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DRY FRICTION INDUCED ATTRACTIVITY OF EQUILIBRIUM SETS IN MECHANICAL MULTIBODY SYSTEMS

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ABSTRACT

The dynamics of mechanical systems with dry friction elements, modelled by set-valued force laws, can be described by differential inclusions. The switching and set-valued nature of the friction force law is responsible for the hybrid character of such models. An equilibrium set of such a differential inclusion corresponds to a stationary mode for which the friction elements are sticking. The attractivity properties of the equilibrium set are of major importance for the overall dynamic behaviour of this type of systems. Conditions for the attractivity of the equilibrium set of linear MDOF mechanical systems with multiple friction elements are presented. These results are obtained by application of a generalisation of LaSalle's principle for differential inclusions of Filippov-type. Besides passive systems, also systems with negative viscous damping are considered. For such systems, only local attractivity of the equilibrium set can be assured under certain conditions. Moreover, an estimate for the region of attraction is given for these cases. The results are illustrated by means of a 2DOF example.

INTRODUCTION

The presence of dry friction can influence the behaviour and performance of mechanical systems as it can induce several phenomena, such as friction-induced limit-cycling, damping of vibrations and stiction. Dry friction in mechanical systems is often modelled using set-valued constitutive models (see Glocker (2001)), such as the set-valued Coulomb's law. Set-valued friction models have the advantage to properly model stiction, since the friction force is allowed to be non-zero at zero relative velocity. The dynamics of mechanical systems with set-valued friction laws are described by differential inclusions. We limit ourselves to set-valued friction laws which lead to Filippov-type systems (Filippov (1988)). Filippov systems, describing systems with friction, can exhibit equilibrium sets, which correspond to the stiction behaviour of those systems.

The overall dynamics of mechanical systems is largely affected by the stability and attractivity properties of the equilibrium sets. For example, the loss of stability of the equilibrium set can, in certain applications, cause limit-cycling. Moreover, the stability and attractivity properties of the equilibrium set can also seriously affect the performance of control systems. In Alvarez et al. (2000); Shevitz and Paden (1994) and Bacciotti and Ceragioli (1999), stability and attractivity properties of (sets of) equilibria in differential inclusions are studied. More specifically, in Alvarez et al. (2000) and Shevitz and Paden (1994) the attractivity of the equilibrium set of a passive, one-degree-of-freedom friction oscillator with one switching boundary (i.e. one dry friction element) is discussed. Moreover, in Shevitz and Paden (1994) and Bacciotti and Ceragioli (1999) the Lyapunov stability of an equilibrium point in the equilibrium set is shown. However, most papers are limited to either one-degree-of-freedom systems or to systems exhibiting only one switching boundary.

We will provide conditions under which the equilibrium set is attractive for multi-degree-of-freedom mechanical systems with an arbitrary number of Coulomb friction elements using Lyapunov-type stability analysis and a generalisation of LaSalle's invariance principle for non-smooth systems. Moreover, passive as well as non-passive systems will be considered. The non-passive systems that will be studied are linear mechanical systems with a non-positive definite damping matrix with additional dry friction elements. The non-positive-definiteness of the damping matrix of linearised systems can be caused by fluid, aeroelastic, control and gyroscopical forces, which can cause instabilities. It will be demonstrated in this paper that the presence of dry friction in such an unstable linear system can (conditionally) ensure the local attractivity of the equilibrium set of the resulting system with dry friction. Moreover, an estimate of the region of attraction for the equilibrium set will be given. A rigid multibody approach is used for the description of mechanical systems with friction, which allows for a natural physical interpretation of the conditions for attractivity.

In the next section, the equations of motion for linear mechanical systems with frictional elements are formulated and the equilibrium set is defined. Subsequently, the attractivity properties of the equilibrium set are studied by means of a generalisation of LaSalle's invariance principle. An example is studied in order to illustrate the theoretical results and to investigate the correspondence between the estimated and actual region of attraction. Finally, a discussion of the obtained results and concluding remarks are given.

MODELLING OF MECHANICAL SYSTEMS WITH COULOMB FRICTION

In this section, we will formulate the equations of motion for linear mechanical systems with m frictional translational joints. These translational joints restrict the motion of the system to a manifold determined by the bilateral holonomic constraint equations imposed by these joints (sliders). Coulomb's friction law is assumed to hold in the tangential direction of the manifold.

Let us formulate the equations of motions for such systems by:

$$\boldsymbol{M}\ddot{\boldsymbol{q}}+\boldsymbol{C}\dot{\boldsymbol{q}}+\boldsymbol{K}\boldsymbol{q}-\boldsymbol{W}_T\boldsymbol{\lambda}_T=\boldsymbol{0}, \quad (1)$$

in which $\boldsymbol{q}$ is a column of independent generalised coordinates, $\boldsymbol{M}$, $\boldsymbol{C}$ and $\boldsymbol{K}$ represent the mass-matrix, damping-matrix and stiffness-matrix, respectively, and $\boldsymbol{\lambda}_T$ is a column of friction forces in the translational joints. These friction forces obey the following set-valued force law:

$$\boldsymbol{\lambda}_T \in -\boldsymbol{\Lambda}\,\mathrm{Sign}(\boldsymbol{\gamma}_T), \quad (2)$$

with

$$\boldsymbol{\Lambda}=\mathrm{diag}\left(\left[\mu_1|\lambda_{N_1}|\ \ldots\ \mu_m|\lambda_{N_m}|\right]\right)$$

and the set-valued sign function

$$\mathrm{Sign}(x)=\begin{cases}-1, & x<0,\\ [-1,1], & x=0,\\ 1, & x>0.\end{cases} \quad (3)$$

Herein, λ_{N_i} and μ_i, $i=1,\ldots,m$, are the normal contact force and the friction coefficient in translational joint i. Moreover, $\boldsymbol{W}_T^{\mathrm{T}}=\frac{\partial\boldsymbol{\gamma}_T}{\partial\dot{\boldsymbol{q}}}$ is a matrix reflecting the generalised force directions of the friction forces. Herein, $\boldsymbol{\gamma}_T$ is a column of relative sliding velocities in the translational joints. Equation (1) forms, together with a set-valued friction law (2), a differential inclusion. Differential inclusions of this type are called Filippov systems which obey Filippov's solution concept (Filippov's convex method). Consequently, the existence of solutions of system (1) is guaranteed. Moreover, due to the fact that $\mu_i \geq 0$, $i=1,\ldots,m$, which excludes the possibility of repulsive sliding modes along the switching boundaries, also uniqueness of solutions in forward time is guaranteed (see Leine and Nijmeijer (2004)).

Due to the set-valued nature of the friction law (2), the system exhibits an equilibrium set. Since we assume that $\boldsymbol{\gamma}_T=\boldsymbol{W}_T^T\dot{\boldsymbol{q}}$, $\dot{\boldsymbol{q}}=\boldsymbol{0}$ implies $\boldsymbol{\gamma}_T=\boldsymbol{0}$. This means that every equilibrium implies sticking in all contact points and obeys the equilibrium inclusion:

$$\boldsymbol{K}\boldsymbol{q}+\boldsymbol{W}_T\boldsymbol{\Lambda}\,\mathrm{Sign}(\boldsymbol{0})\ni\boldsymbol{0}. \quad (4)$$

The equilibrium set is therefore given by

$$\mathcal{E}=\left\{(\boldsymbol{q},\dot{\boldsymbol{q}})\in\mathbb{R}^{2n}\,|\,(\dot{\boldsymbol{q}}=\boldsymbol{0})\wedge\boldsymbol{q}\in-\boldsymbol{K}^{-1}\boldsymbol{W}_T\boldsymbol{\Lambda}\,\mathrm{Sign}(\boldsymbol{0})\right\} \quad (5)$$

and is positively invariant due to the uniqueness of the solutions in forward time.

ATTRACTIVITY ANALYSIS OF THE EQUILIBRIUM SET

Let us now study the attractivity properties of this equilibrium set $\mathcal{E}$. Hereto, we will use LaSalle's principle (Khalil (1996)), but applied to Filippov systems with uniqueness of solutions in forward time (Van de Wouw and Leine (2004)).

Let us consider the stability of linear systems with friction and positive definite matrices $\boldsymbol{M}$, $\boldsymbol{K}$ and a non-positive damping matrix $\boldsymbol{C}$. Note that this implies that the equilibrium point of the linear system without friction is either stable or unstable, but in any case not *asymptotically* stable. In the following theorem we state the condition under which (part of) the equilibrium set of the system with friction is locally attractive.

Theorem 1
Consider system (1) with friction law (2). Assume that the matrices $\boldsymbol{M}$ and $\boldsymbol{K}$ are positive definite and the matrix $\boldsymbol{C}$ is not positive

definite but symmetric. If the following condition is satisfied: $\boldsymbol{U}_{c_i} \in span\{\boldsymbol{W}_T\}$ for $i = 1,\dots,n_q$, where $\boldsymbol{U}_c = \{\boldsymbol{U}_{c_i}\}$ is a matrix containing the n_q eigencolumns corresponding to the eigenvalues of $\boldsymbol{C}$, which lie in the closed left-half complex plane, then a convex subset of the equilibrium set (5) is locally attractive.

Proof: We consider a positive definite function

$$V = \frac{1}{2}\dot{\boldsymbol{q}}^{\mathrm{T}}\boldsymbol{M}\dot{\boldsymbol{q}} + \frac{1}{2}\boldsymbol{q}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{q}. \tag{6}$$

Using friction law (2) and the fact that $\boldsymbol{\gamma}_T = \boldsymbol{W}_T^{\mathrm{T}}\dot{\boldsymbol{q}}$, the time-derivative of V is

$$\begin{aligned}\dot{V} &= \dot{\boldsymbol{q}}^{\mathrm{T}}\left(-\boldsymbol{C}\dot{\boldsymbol{q}} - \boldsymbol{K}\boldsymbol{q} + \boldsymbol{W}_T\boldsymbol{\lambda}_T\right) + \dot{\boldsymbol{q}}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{q} \\ &= -\dot{\boldsymbol{q}}^{\mathrm{T}}\boldsymbol{C}\dot{\boldsymbol{q}} - \boldsymbol{\gamma}_T^{\mathrm{T}}\boldsymbol{\Lambda}\,\mathrm{Sign}(\boldsymbol{\gamma}_T) \\ &= -\dot{\boldsymbol{q}}^{\mathrm{T}}\boldsymbol{C}\dot{\boldsymbol{q}} - \boldsymbol{p}^{\mathrm{T}}|\boldsymbol{\gamma}_T| \\ &= -\dot{\boldsymbol{q}}^{\mathrm{T}}\boldsymbol{C}\dot{\boldsymbol{q}} - \boldsymbol{p}^{\mathrm{T}}|\boldsymbol{W}_T^{\mathrm{T}}\dot{\boldsymbol{q}}|,\end{aligned} \tag{7}$$

where the columns $\boldsymbol{p}$ and $|\boldsymbol{\gamma}_T|$ are defined by $\boldsymbol{p} = \{\Lambda_{ii}\}$, $|\boldsymbol{\gamma}_T| = \{|\dot{g}_{T_i}|\}$, for $i = 1,\dots,m$. Equation (7) implies that $\dot{V}$ is a continuous single-valued function (of $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$). It holds that $\boldsymbol{p} \geq \boldsymbol{0}$ and that if $\dot{\boldsymbol{q}} = \boldsymbol{0}$ then $\boldsymbol{\gamma}_T = \boldsymbol{0}$.

We now apply a spectral decomposition of $\boldsymbol{C} = \boldsymbol{U}_c\boldsymbol{\Omega}_c\boldsymbol{U}_c^{\mathrm{T}}$, where $\boldsymbol{U}_c$ is the orthonormal matrix containing all eigencolumns and $\boldsymbol{\Omega}_c$ is the diagonal matrix containing all eigenvalues of $\boldsymbol{C}$, which are real. Moreover, we introduce coordinates $\boldsymbol{\eta}$ such that $\boldsymbol{q} = \boldsymbol{U}_c\boldsymbol{\eta}$. Consequently, $\dot{V}$ satisfies

$$\begin{aligned}\dot{V} &= -\dot{\boldsymbol{q}}^{\mathrm{T}}\boldsymbol{U}_c\boldsymbol{\Omega}_c\boldsymbol{U}_c^{\mathrm{T}}\dot{\boldsymbol{q}} - \boldsymbol{p}^{\mathrm{T}}|\boldsymbol{W}_T^{\mathrm{T}}\dot{\boldsymbol{q}}| \\ &= -\dot{\boldsymbol{\eta}}^{\mathrm{T}}\boldsymbol{\Omega}_c\dot{\boldsymbol{\eta}} - \boldsymbol{p}^{\mathrm{T}}|\boldsymbol{W}_T^{\mathrm{T}}\boldsymbol{U}_c\dot{\boldsymbol{\eta}}|.\end{aligned} \tag{8}$$

The matrix $\boldsymbol{C}$ has n_q eigenvalues in the closed left-half complex plane; all other eigenvalues lie in the open right-half complex plane. Consequently, $\dot{V}$ obeys the inequality

$$\dot{V} \leq -\sum_{i=1}^{n_q}\lambda_i\dot{\eta}_i^2 - \boldsymbol{p}^{\mathrm{T}}|\boldsymbol{W}_T^{\mathrm{T}}\boldsymbol{U}_c\dot{\boldsymbol{\eta}}| \quad \forall\,\dot{\boldsymbol{\eta}}, \tag{9}$$

where we assumed that the eigenvalues (and eigencolumns) of $\boldsymbol{C}$ are ordered in such a manner that λ_i, $i = 1,\dots,n_q$, correspond to the eigenvalues of $\boldsymbol{C}$ in the closed left-half complex plane. Assume that $\exists\,\alpha > 0$ such that

$$\sum_{i=1}^{n_q}|\boldsymbol{e}_i^{\mathrm{T}}\dot{\boldsymbol{\eta}}| \leq \alpha\boldsymbol{p}^{\mathrm{T}}|\boldsymbol{W}_T^{\mathrm{T}}\boldsymbol{U}_c\dot{\boldsymbol{\eta}}| \quad \forall\,\dot{\boldsymbol{\eta}}. \tag{10}$$

Herein, $\boldsymbol{e}_i$ is a unit-column with a non-zero element on the i-th position. Assuming that such an α can be found, (9) results in

$$\begin{aligned}&\dot{V} \leq -\sum_{i=1}^{n_q}\lambda_i\dot{\eta}_i^2 - \beta\sum_{i=1}^{n_q}|\dot{\eta}_i| \leq 0, \\ &\quad\forall\,\dot{\eta}_i \in \left\{\dot{\eta}_i \mid \frac{\beta}{\lambda_i} \leq \dot{\eta}_i \leq -\frac{\beta}{\lambda_i},\right\} \text{ for } \lambda_i < 0, \\ &\quad\forall\,\dot{\eta}_i \in \mathbb{R} \text{ for } \lambda_i = 0,\end{aligned} \tag{11}$$

for $i = 1,\dots,n_q$ with $\beta = \frac{1}{\alpha}$ and $\dot{\eta}_i = \boldsymbol{e}_i^{\mathrm{T}}\dot{\boldsymbol{\eta}}$. Let us now investigate when $\exists\,\alpha > 0$ such that (10) is satisfied. Note, hereto, that if

$$\boldsymbol{e}_i \in \mathrm{span}\left\{\boldsymbol{U}_c^{\mathrm{T}}\boldsymbol{W}_T\right\},\ \forall\, i \in [1,\dots,n_q],$$

then $\exists\,\boldsymbol{\nu}_i^{\mathrm{T}}$ such that $\boldsymbol{e}_i^{\mathrm{T}} = \boldsymbol{\nu}_i^{\mathrm{T}}\boldsymbol{W}_T^{\mathrm{T}}\boldsymbol{U}_c$. It therefore holds that $|\boldsymbol{e}_i^{\mathrm{T}}\dot{\boldsymbol{\eta}}| = |\boldsymbol{\nu}_i^{\mathrm{T}}\boldsymbol{W}_T^{\mathrm{T}}\boldsymbol{U}_c\dot{\boldsymbol{\eta}}|$ and $|\boldsymbol{e}_i^{\mathrm{T}}\dot{\boldsymbol{\eta}}| \leq |\boldsymbol{\nu}_i^{\mathrm{T}}||\boldsymbol{W}_T^{\mathrm{T}}\boldsymbol{U}_c\dot{\boldsymbol{\eta}}|$. Choose the smallest $\tilde{\alpha}_i$ such that $|\boldsymbol{\nu}_i^{\mathrm{T}}| \leq \tilde{\alpha}_i\boldsymbol{p}^{\mathrm{T}}$, where the sign $\leq$ has to be understood component-wise. Then it holds that $|\boldsymbol{e}_i^{\mathrm{T}}\dot{\boldsymbol{\eta}}| \leq \tilde{\alpha}_i\boldsymbol{p}^{\mathrm{T}}|\boldsymbol{W}_T^{\mathrm{T}}\boldsymbol{U}_c\dot{\boldsymbol{\eta}}|\ \ \forall\,\dot{\boldsymbol{\eta}},\ \forall\, i \in [1,\dots,n_q]$. Note that α in (10) can be taken as $\alpha = \sum_{i=1}^{n_q}\tilde{\alpha}_i$. Finally, one should realise that if and only if

$$\boldsymbol{U}_c\boldsymbol{e}_i \in \mathrm{span}\left\{\boldsymbol{W}_T\right\}, \tag{12}$$

or, in other words, if the i-th column $\boldsymbol{U}_{c_i}$ of $\boldsymbol{U}_c$ satisfies $\boldsymbol{U}_{c_i} \in \mathrm{span}\{\boldsymbol{W}_T\}$ (note in this respect that $\boldsymbol{U}_c$ is real and symmetric), then it holds that $\boldsymbol{e}_i \in \mathrm{span}\left\{\boldsymbol{U}_c^{\mathrm{T}}\boldsymbol{W}_T\right\}$. Therefore, a sufficient condition for the validity of (11) can be given by

$$\boldsymbol{U}_{c_i} \in \mathrm{span}\left\{\boldsymbol{W}_T\right\},\ \ \forall\, i \in [1,\dots,n_q]. \tag{13}$$

Now, we apply LaSalle's Invariance Principle. Let us, hereto, define a set $\mathcal{C}$ by

$$\mathcal{C} = \left\{(\boldsymbol{q},\dot{\boldsymbol{q}}) \mid |\left(\boldsymbol{U}_c^{\mathrm{T}}\dot{\boldsymbol{q}}\right)_i| \leq -\frac{\beta}{\lambda_i},\ i = 1,\dots,n_q\right\}, \tag{14}$$

where $\left(\boldsymbol{U}_c^{\mathrm{T}}\dot{\boldsymbol{q}}\right)_i$ denotes the i-th element of the column $\boldsymbol{U}_c^{\mathrm{T}}\dot{\boldsymbol{q}}$. Moreover, let us define a set $\mathcal{I}_\rho$ such that $\mathcal{I}_\rho = \left\{(\boldsymbol{q},\dot{\boldsymbol{q}}) \mid V(\boldsymbol{q},\dot{\boldsymbol{q}}) \leq \rho\right\}$ and choose the constant ρ such that $\mathcal{I}_\rho \subset \mathcal{C}$. Moreover, we define a set $\mathcal{S} \subset \mathcal{I}_\rho$ by $\mathcal{S} = \left\{(\boldsymbol{q},\dot{\boldsymbol{q}}) \in \mathcal{I}_\rho : \dot{\boldsymbol{q}} = \boldsymbol{0}\right\}$. Furthermore, the largest invariant set in $\mathcal{S}$ is a subset $\tilde{\mathcal{E}}$ of the equilibrium set $\mathcal{E}$, where $\tilde{\mathcal{E}} = \mathcal{E} \bigcap \mathrm{int}(\mathcal{I}_{\rho^*})$ and

$$\rho^* = \max_{\left\{\rho : \mathcal{I}_\rho \subset \mathcal{C}\right\}} \rho. \tag{15}$$

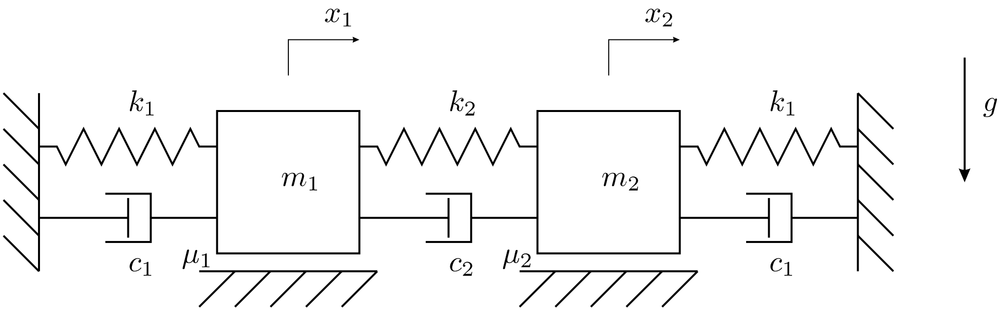

Figure 1. 2DOF mass-spring-damper system with Coulomb friction.

Note that $\dot{V} = 0$ if and only if $(\boldsymbol{q}, \dot{\boldsymbol{q}}) \in \mathcal{S}$ and $\dot{V} < 0$ otherwise. Application of LaSalle's invariance principle concludes the proof of the local attractivity of $\tilde{\mathcal{E}}$ under condition (13). $\square$

At this point several remarks should be made:

1. It should be noted that the proof of Theorem 1 provides us with a conservative estimate of the region of attraction $\mathcal{A}$ of the locally attractive equilibrium set $\mathcal{E}$. The estimate $\mathcal{B}$ can be formulated in terms of the generalised displacements and velocities: $\mathcal{B} = I_{\rho^*}$, where ρ^* satisfies (15), the set $\mathcal{C}$ is given by (14) and V is given by (6); In Van de Wouw and Leine (2004), an explicit expression for ρ^* is provided which allows to estimate the region of attraction of the equilibrium set:

$$\begin{aligned} &\rho^* = \min_{i=1,\ldots,n_q} \rho_i, \\ &\text{with} \\ &\rho_i = \frac{\beta^2}{2\lambda_i^2} \frac{1}{\|\boldsymbol{e}_{n+i}^T \boldsymbol{S}^{-1}\|^2}, \end{aligned} \tag{16}$$

   where $\boldsymbol{S}$ is the square root of $\boldsymbol{P}$ ($\boldsymbol{P} = \boldsymbol{S}^T\boldsymbol{S}$) and $\boldsymbol{P}$ is given by

$$\boldsymbol{P} = \begin{bmatrix} \boldsymbol{U}_c^T \boldsymbol{K} \boldsymbol{U}_c & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{U}_c^T \boldsymbol{M} \boldsymbol{U}_c \end{bmatrix}. \tag{17}$$

2. The proof of Theorem 1 also shows that boundedness of solutions (starting in $\mathcal{B}$) is ensured and that the equilibrium *point* $(\boldsymbol{q}, \dot{\boldsymbol{q}}) = (\boldsymbol{0}, \boldsymbol{0})$ is Lyapunov stable.
3. It can be shown that if it holds that $\boldsymbol{\Lambda}^{\mathrm{T}} \boldsymbol{W}_T^{\mathrm{T}} \boldsymbol{K}^{-\mathrm{T}} \boldsymbol{W}_T \boldsymbol{\Lambda} < 2\rho^*$, then $\mathcal{E} \subset I_{\rho^*}$. In that case the entire equilibrium set $\mathcal{E}$ is locally attractive.
4. An important consequence of Theorem 1 is that when the damping-matrix $\boldsymbol{C}$ is positive definite, global attractivity of the equilibrium set is assured. Note, hereto, that in the proof of Theorem 1, (13) is automatically satisfied and ρ can be taken arbitrarily large in that case.

ILLUSTRATING EXAMPLE

In this section, we will illustrate the results of the previous section by means of an example of a 2DOF mass-spring-damper system, see Figure 1. The equation of motion of this system can be written in the form (1), with $\boldsymbol{q}^{\mathrm{T}} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$ and the generalised friction forces $\boldsymbol{\lambda}_T$ given by the Coulomb friction law (2). Herein the matrices $\boldsymbol{M}$, $\boldsymbol{C}$, $\boldsymbol{K}$, $\boldsymbol{W}_T$ and $\boldsymbol{\Lambda}$ are given by

$$\begin{aligned} \boldsymbol{M} &= \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \ \boldsymbol{C} = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_1 + c_2 \end{bmatrix}, \\ \boldsymbol{K} &= \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_1 + k_2 \end{bmatrix}, \\ \boldsymbol{W}_T &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \boldsymbol{\Lambda} = \begin{bmatrix} \mu_1 m_1 g & 0 \\ 0 & \mu_2 m_2 g \end{bmatrix}, \end{aligned} \tag{18}$$

with $m_1, m_2, k_1, k_2 > 0$ and $\mu_1, \mu_2 \geq 0$. Moreover, the tangential velocity $\boldsymbol{\gamma}_T$ in the frictional contacts is given by $\boldsymbol{\gamma}_T = \begin{bmatrix} \dot{x}_1 & \dot{x}_2 \end{bmatrix}^{\mathrm{T}}$. Let us first compute the spectral decomposition of the damping-matrix, $\boldsymbol{C} = \boldsymbol{U}_c \boldsymbol{\Omega}_c \boldsymbol{U}_c^{\mathrm{T}}$, with (for non-singular $\boldsymbol{C}$):

$$\boldsymbol{U}_c = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \ \boldsymbol{\Omega}_c = \begin{bmatrix} c_1 & 0 \\ 0 & c_1 + 2c_2 \end{bmatrix}. \tag{19}$$

The equilibrium set $\mathcal{E}$, as defined by (5), is given by

$$\begin{aligned} \mathcal{E} = \Big\{ (x_1, x_2, \dot{x}_1, \dot{x}_2) \mid \dot{x}_1 = 0 \wedge \dot{x}_2 = 0 \wedge \\ |x_1| \leq \frac{(k_1 + k_2)\mu_1 m_1 g + k_2 \mu_2 m_2 g}{k_1^2 + 2k_1 k_2} \wedge \\ |x_2| \leq \frac{(k_1 + k_2)\mu_2 m_2 g + k_2 \mu_1 m_1 g}{k_1^2 + 2k_1 k_2} \Big\}. \end{aligned} \tag{20}$$

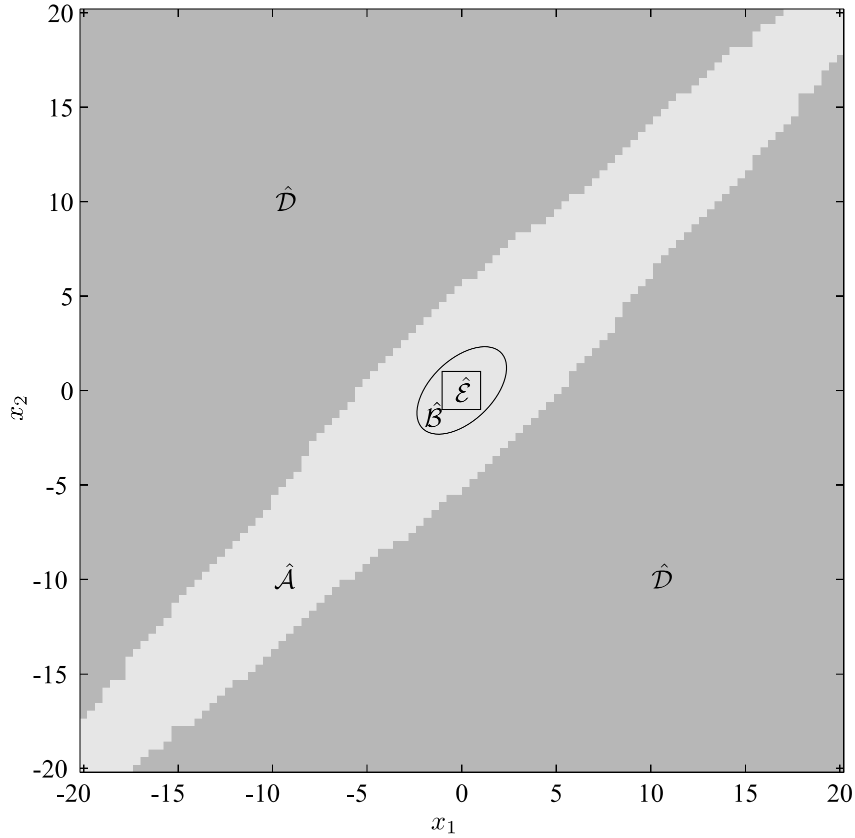

Figure 2. Cross-section of the region of attraction $\mathcal{A}$ with the plane defined by $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$.

Let us now consider two different cases for the damping parameters c_1 and c_2:
Firstly, we consider the case that $\mathbf{c_1 > 0}$ **and** $\mathbf{c_2 > -c_1/2}$. Note that $\boldsymbol{C} > 0$ if and only if $c_1 > 0$ and $c_2 > -c_1/2$. Consequently, the global attractivity of the equilibrium set $\mathcal{E}$ is assured. It should be noted that this is also the case when one or both of the friction coefficients μ_1 and μ_2 vanish.

Secondly, we consider the case that $\mathbf{c_1 > 0}$ **and** $\mathbf{c_2 < -c_1/2}$. Clearly, the damping matrix is not positive definite in this case. As a consequence, the equilibrium point of the system without friction is unstable. Still the equilibrium set of the system with friction can be locally attractive. Therefore, Theorem 1 can be used to investigate the attractivity properties of (a subset of) the equilibrium set. For the friction situation depicted in Figure 1, condition (13) is satisfied if $\mu_1 > 0$ and $\mu_2 > 0$. Namely, $\boldsymbol{W}_T$ spans the two-dimensional space and, consequently, the eigencolumn of the damping matrix corresponding to the unstable eigenvalue $c_1 + 2c_2$, namely $\begin{bmatrix}-1 & 1\end{bmatrix}^{\mathrm{T}}$, lies in the space spanned by the columns of $\boldsymbol{W}_T$.

Since the attractivity is only local, it is desirable to provide an estimate $\mathcal{B}$ of the region of attraction $\mathcal{A}$ of (a subset of) the equilibrium set. Here, we present a comparison between the actual region of attraction (obtained by numerical simulation) and the estimate $\mathcal{B}$ for the following parameter set: $m_1 = m_2 = 1\,\mathrm{kg}$, $k_1 = k_2 = 1\,\mathrm{N/m}$, $c_1 = 0.5\,\mathrm{Ns/m}$, $c_2 = -0.375\,\mathrm{Ns/m}$, $\mu_1 = \mu_2 = 0.1$ and $g = 10\mathrm{m/s}^2$. The numerical simulations are performed using an event-driven integration method as described in Pfeiffer and Glocker (1996). The event-driven integration method is a hybrid integration technique that uses a standard ODE solver for the integration of smooth phases of the system dynamics and a LCP (Linear Complementarity Problem) formulation to determine the next hybrid mode at the switching boundaries. For these parameter settings, $\mathcal{E} \subset \mathrm{int}(\mathcal{I}_{\rho^*})$ and the local attractivity of the entire equilibrium set $\mathcal{E}$ is ensured. In Figure 2, we show a cross-section of $\mathcal{A}$ with the plane $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$, denoted by $\hat{\mathcal{A}}$, which was obtained numerically. Hereto, a grid of initial conditions in the plane $\dot{x}_1 = \dot{x}_2 = 0$ was defined, for which the solutions were obtained by numerically integrating the system over a given time span T. Subsequently, a check was performed to inspect whether the state of the system at time T was in the equilibrium set $\mathcal{E}$. Initial conditions corresponding to attractive solutions are depicted with a light colour (set $\hat{\mathcal{A}}$) and initial conditions corresponding to non-attractive solutions are depicted with a dark grey colour (set $\hat{\mathcal{D}}$). Moreover, $\hat{\mathcal{E}}$ and $\hat{\mathcal{B}}$ are also shown in the figure, where the ˆ indicates that we are referring to cross-sections of the sets. It should be noted that $\hat{\mathcal{E}} \subset \hat{\mathcal{B}}$. As expected the set $\mathcal{B}$ is a conservative estimate for the region of attraction $\mathcal{A}$. In Van de Wouw and Leine (2004), more examples are discussed in which the crucial condition for local attractivity (13) is not satisfied.

CONCLUSIONS

Conditions for the (local) attractivity of (subsets of) equilibrium sets of mechanical systems with friction are derived. The systems are allowed to have multiple degrees-of-freedom and multiple switching boundaries (friction elements). It is shown that the equilibrium set $\mathcal{E}$ of a linear mechanical system, which without friction exhibits a stable equilibrium point E, will always be attractive when Coulomb friction elements are added. Moreover, it has been shown that even if the system without friction has an unstable equilibrium point E, then (a subset of) the equilibrium set $\mathcal{E}$ of the system with friction can under certain conditions be locally attractive and the equilibrium point $E \subset \mathcal{E}$ is stable. The crucial condition can be interpreted as follows: the space spanned by the eigendirections of the damping matrix, related to non-positive eigenvalues, lies in the space spanned by the generalised force directions of the dry friction elements.

Lyapunov stability of the equilibrium set of non-passive systems is not addressed, however, the combination of the attractivity property of the equilibrium set and the boundedness of solutions within $\mathcal{B}$ can be a valuable characteristic when the equilibrium set is a desired steady state of the system. Moreover, an estimate of the region of attraction of the equilibrium set is provided.

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