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An extended version of the Intermediate Axis Theorem for a freely rotating rigid body

Remco Leine[1,*] and **Giuseppe Capobianco**[1]

[1] Institute for Nonlinear Mechanics, University of Stuttgart, Germany

Current numerical schemes for 3D rockfall simulation are not able to correctly represent the stability properties of a freely rotating body. Here, we give a proof using Lyapunov functions of an extended intermediate axis theorem, which not only involves the angular momentum equations but also the orientation of the body. Inspired by the stability proof, we present a novel scheme which respects the stability properties of a freely rotating body and which can be incorporated in numerical schemes for the simulation of rigid bodies with frictional unilateral constraints.

1 Introduction

A full 3D simulation technique for rockfall dynamics, taking rock shape into account and using the state-of-the-art methods of multibody dynamics and nonsmooth contact dynamics, has been developed in [2]. The rockfall simulation is based on the nonsmooth contact dynamics method with hard contact laws. The rock is modeled as an arbitrary convex polyhedron and the terrain model is based on a high resolution digital elevation model. The developed numerical methods have been implemented in the code RAMMS::ROCKFALL (Figure 1), which is being actively used in the natural hazards research community [1]. Field observations of natural rockfall events as well as high precision measurements with instrumented rocks have shown that platy disk-shaped rocks have the tendency to roll and bump down the slope around their major principal axis. Simulations with the present implementation of RAMMS::ROCKFALL , however, fail to represent the observed rolling phenomenon.

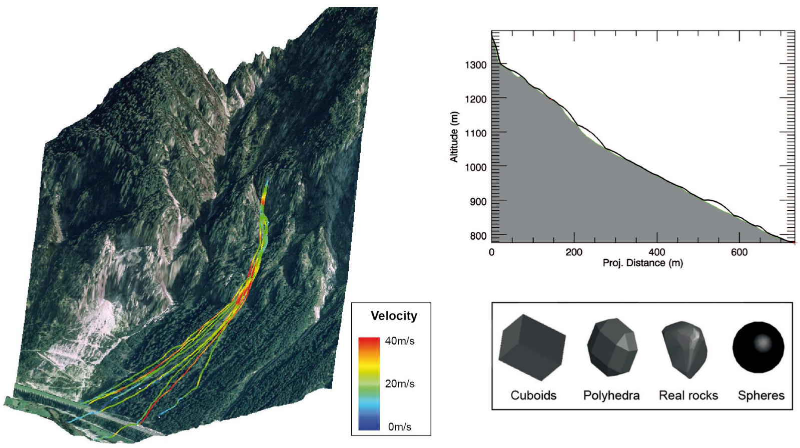

Fig. 1: RAMMS: Topography with rockfall trajectories (left) and XY-plot of a single trajectory (right)

The intermediate axis theorem is a result of the Euler equations

$$\boldsymbol{\Theta}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\boldsymbol{\Theta}\boldsymbol{\omega}) = \mathbf{0}$$

describing the movement of a rigid body with three distinct principal moments of inertia. The theorem describes the following effect: rotation of a rigid body around its minor and major principal axes is stable, while rotation around its intermediate principal axis is unstable. The classical intermediate axis theorem, however, only involves the Euler equations for the three components of the angular velocity. In this paper, we describe the dynamics of a freely rotating body in state-space form using as states the three angular velocity components $\boldsymbol{\omega}$ and an arbitrary parametrization of the orientation of the body with respect to the inertial frame, (i.e. the transformation matrix $\boldsymbol{A}_{IK}$). Using the method of Lyapunov functions we rigorously prove an extended version of the intermediate axis theorem which considers the stability of the motion in the full state-space $(\boldsymbol{A}_{IK}, \boldsymbol{\omega})$.

In [3] paper we show that the present scheme, which is fully explicit during flight phases of the rock, does not respect the intermediate axis theorem. Furthermore, an implicit scheme is presented which correctly describes the stability properties of a freely rotating body.

* Corresponding author: e-mail leine@inm.uni-stuttgart.de

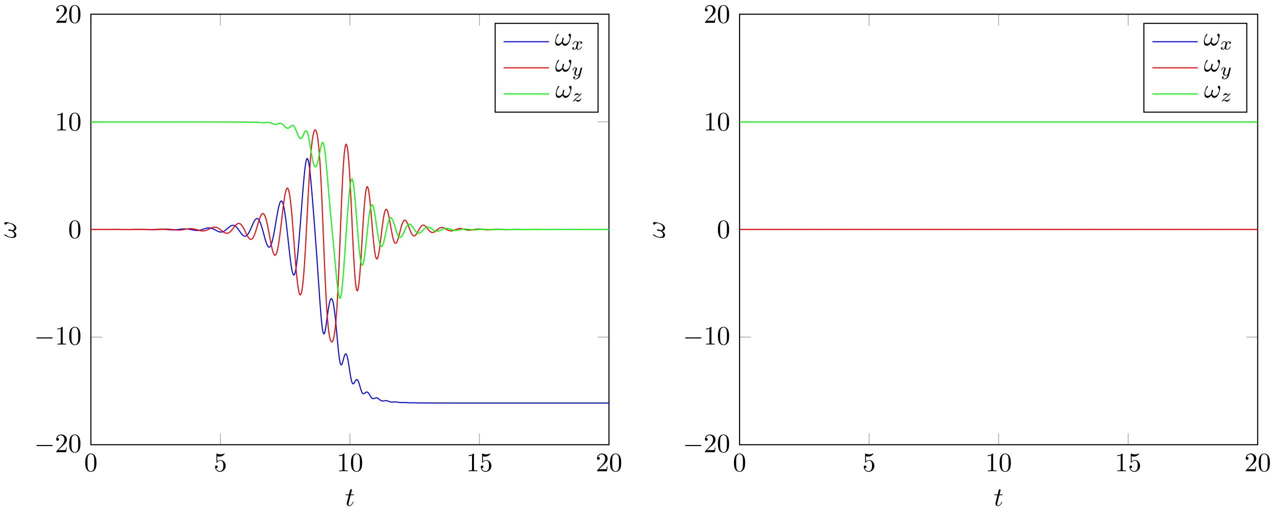

Fig. 2: Angular velocities: left the explicit scheme, right the implicit scheme

2 A stability preserving implicit scheme

The alternative scheme for rockfall simulation is implicit and preserves the stability properties of the principal axes of rotation in accordance with the intermediate axis theorem. The implicit scheme consists of two parts:

1. as update rule for the angular velocity, we use an implicit scheme which preserves the kinetic energy as well as the magnitude of the spin (i.e. the angular momentum with respect to the center of mass),
2. as update rule for the orientation parametrization, we propose a explicit rule which preserves the direction of the spin.

The stability properties of the implicit scheme are proven using the Lyapunov function which has been derived to prove the extended intermediate axis theorem.

We compare the explicit and the implicit scheme on a numerical example by considering a cuboid with principal moments of inertia such that $A < B < C$. Rotation in the neighborhood of stationary rotation $\boldsymbol{\omega}_\star = \Omega \boldsymbol{e}_3$ around the major principal axis $\boldsymbol{e}_3$ is considered. We simulate 20 s using a time-step of $\Delta t = 0.01$ s for both the explicit and implicit scheme (Figure 2). The body initially rotates in the vicinity of the major principal axis with angular velocity $\omega_z = \Omega = 10$ rad/s, which is stable as follows from the extended intermediate axis theorem. However, in the numerical solution of the explicit scheme, the body deviates from stationary rotation around the major principal axis (approximately at $t = 10$ s) and tends to stable rotation around the minor principal axis with angular speed $\omega_x = -\sqrt{C/A}\,\Omega$. If a smaller time-step is taken in the explicit scheme, then the change of axis will be slower and will take place at a later point in time. The solution of the implicit scheme remains very close to the major principal axis, both in angular velocity and in orientation and is therefore much more accurate.

The computational cost per time-step is larger for the implicit scheme than the explicit scheme, as Newton iterations are needed to solve the implicit equations. However, numerical simulations show that the implicit scheme is far more accurate as it respects both the energy conservation and the invariance of the spin. The implicit scheme therefore allows to take larger time-steps without excessive error, making it a suitable scheme for rockfall simulation.

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