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STABILITY AND ATTRACTIVITY OF MECHANICAL SYSTEMS WITH UNILATERAL CONSTRAINTS

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Abstract. *In this paper we will give conditions under which the equilibrium set of multi-degree-of-freedom nonlinear mechanical systems with an arbitrary number of frictional unilateral constraints is attractive. The theorems for attractivity are proved by using the framework of measure differential inclusions together with a Lyapunov-type stability analysis and a generalisation of LaSalle's invariance principle for non-smooth systems. The special structure of mechanical multi-body systems allows for a natural Lyapunov function and an elegant derivation of the proof. These results are illustrated by means of examples with unilateral frictional constraints.*

1 INTRODUCTION

Dry friction can seriously affect the performance of a wide range of systems. More specifically, the stiction phenomenon in friction can induce the presence of equilibrium sets, see for example [25]. The stability properties of such equilibrium sets is of major interest when analysing the global dynamic behaviour of these systems.

The aim of the paper is to present a number of theoretical results which can be used to rigourously prove the conditional attractivity of the equilibrium set for nonlinear mechanical systems with frictional unilateral constraints (including impact) using Lyapunov stability theory and LaSalle's invariance principle.

The dynamics of mechanical systems with set-valued friction laws are described by differential inclusions of Filippov-type [13, 16]. Filippov systems, describing systems with friction, can exhibit equilibrium sets, which correspond to the stiction behaviour of those systems. Many publications deal with stability and attractivity properties of (sets of) equilibria in differential inclusions [22, 2, 3, 26, 1, 11]. In previous publications [23, 24], we provided conditions under which the equilibrium set of multi-degree-of-freedom *linear* mechanical systems with an arbitrary number of Coulomb friction elements is attractive using Lyapunov-type stability analysis and a generalisation of LaSalle's invariance principle for non-smooth systems. Dissipative as well as non-dissipative linear systems have been considered, but the analysis was restricted to bilateral frictional constraints and linear systems.

Systems with impact between rigid bodies undergo instantaneous changes in the velocities of the bodies. Impact systems, with or without friction, can be properly described by measure differential inclusions as introduced by Moreau [18, 19] (see also [10, 16, 4]), which allow for discontinuities in the state of the system. Measure differential inclusions, being more general than Filippov systems, can exhibit equilibrium sets as well.

The Lagrange-Dirichlet stability theorem is extended by Brogliato [5] to measure differential inclusions describing mechanical systems with frictionless impact. The idea to use Lyapunov functions involving indicator functions associated with unilateral constraints is most probably due to [5]. It is clearly explained in the work of Chareyron and Wieber [7, 8] why the Lyapunov function has to be globally positive definite, in order to prove stability in the presence of state-discontinuities (when no further assumptions on the system or the form of the Lyapunov function are made). LaSalle's invariance principle is generalised in [6] to differential inclusions and in [7, 8] to measure differential inclusions describing mechanical systems with frictionless impact. The proof of LaSalle's invariance principle strongly relies on the positive invariance of limit sets. It is assumed in [7, 8] that the system enjoys continuity of the solution with respect to the initial condition which is a sufficient condition for positive invariance of limit sets. In [8], an extension of LaSalle invariance principle to systems with unilateral constraints is presented (more specifically it is applied to mechanical systems with frictionless unilateral contacts).

In this paper (see also [17]) we give conditions under which the equilibrium set of multi-degree-of-freedom *nonlinear* mechanical systems with an arbitrary number of frictional *unilateral* constraints (i.e. systems with friction and impact) is attractive. The theorems for attractivity are proved by using the framework of measure differential inclusions together with a Lyapunov-type stability analysis and a generalisation of LaSalle's invariance principle for non-smooth systems, which is based on the assumption that every limit set is positively invariant (see also [14]). The special structure of mechanical multi-body systems allows for a natural choice of the Lyapunov function and a systematic derivation of the proof for this large class of systems.

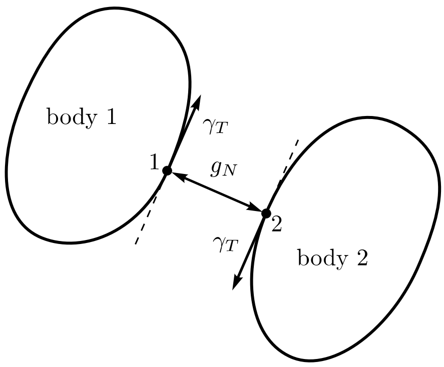


Figure 1: Contact distance g_N and tangential velocity γ_T between two rigid bodies.

In Sections 2 and 3, the constitutive laws for frictional unilateral contact and impact are formulated as set-valued force laws. The modelling of mechanical systems with dry friction and impact by measure differential inclusions is discussed in Section 4. Subsequently, the attractivity properties of the equilibrium set of a system with frictional unilateral contact are studied in Section 5. In Section 6, two examples are studied in order to illustrate the theoretical results of Section 5. A discussion of the results and concluding remarks are given in Section 7.

2 FRICTIONAL CONTACT LAWS IN THE FORM OF SET-VALUED FORCE LAWS

In this section we formulate the constitutive laws for frictional unilateral contact formulated as set-valued force laws (see [10] for an extensive treatise on the subject). Normal contact between rigid bodies is described by a set-valued force law called Signorini's law. Consider two convex rigid bodies at a relative distance g_N from each other (Fig. 1). The normal contact distance g_N is uniquely defined for convex bodies and is such, that the points 1 and 2 have parallel tangent planes (shown as dashed lines in Fig. 1). The normal contact distance g_N is nonnegative because the bodies do not penetrate into each other. The bodies touch when $g_N = 0$. The normal contact force λ_N between the bodies is nonnegative because the bodies can exert only repelling forces on each other, i.e. the constraint is unilateral. The normal contact force vanishes when there is no contact, i.e. $g_N > 0$, and can only be positive when contact is present, i.e. $g_N = 0$. Under the assumption of impenetrability only two situations may occur:

$$\begin{aligned} &g_N = 0 \wedge \lambda_N \geq 0 \quad \text{contact}, \\ &g_N > 0 \wedge \lambda_N = 0 \quad \text{no contact}. \end{aligned} \tag{1}$$

From (1) we see that the normal contact law shows a complementarity behaviour: the product of the contact force and normal contact distance is always zero, i.e. $g_N\lambda_N = 0$. The relation between the normal contact force and the normal contact distance is therefore described by

$$g_N \geq 0, \quad \lambda_N \geq 0, \quad g_N\lambda_N = 0, \tag{2}$$

which is the inequality complementarity condition between g_N and λ_N. The inequality complementarity behaviour of the normal contact law is depicted in the lower-right figure of Fig. 2 and shows a set-valued graph of admissible combinations of g_N and λ_N. The magnitude of the contact force is denoted by λ_N and the direction of the contact force is normal to the bodies, i.e. along the line 1–2 in Fig. 1.

The normal contact law, also called Signorini's law, can be expressed by the subdifferential (see [21]) of a non-smooth conjugate potential $\Psi^*_{C_N}(g_N)$

$$-\lambda_N \in \partial\Psi^*_{C_N}(g_N) \Longleftrightarrow g_N \in \partial\Psi_{C_N}(-\lambda_N), \tag{3}$$

where $C_N = \{-\lambda_N \in \mathbb{R} | \lambda_N \geq 0\} = \mathbb{R}^-$ is the admissible set of negative contact forces $-\lambda_N$ and Ψ_{C_N} is the indicator function of C_N. The potential Ψ_{C_N} is depicted in the upper-left figure of Fig. 2 and is the indicator function of $C_N = \mathbb{R}^-$. Taking the subdifferential of the indicator function gives the set-valued relation $g_N \in \partial\Psi_{C_N}(-\lambda_N)$, depicted in the lower left figure. Interchanging the axis gives the lower right figure which expresses (3) and is equivalent to the lower-right graph of Fig. 2. Integration of the latter relation gives the support function $\Psi^*_{C_N}(g_N)$, which is the conjugate of the indicator function on C_N. The normal contact law for a number of contact points $i = 1, \ldots, n_C$ can formally be stated as

$$-\boldsymbol{\lambda}_N \in \partial\Psi^*_{C_N}(\boldsymbol{g}_N) \iff \boldsymbol{g}_N \in \partial\Psi_{C_N}(-\boldsymbol{\lambda}_N), \qquad C_N = \{-\boldsymbol{\lambda}_N \in \mathbb{R}^n | \boldsymbol{\lambda}_N \geq \mathbf{0}\}, \tag{4}$$

where $\boldsymbol{\lambda}_N$ is the vector containing the normal contact forces λ_{Ni} and $\boldsymbol{g}_N$ is the vector of normal contact distances g_{Ni}. Signorini's law, which is a set-valued law for normal contact on displacement level, can for closed contacts with $g_N = 0$ be expressed on velocity level:

$$-\lambda_N \in \partial\Psi^*_{C_N}(\gamma_N) \iff \gamma_N \in \partial\Psi_{C_N}(-\lambda_N), \qquad g_N = 0, \tag{5}$$

where γ_N is the relative normal contact velocity, i.e. $\gamma_N = \dot{g}_N$ for non-impulsive motion.

Coulomb's friction law is another classical example of a force law that can be described by a non-smooth potential. Consider two bodies as depicted in Fig. 1 with Coulomb friction at the contact point. We denote the relative velocity of point 1 with respect to point 2 along their tangent plane by γ_T. If contact is present between the bodies ($g_N = 0$), then the friction between the bodies imposes a force λ_T along the tangent plane of the contact point. If the bodies are sliding over each other, then the friction force λ_T has the magnitude $\mu\lambda_N$ and acts in the direction of $-\gamma_T$, i.e. $-\lambda_T = \mu\lambda_N \operatorname{sign}(\gamma_T)$ for $\gamma_T \neq 0$, where μ is the friction coefficient and λ_N is the normal contact force. If the relative tangential velocity vanishes ($\gamma_T = 0$), then the bodies purely roll over each other without slip. Pure rolling, or no slip for locally flat objects, is denoted by *stick*. If the bodies stick, then the friction force must lie in the interval $-\mu\lambda_N \leq \lambda_T \leq \mu\lambda_N$. For unidirectional friction, the following three cases are possible:

$$\begin{array}{llll} \gamma_T = 0 & \Rightarrow & |\lambda_T| \leq \mu\lambda_N & \text{sticking}, \\ \gamma_T < 0 & \Rightarrow & \lambda_T = +\mu\lambda_N & \text{negative sliding}, \\ \gamma_T > 0 & \Rightarrow & \lambda_T = -\mu\lambda_N & \text{positive sliding}. \end{array} \tag{6}$$

We can express the friction force by a potential $\pi_T(\gamma_T)$, which we mechanically interpret as a dissipation function, i.e. $-\lambda_T \in \partial\pi_T(\gamma_T)$ with $\pi_T(\gamma_T) = \mu\lambda_N|\gamma_T|$, from which follows the set-valued force law which is depicted in the lower-right graph of Fig. 3. A non-smooth convex potential therefore leads to a maximal monotone set-valued force law. The admissible values of the negative tangential force λ_T form a convex set $C_T = \{-\lambda_T \mid -\mu\lambda_N \leq \lambda_T \leq +\mu\lambda_N\}$ which is bounded by the values of the normal force. Coulomb's law can be expressed with the aid of the indicator function of C_T as $\gamma_T \in \partial\Psi_{C_T}(-\lambda_T)$ where the indicator function Ψ_{C_T} is the conjugate potential of the support function $\pi_T(\gamma_T) = \Psi^*_{C_T}(\gamma_T)$ [10], see Fig. 3.

The classical Coulomb's friction law for spatial contact formulates a two-dimensional friction force $\boldsymbol{\lambda}_T \in \mathbb{R}^2$ which lies in the tangent-plane of the contacting bodies. The set of negative admissible friction forces is a disk $C_T = \{-\boldsymbol{\lambda}_T \mid \|\boldsymbol{\lambda}_T\| \leq \mu\lambda_N\} \subset \mathbb{R}^2$ for isotropic Coulomb friction. Using the set C_T, the spatial Coulomb friction law can be formulated as

$$-\boldsymbol{\lambda}_T \in \partial\Psi^*_{C_T}(\boldsymbol{\gamma}_T) \iff \boldsymbol{\gamma}_T \in \partial\Psi_{C_T}(-\boldsymbol{\lambda}_T), \tag{7}$$

in which $\boldsymbol{\gamma}_T \in \mathbb{R}^2$ is the relative sliding velocity. Similarly, an elliptic choice of C_T would result in an orthotropic Coulomb friction law.

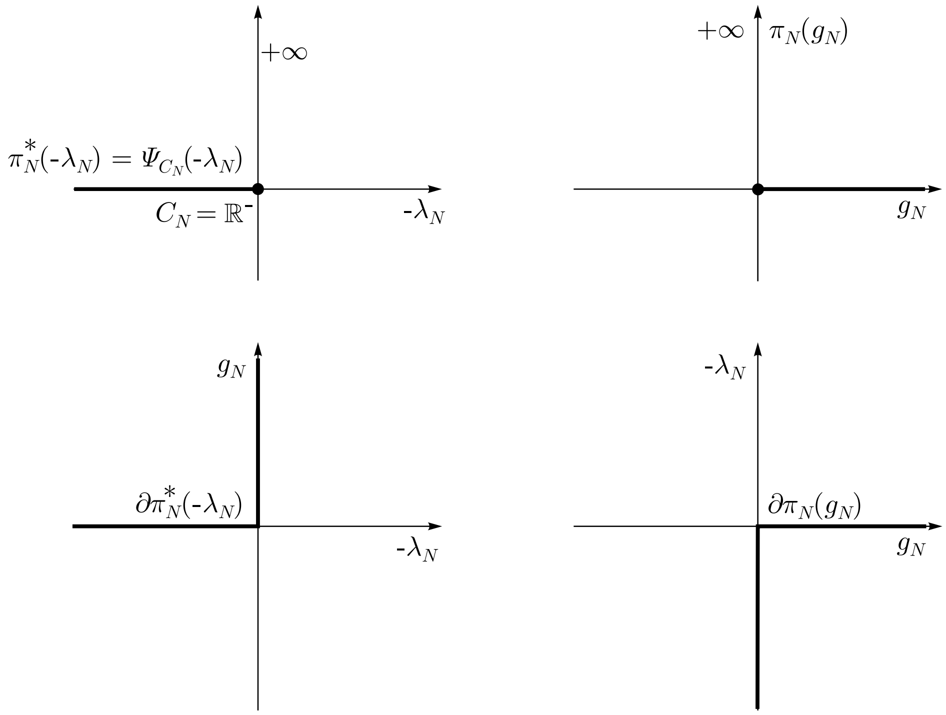


Figure 2: Potential, conjugate potential and subdifferential of the normal contact problem $C = C_N = \mathbb{R}^-$.

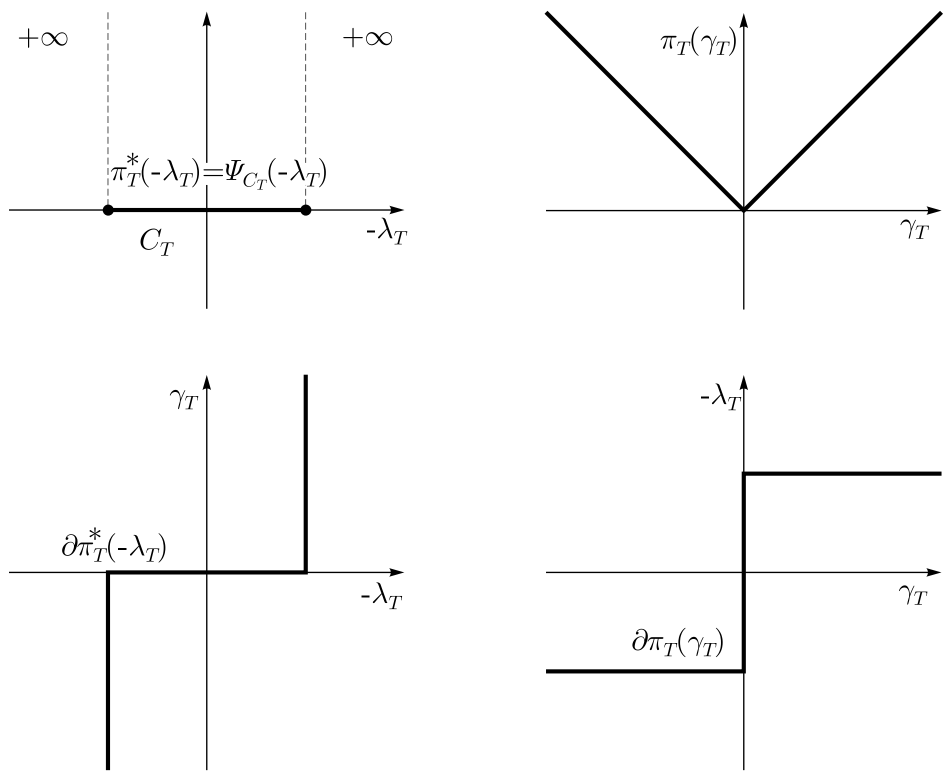


Figure 3: Potential, conjugate potential and subdifferential of the tangential contact problem $C = C_T$.

3 IMPACT LAWS

Signorini's law and Coulomb's friction law are set-valued force laws for non-impulsive forces. In order to describe impact, we need to introduce impact laws for the contact impulses. We will consider a Newton-type of restitution law,

$$\gamma_N^+ = -e_N \gamma_N^-, \quad g_N = 0, \tag{8}$$

which relates the post-impact velocity γ_N^+ of a contact point to the pre-impact velocity γ_N^- by Newton's coefficient of restitution e_N. The case $e_N = 1$ corresponds to a completely elastic contact, whereas $e_N = 0$ corresponds to a completely inelastic contact. The impact, which causes the sudden change in relative velocity, is accompanied by a normal contact impulse $\Lambda_N > 0$. Following [9], suppose that, for any reason, the contact does not participate in the impact, i.e. that the value of the normal contact impulse Λ_N is zero, although the contact is closed. This happens normally for multi-contact situations. For this case we allow the post-impact relative velocities to be higher than the value prescribed by Newtons impact law, $\gamma_N^+ > -e_N \gamma_N^-$, in order to express that the contact is superfluous and could be removed without changing the contact-impact process. We can therefore express the impact law as an inequality complementarity on velocity–impulse level: $\Lambda_N \geq 0$, $\xi_N \geq 0$, $\Lambda_N \xi_N = 0$, with $\xi_N = \gamma_N^+ + e_N \gamma_N^-$ (see [9]). Similarly to Signorini's law on velocity level, we can write the impact law in normal direction as

$$-\Lambda_N \in \partial \Psi^*_{C_N}(\xi_N), \quad g_N = 0. \tag{9}$$

A normal contact impulse Λ_N at a frictional contact leads to a tangential contact impulse $\boldsymbol{\Lambda}_T$ with $\|\boldsymbol{\Lambda}_T\| \leq \mu \Lambda_N$. We therefore have to specify a tangential impact law as well. The tangential impact law can be formulated in a similar way as has been done for the normal impact law:

$$-\boldsymbol{\Lambda}_T \in \partial \Psi^*_{C_T(\Lambda_N)}(\boldsymbol{\xi}_T), \quad g_N = 0, \tag{10}$$

with $\boldsymbol{\xi}_T = \boldsymbol{\gamma}_T^+ + e_T \boldsymbol{\gamma}_T^-$. This impact law involves the tangential restitution coefficient e_T, which is normally considered to be zero, but can be used to model the tangential velocity reversal as observed in the motion of the Super Ball, being a very elastic ball used on play grounds.

4 MODELLING OF SYSTEMS WITH DRY FRICTION AND IMPACT

In this section, we define the class of nonlinear time-autonomous mechanical systems with unilateral frictional contact for which the stability results are derived in Section 5. We first derive a measure differential inclusion describing the dynamics of mechanical systems with discontinuities in the velocity and, subsequently, we study the equilibrium set.

4.1 The measure differential inclusion

We assume that the mechanical systems exhibit only bilateral holonomic frictionless constraints and unilateral constraints in which dry friction can be present. Furthermore, we assume that a set of independent generalised coordinates, $\boldsymbol{q} \in \mathbb{R}^n$, for which these bilateral constraints are eliminated from the formulation of the dynamics of the system, is known. The generalised coordinates $\boldsymbol{q}(t)$ are assumed to be absolutely continuous functions of time t. Also, we assume the generalised velocities, $\boldsymbol{u}(t) = \dot{\boldsymbol{q}}(t)$ for almost all t, to be functions of locally bounded variation. At each time-instance it is therefore possible to define a left limit $\boldsymbol{u}^-$ and a right limit $\boldsymbol{u}^+$ of the velocity. The generalised accelerations $\dot{\boldsymbol{u}}$ are therefore not for all t defined. The set of discontinuity points $\{t_j\}$ for which $\dot{\boldsymbol{u}}$ is not defined is assumed to be Lebesgue negligible. We formulate the dynamics of the system using a Lagrangian approach, resulting in

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}(T_{,\boldsymbol{u}}) - T_{,\boldsymbol{q}} + U_{,\boldsymbol{q}}\right)^{\mathrm{T}} = \boldsymbol{f}^{\mathrm{nc}}(\boldsymbol{q},\boldsymbol{u}) + \boldsymbol{W}_N(\boldsymbol{q})\boldsymbol{\lambda}_N + \boldsymbol{W}_T(\boldsymbol{q})\boldsymbol{\lambda}_T, \tag{11}$$

or, alternatively,

$$\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{u}} - \boldsymbol{h}(\boldsymbol{q},\boldsymbol{u}) = \boldsymbol{W}_N(\boldsymbol{q})\boldsymbol{\lambda}_N + \boldsymbol{W}_T(\boldsymbol{q})\boldsymbol{\lambda}_T, \tag{12}$$

which is a differential equation for the non-impulsive part of the motion. Herein, $\boldsymbol{M}(\boldsymbol{q}) = \boldsymbol{M}^{\mathrm{T}}(\boldsymbol{q}) > 0$ is the mass-matrix. The scalar T represents kinetic energy and it is assumed that it can be written as $T = \frac{1}{2}\boldsymbol{u}^{\mathrm{T}}\boldsymbol{M}(\boldsymbol{q})\boldsymbol{u}$. Moreover, U denotes the potential energy. The column-vector $\boldsymbol{f}^{\mathrm{nc}}$ in (11) represents all smooth generalised non-conservative forces. The state-dependent column-vector $\boldsymbol{h}(\boldsymbol{q},\boldsymbol{u})$ in (12) contains all differentiable forces, such as spring forces, gravitation, smooth damper forces and gyroscopic terms.

We introduce the following index sets:

$$\begin{aligned} &I_G = \{1,\ldots,n_C\} && \text{the set of all contacts,}\\ &I_N = \{i \in I_G \mid g_{Ni}(\boldsymbol{q}) = 0\} && \text{the set of all closed contacts,} \end{aligned} \tag{13}$$

and set up the force laws and impact laws of each contact as has been elaborated in Sections 2 and 3. The normal contact distances $g_{Ni}(\boldsymbol{q})$ depend on the generalised coordinates $\boldsymbol{q}$ and are gathered in a vector $\boldsymbol{g}_N(\boldsymbol{q})$.

During a non-impulsive part of the motion, the normal contact force $-\lambda_{Ni} \in C_N$ and friction force $-\boldsymbol{\lambda}_{Ti} \in C_{Ti} \subset \mathbb{R}^p$ of each closed contact $i \in I_N$, are assumed to be associated with a non-smooth potential, being the support function of a convex set, i.e.

$$-\lambda_{Ni} \in \partial\Psi^*_{C_N}(\gamma_{Ni}), \qquad -\boldsymbol{\lambda}_{Ti} \in \partial\Psi^*_{C_{Ti}}(\boldsymbol{\gamma}_{Ti}), \tag{14}$$

where $C_N = \mathbb{R}^-$ and the set C_{Ti} can be dependent on the normal contact force $\lambda_{Ni} \geq 0$. The normal and tangential contact forces of all n_C contacts are gathered in columns $\boldsymbol{\lambda}_N = \{\lambda_{Ni}\}$ and $\boldsymbol{\lambda}_T = \{\boldsymbol{\lambda}_{Ti}\}$ and the corresponding normal and tangential relative velocities are gathered in columns $\boldsymbol{\gamma}_N = \{\gamma_{Ni}\}$ and $\boldsymbol{\gamma}_T = \{\boldsymbol{\gamma}_{Ti}\}$, for $i \in I_G$. We assume that these contact velocities are related to the generalised velocities through:

$$\boldsymbol{\gamma}_N(\boldsymbol{q},\boldsymbol{u}) = \boldsymbol{W}_N^{\mathrm{T}}(\boldsymbol{q})\boldsymbol{u}, \qquad \boldsymbol{\gamma}_T(\boldsymbol{q},\boldsymbol{u}) = \boldsymbol{W}_T^{\mathrm{T}}(\boldsymbol{q})\boldsymbol{u}. \tag{15}$$

It should be noted that $\boldsymbol{W}_X^{\mathrm{T}}(\boldsymbol{q}) = \frac{\partial \boldsymbol{\gamma}_X}{\partial \boldsymbol{u}}$ for $X = N, T$. This assumption is very important as it excludes rheonomic contacts.

Equation (12) together with the set-valued force laws (14) form a differential inclusion

$$\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{u}} - \boldsymbol{h}(\boldsymbol{q},\boldsymbol{u}) \in -\sum_{i\in I_N} \boldsymbol{W}_{Ni}(\boldsymbol{q})\partial\Psi^*_{C_N}(\boldsymbol{\gamma}_{Ni}) - \boldsymbol{W}_{Ti}(\boldsymbol{q})\partial\Psi^*_{C_{Ti}}(\boldsymbol{\gamma}_{Ti}), \quad \text{a.e.} \tag{16}$$

Differential inclusions of this type are called Filippov systems. The differential inclusion (16) only holds for impact free motion.

Subsequently, we define for each contact point the constitutive impact laws

$$-\Lambda_{Ni} \in \partial\Psi^*_{C_N}(\xi_{Ni}), \quad -\boldsymbol{\Lambda}_{Ti} \in \partial\Psi^*_{C_{Ti}(\Lambda_{Ni})}(\boldsymbol{\xi}_{Ti}), \quad i \in I_N, \tag{17}$$

with $\xi_{Ni} = \gamma_{Ni}^+ + e_{Ni}\gamma_{Ni}^-$ and $\boldsymbol{\xi}_{Ti} = \boldsymbol{\gamma}_{Ti}^+ + e_{Ti}\boldsymbol{\gamma}_{Ti}^-$ in which e_{Ni} and e_{Ti} are the normal and tangential restitution coefficients respectively. The inclusions (17) form very complex set-valued mappings representing the contact laws at the impulse level. The force laws for non-impulsive motion can be put in the same form because $\boldsymbol{u}^+ = \boldsymbol{u}^-$ holds in the absence of impacts and because of the positive homogeneity of the support function (see [21]), i.e. $-\lambda_{Ni} \in \partial\Psi^*_{C_N}(\xi_{Ni})$ and $-\boldsymbol{\lambda}_{Ti} \in \partial\Psi^*_{C_{Ti}(\lambda_{Ni})}(\boldsymbol{\xi}_{Ti})$. We now replace the differential inclusion (16), which holds for almost all t, by an equality of measures

$$\boldsymbol{M}(\boldsymbol{q})\mathrm{d}\boldsymbol{u} - \boldsymbol{h}(\boldsymbol{q},\boldsymbol{u})\mathrm{d}t = \boldsymbol{W}_N(\boldsymbol{q})\mathrm{d}\boldsymbol{P}_N + \boldsymbol{W}_T(\boldsymbol{q})\mathrm{d}\boldsymbol{P}_T \quad \forall t, \tag{18}$$

which holds for all time-instances t. The differential measure of the contact impulsions $\mathrm{d}\boldsymbol{P}_N$ and $\mathrm{d}\boldsymbol{P}_T$ contains a Lebesgue measurable part $\boldsymbol{\lambda}\mathrm{d}t$ and an atomic part $\boldsymbol{\Lambda}\mathrm{d}\eta$

$$\mathrm{d}\boldsymbol{P}_N = \boldsymbol{\lambda}_N\mathrm{d}t + \boldsymbol{\Lambda}_N\mathrm{d}\eta, \qquad \mathrm{d}\boldsymbol{P}_T = \boldsymbol{\lambda}_T\mathrm{d}t + \boldsymbol{\Lambda}_T\mathrm{d}\eta, \tag{19}$$

which can be expressed as inclusions

$$-\mathrm{d}P_{Ni} \in \partial\Psi^*_{C_N}(\xi_{Ni})(\mathrm{d}t+\mathrm{d}\eta), \qquad -\mathrm{d}\boldsymbol{P}_{Ti} \in \partial\Psi^*_{C_{Ti}(\lambda_{Ni})}(\boldsymbol{\xi}_{Ti})\mathrm{d}t + \partial\Psi^*_{C_{Ti}(\Lambda_{Ni})}(\boldsymbol{\xi}_{Ti})\mathrm{d}\eta. \tag{20}$$

As an abbreviation we write

$$\boldsymbol{M}(\boldsymbol{q})\mathrm{d}\boldsymbol{u} - \boldsymbol{h}(\boldsymbol{q},\boldsymbol{u})\mathrm{d}t = \boldsymbol{W}(\boldsymbol{q})\mathrm{d}\boldsymbol{P} \quad \forall t, \tag{21}$$

using short-hand notation

$$\boldsymbol{\lambda} = \begin{bmatrix}\boldsymbol{\lambda}_N\\ \boldsymbol{\lambda}_T\end{bmatrix}, \quad \boldsymbol{\Lambda} = \begin{bmatrix}\boldsymbol{\Lambda}_N\\ \boldsymbol{\Lambda}_T\end{bmatrix}, \quad \boldsymbol{P} = \begin{bmatrix}\boldsymbol{P}_N\\ \boldsymbol{P}_T\end{bmatrix}, \quad \boldsymbol{W} = \begin{bmatrix}\boldsymbol{W}_N & \boldsymbol{W}_T\end{bmatrix}, \quad \boldsymbol{\gamma} = \begin{bmatrix}\gamma_N\\ \gamma_T\end{bmatrix}. \tag{22}$$

Furthermore we introduce the quantities

$$\boldsymbol{\xi} \equiv \boldsymbol{\gamma}^+ + \boldsymbol{E}\boldsymbol{\gamma}^-, \qquad \boldsymbol{\delta} \equiv \boldsymbol{\gamma}^+ - \boldsymbol{\gamma}^-, \tag{23}$$

with $\boldsymbol{E} = \mathrm{diag}(\{e_{Ni}, e_{Ti}\})$ from which we deduce $\boldsymbol{\gamma}^+ = (\boldsymbol{I}+\boldsymbol{E})^{-1}(\boldsymbol{\xi}+\boldsymbol{E}\boldsymbol{\delta})$ and $\boldsymbol{\gamma}^- = (\boldsymbol{I}+\boldsymbol{E})^{-1}(\boldsymbol{\xi}-\boldsymbol{\delta})$. The equality of measures (21) together with the set-valued force laws (20) form a measure differential inclusion which describes the time-evolution of a mechanical system with discontinuities in the generalised velocities. Such a measure differential inclusion does not necessarily have existence and uniqueness of solutions for all admissible initial conditions. Indeed, if the friction coefficient is large, then the coupling between motion normal to the constraint and tangential to the constraint can cause existence and uniqueness problem (known as the Painlevé problem [4, 15]). In the following, we assume existence and uniqueness of solutions in forward time. The contact laws guarantee that the generalised positions $\boldsymbol{q}(t)$ are such that penetration is avoided ($g_{Ni} \geq 0$) and the generalised positions therefore remain within the admissible set

$$\mathcal{K} = \{\boldsymbol{q} \in \mathbb{R}^n \mid g_{Ni}(\boldsymbol{q}) \geq 0\, \forall i \in I_G\}, \tag{24}$$

for all t. The condition $\boldsymbol{q}(t) \in \mathcal{K}$ follows from the assumption of existence of solutions. We remark that the following theorems can be relaxed to systems with non-uniqueness of solutions.

4.2 Equilibrium set

The measure differential inclusion described by (21) and (20) exhibits an equilibrium set. Note that the assumption of scleronomic contacts implies that implies $\boldsymbol{\gamma}_T = \mathbf{0}$ for $\boldsymbol{u} = \mathbf{0}$, see (15). This means that every equilibrium implies sticking in all closed contact points. Every equilibrium position has to obey the equilibrium inclusion

$$\boldsymbol{h}(\boldsymbol{q}, \mathbf{0}) - \sum_{i \in I_N} \left(\boldsymbol{W}_{Ni}(\boldsymbol{q}) \partial \Psi^*_{C_N}(\mathbf{0}) + \boldsymbol{W}_{Ti}(\boldsymbol{q}) \partial \Psi^*_{C_{Ti}}(\mathbf{0}) \right) \ni \mathbf{0}, \tag{25}$$

which, using $C = \partial \Psi^*_C(0)$, simplifies to $\boldsymbol{h}(\boldsymbol{q}, \mathbf{0}) - \sum_{i \in I_N} (\boldsymbol{W}_{Ni}(\boldsymbol{q}) C_{Ni} + \boldsymbol{W}_{Ti}(\boldsymbol{q}) C_{Ti}) \ni \mathbf{0}$. An equilibrium set, being a simply connected set of equilibrium points, is therefore given by

$$\mathcal{E} \subset \left\{ (\boldsymbol{q}, \boldsymbol{u}) \in \mathbb{R}^n \times \mathbb{R}^n | \, (\boldsymbol{u} = \mathbf{0}) \wedge \boldsymbol{h}(\boldsymbol{q}, \mathbf{0}) + \sum_{i \in I_N} \left(\boldsymbol{W}_{Ni}(\boldsymbol{q}) \mathbb{R}^+ - \boldsymbol{W}_{Ti}(\boldsymbol{q}) C_{Ti} \right) \ni \mathbf{0} \right\} \tag{26}$$

and is positively invariant if we assume uniqueness of the solutions in forward time. With $\mathcal{E}$ we denote an equilibrium set of the measure differential inclusion in the state-space $(\boldsymbol{q}, \boldsymbol{u})$, while $\mathcal{E}_{\boldsymbol{q}}$ is reserved for the union of equilibrium positions $\boldsymbol{q}^*$, i.e. $\mathcal{E} = \{(\boldsymbol{q}, \boldsymbol{u}) \in \mathbb{R}^n \times \mathbb{R}^n \mid \boldsymbol{q} \in \mathcal{E}_{\boldsymbol{q}}, \boldsymbol{u} = \mathbf{0}\}$. Note that nonlinear mechanical systems without dry friction can exhibit multiple equilibria. Similarly, a system with dry friction may exhibit multiple equilibrium sets.

Let us now state some consequences of the assumptions made, which will be used in the next section. Due to the fact that the kinetic energy can be described by

$$T = \frac{1}{2} \boldsymbol{u}^{\mathrm{T}} \boldsymbol{M}(\boldsymbol{q}) \boldsymbol{u} = \frac{1}{2} M_{rs} u^r u^s, \tag{27}$$

with $\boldsymbol{M}(\boldsymbol{q}) = \boldsymbol{M}^{\mathrm{T}}(\boldsymbol{q})$, we can write in tensorial language

$$\begin{aligned} \frac{\partial T}{\partial q^k} &= \frac{1}{2} \left(\frac{\partial M_{rs}}{\partial q^k} \right) u^r u^s, \qquad \frac{\partial T}{\partial u^k} = M_{kr} u^r, \\ \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial T}{\partial u^k} \right) &= M_{kr} \dot{u}^r + \left(\frac{\partial M_{kr}}{\partial q^s} \right) u^r u^s = M_{kr} \dot{u}^r + 2 \frac{\partial T}{\partial q^k} + \left(\frac{\partial M_{kr}}{\partial q^s} - \frac{\partial M_{rs}}{\partial q^k} \right) u^r u^s \\ \frac{\mathrm{d}}{\mathrm{d}t} (T_{,\boldsymbol{u}}) &= \dot{\boldsymbol{u}}^{\mathrm{T}} \boldsymbol{M}(\boldsymbol{q}) + 2 T_{,\boldsymbol{q}} - (\boldsymbol{f}^{\mathrm{gyr}})^{\mathrm{T}} \qquad \text{for almost all } t \end{aligned} \tag{28}$$

with the gyroscopic forces

$$\boldsymbol{f}^{\mathrm{gyr}} = \{ f_k^{\mathrm{gyr}} \}, \qquad f_k^{\mathrm{gyr}} = - \left(\frac{\partial M_{kr}}{\partial q^s} - \frac{\partial M_{rs}}{\partial q^k} \right) u^r u^s. \tag{29}$$

In the next section we will exploit that the gyroscopic forces $\boldsymbol{f}_{\mathrm{gyr}}$ have zero power, i.e. $\boldsymbol{u}^{\mathrm{T}} \boldsymbol{f}^{\mathrm{gyr}} = u^k f_k^{\mathrm{gyr}} = 0$. In the same way as before, we can write the differential measure of $T_{,\boldsymbol{u}}$ as $\mathrm{d}\,(T_{,\boldsymbol{u}}) = \mathrm{d}\boldsymbol{u}^{\mathrm{T}} \boldsymbol{M}(\boldsymbol{q}) + 2 T_{,\boldsymbol{q}} \mathrm{d}t - (\boldsymbol{f}^{\mathrm{gyr}})^{\mathrm{T}} \mathrm{d}t$ for all t. Comparison with (12) and (11) yields

$$\boldsymbol{h} = \boldsymbol{f}^{\mathrm{nc}} + \boldsymbol{f}^{\mathrm{gyr}} - \left(T_{,\boldsymbol{q}} + U_{,\boldsymbol{q}} \right)^{\mathrm{T}}. \tag{30}$$

5 ATTRACTIVITY OF EQUILIBRIUM SETS FOR NONLINEAR SYSTEMS

In this section, we will investigate the attractivity properties of the equilibrium sets defined in the previous section. We define the following nonlinear functionals $\mathbb{R}^n \to \mathbb{R}$ on $\boldsymbol{u} \in \mathbb{R}^n$:

- $D^{\mathrm{nc}}_{\boldsymbol{q}}(\boldsymbol{u}) := -\boldsymbol{u}^{\mathrm{T}} \boldsymbol{f}^{\mathrm{nc}}(\boldsymbol{q}, \boldsymbol{u})$ is the dissipation rate function of the smooth non-conservative forces.
- $D^{\lambda_T}_{\boldsymbol{q}}(\boldsymbol{u}) := \sum_{i \in I_N} \frac{1}{1+e_{Ti}} \Psi^*_{C_{Ti}(\lambda_{Ni})}(\boldsymbol{\xi}_{Ti}(\boldsymbol{q}, \boldsymbol{u}))$ is the dissipation rate function of the tangential contact forces.
- $D^{\Lambda_T}_{\boldsymbol{q}}(\boldsymbol{u}) := \sum_{i \in I_N} \frac{1}{1+e_{Ti}} \Psi^*_{C_{Ti}(\Lambda_{Ni})}(\boldsymbol{\xi}_{Ti}(\boldsymbol{q}, \boldsymbol{u}))$ is the dissipation rate function of the tangential contact impulses.

For non-impulsive motion it holds that $\boldsymbol{\gamma}_T = \boldsymbol{\gamma}_T^+ = \boldsymbol{\gamma}_T^-$ and $\boldsymbol{\xi}_T = (1 + e_T)\boldsymbol{\gamma}_T$. Due to the fact that the support function is positively homogeneous it follows that

$$D^{\lambda_T}_{\boldsymbol{q}}(\boldsymbol{u}) = \sum_{i \in I_N} \Psi^*_{C_{Ti}(\lambda_{Ni})}(\boldsymbol{\gamma}_{Ti}(\boldsymbol{q}, \boldsymbol{u})) = \sum_{i \in I_N} -\boldsymbol{\lambda}_{Ti}\boldsymbol{\gamma}_{Ti}(\boldsymbol{q}, \boldsymbol{u}), \tag{31}$$

from which we see that the dissipation rate function of the tangential contact forces does not depend on the restitution coefficient e_T. The above dissipation rate functions are of course functions of $(\boldsymbol{q}, \boldsymbol{u})$, but we write them as nonlinear functionals on $\boldsymbol{u}$ for every fixed $\boldsymbol{q}$ so that we can speak of the zero set $D^{-1}_{\boldsymbol{q}}(0) = \{\boldsymbol{u} \in \mathbb{R}^n \mid D_{\boldsymbol{q}}(\boldsymbol{u}) = 0\}$ of the functional $D_{\boldsymbol{q}}(\boldsymbol{u})$.

As stated before, the type of systems under investigation may exhibit multiple equilibrium sets. Here, we will study the attractivity properties of a specific given equilibrium set. By $\boldsymbol{q}_{\mathrm{e}}$ we denote an equilibrium position of the system with unilateral *frictionless* contacts

$$\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{u}} - \boldsymbol{h}(\boldsymbol{q}, \boldsymbol{u}) - \boldsymbol{W}_N(\boldsymbol{q})\boldsymbol{\lambda}_N = \boldsymbol{0}, \tag{32}$$

from which follows that the equilibrium position $\boldsymbol{q}_{\mathrm{e}}$ is determined by the inclusion $\boldsymbol{h}(\boldsymbol{q}_{\mathrm{e}}, \boldsymbol{0}) - \sum_{i \in I_G} \boldsymbol{W}_{Ni}(\boldsymbol{q}_{\mathrm{e}}) \partial \Psi^*_{C_N}(g_{Ni}(\boldsymbol{q}_{\mathrm{e}})) \ni \boldsymbol{0}$ or

$$\boldsymbol{h}(\boldsymbol{q}_{\mathrm{e}}, \boldsymbol{0}) - \sum_{i \in I_N} \boldsymbol{W}_{Ni}(\boldsymbol{q}_{\mathrm{e}}) \partial \Psi^*_{C_N}(\underbrace{\gamma_{Ni}(\boldsymbol{q}_{\mathrm{e}}, \boldsymbol{0})}_{=0}) \ni \boldsymbol{0}, \tag{33}$$

which is equivalent to $\boldsymbol{h}(\boldsymbol{q}_{\mathrm{e}}, \boldsymbol{0}) + \boldsymbol{W}_N(\boldsymbol{q}_{\mathrm{e}})\mathbb{R}^+ \ni \boldsymbol{0}$ with $\boldsymbol{W}_N = \{\boldsymbol{W}_{Ni}\},\ i \in I_N$. Let the potential $Q(\boldsymbol{q})$ be the *total* potential energy of the system

$$Q(\boldsymbol{q}) = U(\boldsymbol{q}) + \sum_{i \in I_G} \Psi^*_{C_N}(g_{Ni}(\boldsymbol{q})), \tag{34}$$

which is the sum of the potential energy of all smooth potential forces and the support functions of the normal contact forces. Moreover, we assume that the equilibrium position $\boldsymbol{q}_{\mathrm{e}}$ is a local minimum of the total potential energy $Q(\boldsymbol{q})$. The subset $\mathcal{U}$ is assumed to enclose the equilibrium set $\mathcal{E}_q$ under investigation. Notice that the equilibrium point $\boldsymbol{q}_{\mathrm{e}}$ of the system without friction is also an equilibrium point of the system with friction, $(\boldsymbol{q}_{\mathrm{e}}, \boldsymbol{0}) \in \mathcal{E}$. If the system does exhibit multiple equilibrium sets, then the attractivity of $\mathcal{E}$ will be only local for obvious reasons. In the following we will make use of the Lyapunov candidate function

$$V = T(\boldsymbol{q}, \boldsymbol{u}) + Q(\boldsymbol{q}) = T(\boldsymbol{q}, \boldsymbol{u}) + U(\boldsymbol{q}) + \sum_{i \in I_G} \Psi^*_{C_N}(g_{Ni}(\boldsymbol{q})), \tag{35}$$

being the sum of kinetic and total potential energy. The function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is an extended lower semi-continuous function. Moreover, the function $V(t) = V(\boldsymbol{q}(t), \boldsymbol{u}(t))$ is of locally bounded variation in time because $\boldsymbol{q}(t)$ is absolutely continuous and remains in the admissible set $\mathcal{K}$ defined in (24), $\boldsymbol{u} \in \mathrm{lbv}(I, \mathbb{R}^n)$, and T is a Lipschitz continuous function and Q is an extended lower semi-continuous function but only dependent on $\boldsymbol{q}(t)$. In the following we will make use of the differential measure $\mathrm{d}V$ of $V(t)$. If it holds that $\mathrm{d}V \leq 0$, then it follows that

$$V^+(t) - V^-(t_0) = \int_{[t_0,t]} \mathrm{d}V \leq 0, \tag{36}$$

which means that $V(t)$ is non-increasing. Similarly, $\mathrm{d}V < 0$ implies a strict decrease of $V(t)$. We now formulate a technical result which states conditions under which the equilibrium set can be shown to be (locally) attractive.

Theorem 1 (Attractivity of the equilibrium set)
Consider an equilibrium set $\mathcal{E}$ of the system (21), with constitutive laws (14) and (20). If

1. $T = \frac{1}{2}\boldsymbol{u}^{\mathrm{T}}\boldsymbol{M}(\boldsymbol{q})\boldsymbol{u}$, *with* $\boldsymbol{M}(\boldsymbol{q}) = \boldsymbol{M}^{\mathrm{T}}(\boldsymbol{q}) > 0$,
2. *the equilibrium position* $\boldsymbol{q}_{\mathrm{e}}$ *is a local minimum of the total potential energy* $Q(\boldsymbol{q})$ *and* $Q(\boldsymbol{q})$ *has a non-vanishing generalised gradient for all* $\boldsymbol{q} \in \mathcal{U}\backslash\{\boldsymbol{q}_{\mathrm{e}}\}$, *i.e.* $\boldsymbol{0} \notin \partial Q(\boldsymbol{q}) \, \forall \boldsymbol{q} \in \mathcal{U}\backslash\{\boldsymbol{q}_{\mathrm{e}}\}$, *and the equilibrium set* $\mathcal{E}_{\boldsymbol{q}}$ *is contained in* $\mathcal{U}$, *i.e.* $\mathcal{E}_{\boldsymbol{q}} \subset \mathcal{U}$,
3. $D^{\mathrm{nc}}_{\boldsymbol{q}}(\boldsymbol{u}) = -\boldsymbol{u}^{\mathrm{T}}\boldsymbol{f}^{\mathrm{nc}} \geq 0$, *i.e. the smooth non-conservative forces are dissipative, and* $\boldsymbol{f}^{\mathrm{nc}} = \boldsymbol{0}$ *for* $\boldsymbol{u} = \boldsymbol{0}$,
4. *there exists a non-empty set* $I_C \subset I_G$ *and an open neighbourhood* $\mathcal{V} \subset \mathbb{R}^n \times \mathbb{R}^n$ *of the equilibrium set, such that* $\dot{\gamma}_{Ni}(\boldsymbol{q}, \boldsymbol{u}) < 0$ *(a.e.) for* $\forall i \in I_C\backslash I_N$ *and* $(\boldsymbol{q}, \boldsymbol{u}) \in \mathcal{V}$,
5. $D^{\mathrm{nc}\,-1}_{\boldsymbol{q}}(0) \cap D^{\lambda_{TC}\,-1}_{\boldsymbol{q}}(0) \cap \ker \boldsymbol{W}^{\mathrm{T}}_{NC}(\boldsymbol{q}) = \{\boldsymbol{0}\} \quad \forall \boldsymbol{q} \in \mathcal{C}$ *with*

   $\boldsymbol{g}_{NC} = \{\boldsymbol{g}_{Ni}\}$, $\boldsymbol{W}_{NC} = \{\boldsymbol{w}_{Ni}\}$ *for* $i \in I_C$ *as defined in 4.*,

   $$\mathcal{C} = \{\boldsymbol{q} \mid \boldsymbol{g}_{NC}(\boldsymbol{q}) = \boldsymbol{0}\}, \qquad D^{\lambda_{TC}}_{\boldsymbol{q}} = \sum_{i \in I_C \cap I_N} \Psi^*_{C_{Ti}(\lambda_{Ni})}(\boldsymbol{\gamma}_{Ti}(\boldsymbol{q}, \boldsymbol{u})),$$

6. $0 \leq e_{Ni} < 1$, $|e_{Ti}| < 1 \; \forall i \in I_G$,
7. *one of the following conditions holds*
   (a) *the restitution coefficients are small in the sense that* $\frac{2e_{\max}}{1+e_{\max}} < \frac{1}{\mathrm{cond}(\boldsymbol{G}(\boldsymbol{q}))} \; \forall \boldsymbol{q} \in \mathcal{C}$ *where* $\boldsymbol{G}(\boldsymbol{q}) := \boldsymbol{W}(\boldsymbol{q})^{\mathrm{T}}\boldsymbol{M}(\boldsymbol{q})^{-1}\boldsymbol{W}(\boldsymbol{q})$ *and* $e_{\max}$ *is the largest restitution coefficient, i.e.* $e_{\max} \geq \max(e_{Ni}, e_{Ti}) \, \forall i \in I_G$,
   (b) *all restitution coefficients are equal, i.e.* $e = e_{Ni} = e_{Ti} \forall i \in I_G$,
   (c) *friction is absent, i.e.* $\mu_i = 0 \, \forall i \in I_G$,

8. *$\mathcal{E} \subset \mathcal{I}_{\rho^*}$ in which the set $\mathcal{I}_{\rho^*}$, with $\mathcal{I}_\rho = \{(\boldsymbol{q},\boldsymbol{u}) \in \mathbb{R}^n \times \mathbb{R}^n \mid V(\boldsymbol{q},\boldsymbol{u}) < \rho\}$, is the largest level set of V, given by (35), that is contained in $\mathcal{V}$ and $\mathcal{Q} = \{(\boldsymbol{q},\boldsymbol{u}) \in \mathbb{R}^n \times \mathbb{R}^n \mid \boldsymbol{q} \in \mathcal{U}\}$, i.e.*

$$\rho^* = \max_{\{\rho : \mathcal{I}_\rho \subset (\mathcal{V} \cap \mathcal{Q})\}} \rho, \tag{37}$$

9. *each limit set in $\mathcal{I}_{\rho^*}$ is positively invariant,*

then the equilibrium set $\mathcal{E}$ is locally attractive and $\mathcal{I}_{\rho^*}$ is a conservative estimate for the region of attraction.

Proof: Note that V is positive definite around the equilibrium point $(\boldsymbol{q},\boldsymbol{u}) = (\boldsymbol{q}_\mathrm{e}, \boldsymbol{0})$ due to conditions 1 and 2 in the theorem. Classically, we seek the time-derivative of V in order to prove the decrease of V along solutions of the system. However, $\dot{\boldsymbol{u}}$ is not defined for all t and $\boldsymbol{u}$ can undergo jumps. We therefore compute the differential measure $\mathrm{d}V = \mathrm{d}T + \mathrm{d}Q$. The total potential energy, being an extended lower semi-continuous function, is only a function of the generalised displacements $\boldsymbol{q}$, which are absolutely continuous, and it therefore holds that

$$\mathrm{d}Q = dQ(\boldsymbol{q})(\mathrm{d}\boldsymbol{q}) = U_{,\boldsymbol{q}}\mathrm{d}\boldsymbol{q} + d\Psi_{\mathcal{K}}(\boldsymbol{q})(\mathrm{d}\boldsymbol{q}), \tag{38}$$

where $dQ(\boldsymbol{q})(\mathrm{d}\boldsymbol{q})$ is the subderivative (see [21]) of Q at $\boldsymbol{q}$ in the direction $\mathrm{d}\boldsymbol{q} = \boldsymbol{u}\mathrm{d}t$. The subderivative $d\Psi_{\mathcal{K}}(\boldsymbol{q})(\mathrm{d}\boldsymbol{q})$ of the indicator function $\Psi_{\mathcal{K}}(\boldsymbol{q})$ equals the indicator function on the associated contingent cone $K_{\mathcal{K}}(\boldsymbol{q})$, i.e. $d\Psi_{\mathcal{K}}(\boldsymbol{q})(\mathrm{d}\boldsymbol{q}) = \Psi_{K_{\mathcal{K}}(\boldsymbol{q})}(\mathrm{d}\boldsymbol{q})$. It holds that $\boldsymbol{u} \in K_{\mathcal{K}}(\boldsymbol{q})$ due to the consistency of the system and the indicator function on the contingent cone therefore vanishes, i.e. $\Psi_{K_{\mathcal{K}}(\boldsymbol{q})}(\boldsymbol{u}\mathrm{d}t) = 0$. Consequently, the differential measure of Q simplifies to

$$\mathrm{d}Q = U_{,\boldsymbol{q}}\mathrm{d}\boldsymbol{q} + \Psi_{K_{\mathcal{K}}(\boldsymbol{q})}(\mathrm{d}\boldsymbol{q}) = U_{,\boldsymbol{q}}\boldsymbol{u}\mathrm{d}t. \tag{39}$$

The kinetic energy $T(\boldsymbol{q},\boldsymbol{u}) = \frac{1}{2}\boldsymbol{u}^\mathrm{T}\boldsymbol{M}(\boldsymbol{q})\boldsymbol{u}$ is a symmetric quadratic form in $\boldsymbol{u}$ and has the differential measure

$$\mathrm{d}T = \frac{1}{2}(\boldsymbol{u}^+ + \boldsymbol{u}^-)^\mathrm{T}\boldsymbol{M}(\boldsymbol{q})\mathrm{d}\boldsymbol{u} + T_{,\boldsymbol{q}}\mathrm{d}\boldsymbol{q}. \tag{40}$$

The differential measure of the Lyapunov candidate V becomes

$$\begin{aligned}\mathrm{d}V &\overset{(39)+(40)}{=} \frac{1}{2}(\boldsymbol{u}^+ + \boldsymbol{u}^-)^\mathrm{T}\boldsymbol{M}(\boldsymbol{q})\mathrm{d}\boldsymbol{u} + (T_{,\boldsymbol{q}} + U_{,\boldsymbol{q}})\,\mathrm{d}\boldsymbol{q} \\ &\overset{(21)}{=} \frac{1}{2}(\boldsymbol{u}^+ + \boldsymbol{u}^-)^\mathrm{T}\left(\boldsymbol{h}(\boldsymbol{q},\boldsymbol{u})\mathrm{d}t + \boldsymbol{W}\mathrm{d}\boldsymbol{\Lambda}\right) + (T_{,\boldsymbol{q}} + U_{,\boldsymbol{q}})\,\boldsymbol{u}\mathrm{d}t.\end{aligned} \tag{41}$$

A term $\frac{1}{2}(\boldsymbol{u}^+ + \boldsymbol{u}^-)^\mathrm{T}\mathrm{d}t$ in front of a Lebesgue measurable term equals $\boldsymbol{u}^\mathrm{T}\mathrm{d}t$. Together with (30), i.e. $\boldsymbol{h} = \boldsymbol{f}^\mathrm{nc} + \boldsymbol{f}^\mathrm{gyr} - (T_{,\boldsymbol{q}} + U_{,\boldsymbol{q}})^\mathrm{T}$, and (19) with (22) we obtain

$$\mathrm{d}V = \boldsymbol{u}^\mathrm{T}\boldsymbol{f}^\mathrm{nc}\mathrm{d}t + \boldsymbol{u}^\mathrm{T}\boldsymbol{f}^\mathrm{gyr}\mathrm{d}t + \boldsymbol{u}^\mathrm{T}\boldsymbol{W}\boldsymbol{\lambda}\mathrm{d}t + \frac{1}{2}(\boldsymbol{u}^+ + \boldsymbol{u}^-)^\mathrm{T}\boldsymbol{W}\boldsymbol{\Lambda}\mathrm{d}\eta. \tag{42}$$

The gyroscopic forces have zero power $\boldsymbol{u}^{\mathrm{T}}\boldsymbol{f}^{\mathrm{gyr}} = 0$. Moreover, the constraints are assumed to be scleronomic and according to (15) it therefore holds that $\boldsymbol{\gamma} = \boldsymbol{W}^{\mathrm{T}}\boldsymbol{u}$, which gives

$$\begin{aligned}
\mathrm{d}V &= \boldsymbol{u}^{\mathrm{T}}\boldsymbol{f}^{\mathrm{nc}}\mathrm{d}t + \boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{\lambda}\mathrm{d}t + \frac{1}{2}(\boldsymbol{\gamma}^{+} + \boldsymbol{\gamma}^{-})^{\mathrm{T}}\boldsymbol{\Lambda}\mathrm{d}\eta \\
&\overset{(23)}{=} \boldsymbol{u}^{\mathrm{T}}\boldsymbol{f}^{\mathrm{nc}}\mathrm{d}t + \boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{\lambda}\mathrm{d}t + \frac{1}{2}\left((\boldsymbol{I}+\boldsymbol{E})^{-1}\left(2\boldsymbol{\xi} - (\boldsymbol{I}-\boldsymbol{E})\boldsymbol{\delta}\right)\right)^{\mathrm{T}}\boldsymbol{\Lambda}\mathrm{d}\eta \\
&= \boldsymbol{u}^{\mathrm{T}}\boldsymbol{f}^{\mathrm{nc}}\mathrm{d}t + \boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{\lambda}\mathrm{d}t + \boldsymbol{\xi}^{\mathrm{T}}(\boldsymbol{I}+\boldsymbol{E})^{-1}\boldsymbol{\Lambda}\mathrm{d}\eta - \frac{1}{2}\boldsymbol{\delta}^{\mathrm{T}}(\boldsymbol{I}-\boldsymbol{E})(\boldsymbol{I}+\boldsymbol{E})^{-1}\boldsymbol{\Lambda}\mathrm{d}\eta \\
&\overset{(19)+(23)}{=} \boldsymbol{u}^{\mathrm{T}}\boldsymbol{f}^{\mathrm{nc}}\mathrm{d}t + \boldsymbol{\xi}^{\mathrm{T}}(\boldsymbol{I}+\boldsymbol{E})^{-1}\mathrm{d}\boldsymbol{P} - \frac{1}{2}\boldsymbol{\delta}^{\mathrm{T}}(\boldsymbol{I}-\boldsymbol{E})(\boldsymbol{I}+\boldsymbol{E})^{-1}\boldsymbol{\Lambda}\mathrm{d}\eta \\
&= \boldsymbol{u}^{\mathrm{T}}\boldsymbol{f}^{\mathrm{nc}}\mathrm{d}t + \sum_{i\in I_N}\left(\frac{\xi_{Ni}\mathrm{d}P_{Ni}}{1+e_{Ni}} + \frac{\boldsymbol{\xi}_{Ti}^{\mathrm{T}}\mathrm{d}\boldsymbol{P}_{Ti}}{1+e_{Ti}}\right) - \frac{1}{2}\boldsymbol{\delta}^{\mathrm{T}}(\boldsymbol{I}-\boldsymbol{E})(\boldsymbol{I}+\boldsymbol{E})^{-1}\boldsymbol{\Lambda}\mathrm{d}\eta.
\end{aligned} \tag{43}$$

Using (20) and $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}^{*} = \Psi_C^{*}(\boldsymbol{x}^{*})$, we obtain

$$\begin{aligned}
\xi_{Ni}\mathrm{d}P_{Ni} &= -\Psi^{*}_{C_N}(\xi_{Ni})(\mathrm{d}t + \mathrm{d}\eta) = 0 \\
\boldsymbol{\xi}_{Ti}^{\mathrm{T}}\mathrm{d}\boldsymbol{P}_{Ti} &= -\Psi^{*}_{C_{Ti}(\lambda_{Ni})}(\boldsymbol{\xi}_{Ti})\mathrm{d}t - \Psi^{*}_{C_{Ti}(\Lambda_{Ni})}(\boldsymbol{\xi}_{Ti})\mathrm{d}\eta \le 0,
\end{aligned} \tag{44}$$

because of $\Psi^{*}_{C_{Ti}}(\boldsymbol{\xi}_{Ti}) \ge 0$ and $\Psi^{*}_{C_N}(\xi_{Ni}) = \Psi_{\mathbb{R}^{+}}(\xi_{Ni}) = 0$ for admissible $\xi_{Ni} \ge 0$. Moreover, applying (15) to (23) gives

$$\boldsymbol{\delta} := \boldsymbol{\gamma}^{+} - \boldsymbol{\gamma}^{-} = \boldsymbol{W}^{\mathrm{T}}(\boldsymbol{u}^{+} - \boldsymbol{u}^{-}) = \boldsymbol{W}^{\mathrm{T}}\boldsymbol{M}^{-1}\boldsymbol{W}\boldsymbol{\Lambda} = \boldsymbol{G}\boldsymbol{\Lambda}, \tag{45}$$

in which we used the abbreviation $\boldsymbol{G} := \boldsymbol{W}^{\mathrm{T}}\boldsymbol{M}^{-1}\boldsymbol{W}$, which is known as the Delassus matrix [19]. The matrix $\boldsymbol{G}$ is positive definite when $\boldsymbol{W}$ has full rank, because $\boldsymbol{M} > 0$. The matrix $\boldsymbol{G}$ is only positive semi-definite if the matrix $\boldsymbol{W}$ does not have full rank, meaning that the generalised force directions of the contact forces are linearly dependent. However, we assume that the matrix $\boldsymbol{W}$ only contains the generalised force directions of unilateral constraints, and that these unilateral constraints do not constitute a bilateral constraint. It therefore holds that there exists no $\boldsymbol{\Lambda}_N \neq \boldsymbol{0}$ such that $\boldsymbol{W}_N\boldsymbol{\Lambda}_N = \boldsymbol{0}$. The impact law requires that $\boldsymbol{\Lambda}_N \geq \boldsymbol{0}$. Hence, it holds that $\boldsymbol{\Lambda}_N^{\mathrm{T}}\boldsymbol{W}_N^{\mathrm{T}}\boldsymbol{M}^{-1}\boldsymbol{W}_N\boldsymbol{\Lambda}_N > 0$ for all $\boldsymbol{\Lambda}_N \neq \boldsymbol{0}$ with $\boldsymbol{\Lambda}_N \geq \boldsymbol{0}$, even if the unilateral constraints are linearly dependent. Moreover, $\boldsymbol{\Lambda}_T \neq \boldsymbol{0}$ implies $\boldsymbol{\Lambda}_N \neq \boldsymbol{0}$. The inequality $\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{G}\boldsymbol{\Lambda} > 0$ therefore holds for all $\boldsymbol{\Lambda} \neq \boldsymbol{0}$ which obey the impact law (9), even if dependent unilateral constraints are considered.

Using (45), we can put the last term in (43) in the following quadratic form

$$\frac{1}{2}\boldsymbol{\delta}^{\mathrm{T}}(\boldsymbol{I}-\boldsymbol{E})(\boldsymbol{I}+\boldsymbol{E})^{-1}\boldsymbol{\Lambda}\mathrm{d}\eta = \frac{1}{2}\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{G}(\boldsymbol{I}-\boldsymbol{E})(\boldsymbol{I}+\boldsymbol{E})^{-1}\boldsymbol{\Lambda}\mathrm{d}\eta. \tag{46}$$

in which $\boldsymbol{G}(\boldsymbol{I}-\boldsymbol{E})(\boldsymbol{I}+\boldsymbol{E})^{-1}$ is a square matrix. The matrix $(\boldsymbol{I}-\boldsymbol{E})(\boldsymbol{I}+\boldsymbol{E})^{-1}$ is a diagonal matrix which is positive definite if the contacts are not purely elastic, i.e. $0 \le e_{Ni} < 1$ and $0 \le e_{Ti} < 1$ for all i. The smallest diagonal element of $(\boldsymbol{I}-\boldsymbol{E})(\boldsymbol{I}+\boldsymbol{E})^{-1}$ is $\frac{1-e_{\max}}{1+e_{\max}}$. Using Proposition 1 in Appendix A, we deduce that if $\boldsymbol{G}$ is positive definite and if condition 7a holds, then the positive definiteness of $\boldsymbol{G}(\boldsymbol{I}-\boldsymbol{E})(\boldsymbol{I}+\boldsymbol{E})^{-1}$ implies

$$\frac{1}{2}\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{G}(\boldsymbol{I}-\boldsymbol{E})(\boldsymbol{I}+\boldsymbol{E})^{-1}\boldsymbol{\Lambda} > 0, \quad \forall\boldsymbol{\Lambda} \neq \boldsymbol{0}. \tag{47}$$

If the generalised force directions are linearly dependent, then the Delassus matrix $\boldsymbol{G}$ is singular and $\mathrm{cond}(\boldsymbol{G})$ is infinity. Condition 7a can therefore not hold.

If $\boldsymbol{G}$ is positive semi-definite (or even positive definite) and all restitution coefficients are equal to e (condition 7b), then the product $\frac{1}{2}\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{G}(\boldsymbol{I}-\boldsymbol{E})(\boldsymbol{I}+\boldsymbol{E})^{-1}\boldsymbol{\Lambda}$ simplifies to $\frac{1}{2}\frac{1-e}{1+e}\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{G}\boldsymbol{\Lambda}$ which is in general non-negative. Again, we can show that (47) still holds for dependent unilateral constraints if we consider $\boldsymbol{\Lambda} \neq \boldsymbol{0}$ with $\boldsymbol{\Lambda} \geq \boldsymbol{0}$.

If $\boldsymbol{G}$ is positive semi-definite (or even positive definite) and friction is absent (condition 7c: $\mu_i = 0 \forall i \in I_G$), then it holds that $\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{G}(\boldsymbol{I}-\boldsymbol{E})(\boldsymbol{I}+\boldsymbol{E})^{-1}\boldsymbol{\Lambda} = (\boldsymbol{\gamma}_N^+ - \boldsymbol{\gamma}_N^-)^{\mathrm{T}}(\boldsymbol{I}-\boldsymbol{E})(\boldsymbol{I}+\boldsymbol{E})^{-1}\boldsymbol{\Lambda}_N = \sum_i(\gamma_{Ni}^+ - \gamma_{Ni}^-)\frac{1-e_{Ni}}{1+e_{Ni}}\Lambda_{Ni}$. The impact law requires that $\gamma_{Ni}^+ + e_{Ni}\gamma_{Ni}^- > 0$ and $\Lambda_{Ni} \geq 0$. Moreover, the unilateral contacts did not penetrate before the impact and the pre-impact relative velocities γ_{Ni}^- are therefore non-positive. The post-impact relative velocities $\gamma_{Ni}^+ = -e_{Ni}\gamma_{Ni}^-$ are therefore non-negative for $0 \leq e_{Ni} < 1$. Furthermore, if $\Lambda_{Ni} > 0$, then it must hold that $\gamma_{Ni}^- < 0$. Hence, $\frac{1}{2}\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{G}(\boldsymbol{I}-\boldsymbol{E})(\boldsymbol{I}+\boldsymbol{E})^{-1}\boldsymbol{\Lambda} > 0$ for all $\boldsymbol{\Lambda} \neq \boldsymbol{0}$ with $\boldsymbol{\Lambda} \geq \boldsymbol{0}$.

Looking again at the differential measure of the total energy (43), we realise that (under conditions 6 and 7) all terms related to the contact forces and impulses are dissipative or passive. Moreover, if we consider not purely elastic contacts, then nonzero contact impulses $\boldsymbol{\Lambda}$ strictly dissipate energy.

We can now decompose the differential measure $\mathrm{d}V$ in a Lebesgue part and an atomic part

$$\mathrm{d}V = \dot{V}\mathrm{d}t + (V^+ - V^-)\mathrm{d}\eta, \tag{48}$$

with (see (31) and above) $\dot{V} = \boldsymbol{u}^{\mathrm{T}}\boldsymbol{f}^{\mathrm{nc}} - \sum_{i\in I_N}\frac{1}{1+e_{Ti}}\Psi^*_{C_{Ti}(\lambda_{Ni})}(\boldsymbol{\xi}_{Ti}) = -D^{\mathrm{nc}}_{\boldsymbol{q}}(\boldsymbol{u}) - D^{\lambda_T}_{\boldsymbol{q}}(\boldsymbol{u}) \leq 0$ and $V^+ - V^- = -D^{\Lambda_T}_{\boldsymbol{q}}(\boldsymbol{u}) - \frac{1}{2}\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{G}(\boldsymbol{I}+\boldsymbol{E})^{-1}(\boldsymbol{I}-\boldsymbol{E})\boldsymbol{\Lambda} \leq 0$. For positive differential measures $\mathrm{d}t$ and $\mathrm{d}\eta$ we deduce that the differential measure of V (48) is non-positive, $\mathrm{d}V \leq 0$. There are a number of cases for $\mathrm{d}V$ to distinguish:

- **Case $\boldsymbol{u} = \boldsymbol{0}$:** It directly follows that $\mathrm{d}V = 0$.

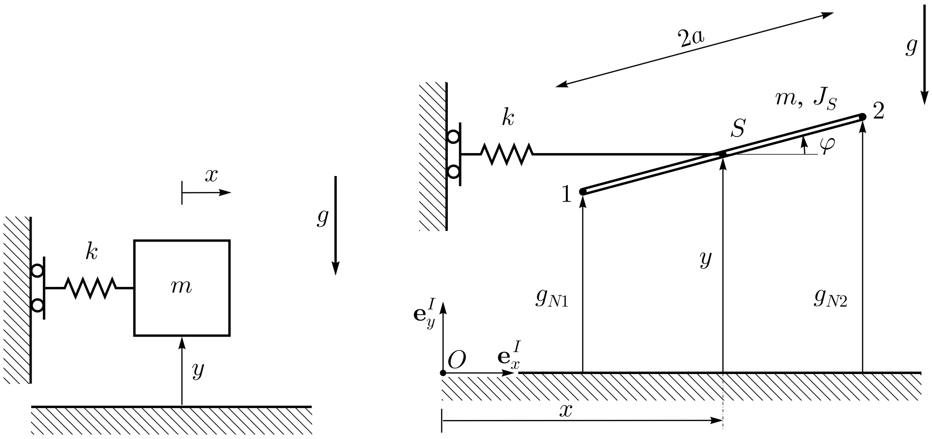
- **Case $g_{Ni} = 0$ and $\gamma_{Ni}^- < 0$ for some $i \in I_N$:** One or more contacts are closing, i.e. there are impacts. It follows from (47) that $V^+ - V^- < 0$ and therefore that $\mathrm{d}V < 0$.

- **Case $\boldsymbol{g}_{NC} = \boldsymbol{0}$, $\boldsymbol{u} \in \ker \boldsymbol{W}_{NC}^{\mathrm{T}}$ and $\boldsymbol{u} = \boldsymbol{u}^- = \boldsymbol{u}^+$ with $\boldsymbol{g}_{NC} = \{g_{Ni}\}$ for $i \in I_C$:** It then holds that all contacts in I_C are closed and remain closed, $I_C \subset I_N$. We now consider $\dot{V}$ as a nonlinear operator on $\boldsymbol{u}$ and write $\dot{V} = 0$ for $\boldsymbol{u} \in \dot{V}_{\boldsymbol{q}}^{-1}(\boldsymbol{0})$ and $\dot{V} < 0$ for $\boldsymbol{u} \notin \dot{V}_{\boldsymbol{q}}^{-1}(\boldsymbol{0})$ with $\dot{V}_{\boldsymbol{q}}^{-1}(\boldsymbol{0}) = D_{\boldsymbol{q}}^{\mathrm{nc}\,-1}(0) \cap D_{\boldsymbol{q}}^{\lambda_T\,-1}(0) \subset D_{\boldsymbol{q}}^{\mathrm{nc}\,-1}(0) \cap D_{\boldsymbol{q}}^{\lambda_{TC}\,-1}(0)$. Condition 5 of the theorem states that, if the contacts in I_C are persistent ($\boldsymbol{W}_{NC}^{\mathrm{T}}\boldsymbol{u} = \boldsymbol{0}$), then dissipation can only vanish if $\boldsymbol{u} = \boldsymbol{0}$, i.e. $D_{\boldsymbol{q}}^{\mathrm{nc}\,-1}(0) \cap D_{\boldsymbol{q}}^{\lambda_{TC}\,-1}(0) = \{\boldsymbol{0}\}$. In other words, if all contacts in I_C are closed and remain closed and $\boldsymbol{u} \neq \boldsymbol{0}$ then dissipation is present. Using condition 5 and $\boldsymbol{u} \in \ker \boldsymbol{W}_{NC}^{\mathrm{T}} \setminus \{\boldsymbol{0}\}$, it follows that $\dot{V}_{\boldsymbol{q}}^{-1}(\boldsymbol{0}) = \{\boldsymbol{0}\}$ and hence $\dot{V} = 0$ for $\boldsymbol{u} = \boldsymbol{0}$ and $\dot{V} < 0$ for $\boldsymbol{u} \neq \boldsymbol{0}$. Impulsive motion for this case is excluded. For a strictly positive differential measure $\mathrm{d}t$ we obtain the differential measure of V as given in (48) and write $\mathrm{d}V = 0$ for $\boldsymbol{u} = \boldsymbol{0}$ and $\mathrm{d}V < 0$ and $\boldsymbol{u} \neq \boldsymbol{0}$.

- **Case $\boldsymbol{g}_{NC} = \boldsymbol{0}$, $\boldsymbol{u} \notin \ker \boldsymbol{W}_{NC}^{\mathrm{T}} \setminus \{\boldsymbol{0}\}$ and $\boldsymbol{W}_{Ni}\boldsymbol{u} > 0$ for some $i \in I_C$:** It then holds that one or more contacts will open. All we can say is that $\mathrm{d}V \leq 0$.

- **Case $g_{Ni} > 0$ for some $i \in I_C$:** One or more contacts are open. All we can say is that $\mathrm{d}V \leq 0$.

We conclude that $\mathrm{d}V = 0$ for $\boldsymbol{u} = \boldsymbol{0}$, $\mathrm{d}V \leq 0$ for $\boldsymbol{g}_{NC} \neq \boldsymbol{0}$ and $\mathrm{d}V < 0$ for $\boldsymbol{g}_{NC} = \boldsymbol{0}$ and $\boldsymbol{u}^- \neq \boldsymbol{0}$.



(a) Falling block (b) Rocking bar

Figure 4: Example systems.

We apply a generalisation of LaSalle's invariance principle, which is valid when every limit set is a positively invariant set [8, 14]. A sufficient condition for the latter is continuity of the solution with respect to the initial condition, which is generally not satisfied by non-smooth mechanical systems with multiple impacts. It is therefore explicitly stated in Condition 9 of Theorem 1 that every limit set in $\mathcal{I}_{\rho^*}$ is positively invariant. Hence, under this assumption, the generalisation of LaSalle's invariance principle can be applied.

Let us consider the set $\mathcal{I}_{\rho^*}$ where ρ^* is chosen such that $\mathcal{I}_{\rho^*} \subset (\mathcal{V} \cap \mathcal{Q})$, see (37). Note that $\mathcal{I}_{\rho^*}$ is a positively invariant set due to the choice of V. Moreover, the set $\mathcal{S} = \{(\boldsymbol{q}, \boldsymbol{u}) \mid \mathrm{d}V = 0\}$ generally has a nonzero intersection with $\mathcal{P} = \{(\boldsymbol{q}, \boldsymbol{u}) \mid \boldsymbol{g}_{NC} \neq \mathbf{0}, \boldsymbol{g}_{NC} \geq \mathbf{0}\}$.

Consider a solution curve with an arbitrary initial condition in $\mathcal{P}$ for $t = t_0$. Due to condition 4 of the theorem, which requires that $\dot{\gamma}_{Ni} < 0$ (a.e.) for $\forall i \in I_C \backslash I_N$, at least one impact will occur for some $t > t_0$. The impact does not necessarily occur at a contact in I_C. In any case, the impact will cause $\mathrm{d}V < 0$ at the impact time. Therefore, there exists no solution curve with initial condition in $\mathcal{P}$ that remains in the intersection $\mathcal{P} \cap \mathcal{S}$. Hence, it holds that the intersection $\mathcal{P} \cap \mathcal{S}$ does not contain any invariant subset. We therefore seek the largest invariant set in $\mathcal{T} = \{(\boldsymbol{q}, \boldsymbol{u}) \mid \boldsymbol{g}_{NC}(\boldsymbol{q}) = \mathbf{0}, \boldsymbol{u} = \mathbf{0}\}$ which clearly is the equilibrium set $\mathcal{E}$. Consequently, we can conclude that the largest invariant set in $\mathcal{S}$ is the equilibrium set $\mathcal{E}$. Hence, it can be concluded from LaSalle's invariance principle that $\mathcal{E}$ is an attractive set. □

Remark: If no conditions on the restitution coefficients exist (other than $0 \leq e_{Ni} < 1$ and $|e_{Ti}| < 1 \forall i$) and if friction is present, then the impact laws (20) can, under circumstances, lead to an energy increase. Such an energetic inconsistency has been reported by Kane and Levinson [12]. In the proof of Theorem 1, we derived sufficient conditions for the energetical consistency (dissipativity) of the adopted impact laws.

6 EXAMPLES

In this section we show how the above theorems can be used to prove the attractivity of an equilibrium set of two mechanical systems.

6.1 Falling block

Consider a planar rigid block (see Fig. 4a) with mass m under the action of gravity (gravitational acceleration g), which is attached to a vertical wall with a spring. The block can freely move in the vertical direction but is not able to undergo a rotation. The coordinates x and y describe the position of the block. The spring is unstressed for $x = 0$. The block comes into contact with a horizontal floor when the contact distance $g_N = y$ becomes zero. The constitutive properties of the contact are the friction coefficient μ and the restitution coefficients $0 \leq e_N < 1$ and $e_T = 0$. The equations of motion for impact free motion read as

$$\begin{aligned} m\ddot{x} + kx &= \lambda_T, \\ m\ddot{y} &= -mg + \lambda_N. \end{aligned} \tag{49}$$

Using generalised coordinates $\boldsymbol{q} = \begin{bmatrix} x & y \end{bmatrix}^{\mathrm{T}}$, we describe the system in the form (18) with

$$\boldsymbol{M} = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}, \quad \boldsymbol{h} = \begin{bmatrix} -kx \\ -mg \end{bmatrix}, \quad \boldsymbol{W}_N = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \boldsymbol{W}_T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{50}$$

The system for $\mu = 0$ admits a unique equilibrium position $\boldsymbol{q}_{\mathrm{e}} = \mathbf{0}$. For $\mu > 0$ there exists an equilibrium set $\mathcal{E} = \{(x, y, \dot{x}, \dot{y}) \mid k|x| \leq \mu mg,\ y = 0,\ \dot{x} = \dot{y} = 0\}$ and it holds that $(\boldsymbol{q}_{\mathrm{e}}, \mathbf{0}) \in \mathcal{E}$. The total potential energy function used in condition 2 of Theorem 1 reads as

$$Q(\boldsymbol{q}) = U(\boldsymbol{q}) + \Psi^*_{C_N}(g_N(\boldsymbol{q})) = \frac{1}{2}kx^2 + mgy + \Psi^*_{\mathbb{R}^-}(y) = \frac{1}{2}kx^2 + mgy + \Psi_{\mathbb{R}^+}(y). \tag{51}$$

Notice that the term $mgy + \Psi_{\mathbb{R}^+}(y)$ is a positive definite term in y. It holds that Q is a positive definite function in $\boldsymbol{q}$, because it is above or equal to another positive definite function $Q(\boldsymbol{q}) \geq \frac{1}{2}kx^2 + mg|y|$. Moreover, the minimum of Q is located at the equilibrium point $\boldsymbol{q}_{\mathrm{e}} = \mathbf{0}$, because $\partial Q(\boldsymbol{q}_{\mathrm{e}}) \ni \mathbf{0}$ and is unique because of the convexity of Q. Condition 2 of Theorem 1 is therefore fulfilled for all $\boldsymbol{q} \in \mathbb{R}^n$. The system does not contain smooth non-conservative forces, i.e. $\boldsymbol{f}^{\mathrm{nc}} = \mathbf{0}$, which fulfills condition 3 of Theorem 1. Denote the contact between block and floor as contact 1 and take $I_C = I_G = \{1\}$. It holds that $\dot{\gamma}_N = -g$ for $g_N = y > 0$, which guarantees the satisfaction of condition 4 of Theorem 1. Furthermore, it holds that $D_{\boldsymbol{q}}^{\mathrm{nc}-1}(0) = \mathbb{R}^n$ and $D_{\boldsymbol{q}}^{\lambda_{TC}-1}(0) = \ker \boldsymbol{W}_T^{\mathrm{T}}$. Because the vectors $\boldsymbol{W}_N$ and $\boldsymbol{W}_T$ are linearly independent it holds that $\ker \boldsymbol{W}_T^{\mathrm{T}} \cap \ker \boldsymbol{W}_N^{\mathrm{T}} = \{\mathbf{0}\}$ and condition 5 of Theorem 1 is therefore fulfilled. Consequently, Theorem 1 proves that the equilibrium set $\mathcal{E}$ is globally attractive.

6.2 Rocking bar

Consider a planar rigid bar with mass m and inertia J_S around the centre of mass S, which is attached to a vertical wall with a spring (Fig. 4b). The gravitational acceleration is denoted by g. The position and orientation of the bar are described by the generalised coordinates $\boldsymbol{q} = \begin{bmatrix} x & y & \varphi \end{bmatrix}^{\mathrm{T}}$, where x and y are the displacements of the centre of mass S with respect to the coordinate frame $(\mathbf{e}_x^I, \mathbf{e}_y^I)$ and φ is the inclination angle. The spring is unstressed for $x = 0$. The bar has length $2a$ and two endpoints which can come into contact with the floor. The contact between bar and floor is described by a friction coefficient $\mu > 0$ and a normal restitution coefficient $0 \leq e_N < 1$ which is equal to the tangential restitution $e_T = e_N$. The contact distances, indicated in Fig. 4b, are $g_{N1} = y - a\sin\varphi$ and $g_{N2} = y + a\sin\varphi$, whereas the relative velocities of contact points 1 and 2 with respect to the floor read as $\gamma_{T1} = \dot{x} + a\dot{\varphi}\sin\varphi$

and $\gamma_{T2} = \dot{x} - a\dot{\varphi}\sin\varphi$. We can describe the system in the form (18) with

$$\boldsymbol{M} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & J_S \end{bmatrix}, \boldsymbol{h} = \begin{bmatrix} -kx \\ -mg \\ 0 \end{bmatrix}, \boldsymbol{W}_N^{\mathrm{T}} = \begin{bmatrix} 0 & 1 & -a\cos\varphi \\ 0 & 1 & a\cos\varphi \end{bmatrix}, \boldsymbol{W}_T^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & a\sin\varphi \\ 1 & 0 & -a\sin\varphi \end{bmatrix}. \tag{52}$$

The system contains a number of equilibrium sets. We will consider the equilibrium set

$$\mathcal{E} = \{(x, y, \varphi, \dot{x}, \dot{y}, \dot{\varphi}) \mid k|x| \leq \mu mg,\ y = 0,\ \varphi = 0,\ \dot{x} = \dot{y} = \dot{\varphi} = 0\}, \tag{53}$$

for which $g_{N1} = g_{N2} = 0$. The total potential energy function (using $\Psi^*_{\mathbb{R}^-}(g_{N1}) = \Psi_{\mathbb{R}^+}(g_{N1})$)

$$Q(\boldsymbol{q}) = U(\boldsymbol{q}) + \Psi^*_{C_N}(g_{N1}(\boldsymbol{q})) + \Psi^*_{C_N}(g_{N2}(\boldsymbol{q})) = \frac{1}{2}kx^2 + mgy + \Psi_{\mathbb{R}^+}(g_{N1}) + \Psi_{\mathbb{R}^+}(g_{N2}) \tag{54}$$

contains a quadratic term in x, a linear term in y and two indicator functions on the contact distances. Notice that $Q(\boldsymbol{q}) = 0$ for $\boldsymbol{q} = 0$. Moreover, it holds that if $g_{N1} \geq 0$ and $g_{N2} \geq 0$ then $y \geq 0$ and $a|\sin\varphi| \leq y$. We therefore deduce that if $g_{N1} \geq 0 \wedge g_{N2} \geq 0$ then $Q(\boldsymbol{q}) = \frac{1}{2}kx^2 + mgy = \frac{1}{2}kx^2 + \frac{mg}{2}(|y| + y)$ and if $g_{N1} < 0 \vee g_{N2} < 0$ then $Q(\boldsymbol{q}) = +\infty$. The function $f(\boldsymbol{q}) = \frac{1}{2}kx^2 + \frac{mg}{2}(|y| + a|\sin\varphi|)$ is locally positive definite in the set $\mathcal{U} = \{\boldsymbol{q} \in \mathbb{R}^n \mid |\varphi| < \frac{\pi}{2}\}$. Consequently, the total potential energy function $Q(\boldsymbol{q}) \geq f(\boldsymbol{q})$ is locally positive definite in the set $\mathcal{U}$ as well. It can be easily checked that the generalised gradient $\partial Q(\boldsymbol{q})$ can only vanish in the set $\mathcal{U}$ for $\boldsymbol{q} = \boldsymbol{q}_{\mathrm{e}}$, i.e. $\boldsymbol{0} \notin \partial Q(\boldsymbol{q})\, \forall \boldsymbol{q} \in \mathcal{U}\backslash\{\boldsymbol{q}_{\mathrm{e}}\}$ and $\boldsymbol{0} \in \partial Q(\boldsymbol{q}_{\mathrm{e}})$.

Smooth non-conservative forces are absent in this system, i.e. $\boldsymbol{f}^{\mathrm{nc}} = \boldsymbol{0}$ and $D^{\mathrm{nc}}_{\boldsymbol{q}}(\boldsymbol{u}) = 0$. We now want to prove that condition 4 of Theorem 1 holds with $I_C = \{1, 2\}$. Consider the open subset $\mathcal{V} = \{(\boldsymbol{q}, \boldsymbol{u}) \in \mathbb{R}^n \times \mathbb{R}^n \mid \mu|\tan\varphi| < 1, a\dot{\varphi}^2 < g\}$ which contains the equilibrium set, i.e. $\mathcal{E} \subset \mathcal{V}$. We consider the following cases with $(\boldsymbol{q}, \boldsymbol{u}) \in \mathcal{V}$:

- $I_N = \emptyset$: both contacts are open, i.e. $g_{N1} > 0$ and $g_{N2} > 0$. It holds for $(\boldsymbol{q}, \boldsymbol{u}) \in \mathcal{V}$ that

$$\begin{aligned} \dot{\gamma}_{N1} &= \ddot{y} - a\ddot{\varphi}\cos\varphi + a\dot{\varphi}^2\sin\varphi = -g + a\dot{\varphi}^2\sin\varphi < 0 \\ \dot{\gamma}_{N2} &= \ddot{y} + a\ddot{\varphi}\cos\varphi - a\dot{\varphi}^2\sin\varphi = -g - a\dot{\varphi}^2\sin\varphi < 0. \end{aligned} \tag{55}$$

- $I_N = \{1\}$: contact 1 is closed and contact 2 is open, i.e. $g_{N1} = 0$ and $g_{N2} > 0$. We consider contact 1 to be closed for a nonzero time-interval. The normal contact acceleration of the closed contact 1 must vanish:

$$\begin{aligned} \dot{\gamma}_{N1} &= \ddot{y} - a\ddot{\varphi}\cos\varphi + a\dot{\varphi}^2\sin\varphi \\ 0 &= -g + \frac{1}{m}\lambda_{N1} + \frac{a^2}{J_S}\cos^2\varphi\,\lambda_{N1} - \frac{a^2}{J_S}\cos\varphi\sin\varphi\,\lambda_{T1} + a\dot{\varphi}^2\sin\varphi \\ 0 &= -g + \left(\frac{1}{m} + \frac{a^2}{J_S}\cos\varphi(\cos\varphi - \bar{\mu}\sin\varphi)\right)\lambda_{N1} + a\dot{\varphi}^2\sin\varphi, \end{aligned} \tag{56}$$

  with $\lambda_{T1} = \bar{\mu}\lambda_{N1}$, i.e. $\bar{\mu} \in -\mu\,\mathrm{Sign}(\gamma_{T1})$. It follows from (56) that the normal contact force λ_{N1} is a function of φ and $\dot{\varphi}$. The contact acceleration of contact 2 therefore becomes

$$\begin{aligned} \dot{\gamma}_{N2} &= \ddot{y} + a\ddot{\varphi}\cos\varphi - a\dot{\varphi}^2\sin\varphi \\ &= -g + \frac{1}{m}\lambda_{N1} - \frac{a^2}{J_S}\cos^2\varphi\,\lambda_{N1} + \frac{a^2}{J_S}\cos\varphi\sin\varphi\,\lambda_{T1} - a\dot{\varphi}^2\sin\varphi \\ &= -2g\frac{a^2\frac{m}{J_S}\cos\varphi(\cos\varphi - \bar{\mu}\sin\varphi)}{1 + a^2\frac{m}{J_S}\cos\varphi(\cos\varphi - \bar{\mu}\sin\varphi)} - \frac{2a\dot{\varphi}^2\sin\varphi}{1 + a^2\frac{m}{J_S}\cos\varphi(\cos\varphi - \bar{\mu}\sin\varphi)}. \end{aligned} \tag{57}$$

Using $|\bar{\mu}| \leq \mu$ and $(\boldsymbol{q}, \boldsymbol{u}) \in \mathcal{V}$ it follows that $\dot{\gamma}_{N2} < 0$.

- $I_N = \{2\}$: contact 1 is open and contact 2 is closed, i.e. $g_{N1} > 0$ and $g_{N2} = 0$. Similar to the previous case we can prove that $\dot{\gamma}_{N1} < 0$.

Hence, there exists a non-empty set $I_C = \{1, 2\}$, such that $\dot{\gamma}_{Ni}(\boldsymbol{q}, \boldsymbol{u}) < 0$ (a.e.) for $\forall i \in I_C \backslash I_N$ and $\forall(\boldsymbol{q}, \boldsymbol{u}) \in \mathcal{V}$. Condition 4 of Theorem 1 is therefore fulfilled.

It holds that $D_{\boldsymbol{q}}^{\mathrm{nc}-1}(0) = \mathbb{R}^n$ and it follows that $D_{\boldsymbol{q}}^{\lambda_T}{}^{-1}(0) = \ker \boldsymbol{W}_T^{\mathrm{T}}(\boldsymbol{q})$. Furthermore, for $\boldsymbol{q} \in \mathcal{C} = \{\boldsymbol{q} \in \mathbb{R}^n \mid g_{N1} = g_{N2} = 0\}$ follows the implication $\boldsymbol{W}_N^{\mathrm{T}}(\boldsymbol{q})\boldsymbol{u} = 0 \Longrightarrow \dot{y} = 0 \wedge \dot{\varphi} = 0$ and similarly $\boldsymbol{W}_T^{\mathrm{T}}(\boldsymbol{q})\boldsymbol{u} = 0 \Longrightarrow \dot{x} = 0$. We conclude that there is always dissipation when both contacts are closed and $\boldsymbol{u} \neq \boldsymbol{0}$ because $\ker \boldsymbol{W}_T^{\mathrm{T}}(\boldsymbol{q}) \cap \ker \boldsymbol{W}_N^{\mathrm{T}}(\boldsymbol{q}) = \{\boldsymbol{0}\}$ for all $\boldsymbol{q} \in \mathcal{C}$, and condition 5 of Theorem 1 is therefore fulfilled. The largest level set of $V = T(\boldsymbol{q}, \boldsymbol{u}) + Q(\boldsymbol{q})$ which lies entirely in $\mathcal{Q} = \{(\boldsymbol{q}, \boldsymbol{u}) \in \mathbb{R}^n \times \mathbb{R}^n \mid \boldsymbol{q} \in \mathcal{U}\}$ is given by $V(\boldsymbol{q}, \boldsymbol{u}) < mga$. The largest level set of V which lies entirely in $\mathcal{V}$ is determined by $V(\boldsymbol{q}, \boldsymbol{u}) < \frac{1}{2} J_S \frac{g}{a}$ and $V(\boldsymbol{q}, \boldsymbol{u}) < \frac{mga}{\sqrt{1+\mu^2}}$. We therefore choose the set $\mathcal{I}_{\rho^*}$ as

$$\mathcal{I}_{\rho^*} = \{(\boldsymbol{q}, \boldsymbol{u}) \in \mathbb{R}^n \times \mathbb{R}^n \mid V(\boldsymbol{q}, \boldsymbol{u}) < \rho^*\}, \text{ with } \rho^* = \min\left(\frac{1}{2} J_S \frac{g}{a}, \frac{mga}{\sqrt{1+\mu^2}}\right). \quad (58)$$

If additionally $\frac{1}{2}\frac{(\mu m g)^2}{k} < \rho^*$ then it holds that $\mathcal{E} \subset \mathcal{I}_{\rho^*}$. We conclude that Theorem 1 proves conditionally the local attractivity of the equilibrium set $\mathcal{E}$ and that $\mathcal{I}_{\rho^*}$ is a conservative estimate of the region of attraction. Naturally, the attractivity is only local, because the system has also other attractive equilibrium sets for $\varphi = n\pi$ with $n \in \mathbb{Z}$ and unstable equilibrium sets around $\varphi = \frac{\pi}{2} + n\pi$.

7 CONCLUSIONS

In this paper conditions are given under which the equilibrium set of a multi-degree-of-freedom nonlinear mechanical system with an arbitrary number of frictional unilateral constraints is attractive. The theorems for attractivity are proved by using the framework of measure differential inclusions together with a Lyapunov-type stability analysis and a generalisation of LaSalle's invariance principle for non-smooth systems. The total mechanical energy of the system, including the support function of the normal contact forces, is chosen as Lyapunov function. It has been proved that, under some conditions, the differential measure of the Lyapunov function is non-positive, which is basically a dissipativity argument. Sufficient conditions for the dissipativity of frictional unilateral constraints are given. If we do not consider dependent constraints, then the restitution coefficients must either be small enough, or, be all equal to each other. The latter condition has also been stated in [20]. Attractivity of the equilibrium set is proved in Theorem 1 under a number of conditions. Condition 4 is a condition which is difficult to satisfy and check. It guarantees that there exists no invariant set when one or more contacts are open. Still, we are able to use Theorem 1 to prove the attractivity of equilibrium sets in the example systems of Section 6. Moreover, we provide conservative estimates for the region of attraction of the equilibrium set.

The theorems presented in this paper have been proved for dissipative systems and form the stepping stone to the analysis of non-dissipative systems for which the equilibrium set might still be attractive due to the dissipation of the frictional impacts (see also [23]). The results of this paper will be used in further research to develop control methods for systems with unilateral constraints.

A A RESULT ON POSITIVE DEFINITE MATRICES

Proposition 1 (see [17])
Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ be a symmetric positive definite matrix and $\boldsymbol{B} \in \mathbb{R}^{n\times n}$ be a diagonal positive definite matrix with the diagonal elements b_{ii} which fulfil $1 \geq b_{ii} \geq b_{\min} > 0$, $i = 1, \ldots, n$. If $1 - b_{\min} < \frac{1}{\mathrm{cond}(A)}$ then it holds that the matrix $\boldsymbol{AB}$ is positive definite.

Proof: The matrix $\boldsymbol{A} = \boldsymbol{A}^{\mathrm{T}} > 0$ has real positive eigenvalues and it therefore holds that $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x} \geq \lambda_{\min}\|\boldsymbol{x}\|^2$, where $\lambda_{\min}$ is the smallest eigenvalue of $\boldsymbol{A}$. Moreover, it holds that

$$\boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}(\boldsymbol{I}-\boldsymbol{B})\boldsymbol{x} \leq |\boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}(\boldsymbol{I}-\boldsymbol{B})\boldsymbol{x}| \leq |\boldsymbol{A}|\,|\boldsymbol{I}-\boldsymbol{B}|\,\|\boldsymbol{x}\|^2 \leq \lambda_{\max}(1-b_{\min})\|\boldsymbol{x}\|^2\,, \tag{59}$$

where $\lambda_{\max}$ is the largest eigenvalue of $\boldsymbol{A}$ and $b_{\min}$ is the smallest diagonal element of $\boldsymbol{B}$. Using the above inequalities, we deduce that

$$\boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{B}\boldsymbol{x} = \boldsymbol{x}^{\mathrm{T}}(\boldsymbol{A}-\boldsymbol{A}(\boldsymbol{I}-\boldsymbol{B}))\boldsymbol{x} \geq (\lambda_{\min} - \lambda_{\max}(1-b_{\min}))\,\|\boldsymbol{x}\|^2\,. \tag{60}$$

Hence, if it holds that $1 - b_{\min} < \frac{\lambda_{\min}}{\lambda_{\max}} =: \frac{1}{\mathrm{cond}(\boldsymbol{A})}$, then it follows that $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{B}\boldsymbol{x} > 0$ holds for all $\boldsymbol{x} \neq \boldsymbol{0}$. □

REFERENCES

[1] S. Adly and D. Goeleven. A stability theory for second-order nonsmooth dynamical systems with application to friction problems. *Journal de Mathématiques Pures et Appliquées*, 83:17–51, 2004.

[2] D. V. Anosov. Stability of the equilibrium positions in relay systems. *Automation and Remote Control*, 20:130–143, 1959.

[3] A. Bacciotti and F. Ceragioli. Stability and stabilization of discontinuous systems and nonsmooth Lyapunov functions. *ESIAM: Control, Optimisation and Calculus of Variations*, 4:361–376, 1999.

[4] B. Brogliato. *Nonsmooth Mechanics*. Springer, London, 2 edition, 1999.

[5] B. Brogliato. Absolute stability and the Lagrange-Dirichlet theorem with monotone multivalued mappings. *Systems & Control Letters*, 51:343–353, 2004.

[6] B. Brogliato and D. Goeleven. The Krakovskii-LaSalle invariance principle for a class of unilateral dynamical systems. *Mathematics of Control, Signals, and Systems*, 17(1):57–76, 2005.

[7] S. Chareyron and P. B. Wieber. A LaSalle's invariance theorem for nonsmooth Lagrangian dynamical systems. *Proc. of the ENOC-2005 Conference, 7–12 August, Eindhoven*, 2005.

[8] S. Chareyron and P. B. Wieber. Stability and regulation of nonsmooth dynamical systems. *IEEE Transactions on Automatic Control*, 2007. submitted.

[9] Ch. Glocker. On frictionless impact models in rigid-body systems. *Philosophical Transactions of the Royal Society of London A*, 359:2385–2404, 2001.

[10] Ch. Glocker. *Set-Valued Force Laws, Dynamics of Non-Smooth Systems*, volume 1 of *Lecture Notes in Applied Mechanics*. Springer-Verlag, Berlin, 2001.

[11] D. Goeleven and B. Brogliato. Necessary conditions of asymptotic stability for unilateral dynamical systems. *Nonlinear Analysis*, 61(6):961–1004, 2005.

[12] T. R. Kane and D. A. Levinson. *Dynamics, theory and applications*. McGraw-Hill, New York, 1985.

[13] R. I. Leine. *Bifurcations in Discontinuous Mechanical Systems of Filippov-Type*. Ph.D. thesis, Eindhoven University of Technology, The Netherlands, 2000.

[14] R. I. Leine. *On the stability of motion in non-smooth mechanical systems*. Habilitation thesis, ETH Zürich, Switzerland, 2007.

[15] R. I. Leine, B. Brogliato, and H. Nijmeijer. Periodic motion and bifurcations induced by the Painlevé paradox. *European Journal of Mechanics – A/Solids*, 21(5):869–896, 2002.

[16] R. I. Leine and H. Nijmeijer. *Dynamics and Bifurcations of Non-Smooth Mechanical Systems*, volume 18 of *Lecture Notes in Applied and Computational Mechanics*. Springer Verlag, Berlin, 2004.

[17] R. I. Leine and N. van de Wouw. Stability properties of equilibrium sets of nonlinear mechanical systems with dry friction and impact. *Nonlinear Dynamics*, 2007. online.

[18] J. J. Moreau. Liaisons unilatérales sans frottement et chocs inelastiques. *Comptes Rendu des Séances de l'Academie des Sciences*, 296:1473–1476, 1983.

[19] J. J. Moreau. Unilateral contact and dry friction in finite freedom dynamics. In J. J. Moreau and P. D. Panagiotopoulos, editors, *Non-Smooth Mechanics and Applications*, volume 302 of *CISM Courses and Lectures*, pages 1–82. Springer, Wien, 1988.

[20] J. J. Moreau. Some numerical methods in multibody dynamics: Application to granular materials. *European Journal of Mechanics A/Solids*, 13(4, suppl.):93–114, 1994.

[21] R. T. Rockafellar and R.-B. Wets. *Variational Analysis*. Springer, Berlin, 1998.

[22] D. Shevitz and B. Paden. Lyapunov stability of nonsmooth systems. *IEEE Transactions on Automatic Control*, 39(9):1910–1914, 1994.

[23] N. Van de Wouw and R. I. Leine. Attractivity of equilibrium sets of systems with dry friction. *Nonlinear Dynamics*, 35(1):19–39, 2004.

[24] N. Van de Wouw and R. I. Leine. Stability of stationary sets in nonlinear systems with set-valued friction. *Proc. 45th IEEE CDC 2006, San Diego*, pages 4271–4276, 2006.

[25] N. van de Wouw, M. van den Heuvel, J. van Rooij, and H. Nijmeijer. Performance of an automatic ball balancer with dry friction. *Int. Jnl. Bifurcation and Chaos*, 15(1):65–82, 2005.

[26] V. A. Yakubovich, G. A. Leonov, and A. K. Gelig. *Stability of Stationary Sets in Control Systems with Discontinuous Nonlinearities*, volume 14 of *Series on Stability, Vibration and Control of Systems. Series A.* World Scientific, Singapore, 2004.