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A WEAK FORM OF HAMILTON'S PRINCIPLE AS VARIATIONAL INEQUALITY

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Abstract

The classical form of Hamilton's principle holds for conservative systems with perfect bilateral constraints. In this paper we derive Hamilton's principle for perfect unilateral constraints (involving impulsive motion) using weak variations. The resulting principle has the form of a variational inequality in Hilbert space.

Key words

unilateral constraint, variational inequality, variational principle, principle of d'Alembert-Lagrange.

1 Introduction

Classical Analytical Mechanics is concerned with mechanical systems with perfect bilateral (mostly holonomic) constraints (Papastavridis, 2002) and is closely related with the calculus of variations as many principles of statics and dynamics are formulated in terms of variational problems, e.g. the principle of virtual work and Hamilton's principle (see for instance (Lanczos, 1962)). Unilateral constraints, which are basically inequality constraints, are completely ignored in classical Analytical Mechanics because inequalities are not discussed by the classical calculus of variations and, also, because unilateral constraints in dynamics lead to shocks with discontinuities in the velocity. The mathematical tools to handle unilateral problems in statics, and later also in dynamics, have only been developed since the last four decades. The field of Nonsmooth Dynamics is now rapidly developing. We refer the reader to the textbooks (Leine and Nijmeijer, 2004; Glocker, 2001; Brogliato, 1999).

Variational problems involving convex inequality constraints are described by variational inequalities and where first introduced by Hartman and Stampacchia (Hartman and Stampacchia, 1966) to study partial differential equations. The applicability of the theory has since been expanded to include problems from mechanics, finance, optimisation and game theory. References on variational inequalities can be found in the standard reference (Kinderlehrer and Stampacchia, 1980) or in the more recent books (Goeleven *et al.*, 2003*a*; Goeleven *et al.*, 2003*b*). Variational problems with non-convex inequality constraints are described by hemi-variational inequalities and are discussed in the seminal work of Panagiotopoulos (Panagiotopoulos, 1993). Moreover, (hemi-)variational inequalities are related to (non-)convex superpotentials through the subderivative known from Nonsmooth Analysis, see the work of Moreau (Moreau, 1968) and Panagiotopoulos (Panagiotopoulos, 1981).

The extension of classical Analytical Mechanics to perfect unilateral constraints asks for a reformulation of the variational principles of mechanics in terms of variational inequalities. The principle of d'Alembert-Lagrange in inequality form has been discussed in (Panagiotopoulos and Glocker, 1998; Panagiotopoulos and Glocker, 2000; Goeleven *et al.*, 1997; Goeleven *et al.*, 1999; May, 1984*a*; May, 1984*b*). Various forms of the principle of Hamilton as variational inequality form can be derived from the principle of d'Alembert-Lagrange as variational inequality (Panagiotopoulos and Glocker, 1998; Panagiotopoulos and Glocker, 2000).

The current paper puts one existing form of the principle of Hamilton as variational inequality, which has already been derived in (Panagiotopoulos and Glocker, 1998; Panagiotopoulos and Glocker, 2000), within the context of weak and strong extrema (Cesar, 1984; Troutman, 1996). It therefore becomes much more clear how these different forms of Hamilton's principle have to be understood. Furthermore, the weak form does not impose any requirements on the energy dissipation (or conservation).

2 Principle of d'Alembert-Lagrange as variational inequality

In this section we discuss the principle of d'Alembert-Lagrange in inequality form (Goeleven *et al.*, 1997; Panagiotopoulos and Glocker, 1998). Consider a point-mass m with position $\boldsymbol{r} \in \mathbb{R}^3$ which is subjected to a constraint force $\boldsymbol{R}$ and an external force $\boldsymbol{F}$. The principle of virtual work states that the virtual work vanishes for all virtual displacements $\delta\boldsymbol{r}$, i.e.

$$\delta W = (m\ddot{\boldsymbol{r}} - \boldsymbol{F} - \boldsymbol{R})^{\mathrm{T}}\delta\boldsymbol{r} = 0 \quad \forall\delta\boldsymbol{r}. \tag{1}$$

The virtual displacements $\delta\boldsymbol{r} = \boldsymbol{r}^* - \boldsymbol{r}$ are infinitely small differences of arbitrary *virtual* positions $\boldsymbol{r}^*$ and the actual position $\boldsymbol{r}$, such that time is kept constant.

We first consider the case where the constraint is a bilateral holonomic scleronomic constraint $g(\boldsymbol{r}) = 0$ with $g \in C^1(\mathbb{R}^3, \mathbb{R})$, in other words the position of the point-mass is constrained to the manifold $K = \{\boldsymbol{r} \in \mathbb{R}^3 \mid g(\boldsymbol{r}) = 0\}$ (see Figure 1). Virtual displacements $\delta\boldsymbol{r}$ are admissible with respect to the constraint if they belong to the tangent space $\mathcal{T}_K(\boldsymbol{r}) = \{\boldsymbol{z} \in \mathbb{R}^3 \mid \frac{\partial g}{\partial \boldsymbol{r}}\boldsymbol{z} = 0\}$. If the virtual work of the constraint force vanishes for all virtual displacements which are admissible with respect to the constraint, i.e.

$$\boldsymbol{R}^{\mathrm{T}}\delta\boldsymbol{r} = 0 \quad \forall\delta\boldsymbol{r} \in \mathcal{T}_K(\boldsymbol{r}), \tag{2}$$

then we speak of a *perfect* bilateral constraint. A perfect constraint force is therefore normal to the constraint manifold in the sense that

$$-\boldsymbol{R} \in \mathcal{N}_K(\boldsymbol{r}), \tag{3}$$

where $\mathcal{N}_K(\boldsymbol{r})$ is the set of all vectors which are normal to the tangent space $\mathcal{T}_K(\boldsymbol{r})$. Equations (2) and (3) are equivalent if K is a manifold. Instead of taking (2) as definition of a perfect constraint, we therefore could also have chosen to take the normality condition (2) of the constraint force as definition. The assumption of perfect bilateral constraints in classical mechanics, and therefore the normality of the constraint force to the constraint manifold, excludes phenomena as friction. Combining (2) and (1) gives the variational equality

$$(m\ddot{\boldsymbol{r}} - \boldsymbol{F})^{\mathrm{T}}\delta\boldsymbol{r} = 0 \quad \forall\delta\boldsymbol{r} \in \mathcal{T}_K(\boldsymbol{r}), \tag{4}$$

which is usually referred to as the classical principle of d'Alembert-Lagrange.

We now replace the bilateral constraint by unilateral holonomic (geometric) scleronomic constraints $g_i(\boldsymbol{r}) \geq 0$, $i = 1 \ldots m$, see Figure 2. The position of the point-mass is in this case constrained to the set $K = \{\boldsymbol{r} \in \mathbb{R}^3 \mid g_i(\boldsymbol{r}) \geq 0, i = 1 \ldots m\}$. In the following, we will confine ourselves to the case that K is a convex set. Virtual displacements $\delta\boldsymbol{r}$ are admissible

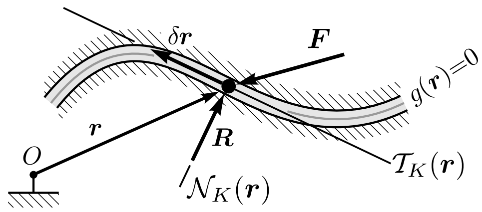

Figure 1. Point mass in contact with a bilateral constraint.

with respect to the unilateral constraint if $\delta\boldsymbol{r} \in \mathcal{T}_K(\boldsymbol{r})$, where $\mathcal{T}_K(\boldsymbol{r})$ is now the tangent *cone*. We define a unilateral constraint as *perfect* if the constraint force satisfies the normality condition (3)

$$-\boldsymbol{R} \in \mathcal{N}_K(\boldsymbol{r}), \tag{5}$$

where $\mathcal{N}_K(\boldsymbol{r})$ is the normal cone to the convex set K. The tangent cone $\mathcal{T}_K(\boldsymbol{r})$ and normal cone $\mathcal{N}_K(\boldsymbol{r})$ are polar in the sense that for all $\boldsymbol{R}$ and $\delta\boldsymbol{r}$ satisfying

$$-\boldsymbol{R} \in \mathcal{N}_K(\boldsymbol{r}), \quad \delta\boldsymbol{r} \in \mathcal{T}_K(\boldsymbol{r}) \tag{6}$$

it holds that

$$-\boldsymbol{R}^{\mathrm{T}}\delta\boldsymbol{r} \leq 0. \tag{7}$$

Hence, for perfect unilateral constraints it holds that the virtual work of the constraint force is nonnegative for admissible virtual displacements, i.e.

$$\boldsymbol{R}^{\mathrm{T}}\delta\boldsymbol{r} \geq 0 \quad \forall\delta\boldsymbol{r} \in \mathcal{T}_K(\boldsymbol{r}). \tag{8}$$

Combining (2) and (1) gives the variational inequality

$$(m\ddot{\boldsymbol{r}} - \boldsymbol{F})^{\mathrm{T}}\delta\boldsymbol{r} \geq 0 \quad \forall\delta\boldsymbol{r} \in \mathcal{T}_K(\boldsymbol{r}), \tag{9}$$

which we will refer to as the principle of d'Alembert-Lagrange in inequality form.

3 The Principle of Hamilton for Non-Impulsive Motion

Consider a system S of which we can address each mass-element $\mathrm{d}m$ by its position vector $\boldsymbol{\xi} \in \mathbb{R}^3$. The mass-element $\mathrm{d}m$ is subjected to external and internal forces $\mathrm{d}\boldsymbol{F}$, which consist of elastic forces, gravitational forces and bilateral holonomic scleronomic constraint forces. Furthermore, the mass-element $\mathrm{d}m$ is subjected to unilateral constraint forces $\mathrm{d}\boldsymbol{R}$ which impose the unilateral holonomic scleronomic constraint $g \geq 0$. The gap function g depends on the state of the system S and therefore on the position vector $\boldsymbol{\xi} \in \mathbb{R}^3$ of all

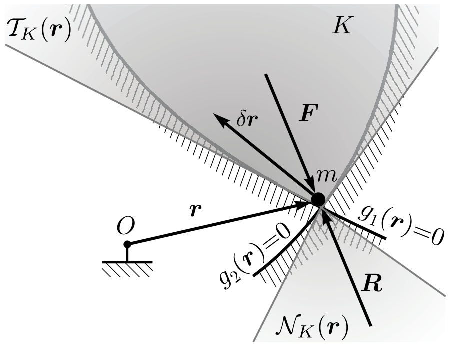

Figure 2. Point mass in contact with a unilateral constraint.

mass-elements of S. The unilateral constraint $g \geq 0$ therefore constrains each mass-element addressed by $\boldsymbol{\xi}$ to a set K_ξ. The principle of virtual work states that if the virtual work

$$\delta W = \int_S \delta\boldsymbol{\xi}^{\mathrm{T}}(\ddot{\boldsymbol{\xi}}\mathrm{d}m - \mathrm{d}\boldsymbol{F} - \mathrm{d}\boldsymbol{R}) \tag{10}$$

vanishes for all virtual variations $\delta\boldsymbol{\xi}$, i.e.

$$\delta W = 0 \quad \forall \delta\boldsymbol{\xi}, \tag{11}$$

then the system is in dynamic equilibrium. The contact forces $\mathrm{d}\boldsymbol{R}$ are assumed to be perfect and it therefore holds that

$$\mathrm{d}\boldsymbol{R}^{\mathrm{T}}\delta\boldsymbol{\xi} \geq 0 \quad \forall \delta\boldsymbol{\xi} \in \mathcal{T}_{K_\xi}(\boldsymbol{\xi}), \tag{12}$$

where $\delta\boldsymbol{\xi} \in \mathcal{T}_{K_\xi}(\boldsymbol{\xi})$ denotes a virtual displacement of $\boldsymbol{\xi}$ which is admissible with respect to the unilateral constraint $g \geq 0$. The principle of d'Alembert-Lagrange in inequality form (9) for the system S therefore reads as

$$\int_S \delta\boldsymbol{\xi}^{\mathrm{T}}(\ddot{\boldsymbol{\xi}}\mathrm{d}m - \mathrm{d}\boldsymbol{F}) \geq 0 \quad \forall \delta\boldsymbol{\xi} \in \mathcal{T}_{K_\xi}(\boldsymbol{\xi}). \tag{13}$$

We now choose a set of generalised coordinates $\boldsymbol{q} \in \mathbb{R}^n$ which uniquely describes the state of the system and which is a minimal set of coordinates with respect to all bilateral constraints. Each particle coordinate $\boldsymbol{\xi}(\boldsymbol{q})$ is therefore completely determined by $\boldsymbol{q}$. An arbitrary variation $\delta\boldsymbol{q}$ causes a variation $\delta\boldsymbol{\xi} = \frac{\partial\boldsymbol{\xi}}{\partial\boldsymbol{q}}\delta\boldsymbol{q}$ which is admissible with respect to all bilateral constraints. The unilateral constraint $g \geq 0$ constrains the generalised coordinates $\boldsymbol{q}$ to a set $K = \{\boldsymbol{q} \in \mathbb{R}^n \mid g \geq 0\}$ which we assume to be convex. A virtual displacement $\delta\boldsymbol{q} \in \mathcal{T}_K(\boldsymbol{q})$ induces a virtual displacement of the particle $\delta\boldsymbol{\xi} = \frac{\partial\boldsymbol{\xi}}{\partial\boldsymbol{q}}\delta\boldsymbol{q} \in \mathcal{T}_{K_\xi}(\boldsymbol{\xi})$ which is admissible with respect to the unilateral constraint $g \geq 0$.

The virtual work of the inertia forces can be rewritten as

$$\int_S \delta\boldsymbol{\xi}^{\mathrm{T}}\ddot{\boldsymbol{\xi}}\mathrm{d}m = \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_S \delta\boldsymbol{\xi}^{\mathrm{T}}\dot{\boldsymbol{\xi}}\mathrm{d}m\right) - \int_S \delta\dot{\boldsymbol{\xi}}^{\mathrm{T}}\dot{\boldsymbol{\xi}}\mathrm{d}m, \tag{14}$$

or, by using the generalised coordinates $\boldsymbol{q}$, as

$$\int_S \delta\boldsymbol{\xi}^{\mathrm{T}}\ddot{\boldsymbol{\xi}}\mathrm{d}m = \frac{\mathrm{d}}{\mathrm{d}t}\left(\delta\boldsymbol{q}^{\mathrm{T}}\boldsymbol{p}\right) - \delta T \tag{15}$$

where

$$\boldsymbol{p} = \int_S \left(\frac{\partial\boldsymbol{\xi}}{\partial\boldsymbol{q}}\right)^{\mathrm{T}}\dot{\boldsymbol{\xi}}\mathrm{d}m \tag{16}$$

is the generalised momentum and δT is the variation of the kinetic energy

$$T = \int_S \frac{1}{2}\dot{\boldsymbol{\xi}}^{\mathrm{T}}\mathrm{d}m\dot{\boldsymbol{\xi}} = \frac{1}{2}\dot{\boldsymbol{q}}^{\mathrm{T}}\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{q}}, \tag{17}$$

with the mass matrix

$$\boldsymbol{M}(\boldsymbol{q}) = \int_S \left(\frac{\partial\boldsymbol{\xi}}{\partial\boldsymbol{q}}\right)^{\mathrm{T}}\mathrm{d}m\left(\frac{\partial\boldsymbol{\xi}}{\partial\boldsymbol{q}}\right). \tag{18}$$

Moreover, we introduce

$$\boldsymbol{f} = \int_S \left(\frac{\partial\boldsymbol{\xi}}{\partial\boldsymbol{q}}\right)^{\mathrm{T}}\mathrm{d}\boldsymbol{F} \tag{19}$$

as the generalised force. Substitution of (15) and (19) in the principle of d'Alembert-Lagrange (13) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\delta\boldsymbol{q}^{\mathrm{T}}\boldsymbol{p}\right) - \delta T - \delta\boldsymbol{q}^{\mathrm{T}}\boldsymbol{f} \geq 0 \quad \forall \delta\boldsymbol{q} \in \mathcal{T}_K(\boldsymbol{q}), \tag{20}$$

which is the inequality form of the well known Lagrange central equation (Hamel, 1912; Papastavridis, 2002). The Lagrange central equation (20) holds at each non-impulsive time instance t for which the generalised velocities $\dot{\boldsymbol{q}}$ exist. Hence, we can integrate the central equation (20) over a non-impulsive time-interval $I = [t_0, t_f]$ which gives

$$\left[\delta\boldsymbol{q}^{\mathrm{T}}\boldsymbol{p}\right]_{t_0}^{t_f} - \int_I \delta T + \delta\boldsymbol{q}^{\mathrm{T}}\boldsymbol{f}\mathrm{d}t \geq 0 \quad \forall \delta\boldsymbol{q} \in \mathcal{T}_K(\boldsymbol{q}). \tag{21}$$

Moreover, we assume the generalised force $\boldsymbol{f}$ to be a potential force $-\boldsymbol{f} = \nabla V(\boldsymbol{q})$ where $V(\boldsymbol{q})$ is the potential energy and $\delta V = -\delta\boldsymbol{q}^{\mathrm{T}}\boldsymbol{f}$. Hence, by defining the Lagrange function $L = T - V$ we arrive at

$$\left[\delta\boldsymbol{q}^{\mathrm{T}}\boldsymbol{p}\right]_{t_0}^{t_f} - \int_I \delta L\mathrm{d}t \geq 0 \quad \forall \delta\boldsymbol{q} \in \mathcal{T}_K(\boldsymbol{q}). \tag{22}$$

If the boundary conditions are fixed, then the variation $\delta\boldsymbol{q}(t)$ vanishes at $t = t_0$ and $t = t_f$ and we are allowed to interchanging the order of integration and variation such that

$$-\delta \int_I L \mathrm{d}t \geq 0 \quad \forall \delta\boldsymbol{q} \in \mathcal{T}_K(\boldsymbol{q}), \tag{23}$$

with the boundary conditions $\boldsymbol{q}(t_0) = \boldsymbol{q}_0$, $\boldsymbol{q}(t_f) = \boldsymbol{q}_f$, which is the principle of Hamilton in inequality form for a non-impulsive time-interval $I = [t_0, t_f]$. From the principle of Hamilton we can derive the Euler-Lagrange equations in inequality form by evaluating the variation in (23) as

$$-\int_I \left(\frac{\partial L}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\boldsymbol{q}}} \right) \delta\boldsymbol{q} \mathrm{d}t \geq 0 \quad \forall \delta\boldsymbol{q} \in \mathcal{T}_K(\boldsymbol{q}). \tag{24}$$

Hence, for each time-instance $t \in I$ the variational inequality

$$-\left(\frac{\partial L}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\boldsymbol{q}}} \right) \delta\boldsymbol{q} \geq 0 \quad \forall \delta\boldsymbol{q} \in \mathcal{T}_K(\boldsymbol{q}) \tag{25}$$

should hold, which can be cast into the differential inclusion

$$\boldsymbol{M}(\boldsymbol{q})\ddot{\boldsymbol{q}} - \boldsymbol{h}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \boldsymbol{f}_R = \boldsymbol{0}, \quad -\boldsymbol{f}_R \in \mathcal{N}_K(\boldsymbol{q}), \tag{26}$$

where the mass matrix $\boldsymbol{M}(\boldsymbol{q})$ is defined by (17) and the vector

$$\boldsymbol{h} = -\left(\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{M}(\boldsymbol{q}) \right) \dot{\boldsymbol{q}} + \left(\frac{\partial T}{\partial \boldsymbol{q}} - \frac{\partial V}{\partial \boldsymbol{q}} \right)^{\mathrm{T}} \tag{27}$$

contains all smooth forces.

4 A Weak Principle of Hamilton

In this section we derive a weak principle of Hamilton in inequality form for impulsive motion by directly incorporating the impulsive dynamics in the principle of virtual work. Concepts of measure and integration theory appear to be very useful in this respect.

We assume that the position $\boldsymbol{\xi}(t)$ is an absolutely continuous function in time and that the velocity $\boldsymbol{\nu}(t)$ of the mass element is a function of locally bounded variation without singular terms (which is sometimes called a function of 'special bounded variation'). This implies the following:

1. At each time-instance t we can define a left and right velocity

$$\begin{aligned} \boldsymbol{\nu}^+(t) &= \lim_{\tau \downarrow 0} \frac{\boldsymbol{\xi}(t+\tau) - \boldsymbol{\xi}(t)}{\tau}, \\ \boldsymbol{\nu}^-(t) &= \lim_{\tau \uparrow 0} \frac{\boldsymbol{\xi}(t+\tau) - \boldsymbol{\xi}(t)}{\tau}. \end{aligned} \tag{28}$$

2. The differential measure $\mathrm{d}\boldsymbol{\xi}$ of the position $\boldsymbol{\xi}(t)$ contains only a density $\boldsymbol{\nu}$ with respect to the Lebesgue measure $\mathrm{d}t$

$$\mathrm{d}\boldsymbol{\xi} = \boldsymbol{\nu}\mathrm{d}t. \tag{29}$$

   For almost all t we can define a velocity $\boldsymbol{\nu}(t) = \dot{\boldsymbol{\xi}}(t)$.
3. The differential measure $\mathrm{d}\boldsymbol{\nu}$ of the velocity $\boldsymbol{\nu}(t)$ contains a density with respect to the Lebesgue measure $\mathrm{d}t$ and with respect to the atomic measure $\mathrm{d}\eta$, i.e.

$$\mathrm{d}\boldsymbol{\nu} = \dot{\boldsymbol{\nu}}\mathrm{d}t + (\boldsymbol{\nu}^+ - \boldsymbol{\nu}^-)\mathrm{d}\eta. \tag{30}$$

Let as before $\mathrm{d}\boldsymbol{R}$ be the unilateral contact force, associated with a unilateral holonomic scleronomic constraint, and $\mathrm{d}\boldsymbol{F}$ be the external force on a mass element $\mathrm{d}m$ with position $\boldsymbol{\xi}$. The non-impulsive dynamics of this mass element is therefore described by the equation of motion

$$\mathrm{d}m\ddot{\boldsymbol{\xi}} - \mathrm{d}\boldsymbol{F} - \mathrm{d}\boldsymbol{R} = \boldsymbol{0}. \tag{31}$$

The impulsive dynamics is described by the impact equation

$$\mathrm{d}m(\boldsymbol{\nu}^+ - \boldsymbol{\nu}^-) - \mathrm{d}\boldsymbol{P} = \boldsymbol{0}, \tag{32}$$

where $\mathrm{d}\boldsymbol{P}$ is the unilateral impulsive force on the mass element $\mathrm{d}m$. The introduction of the differential measure $\mathrm{d}\boldsymbol{\nu}$ allows us to combine the equation of motion (31) and the impact equation (32) in a single formula

$$\mathrm{d}m\,\mathrm{d}\boldsymbol{\nu} - \mathrm{d}\boldsymbol{F}\,\mathrm{d}t - \mathrm{d}\boldsymbol{R}\,\mathrm{d}t - \mathrm{d}\boldsymbol{P}\,\mathrm{d}\eta = \boldsymbol{0}, \tag{33}$$

which is an equality of measures and which should be understood in the sense of integration. The measure $\mathrm{d}\boldsymbol{\nu}$, which has a density with respect to the atomic measure, is by definition a mapping on the space of continuous functions, i.e. $\int_I \boldsymbol{f}^{\mathrm{T}}\mathrm{d}\boldsymbol{\nu}$ only makes sense if $\boldsymbol{f} \in C^0(I, \mathbb{R}^3)$. The equality of measures immediately leads to a principle of virtual work in differential measures

$$\int_S \delta\boldsymbol{\xi}^{\mathrm{T}}(\mathrm{d}m\,\mathrm{d}\boldsymbol{\nu} - \mathrm{d}\boldsymbol{F}\,\mathrm{d}t - \mathrm{d}\boldsymbol{R}\,\mathrm{d}t - \mathrm{d}\boldsymbol{P}\,\mathrm{d}\eta) = \boldsymbol{0} \tag{34}$$

for all $\delta\boldsymbol{\xi} \in D^1(I, \mathbb{R}^3)$. The class $D^1(\Omega, \mathbb{R}) = \{y \in AC(\Omega, \mathbb{R}),\ y' \in BV(\Omega, \mathbb{R})\}$ comprises all functions on the domain Ω which are absolutely continuous and piecewise C^1 as well as absolutely continuous functions with an accumulation point. We put $\hat{\boldsymbol{\xi}}(\varepsilon, t) =$

$\boldsymbol{\xi}(t)+\varepsilon\boldsymbol{\omega}(t)$ with $\boldsymbol{\omega}\in D^1(I,\mathbb{R}^3)$. Often, we will suppress the explicit notation of the time-dependence and simply write $\hat{\boldsymbol{\xi}}(\varepsilon)=\boldsymbol{\xi}+\varepsilon\boldsymbol{\omega}$. For this family of test functions it holds that $\delta\boldsymbol{\xi}=\hat{\boldsymbol{\xi}}_\varepsilon(0)\delta\varepsilon=\boldsymbol{\omega}\delta\varepsilon$ is continuous in time wheras $\delta\boldsymbol{\nu}=\dot{\boldsymbol{\omega}}\delta\varepsilon$ is not. Moreover, note that $\hat{\boldsymbol{\xi}}(\varepsilon)$ converges for $\varepsilon\to 0$ to $\boldsymbol{\xi}$ in the weak norm $\|\cdot\|_1$, with

$$\|y\|_1=\max_{x\in\Omega}|y(x)|+\operatorname*{ess\,sup}_{x\in\Omega}|y'(x)|. \tag{35}$$

We now make the assumption that both the contact force $\mathrm{d}\boldsymbol{R}$ and the contact impulse $\mathrm{d}\boldsymbol{P}$ are perfect unilateral constraint forces/impulses, i.e.

$$-\mathrm{d}\boldsymbol{R}\in\mathcal{N}_{K_\xi}(\boldsymbol{\xi}),\qquad -\mathrm{d}\boldsymbol{P}\in\mathcal{N}_{K_\xi}(\boldsymbol{\xi}). \tag{36}$$

The virtual work of the contact force $\mathrm{d}\boldsymbol{R}$ and the contact impulse $\mathrm{d}\boldsymbol{P}$ is therefore non-negative for admissible virtual displacements

$$\delta\boldsymbol{\xi}^{\mathrm{T}}\mathrm{d}\boldsymbol{R}\geq 0,\quad \delta\boldsymbol{\xi}^{\mathrm{T}}\mathrm{d}\boldsymbol{P}\geq 0\quad \forall\delta\boldsymbol{\xi}\in\mathcal{T}_{K_\xi}(\boldsymbol{\xi}). \tag{37}$$

This bring us to the principle of d'Alembert-Lagrange in inequality form for differential measures

$$\int_S\delta\boldsymbol{\xi}^{\mathrm{T}}(\mathrm{d}m\,\mathrm{d}\boldsymbol{\nu}-\mathrm{d}\boldsymbol{F}\,\mathrm{d}t)\geq 0\quad\forall\delta\boldsymbol{\xi}=\boldsymbol{\omega}\delta\varepsilon\in\mathcal{T}_{K_\xi}(\boldsymbol{\xi}) \tag{38}$$

in which we explicitly write that the virtual displacements $\delta\boldsymbol{\xi}$ are of the form $\boldsymbol{\omega}\delta\varepsilon$. A mass element has a constant mass $\mathrm{d}m$. Taking the differential measure-in-time of the term $\delta\boldsymbol{\xi}^{\mathrm{T}}\mathrm{d}m\,\boldsymbol{\nu}$ and applying the chain rule gives

$$\mathrm{d}(\delta\boldsymbol{\xi}^{\mathrm{T}}\mathrm{d}m\,\boldsymbol{\nu})=\delta\boldsymbol{\nu}^{\mathrm{T}}\mathrm{d}m\frac{1}{2}(\boldsymbol{\nu}^++\boldsymbol{\nu}^-)\mathrm{d}t+\delta\boldsymbol{\xi}^{\mathrm{T}}\mathrm{d}m\,\mathrm{d}\boldsymbol{\nu}, \tag{39}$$

in which we used the equality $\mathrm{d}(\delta\boldsymbol{\xi})=\mathrm{d}\boldsymbol{\omega}\delta\varepsilon=\dot{\boldsymbol{\omega}}\mathrm{d}t\delta\varepsilon=\delta\boldsymbol{\nu}\,\mathrm{d}t$. Moreover, because $\boldsymbol{\nu}(t)=\frac{1}{2}(\boldsymbol{\nu}^+(t)+\boldsymbol{\nu}^-(t))$ for almost all t, it holds that $\frac{1}{2}(\boldsymbol{\nu}^++\boldsymbol{\nu}^-)\mathrm{d}t=\boldsymbol{\nu}\mathrm{d}t$ for Lebesgue integration. We therefore arrive at the variational inequality

$$\mathrm{d}\left(\int_S\delta\boldsymbol{\xi}^{\mathrm{T}}\mathrm{d}m\,\boldsymbol{\nu}\right)-\int_S\delta\boldsymbol{\nu}^{\mathrm{T}}\mathrm{d}m\,\boldsymbol{\nu}\mathrm{d}t-\int_S\delta\boldsymbol{\xi}^{\mathrm{T}}\mathrm{d}\boldsymbol{F}\mathrm{d}t\geq 0 \tag{40}$$

$\forall\delta\boldsymbol{\xi}=\boldsymbol{\omega}\delta\varepsilon\in\mathcal{T}_{K_\xi}(\boldsymbol{\xi})$. We recognise the second term

$$\delta T=\int_S\delta\boldsymbol{\nu}^{\mathrm{T}}\mathrm{d}m\,\boldsymbol{\nu},\qquad T=\int_S\frac{1}{2}\boldsymbol{\nu}^{\mathrm{T}}\mathrm{d}m\boldsymbol{\nu} \tag{41}$$

as being the variation of the kinetic energy T. However, the variations $\delta\boldsymbol{\nu}$ are not arbitrary as $\hat{\boldsymbol{\nu}}(t)=\boldsymbol{\nu}(t)+\delta\boldsymbol{\nu}(t)$ is of the form $\hat{\boldsymbol{\nu}}(t)=\boldsymbol{\nu}(t)+\varepsilon\dot{\boldsymbol{\omega}}(t)$. The variation δT therefore reduces to the Gâteaux derivative $dT(\boldsymbol{\nu};\delta\boldsymbol{\nu})=\nabla T\delta\boldsymbol{\nu}$. Similar as before, we choose generalised coordinates $\boldsymbol{q}(t)$ which form a minimal set of coordinates with respect to the bilateral constraints and with which we can uniquely describe the position $\boldsymbol{\xi}(\boldsymbol{q})$ of each mass element $\mathrm{d}m$. Moreover, we introduce generalised velocities $\boldsymbol{u}(t)$, which are assumed to be of locally bounded variation, and which are such that $\mathrm{d}\boldsymbol{q}=\boldsymbol{u}\mathrm{d}t$. The kinetic energy T is a function of $\boldsymbol{\nu}(\boldsymbol{q},\boldsymbol{u})$ and we can therefore write T as a function $T(\boldsymbol{q},\boldsymbol{u})$. Hence, it holds that $\delta T=T_{\boldsymbol{q}}\delta\boldsymbol{q}+T_{\boldsymbol{u}}\delta\boldsymbol{u}$. The variations $\delta\boldsymbol{q}$ and $\delta\boldsymbol{u}$ are not totally arbitrary as they are of the form $\delta\boldsymbol{q}(t)=\boldsymbol{q}(t)+\varepsilon\boldsymbol{w}(t)$ and $\delta\boldsymbol{u}(t)=\boldsymbol{u}(t)+\varepsilon\dot{\boldsymbol{w}}(t)$, where $\boldsymbol{\omega}=\partial\boldsymbol{\xi}/\partial\boldsymbol{q}\,\boldsymbol{w}$. Using the generalised momentum

$$\boldsymbol{p}=\int_S\left(\frac{\partial\boldsymbol{\xi}}{\partial\boldsymbol{q}}\right)^{\mathrm{T}}\boldsymbol{\nu}\mathrm{d}m \tag{42}$$

and the generalised force (19) we transform the principle of d'Alembert-Lagrange into

$$\mathrm{d}\left(\delta\boldsymbol{q}^{\mathrm{T}}\boldsymbol{p}\right)-\delta T\mathrm{d}t-\delta\boldsymbol{q}^{\mathrm{T}}\boldsymbol{f}\mathrm{d}t\geq 0\quad\forall\delta\boldsymbol{q}=\boldsymbol{w}\delta\varepsilon\in\mathcal{T}_K(\boldsymbol{q}), \tag{43}$$

which is the Lagrange central equation in differential measures. As before, we integrate over a time-interval $I=[t_0,t_f]$ and consider the generalised force $\boldsymbol{f}=-\nabla V(\boldsymbol{q})$ to be a potential force, which yields

$$\left[\delta\boldsymbol{q}^{\mathrm{T}}\boldsymbol{p}\right]_{t_0}^{t_f}-\int_I\delta L\mathrm{d}t\geq 0\quad\forall\delta\boldsymbol{q}=\boldsymbol{w}\delta\varepsilon\in\mathcal{T}_K(\boldsymbol{q}), \tag{44}$$

where $L=T-V$. Finally, taking fixed boundary conditions at t_0 and t_f we obtain

$$-\delta\int_I L\mathrm{d}t\geq 0\quad\forall\delta\boldsymbol{q}=\boldsymbol{w}\delta\varepsilon\in\mathcal{T}_K(\boldsymbol{q}). \tag{45}$$

As mentioned before, the variations $\delta\boldsymbol{q}$ are not totally arbitrary and the variation δs of the action in (45) reduces to the Gâteaux derivative $ds(\boldsymbol{q};\delta\boldsymbol{q})$. Consequently, we arrive at a weak form of the principle of Hamilton in inequality form

$$-ds(\boldsymbol{q};\delta\boldsymbol{q})\geq 0\quad\forall\delta\boldsymbol{q}=\boldsymbol{w}\delta\varepsilon\in\mathcal{T}_K(\boldsymbol{q}), \tag{46}$$

with the action integral

$$s(\boldsymbol{q})=\int_I L\mathrm{d}t. \tag{47}$$

This form of the principle of Hamilton is the condition of a weak local extremal of the action $s(\boldsymbol{q})$ with the weak norm $\|\cdot\|_1$.

5 Conclusions

In this paper we derived a weak form of Hamilton's principle as a variational inequality. There are two interesting things to remark at this derivation. First of all, by making use of differential measures we are able to treat the impulsive and non-impulsive dynamics simultaneously. This means that we do not need to use the usual Weierstrass-Erdmann corner conditions for broken extremals (see (Troutman, 1996)) to treat an impact as has been done in (Panagiotopoulos and Glocker, 1998; Panagiotopoulos and Glocker, 2000). Secondly, as we want to use the principle of virtual work for differential measures (34) we are forced to consider test functions $\hat{\boldsymbol{\xi}}(\varepsilon, t) = \boldsymbol{\xi}(t) + \varepsilon\boldsymbol{\omega}(t)$ of which the variation $\delta\boldsymbol{\xi} = \boldsymbol{\omega}\delta\varepsilon$ is time-continuous. A test function of this form can have kinks (and therefore impacts), but this class of test functions does not include a family of curves which only varies the impact time, i.e. it is *weak*. Accordingly, for this weak form of the principle of Hamilton we only have to satisfy the first Weierstrass-Erdmann corner condition. Hence, we are able to prove the validity of (46) by only assuming that the unilateral holonomic constraint is perfect which is expressed by (36). The assumption of energy conservation during the impact, i.e. $T^+ = T^-$, is therefore not a necessary condition for the validity of (46).

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