonlinear due to the presence of the unilateral constraint, its Poincaré map is cone-wise linear. The cone-wise linearity originates from the homogeneity of the linear differential equation and the homogeneity of the impact map.

The practical merit of the paper is that a precise estimation of the critical restitution coefficient can be obtained by calculating the fundamental solutions of the unconstrained Hill's equation using direction numerical integration methods (ODE-solvers). Common approximation methods, such as averaging and Hill's infinite determinant, can be used to give closed form expressions for the critical restitution coefficient.

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