Variational analysis of inequality impact laws

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Summary. Using tools from convex analysis we reveal important mathematical properties of inequality impact laws such as the generalized Newton's impact law. A key issue is the identification of the mathematical entities in which the relationships between these properties become apparent. In this paper we show that the 'mean' contact velocity, being half the sum of the pre- and post-impact contact velocity, is dual to the impact impulsive force. Here, a (set-valued) mapping between these dual variables is introduced and we prove that its maximal monotonicity is equivalent to the non-expansivity of the impact map from pre- to post-impact velocities. Explicit expressions of these mappings are given for the generalized Newton's impact law with global restitution coefficient.

Introduction

Various impact laws are used in multibody dynamics with hard unilateral constraints, such as the generalized Newton's impact law and generalized Poisson's impact law. These impact laws are usually formulated as inequality complementarities on local contact kinematic and kinetic quantities (i.e. relative contact velocities and impulsive contact forces). In [7] an extensive mathematical framework, which has been named *Variational Analysis*, has been presented that unifies problems in optimization, mechanics, control and stability theory. The aim of this paper is to gain a deep understanding of impact laws through a variational analysis. A rigorous mathematical framework for impact laws is essential for the development of new impact laws and forms a basis for further research in nonlinear dynamics and control of mechanical systems with unilateral constraints.
The results of this paper are of direct use in the accompanying paper [1], in which it is shown that the maximal monotonicity property of the impact operator $\mathcal{H}$ leads to a convergence property of a Lagrangian system with unilateral constraints which can be used directly for the design of a synchronization based observer.

Mechanical systems with unilateral constraints

Lagrangian systems such as multibody systems can be described by an equation of motion in the form

$$\boldsymbol{M}(\boldsymbol{q},t)\ddot{\boldsymbol{q}} - \boldsymbol{h}(\boldsymbol{q},\dot{\boldsymbol{q}},t) = \boldsymbol{0}, \tag{1}$$

where $\boldsymbol{M}(\boldsymbol{q},t)$ is the mass matrix and $\boldsymbol{h}$ contains the generalized forces with respect to the minimal coordinates $\boldsymbol{q} \in \mathbb{R}^f$. In addition, we may consider frictionless (scleronomic) contacts between bodies of the system. Let $g_i(\boldsymbol{q})$, $i = 1,\dots n$, denote signed contact distances between contacting partners, which we gather in the vector $\boldsymbol{g} = \{g_i\}$. The addition of contacts invokes contact forces $\boldsymbol{\lambda} = \{\lambda_i\}$ on the right hand side of (1) such that

$$\boldsymbol{M}(\boldsymbol{q},t)\ddot{\boldsymbol{q}} - \boldsymbol{h}(\boldsymbol{q},\dot{\boldsymbol{q}},t) = \boldsymbol{W}(\boldsymbol{q})\boldsymbol{\lambda}, \tag{2}$$

where $\boldsymbol{W} = \{\boldsymbol{w}_i\}$ is the matrix of generalized force directions $\boldsymbol{w}_i^{\mathrm{T}} = \frac{\partial g_i}{\partial \boldsymbol{q}}$ [2]. If the contacting bodies are considered to be impenetrable, then this leads to hard unilateral constraints in the form of inequalities $\boldsymbol{g}(\boldsymbol{q}) \geq 0$. In this setting, the generalized velocity $\boldsymbol{u} = \dot{\boldsymbol{q}}$ (defined for almost all t) jumps at collision time instants from the pre-impact velocity $\boldsymbol{u}^-$ to the post-impact velocity $\boldsymbol{u}^+$. On collision time instants, the impulsive dynamics is described by the impact equation

$$\boldsymbol{M}(\boldsymbol{q},t)(\boldsymbol{u}^+ - \boldsymbol{u}^-) = \boldsymbol{W}(\boldsymbol{q})\boldsymbol{\Lambda} \tag{3}$$

together with an impact law which specifies the dependence of the impulsive contact forces $\boldsymbol{\Lambda}$ on kinematic quantities.
We will assume that the constraint distances $\boldsymbol{g}$ are only a function of the generalized positions $\boldsymbol{q}$ and the constraint velocities are therefore a linear function of the generalized velocities,

$$\boldsymbol{\gamma} = \boldsymbol{W}^{\mathrm{T}}\boldsymbol{u}. \tag{4}$$

A velocity jump due to a collision leads to the pre-impact contact velocity $\boldsymbol{\gamma}^- = \boldsymbol{W}^{\mathrm{T}}\boldsymbol{u}^-$ and post-impact contact velocity $\boldsymbol{\gamma}^+ = \boldsymbol{W}^{\mathrm{T}}\boldsymbol{u}^+$. The impact equation (3) may be expressed in the contact velocities as

$$\boldsymbol{\gamma}^+ - \boldsymbol{\gamma}^- = \boldsymbol{W}^{\mathrm{T}}(\boldsymbol{u}^+ - \boldsymbol{u}^-) = \boldsymbol{W}^{\mathrm{T}}\boldsymbol{M}^{-1}\boldsymbol{W}\boldsymbol{\Lambda} = \boldsymbol{G}\boldsymbol{\Lambda}, \tag{5}$$

where $\boldsymbol{G} = \boldsymbol{W}^{\mathrm{T}}\boldsymbol{M}^{-1}\boldsymbol{W}$ is the Delassus matrix. Here, we will assume that the generalized force directions $\boldsymbol{w}_i$ are linearly independent, and, therefore, that the matrix $\boldsymbol{W}$ is of full rank which implies the positive definiteness of $\boldsymbol{G}$.
The impact equation is to be complemented by an impact law and one of the simplest impact laws for unilaterally constrained multibody systems is the generalized Newton's restitution rule.

Generalized Newton's impact law

The classical Newton's impact law for a closed geometric unilateral constraint i

$$\gamma_i^+ = -\varepsilon_i \gamma_i^-, \qquad g_i = 0, \qquad 0 \le \varepsilon_i \le 1 \tag{6}$$

relates the post-impact velocity γ_i^+ of unilateral constraint i to the pre-impact velocity γ_i^- by a Newtonian coefficient of restitution ε_i. The case $\varepsilon_i = 1$ corresponds to a completely elastic impact, whereas $\varepsilon_i = 0$ corresponds to a completely inelastic impact. The impact, which causes the sudden change in constraint velocity, is accompanied by an constraint impulse $\Lambda_i > 0$ at unilateral constraint i. Suppose that, for any reason, the unilateral constraint does not participate in the impact, i.e. that the value of the constraint impulse Λ_i is zero, although the geometric unilateral constraint is closed ($g_i = 0$). If $\gamma_i^- < 0$, then the occurrence of such a superfluous constraint only happens for multi-constraint situations. Following [3], for superfluous constraints we generalize the classical Newton's impact law by allowing post-impact constraint velocities higher than prescribed by Newton's impact law in the case of a non-vanishing impulse, i.e. $\gamma_i^+ \ge -\varepsilon_i \gamma_i^-$. Summarizing, two cases can occur at a closed unilateral constraint i (i.e. $g_i = 0$):

1. The unilateral constraint is actively participating in the impact process, i.e. $\Lambda_i > 0$ and $\gamma_i^+ = -\varepsilon_i \gamma_i^-$,
2. The unilateral constraint is superfluous, i.e. $\Lambda_i = 0$ and $\gamma_i^+ \ge -\varepsilon_i \gamma_i^-$.

These two cases are combined in an inequality complementarity impact law on velocity–impulse level:

$$\Lambda_i \ge 0, \quad \gamma_i^+ + \varepsilon_i \gamma_i^- \ge 0, \quad \Lambda_i(\gamma_i^+ + \varepsilon_i \gamma_i^-) = 0, \tag{7}$$

whereas $\Lambda_i = 0$ if $g_i > 0$. Using the Upr operator (see the Appendix) we write the generalized Newton's impact law as

$$-\Lambda_i \in \mathrm{Upr}\,(\gamma_i^+ + \varepsilon_i \gamma_i^-) \quad \Leftrightarrow \quad \begin{cases} \Lambda_i > 0 & \text{when } \gamma_i^+ = -\varepsilon_i \gamma_i^-, \\ \Lambda_i = 0 & \text{when } \gamma_i^+ \ge -\varepsilon_i \gamma_i^-, \end{cases} \tag{8}$$

or in vector form as

$$-\boldsymbol{\Lambda} \in \mathrm{Upr}\,(\boldsymbol{\xi}), \tag{9}$$

where $\boldsymbol{\xi} = \boldsymbol{\gamma}^+ + \boldsymbol{E}\boldsymbol{\gamma}^-$ and $\boldsymbol{E} = \mathrm{diag}(\varepsilon_i)$.
The generalized Newton's impact law (8) is the simplest inequality impact law for hard unilateral constraints, as it only uses a single inequality complementarity. If $\gamma_i^- \le 0$ and $\varepsilon_i \ge 0$, then it holds that $\gamma_i^+ \ge -\varepsilon_i \gamma_i^- \ge 0$. The kinematic consistency of the post-impact velocities $\gamma_i^+ \ge 0$ of the generalized Newton's impact law therefore follows from the sign of the pre-impact velocities γ_i^-. This sign condition naturally holds for geometric unilateral constraints as the bodies have to approach each other in order to come into contact.
An impact law, such as the generalized Newton's impact law, is usually expressed in the pre- and post-impact contact velocities, i.e. $\boldsymbol{\gamma}^-$ and $\boldsymbol{\gamma}^+$. In the following, we will propose a different kind of representation which reflects the specific properties of the impact law much better.

Variational analysis for impact laws

A variational analysis of impact laws requires the identification of dual variables which constitute the impact law. Hereto, we consider the kinetic energy dissipated in the impact process

$$T^+ - T^- = \frac{1}{2}\boldsymbol{u}^{+\mathrm{T}}\boldsymbol{M}\boldsymbol{u}^+ - \frac{1}{2}\boldsymbol{u}^{-\mathrm{T}}\boldsymbol{M}\boldsymbol{u}^- = \frac{1}{2}\left(\boldsymbol{u}^+ + \boldsymbol{u}^-\right)^{\mathrm{T}}\boldsymbol{M}\left(\boldsymbol{u}^+ - \boldsymbol{u}^-\right) = \frac{1}{2}\left(\boldsymbol{\gamma}^+ + \boldsymbol{\gamma}^-\right)^{\mathrm{T}}\boldsymbol{\Lambda}, \tag{10}$$

in which we identify the kinematic quantity

$$\bar{\boldsymbol{\gamma}} = \frac{1}{2}\left(\boldsymbol{\gamma}^+ + \boldsymbol{\gamma}^-\right). \tag{11}$$

The kinematic variable $\bar{\boldsymbol{\gamma}}$ and kinetic variable $\boldsymbol{\Lambda}$ are therefore dual variables in the sense that $\delta\bar{\boldsymbol{\gamma}}^{\mathrm{T}}\boldsymbol{\Lambda}$ is the virtual work of the impulsive constraint force. The key idea of this paper is to reveal the mathematical structure of the impact law by formulating the impact law as a set-valued relationship

$$-\boldsymbol{\Lambda} \in \mathcal{H}(\bar{\boldsymbol{\gamma}}) \tag{12}$$

between the dual variables $\bar{\boldsymbol{\gamma}}$ and $\boldsymbol{\Lambda}$, where $\mathcal{H} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is (in general) a set-valued operator. We first recall the definition of (cyclic) maximal monotonicity of a set-valued function [7].

convex l.s.c. dissipation function $\Phi(\bar{\gamma})$

$\Downarrow$

$S:\gamma^- \longmapsto \gamma^+$		$-\mathbf{\Lambda} \in \mathcal{H}(\bar{\gamma})$		$Z:\mathbf{u}^- \longmapsto \mathbf{u}^+$
non-expansive in G^{-1}	$\Longleftrightarrow$	maximal monotone	$\Longleftrightarrow$	non-expansive in M
$\Vert\gamma_1^+ - \gamma_2^+\Vert_{\mathbf{G}^{-1}} \leq \Vert\gamma_1^- - \gamma_2^-\Vert_{\mathbf{G}^{-1}}$		$-(\mathbf{\Lambda}_1 - \mathbf{\Lambda}_2)^T(\bar{\gamma}_1 - \bar{\gamma}_2) \geq 0$		$\Vert\mathbf{u}_1^+ - \mathbf{u}_2^+\Vert_{\mathbf{M}} \leq \Vert\mathbf{u}_1^- - \mathbf{u}_2^-\Vert_{\mathbf{M}}$

Figure 1: Implications of a maximal monotone impact law.

Definition 1 (Maximal monotonicity of a set-valued function)
The set-valued function $\mathcal{F}$ is called monotone if its graph is monotone in the sense that

$$(\boldsymbol{y}_1 - \boldsymbol{y}_2)^{\mathrm{T}}(\boldsymbol{x}_1 - \boldsymbol{x}_2) \geq 0 \tag{13}$$

for all $(\boldsymbol{x}_1, \boldsymbol{y}_1)$ and $(\boldsymbol{x}_2, \boldsymbol{y}_2)$ such that $\boldsymbol{y}_1 \in \mathcal{F}(\boldsymbol{x}_1)$ and $\boldsymbol{y}_2 \in \mathcal{F}(\boldsymbol{x}_2)$. Furthermore, $\mathcal{F}$ is called maximal monotone if it is monotone and if there exists no other monotone set-valued function whose graph strictly contains the graph of $\mathcal{F}$.

Definition 2 (Cyclic maximal monotonicity of a set-valued function)
The set-valued function $\mathcal{F}$ is called cyclically monotone if for any choice of points $\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_m$ (for arbitrary $m > 0$) and $\boldsymbol{y}_i \in \mathcal{F}(\boldsymbol{x}_i)$ one has

$$\boldsymbol{y}_0^{\mathrm{T}}(\boldsymbol{x}_1 - \boldsymbol{x}_0) + \boldsymbol{y}_1^{\mathrm{T}}(\boldsymbol{x}_2 - \boldsymbol{x}_1) + \cdots + \boldsymbol{y}_m^{\mathrm{T}}(\boldsymbol{x}_0 - \boldsymbol{x}_m) \leq 0. \tag{14}$$

Furthermore, $\mathcal{F}$ is called maximal cyclically monotone if its graph cannot be enlarged without destroying this property.

Cyclic maximal monotonicity of $\mathcal{F}$ is a stronger condition than maximal monotonicity. Furthermore, if $\mathcal{F}$ is cyclically maximal monotone, then it can be considered to be the subdifferential of a convex proper lower semicontinuous function $f : \mathbb{R}^n \mapsto \mathbb{R} \cup \{\infty\}$, i.e. $\mathcal{F}(\boldsymbol{x}) = \partial f(\boldsymbol{x})$ [7]. Consequently, if the impact map $\mathcal{H}$ enjoys the cyclic maximal monotonicity property, then there exists a convex proper lower semicontinuous dissipation function Φ, such that

$$-\mathbf{\Lambda} \in \partial\Phi(\bar{\gamma}) = \mathcal{H}(\bar{\gamma}). \tag{15}$$

The dissipation function is not to be confused with the impact work $D(\bar{\gamma})$ given by

$$D(\bar{\gamma}) = -(T^+ - T^-) = -\bar{\gamma}^{\mathrm{T}}\mathbf{\Lambda}, \tag{16}$$

and, if $\mathcal{H}$ is cyclically maximal monotone, then Φ exists and we may write $D(\bar{\gamma}) = \bar{\gamma}^{\mathrm{T}}\partial\Phi(\bar{\gamma})$. Typically, an impact law fulfills that vanishing pre-impact velocities $\boldsymbol{\gamma}^- = \mathbf{0}$ do not lead to impulsive forces, and therefore vanishing post-impact velocities $\boldsymbol{\gamma}^+ = \boldsymbol{\gamma}^- = \mathbf{0}$. This implies the natural condition $\mathbf{0} \in \mathcal{H}(\mathbf{0})$ for the impact map $\mathcal{H}$. The maximal monotonicity of $\mathcal{H}$ together with this condition yields $-(\mathbf{\Lambda} - \mathbf{0})^{\mathrm{T}}(\bar{\gamma} - \mathbf{0}) \geq 0$ and therefore the dissipativity $D(\bar{\gamma}) \geq 0$ of the impact law. Maximal monotonicity of the impact law is therefore a stronger condition than dissipativity.
Other ways to express an impact law is by a mapping S from pre-impact to post-impact constraint velocities

$$\boldsymbol{\gamma}^+ = S(\boldsymbol{\gamma}^-), \tag{17}$$

or by a mapping Z from pre-impact to post-impact generalized velocities

$$\boldsymbol{u}^+ = Z(\boldsymbol{u}^-). \tag{18}$$

The mappings S and Z are single-valued whereas $\mathcal{H}$ is generally set-valued. The introduction of the mappings $\mathcal{H}$, S and Z as expressions for the impact law allows us to reveal important properties which we gather in the following theorem and are illustrated in Figure 1.

Theorem 1
The following properties of the impact law are equivalent

1. *The set-valued impact map $\mathcal{H}$ is maximal monotone, i.e.*

   $$-(\bar{\gamma}_1 - \bar{\gamma}_2)^{\mathrm{T}}(\mathbf{\Lambda}_1 - \mathbf{\Lambda}_2) \geq 0, \qquad \text{where } -\mathbf{\Lambda}_i \in \mathcal{H}(\bar{\gamma}_i),\ i = 1, 2,$$

   and no other point can be added to the graph of $\mathcal{H}$ without destroying the monotonicity property.

2. *The impact map S in local contact velocities is maximal non-expansive in the metric $\boldsymbol{G}^{-1}$, i.e.*

   $$\Vert\boldsymbol{\gamma}_1^+ - \boldsymbol{\gamma}_2^+\Vert_{\boldsymbol{G}^{-1}} \leq \Vert\boldsymbol{\gamma}_1^- - \boldsymbol{\gamma}_2^-\Vert_{\boldsymbol{G}^{-1}}, \qquad \boldsymbol{G} = \boldsymbol{W}^{\mathrm{T}}\boldsymbol{M}^{-1}\boldsymbol{W},$$

   where $\boldsymbol{\gamma}_i^+ = S(\boldsymbol{\gamma}_i^-)$ and the domain of S is $\mathbb{R}^n$.

3. *The impact map Z in generalized velocities is maximal non-expansive in the metric $\boldsymbol{M}$, i.e.*

$$\|\boldsymbol{u}_1^+ - \boldsymbol{u}_2^+\|_{\boldsymbol{M}} \leq \|\boldsymbol{u}_1^- - \boldsymbol{u}_2^-\|_{\boldsymbol{M}},$$

where $\boldsymbol{u}_i^+ = Z(\boldsymbol{u}_i^-)$ and the domain of Z is $\mathbb{R}^f$.

Proof: Using (11) and (5), the monotonicity condition of $\mathcal{H}$ can be rewritten as

$$\begin{aligned}
0 \geq& 2(\bar{\gamma}_1 - \bar{\gamma}_2)^{\mathrm{T}}(\boldsymbol{\Lambda}_1 - \boldsymbol{\Lambda}_2) \\
=&(\gamma_1^+ + \gamma_1^- - \gamma_2^+ - \gamma_2^-)^{\mathrm{T}}\boldsymbol{G}^{-1}(\gamma_1^+ - \gamma_1^- - \gamma_2^+ + \gamma_2^-) \\
=&(\gamma_1^+ - \gamma_2^+)^{\mathrm{T}}\boldsymbol{G}^{-1}(\gamma_1^+ - \gamma_2^+) - (\gamma_1^- - \gamma_2^-)^{\mathrm{T}}\boldsymbol{G}^{-1}(\gamma_1^- - \gamma_2^-) \\
=&\|\gamma_1^+ - \gamma_2^+\|_{\boldsymbol{G}^{-1}}^2 - \|\gamma_1^- - \gamma_2^-\|_{\boldsymbol{G}^{-1}}^2
\end{aligned}$$

from which follows the non-expansivity of S in the metric $\boldsymbol{G}^{-1}$ and vice versa. Furthermore, using (4) together with (11) and (3), the monotonicity condition of $\mathcal{H}$ can be rewritten as

$$\begin{aligned}
0 \geq& 2(\bar{\gamma}_1 - \bar{\gamma}_2)^{\mathrm{T}}(\boldsymbol{\Lambda}_1 - \boldsymbol{\Lambda}_2) \\
=&(\boldsymbol{u}_1^+ + \boldsymbol{u}_1^- - \boldsymbol{u}_2^+ - \boldsymbol{u}_2^-)^{\mathrm{T}}\boldsymbol{W}(\boldsymbol{\Lambda}_1 - \boldsymbol{\Lambda}_2) \\
=&(\boldsymbol{u}_1^+ + \boldsymbol{u}_1^- - \boldsymbol{u}_2^+ - \boldsymbol{u}_2^-)^{\mathrm{T}}\boldsymbol{M}(\boldsymbol{u}_1^+ - \boldsymbol{u}_1^- - \boldsymbol{u}_2^+ + \boldsymbol{u}_2^-) \\
=&\|\boldsymbol{u}_1^+ - \boldsymbol{u}_2^+\|_{\boldsymbol{M}}^2 - \|\boldsymbol{u}_1^- - \boldsymbol{u}_2^-\|_{\boldsymbol{M}}^2
\end{aligned}$$

which is the non-expansivity of Z in the metric $\boldsymbol{M}$.
The equivalence of the maximality remains to be proven. Hereto, use (11) and (5) to arrive at $\boldsymbol{\gamma}^- = \bar{\gamma} - \frac{1}{2}\boldsymbol{G}\boldsymbol{\Lambda}$ from which follows $\boldsymbol{\gamma}^- \in \bar{\gamma} + \frac{1}{2}\boldsymbol{G}\mathcal{H}(\bar{\gamma}) = (I + \frac{1}{2}\boldsymbol{G}\mathcal{H})(\bar{\gamma})$, where I denotes the identity. Moreover, it holds that $\boldsymbol{\gamma}^+ = 2\bar{\gamma} - \boldsymbol{\gamma}^-$ and therefore

$$S = 2(I + \frac{1}{2}\boldsymbol{G}\mathcal{H})^{-1} - I. \tag{19}$$

The equivalence of the maximality follows from the Minty parametrization of a set-valued graph (see Theorem 12.15 in [7]): if $\mathcal{F}$ is maximal monotone, then $(I + \mathcal{F})^{-1}$ is single valued and defined on $\mathbb{R}^n$. Therefore, if $\mathcal{H}$ is maximal monotone, then so is $\frac{1}{2}\boldsymbol{G}\mathcal{H}$ and S must have the $\mathbb{R}^n$ as its domain. Lastly, using (3) we deduce that

$$Z = I + \boldsymbol{M}^{-1}\boldsymbol{W}\boldsymbol{G}^{-1}(S \circ \boldsymbol{W}^{\mathrm{T}} - \boldsymbol{W}^{\mathrm{T}}). \tag{20}$$

Therefore, if S is single-valued with domain $\mathbb{R}^n$, then also Z is single-valued with domain $\mathbb{R}^f$. $\square$

For a cyclically maximal monotone impact map $\mathcal{H}$ which is positively homogeneous we are able to prove the following:

Theorem 2
If Φ exists and is positively homogeneous of degree 2, then it holds that $\Phi(\bar{\gamma}) = \frac{1}{2}D(\bar{\gamma})$.

Proof: If Φ is positively homogeneous of degree 2 then it holds that

$$\Phi(\alpha\bar{\gamma}) = \alpha^2\Phi(\bar{\gamma}) \qquad \forall\alpha > 0. \tag{21}$$

Differentiation with respect to α gives the Euler identity $\partial\Phi(\alpha\bar{\gamma})^{\mathrm{T}}\bar{\gamma} = 2\alpha\Phi(\bar{\gamma})$. Setting $\alpha = 1$ yields $D(\bar{\gamma}) = 2\Phi(\bar{\gamma})$. $\square$

Impact mappings for Newton's impact law with global restitution coefficient

As an example, we show the mappings $\mathcal{H}$, Z and S for the generalized Newton's impact law with a global restitution coefficient ε, i.e. all restitution coefficients ε_i are identical.
Using (11) and (5), we express $\boldsymbol{\xi}$ as

$$\boldsymbol{\xi} = \boldsymbol{\gamma}^+ + \varepsilon\boldsymbol{\gamma}^- = \frac{1+\varepsilon}{2}(\boldsymbol{\gamma}^+ + \boldsymbol{\gamma}^-) + \frac{1-\varepsilon}{2}(\boldsymbol{\gamma}^+ - \boldsymbol{\gamma}^-) = (1+\varepsilon)\bar{\gamma} + \frac{1-\varepsilon}{2}\boldsymbol{G}\boldsymbol{\Lambda}. \tag{22}$$

Newton's impact law $-\boldsymbol{\Lambda} \in \mathrm{Upr}(\boldsymbol{\xi})$ can be inverted to $-\boldsymbol{\xi} \in \mathrm{Upr}(\boldsymbol{\Lambda})$. Together with the positive homogeneity of Upr the latter yields

$$-\bar{\gamma} - \frac{1-\varepsilon}{2(1+\varepsilon)}\boldsymbol{G}\boldsymbol{\Lambda} \in \mathrm{Upr}(\boldsymbol{\Lambda}). \tag{23}$$

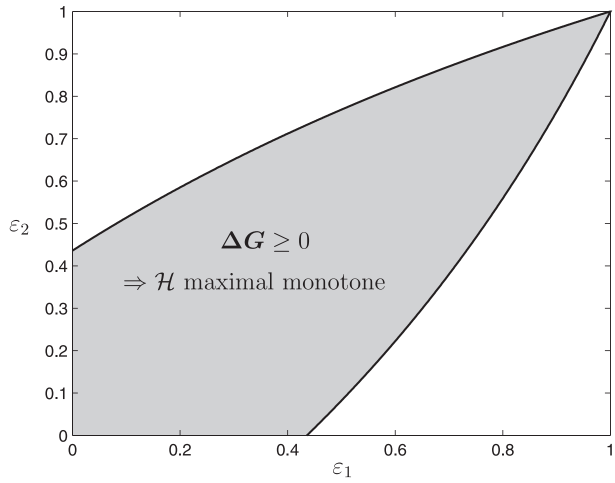

Figure 2: Maximal monotonicity of the generalized Newton's impact law.

The equivalence (37), shown the the Appendix, allows us to separate $\bar{\boldsymbol{\gamma}}$ from $\boldsymbol{\Lambda}$

$$\frac{1-\varepsilon}{2(1+\varepsilon)}\boldsymbol{\Lambda} = \operatorname{prox}^{\boldsymbol{G}}_{\mathbb{R}^+_0}(-\boldsymbol{G}^{-1}\bar{\boldsymbol{\gamma}}). \tag{24}$$

Hence, we arrive at

$$-\boldsymbol{\Lambda} \in \mathcal{H}(\bar{\boldsymbol{\gamma}}) = \begin{cases} \dfrac{2(1+\varepsilon)}{\varepsilon-1}\operatorname{prox}^{\boldsymbol{G}}_{\mathbb{R}^+_0}(-\boldsymbol{G}^{-1}\bar{\boldsymbol{\gamma}}) & 0 \leq \varepsilon < 1, \\ \operatorname{Upr}(\bar{\boldsymbol{\gamma}}) & \varepsilon = 1, \end{cases} \tag{25}$$

Herein, $\operatorname{prox}^{\boldsymbol{G}}_{\mathbb{R}^+_0}(\cdot)$ denotes the proximal point to the set $\mathbb{R}^+_0$ in the metric $\boldsymbol{G}$. For the generalized Newton's impact law, $\mathcal{H}$ is a single-valued proximal point mapping for $\varepsilon < 1$ which degenerates in a set-valued unilateral primitive for $\varepsilon = 1$. Similarly, we can derive the mappings S and Z as

$$\boldsymbol{\gamma}^+ = S(\boldsymbol{\gamma}^-) = (1+\varepsilon)\boldsymbol{G}\operatorname{prox}^{\boldsymbol{G}}_{\mathbb{R}^+_0}(-\boldsymbol{G}^{-1}\boldsymbol{\gamma}^-) + \boldsymbol{\gamma}^-, \tag{26}$$

$$\boldsymbol{u}^+ = Z(\boldsymbol{u}^-) = (1+\varepsilon)\operatorname{prox}^{\boldsymbol{M}}_{\mathcal{T}_C}(\boldsymbol{u}^-) - \varepsilon\boldsymbol{u}^-, \qquad \text{where } \mathcal{T}_C = \{\boldsymbol{u} | \boldsymbol{W}^{\mathrm{T}}\boldsymbol{u} \geq 0\}. \tag{27}$$

Furthermore, we can prove that $-\boldsymbol{\Lambda} \in \mathcal{H}(\bar{\boldsymbol{\gamma}}) = \partial\Phi(\bar{\boldsymbol{\gamma}})$ with the dissipation function

$$\Phi(\bar{\boldsymbol{\gamma}}) = \frac{1+\varepsilon}{1-\varepsilon}\|\operatorname{prox}^{\boldsymbol{G}}_{\mathbb{R}^+_0}(-\boldsymbol{G}^{-1}\bar{\boldsymbol{\gamma}})\|^2_{\boldsymbol{G}} = \frac{1+\varepsilon}{1-\varepsilon}\left(\|\bar{\boldsymbol{\gamma}}\|^2_{\boldsymbol{G}} - \operatorname{dist}^{\boldsymbol{G}}_{\mathbb{R}^+_0}(-\boldsymbol{G}^{-1}\bar{\boldsymbol{\gamma}})^2\right). \tag{28}$$

Indeed, using $\partial(\frac{1}{2}\operatorname{dist}_C(\boldsymbol{x})^2) = \boldsymbol{x} - \operatorname{prox}_C(\boldsymbol{x})$, see [7], we can check that $\mathcal{H}(\bar{\boldsymbol{\gamma}}) = \partial\Phi(\bar{\boldsymbol{\gamma}})$. Hence, the generalized Newton's impact law with a global restitution coefficient has an impact map $\mathcal{H}$ which is cyclically maximal monotone and, correspondingly, a convex dissipation function Φ being positively homogeneous of degree 2.

Maximal monotonicity of Newton's impact law with different restitution coefficients

The more general case of different restitution coefficients renders the problem more complex and the maximal monotonicity of $\mathcal{H}$ is not fulfilled for all values of $0 \leq \varepsilon_i < 1$. However, a sufficient condition can be given.

Theorem 3
A sufficient condition for the maximal monotonicity of the operator $\mathcal{H}(\bar{\boldsymbol{\gamma}})$ of Newton's impact law with different restitution coefficients, is that $\boldsymbol{E}$ is diagonal and that $\boldsymbol{\Delta G}$ is positive semi-definite, where $\boldsymbol{\Delta} := (\boldsymbol{I}+\boldsymbol{E})^{-1}(\boldsymbol{I}-\boldsymbol{E})$.

Proof: Using $\boldsymbol{\xi} = \boldsymbol{\gamma}^+ + \boldsymbol{E}\boldsymbol{\gamma}^-$ and (5) we can express the contact velocities as

$$\boldsymbol{\gamma}^+ = (\boldsymbol{I}+\boldsymbol{E})^{-1}(\boldsymbol{\xi} + \boldsymbol{E}\boldsymbol{G}\boldsymbol{\Lambda}), \qquad \boldsymbol{\gamma}^- = (\boldsymbol{I}+\boldsymbol{E})^{-1}(\boldsymbol{\xi} - \boldsymbol{G}\boldsymbol{\Lambda}), \tag{29}$$

and the mean contact velocity as

$$\bar{\boldsymbol{\gamma}} = \frac{1}{2}(\boldsymbol{\gamma}^+ + \boldsymbol{\gamma}^-) = (\boldsymbol{I}+\boldsymbol{E})^{-1}\boldsymbol{\xi} - \frac{1}{2}\boldsymbol{\Delta G \Lambda}. \tag{30}$$

Evaluation of the monotonicity expression $(\boldsymbol{\Lambda}_2 - \boldsymbol{\Lambda}_1)^{\mathrm{T}}(\bar{\boldsymbol{\gamma}}_2 - \bar{\boldsymbol{\gamma}}_1)$ leads to

$$-(\boldsymbol{\Lambda}_2 - \boldsymbol{\Lambda}_1)^{\mathrm{T}}(\bar{\boldsymbol{\gamma}}_2 - \bar{\boldsymbol{\gamma}}_1) = -(\boldsymbol{\Lambda}_2 - \boldsymbol{\Lambda}_1)^{\mathrm{T}}(\boldsymbol{I}+\boldsymbol{E})^{-1}(\boldsymbol{\xi}_2 - \boldsymbol{\xi}_1) + \frac{1}{2}(\boldsymbol{\Lambda}_2 - \boldsymbol{\Lambda}_1)^{\mathrm{T}}\boldsymbol{\Delta G}(\boldsymbol{\Lambda}_2 - \boldsymbol{\Lambda}_1) \tag{31}$$

The first term on the right-hand side can be written as

$$-(\boldsymbol{\Lambda}_2-\boldsymbol{\Lambda}_1)^{\mathrm{T}}(\boldsymbol{I}+\boldsymbol{E})^{-1}(\boldsymbol{\xi}_2-\boldsymbol{\xi}_1)=-\sum_{i=1}^{n}(1+\varepsilon_i)^{-1}(\Lambda_{2i}-\Lambda_{1i})(\xi_{2i}-\xi_{1i}), \tag{32}$$

where $-\xi_{1i}\in\mathrm{Upr}(\Lambda_{1i})$, $-\xi_{2i}\in\mathrm{Upr}(\Lambda_{2i})$. This term is non-negative because of the maximal monotonicity of the Upr-operator. The second term is maximal monotone if the matrix $\boldsymbol{\Delta G}$ is positive semi-definite. Hence, the right-hand side is non-negative proving that the operator $\mathcal{H}$ is maximal monotone. □

We are able to distinguish two special cases. On the one hand $\boldsymbol{E}=\varepsilon\boldsymbol{I}$ (a global restitution coefficient) and on the other hand $\boldsymbol{G}=\mathrm{diag}\{G_{ii}\}$, which implies that the contacts are uncoupled. Figure 2 illustrates Theorem 3 for

$$\boldsymbol{G}=\begin{pmatrix}1 & 0.9\\ 0.9 & 1\end{pmatrix}. \tag{33}$$

The mapping $\mathcal{H}$ is maximal monotone for values ε_1 and ε_2 which are not too far apart.

Conclusions and Outlook

In this paper, the kinematic quantity $\bar{\gamma}$, being dual to the impulsive contact force $\boldsymbol{\Lambda}$, has been identified as a fundamental mathematical object for a variational analysis of impact laws. A key result is the equivalence of the maximal monotonicity of $\mathcal{H}$ and the maximal non-expansivity of $S:\boldsymbol{\gamma}^-\mapsto\boldsymbol{\gamma}^+$ and $Z:\boldsymbol{u}^-\mapsto\boldsymbol{u}^+$. The existence of a convex lower semi-continuous dissipation function Φ is sufficient for the maximal monotonicity of $\mathcal{H}$. Explicit expressions for the dissipation function and the impact maps have been derived for the generalized Newton's impact law with global restitution coefficient. Furthermore, we have shown that he generalized Newton's impact law with non-identical restitution coefficients is maximal monotone if the product $\boldsymbol{\Delta G}$ is positive semi-definite. Further research will investigate the generalized Poisson's impact law [4].

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Appendix

We make use of the so-called unilateral primitive [2] defined by

$$\mathrm{Upr}(x)=\partial\Psi_{\mathbb{R}_0^+}=\begin{cases}0 & x>0,\\ (-\infty,0] & x=0,\\ \emptyset & x<0,\end{cases} \tag{34}$$

which is the subdifferential of the indicator function $\Psi_{\mathbb{R}_0^+}$ on the set $\mathbb{R}_0^+$. With the unilateral primitive Upr the inequality complementarity is expressed, i.e. $-y\in\mathrm{Upr}(x)$ is identical to $x\geq 0$, $y\geq 0$, $xy=0$ [2, 5], or, if $\boldsymbol{x},\boldsymbol{y}\in\mathbb{R}^n$,

$$-\boldsymbol{y}\in\mathrm{Upr}(\boldsymbol{x})\quad\Longleftrightarrow\quad \boldsymbol{0}\leq\boldsymbol{x}\perp\boldsymbol{y}\geq\boldsymbol{0}. \tag{35}$$

The proximal point function to the closed non-empty convex set C in the metric $\boldsymbol{G}$ is defined as

$$\mathrm{prox}_C^{\boldsymbol{G}}(\boldsymbol{z})=\underset{\boldsymbol{x}\in C}{\mathrm{argmin}}\,\|\boldsymbol{x}-\boldsymbol{z}\|_{\boldsymbol{G}}=\underset{\boldsymbol{x}}{\mathrm{argmin}}\,\frac{1}{2}\|\boldsymbol{x}-\boldsymbol{z}\|_{\boldsymbol{G}}^2+\Psi_C(\boldsymbol{x}). \tag{36}$$

The right-hand side attains its minimum when its subdifferential vanishes, i.e. $\boldsymbol{G}(\boldsymbol{x}-\boldsymbol{z})+\partial\Psi_C(\boldsymbol{x})\ni\boldsymbol{0}$, from which we deduce the equivalence

$$\boldsymbol{G}(\boldsymbol{z}-\boldsymbol{x})\in N_C(\boldsymbol{x})\quad\Leftrightarrow\quad\boldsymbol{x}=\mathrm{prox}_C^{\boldsymbol{G}}(\boldsymbol{z}). \tag{37}$$

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