Impulsive Control of Mechanical Motion Systems with Uncertain Friction

N. van de Wouw, R.I. Leine

Abstract— **In this paper, we consider the robust set-point stabilisation problem for motion systems subject to friction. Robustness aspects are particularly relevant in practice, where uncertainties in the friction model are unavoidable. We propose an impulsive feedback control design that robustly stabilises the set-point for a class of position-, velocity- and time-dependent friction laws with uncertainty. Moreover, it is shown that this control strategy guarantees the finite-time convergence to the set-point which is a favourable characteristic of the resulting closed loop from a transient performance perspective. The results are illustrated by means of an example.**

I. Introduction

In this paper, we consider the robust set-point stabilisation problem for motion control systems with uncertain friction using an impulsive control strategy. It is well known that controlled motion systems with friction exhibit many undesirable effects such as stick-slip limit cycling, large settling times and non-zero steady-state errors, see e.g. [1]–[4]. In the literature many different approaches towards the control of motion systems with friction have been proposed, such as PID control design, friction compensation, dithering-based approaches, adaptive techniques and impulsive control strategies. As shown e.g. in [1], PID control techniques may suffer from an instability phenomenon known as hunting limit cycling. Many friction compensation approaches are available in the literature (see, for example, [1]–[5]) and have successfully been applied in practice, although it is widely recognised that the undercompensation and overcompensation of friction (due to inevitable friction modelling errors) may lead to non-zero steady-state errors and limit cycling [4], [6], [7]. Examples of adaptive compensation approaches are reported in [8], [9]. Dithering-based approaches, see e.g. [1], [10], [11], aim at smoothing the discontinuity induced by (Coulomb) friction by the introduction of high-frequency excitations and thereby aim to avoid non-zero steady-state errors. The basic idea behind impulsive control strategies is the introduction of controlled impulsive forces when the system gets stuck at a non-zero steady-state error (due the stiction effect of friction), see e.g. [1], [12]–[18]. One of the key practical problems faced in any of those 'friction-beating' strategies is the fact that friction is a phenomenon which is particularly hard to model accurately, especially due to e.g. changing environmental conditions such as lubrication conditions, temperature, wear, humidity etc. [1], [2]. It is therefore of the utmost importance to develop stabilising controllers that are robust against uncertainties in the friction.

Here, we propose an impulsive feedback control strategy which guarantees the robust stability of the set-point in the face of frictional uncertainties, where we consider a large class of position-dependent, velocity-dependent, and time-varying friction models. The practical feasibility of impulsive force manipulation for the positioning of motion control systems has been illustrated in [13], [14], [16], [17]. Moreover, different impulsive feedback control strategies have been proposed in [15], [17]–[19]. However, rigorous stability analyses of the closed-loop system are rare, especially when accounting for uncertainties in the friction model. A notable exception is the recent work in [18] in which an impulsive feedback law similar to the one proposed in this paper has been studied. The common idea behind this impulsive control law is that, when the system reaches the stick phase at a non-zero regulation error, an impulsive force is applied, which kicks the system out of the stick phase and whose magnitude is dependent on the positioning error. The current work differs from and extends the work in [18] in the following ways. Firstly, in this paper we provide a proof for the robust set-point stability for a class of set-valued Coulomb friction models where the friction coefficient may be position-dependent, velocity-dependent and time-dependent, whereas in [18] only a stability analysis for uncertain, but *constant*, friction coefficients is given. Given the fact that position-dependencies, velocity-dependencies (think of e.g. the Stribeck effect) and time-dependent frictional characteristics (due to e.g. changing temperature, humidity or lubrication conditions) are always present in practice, such an extension is very relevant for applications. Secondly, in [18] a combination of an impulsive controller with a smooth linear position-error feedback controller is considered. In the current work, we consider an impulsive controller in combination with a more general linear state-feedback controller. As also stated in [18], such an extension is highly desirable from a performance perspective. Finally, in the current paper we present conditions under which *finite-time* stability of the set-point can be achieved, as opposed to mere asymptotic stability in [18].

Resuming, the main contributions of the current paper are as follows. Firstly, we propose an impulsive feedback control design for a motion control system consisting of a controlled inertia subject to friction modelled by a general class of set-valued, position-dependent, velocity-dependent, and time-

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varying friction models. Secondly, we present conditions under which the robust finite-time stability of the set-point can be guaranteed in the face of uncertainties in the friction.

The outline of the paper is as follows. In Section II, the control problem tackled in this paper is formalised. In Section III, the impulsive control design is introduced. The robust (finite-time) stability analysis of the impulsive closed-loop system is presented in Section IV. The effectiveness of the control design and its robustness properties are illustrated by means of an example in Section V. Finally, concluding remarks are presented in Section VI.

II. Control Problem Formulation

Consider a mechanical system consisting of an inertia with mass m which is in frictional contact with a support (see Figure 1). We denote the position of the inertia by z and its velocity by $\dot{z}$. A friction force F_f acts between the mass and the support under the influence of a normal force mg, with g the gravitational acceleration. The control input consists of a finite control force u and an impulsive control force U. The dynamics of the control system is described by the equation of motion $m\ddot{z} = u + F_f(z, \dot{z}, t)$ and the impact equation $m(\dot{z}^+(t_j) - \dot{z}^-(t_j)) = U$, which relates the difference between the post-impact velocity $\dot{z}^+(t_j)$ and the pre-impact velocity $\dot{z}^-(t_j)$ to the impulsive control force U at time t_j.

The friction force $F_f(z, \dot{z}, t)$ is assumed to obey the following set-valued force law:

$$F_f(z, \dot{z}, t) \in -mg\mu(z, \dot{z}, t)\mathrm{Sign}(\dot{z}), \tag{1}$$

where $\mathrm{Sign}(\cdot)$ denotes the set-valued sign function

$$\mathrm{Sign}(y) := \begin{cases} y/|y|, & y \neq 0 \\ [-1, 1], & y = 0 \end{cases}.$$

Moreover, $\mu(z, \dot{z}, t)$ denotes the friction coefficient that may depend on z, $\dot{z}$ and time t. Note that (1) represent a rather large class of friction models including possibly position-dependent friction, velocity-dependent effects, such as the Stribeck effect, and time-dependent friction (which can occur in practice due to changing temperature/humidity of the contact, wear or changing lubrication conditions). Moreover, (1) represents a set-valued friction model to account for the stiction effect induced by dry friction. In the remainder of this paper, we adopt the following assumption on the friction coefficient.

Assumption 1
The friction coefficient $\mu(z, \dot{z}, t)$ is lower bounded by $\underline{\mu}$ and upper bounded by $\overline{\mu}$, i.e. it holds that $\underline{\mu} \leq \mu(z, \dot{z}, t) \leq \overline{\mu}, \ \forall t, z, \dot{z} \in \mathbb{R}$, for some $0 < \underline{\mu} \leq \overline{\mu}$.

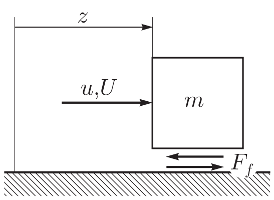

Fig. 1: Mechanical motion system with control input.

The impulsive and non-impulsive dynamics of the system can be represented by a (in general non-autonomous) first-order measure differential inclusion [20], [21]:

$$\begin{aligned} \mathrm{d}x_1 &= x_2 \,\mathrm{d}t \\ \mathrm{d}x_2 &\in -g\mu(x_1, x_2, t)\mathrm{Sign}(x_2)\,\mathrm{d}t + \frac{1}{m}\,\mathrm{d}p \end{aligned} \tag{2}$$

with the state vector $\boldsymbol{x} = [x_1 \ x_2]^{\mathrm{T}} := [z \ \dot{z}]^{\mathrm{T}}$ and where

$$\mathrm{d}p = u\,\mathrm{d}t + U\,\mathrm{d}\eta \tag{3}$$

is the differential measure of the control input, $\mathrm{d}t$ is the Lebesgue measure and $\mathrm{d}\eta$ is a differential atomic measure consisting of a sum of Dirac point measures [22]. The decomposition of the control force as in (3) implies that the differential measure $\mathrm{d}\boldsymbol{x}$ of the state can be decomposed as follows $\mathrm{d}\boldsymbol{x} = \dot{\boldsymbol{x}}\,\mathrm{d}t + (\boldsymbol{x}^+ - \boldsymbol{x}^-)\,\mathrm{d}\eta$. Such a decomposition, implies that $\boldsymbol{x}(t)$ is a special function of locally bounded variation [23]. The state $\boldsymbol{x}(t)$ admits at each time-instant t a left and right limit $\boldsymbol{x}^-(t_j) = \lim_{t\uparrow t_j} \boldsymbol{x}(t)$, $\boldsymbol{x}^+(t_j) = \lim_{t\downarrow t_j} \boldsymbol{x}(t)$, as $\boldsymbol{x}(t)$ is of (special) locally bounded variation. The time-evolution of $\boldsymbol{x}(t)$ is governed by the integration process $\boldsymbol{x}^+(t_1) = \boldsymbol{x}^-(t_0) + \int_{[t_0,t_1]} \mathrm{d}\boldsymbol{x}$, where $[t_0, t_1]$ denotes the compact time-interval between t_0 and $t_1 \geq t_0$.

Now let us state the control problem considered in this paper.

Problem 1
Design a control law for u and U for system (2), (3) such that $\boldsymbol{x} = \boldsymbol{0}$ is a robustly globally uniformly attractively stable equilibrium point of the closed-loop system for a class of uncertain friction models of the form (1) satisfying Assumption 1.

The controller proposed in this paper will induce stability and finite-time attractivity, i.e. symptotic stability[1].

III. Impulsive Feedback Control Design

In order to solve Problem 1, we adopt a proportional-derivative (state-)feedback control law for u in (3):

$$u(x_1, x_2) = -k_1 x_1 - k_2 x_2, \quad k_1, k_2 > 0, \tag{4}$$

together with an impulsive feedback control law for U in (3):

$$U(x_1, x_2^-) = \begin{cases} k_3(x_1), & \text{if } (x_2^- = 0) \wedge (|x_1| \leq \frac{mg\overline{\mu}}{k_1}) \\ 0, & \text{else} \end{cases}, \tag{5}$$

where the constants k_1, k_2 and the function $k_3(x_1)$ are to be designed. The resulting closed-loop dynamics can be formulated in terms of a measure differential inclusion:

$$\begin{aligned} \mathrm{d}x_1 =& x_2\,\mathrm{d}t \\ \mathrm{d}x_2 \in& \left(-\frac{k_1}{m}x_1 - \frac{k_2}{m}x_2 - g\mu(x_1, x_2, t)\mathrm{Sign}(x_2)\right)\mathrm{d}t \\ &+ \frac{1}{m}U(x_1, x_2^-)\,\mathrm{d}\eta. \end{aligned} \tag{6}$$

[1]For a definition of symptotic stability we refer to e.g. [21].

In between impulsive control actions, the non-impulsive dynamics is described by the differential inclusion

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &\in -\frac{k_1}{m}x_1 - \frac{k_2}{m}x_2 - g\mu(x_1,x_2,t)\mathrm{Sign}(x_2).\end{aligned} \tag{7}$$

The state of the system may jump at impulsive time-instants t_j for which $U \neq 0$, i.e. for time instants at which $x_2^-(t_j) = 0$, $|x_1(t_j)| \leq \frac{mg\overline{\mu}}{k_1}$, according to the state reset map

$$x_1^+(t_j) = x_1^-(t_j), \;\; x_2^+(t_j) = x_2^-(t_j) + \frac{k_3(x_1^-(t_j))}{m}. \tag{8}$$

We denote $x_1(t_j) = x_1^-(t_j) = x_1^+(t_j)$, since the position $x_1(t) = z(t)$ is an absolutely continuous function of time, and use the following definitions: $\omega_n := \sqrt{k_1/m}$ and $\zeta := \frac{k_2}{2\sqrt{k_1 m}}$ denote the undamped eigenfrequency and damping ratio of the closed-loop non-impulsive dynamics for $x_2(t) \neq 0$, $\lambda_1 := -\omega_n\zeta + \omega_n\sqrt{\zeta^2-1}$, $\lambda_2 := -\omega_n\zeta - \omega_n\sqrt{\zeta^2-1}$ and

$$\underline{c} := \frac{mg\underline{\mu}}{k_1} = \frac{g\underline{\mu}}{\lambda_1\lambda_2}, \qquad \overline{c} := \frac{mg\overline{\mu}}{k_1} = \frac{g\overline{\mu}}{\lambda_1\lambda_2}. \tag{9}$$

A. Impulsive Controller Design

Let us first explain the rationale behind the design of the controller (3), (4), (5). Hereto, consider the case that $\mu(x_1,x_2,t) = \mu$, with μ a constant, and consider the system without the impulsive part of the controller (i.e. $k_3(x_1) = 0$ in (5)). In this case the closed-loop system is a PD-controlled inertia with Coulomb friction which exhibits an equilibrium set defined by $\{\boldsymbol{x} \in \mathbb{R}^2 \mid |x_1| \leq \frac{mg\mu}{k_1} \wedge x_2 = 0\}$. Clearly, the closed-loop system will then ultimately converge to the equilibrium set and an undesirable non-zero steady-state error will result. Note that for (non-constant) friction coefficients $\mu(x_1,x_2,t)$ satisfying Assumption 1, the closed-loop system without impulsive control will exhibit a time-varying stick set $\mathcal{E}(t)$ that satisfies $\underline{\mathcal{E}} \subseteq \mathcal{E}(t) \subseteq \overline{\mathcal{E}} \quad \forall t$, where $\underline{\mathcal{E}} = \{\boldsymbol{x} \in \mathbb{R}^2 \mid |x_1| \leq \frac{mg\underline{\mu}}{k_1} \wedge x_2 = 0\}$, $\overline{\mathcal{E}} = \{\boldsymbol{x} \in \mathbb{R}^2 \mid |x_1| \leq \frac{mg\overline{\mu}}{k_1} \wedge x_2 = 0\}$ are the minimal and maximal stick sets, respectively. A point $\boldsymbol{x}^* = \begin{bmatrix} x_1^* & x_2^* \end{bmatrix}^T \in \underline{\mathcal{E}}$ remains stationary for all times and is therefore an equilibrium point of the PD-controlled system. The time-varying nature of the stick set $\mathcal{E}(t)$ may, however, destroy the stationarity of points in $\mathcal{E}(t)\backslash\underline{\mathcal{E}}$. The set $\mathcal{E}(t)$ therefore denotes the stick set at time t and not an equilibrium set. The basic idea behind the impulsive controller (3), (4), (5) is to apply an impulsive control force when the state of the system enters the maximal stick-set $\overline{\mathcal{E}}$, i.e. when $\boldsymbol{x}^-(t) \in \overline{\mathcal{E}}$. Loosely speaking, the impulsive force kicks the system out of the stick phase allowing it to further converge (closer) to the set-point. Clearly, the impulsive part of the controller prevents the existence of an equilibrium set (and the occurrence of non-zero steady-state errors). However, energy is added to the system at every time-instant on which an impulsive control action is applied. In this paper, we will provide design rules for k_1, k_2 and $k_3(x_1)$ such that more energy is dissipated (through the derivative action of the controller and the friction) in a time-interval between two impulsive control actions than is provided by the impulsive control action preceding this time-interval.

In order to design the impulsive part of the controller $k_3(x_1)$, we take the following perspective. Consider a time instant t_j for which $\boldsymbol{x}^-(t_j) \in \overline{\mathcal{E}}$, i.e. an impulsive control action $U = k_3(x_1(t_j))$ will be induced by the controller (3), (4), (5) at $t = t_j$. Note that an impulsive control force results only in a jump of the velocity $x_2(t)$ whereas the position $x_1(t)$ is absolutely continuous, see (8). The impulsive control action will cause $\boldsymbol{x}^+(t_j) \notin \overline{\mathcal{E}}$. Let t_{j+1} denote the first time-instant for which $\boldsymbol{x}(t)$ reaches again $\overline{\mathcal{E}}$, i.e. $x_2^-(t_{j+1}) = 0$. Now, we will design $k_3(x_1)$ in (5) such that the velocity will be reset to such a post-impact velocity $x_2^+(t_j)$ that the solution to (7), with $\mu(z,\dot{z},t) = \underline{\mu}$ and initial condition $\big(x_1(t_j), x_2^+(t_j)\big)$, will converge to the origin in finite time $\underline{t}_{j+1}$ without any impulses and/or velocity reversals occurring in the time-interval $(t_j, \underline{t}_{j+1}]$. The impulsive controller design will satisfy the condition

$$k_3(y)\,y < 0, \text{ for } y \neq 0, \text{ and } k_3(0) = 0; \tag{10}$$

in other words, $\boldsymbol{x} = \mathbf{0}$ is an equilibrium point of the controlled system and the impulsive control force U is opposite to the position error $x_1(t_j)$. In Section IV, we will show that this control design also robustly stabilises the closed-loop system with a time-varying and state-dependent friction coefficient $\mu(t) = \mu(x_1(t), x_2(t), t)$ satisfying Assumption 1.

Let us now design the impulsive control law $k_3(x_1)$ that has the above properties. Hereto, consider the case that $x_1(t_j) < 0$ (the case $x_1(t_j) > 0$ can be studied in an analogous fashion). This implies that $k_3(x_1(t_j)) > 0$, see (10), and $x_2^+(t_j) > 0$. On the non-impulsive open time-interval $(t_j, \underline{t}_{j+1})$, the dynamics of (2) for $\mu(x_1,x_2,t) = \underline{\mu}$ is therefore governed by the differential equation

$$\begin{aligned}\dot{\underline{x}}_1(t) &= \underline{x}_2(t)\\ \dot{\underline{x}}_2(t) &= -\omega_n^2\underline{x}_1(t) - 2\zeta\omega_n\underline{x}_2(t) - g\underline{\mu}.\end{aligned} \tag{11}$$

We seek a solution curve of (11) with the boundary conditions $\underline{\boldsymbol{x}}^+(t_j) = \begin{bmatrix} x_1(t_j) & x_2^+(t_j)\end{bmatrix}^{\mathrm{T}}$ and $\underline{\boldsymbol{x}}^-(\underline{t}_{j+1}) = \begin{bmatrix} 0 & 0\end{bmatrix}^{\mathrm{T}}$. The initial position $x_1(t_j)$ and initial time t_j are *a priori* known. The initial velocity $x_2^+(t_j)$ as well as the end time $\underline{t}_{j+1} > t_j$ are yet unknown. We therefore have to solve a mixed boundary value problem for the unknowns $x_2^+(t_j)$ and $\underline{t}_{j+1}$. Hereto, we express the solution for $\mu(x_1,x_2,t) = \underline{\mu}$ in closed form as follows

$$\begin{aligned}\underline{x}_1(t) &= \underline{c}\left(\frac{\lambda_2}{\lambda_2-\lambda_1}e^{\lambda_1(t-\underline{t}_{j+1})} - \frac{\lambda_1}{\lambda_2-\lambda_1}e^{\lambda_2(t-\underline{t}_{j+1})} - 1\right)\\ \underline{x}_2(t) &= \underline{c}\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\left(e^{\lambda_1(t-\underline{t}_{j+1})} - e^{\lambda_2(t-\underline{t}_{j+1})}\right)\end{aligned} \tag{12}$$

for $\zeta > 1$ and with $\underline{c}$ given by (9). Subsequently, using (12) we require that $\underline{x}_1(t)$ at time t_j equals the *a priori* known initial position $x_1(t_j)$. This yields a nonlinear real algebraic equation

$$\mathsf{f}(\underline{t}_{j+1}) = 0 \tag{13}$$

for the unknown end time $\underline{t}_{j+1}$, where the function $\mathsf{f}(t)$ is given by

$$\mathsf{f}(t) = \underline{c}\left(\frac{\lambda_2}{\lambda_2-\lambda_1}e^{\lambda_1(t_j-t)} - \frac{\lambda_1}{\lambda_2-\lambda_1}e^{\lambda_2(t_j-t)} - 1\right) - x_1(t_j). \tag{14}$$

Let us now study the following questions for the system of equations (12), (13), (14):

- For which domain in $x_1(t_j)$ does a unique solution pair $(\underline{t}_{j+1}, x_2^+(t_j))$ exist?
- If a such solution pair exists, can we show that both the time lapse $\underline{t}_{j+1} - t_j$ and $x_2^+(t_j)$ are bounded for bounded $x_1(t_j)$ (i.e. the impulsive control law yields bounded impulses and the resulting flowing response of system (11) converges to the origin in finite time)?

In the following proposition, we propose the impulsive control law that exhibits the above properties. Note that the impulsive control action $k_3(x_1(t_j))$ can be computed from (8) using the fact that $x_2^-(t_j) = 0$:

$$k_3(x_1(t_j)) = m x_2^+(t_j). \tag{15}$$

Proposition 1
Consider the impulsive control law $k_3(x_1(t_j))$ for a given $x_1(t_j)$, with t_j arbitrary, defined by (15)*, where*

1. $\underline{t}_{j+1}$ is the solution of (13), (14)*;*
2. the value of $x_2^+(t_j)$ is determined by the evaluation of $\underline{x}_2(t)$, given by (12) at $t = t_j$.

If $\zeta > 1$, then it holds that $k_3(x_1)$ is uniquely defined and bounded for all $(x_1, x_2) \in \overline{\mathcal{E}}$.

Proof: For the sake of brevity we refer to [24] for a detailed proof. ∎

A schematic representation of the impulsive control law $k_3(x_1)$ for $\zeta > 1$ is given in Figure 2, where we recall that it is only applied for $x_1 \in \overline{\mathcal{E}}^* := \{\boldsymbol{x}_1 \in \mathbb{R} \mid |x_1| \leq \frac{mg\overline{\mu}}{k_1}\}$ (the solid part of the graph). Note that the impulsive control law (15) can be computed *a priori* given the plant properties, the uncertainty bounds $\underline{\mu}$ and $\overline{\mu}$ on the friction coefficient and the gains k_1 and k_2 of the PD-controller.

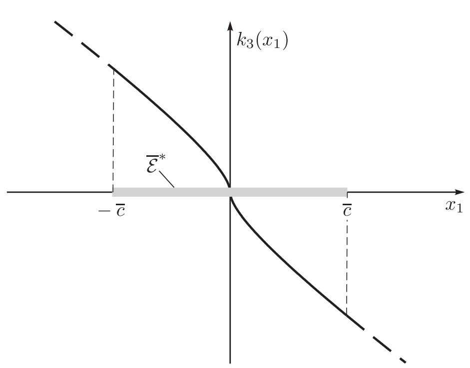

Fig. 2: Schematic representation of the impulsive control law $k_3(x_1)$ for $\zeta \geq 1$ ($\overline{\mathcal{E}}^* = \{\boldsymbol{x}_1 \in \mathbb{R} \mid |x_1| \leq \frac{mg\overline{\mu}}{k_1}\}$).

B. Switching Impulsive Controller Design

We will consider the following switching impulsive control law consisting of three phases:

1) The system starts at an arbitrary initial condition $\boldsymbol{x}(t_0) \in \mathbb{R}^2$. The parameters k_1 and k_2 of the PD-controller are chosen such that the closed loop system without friction is an undercritically damped oscillator (i.e. $\zeta < 1$). We assume that the solution $\boldsymbol{x}(t)$ is attracted in a finite time (denoted by t_1) to $\overline{\mathcal{E}}$. In the next section, we will formalise this assumption and provide sufficient conditions under which this assumption is satisfied, which will explicate the motivation for the choice of $\zeta < 1$ in ensuring finite-time attractivity to the stick-set.
2) The impulsive controller turns on at $t = t_1 \geq t_0$ when $\boldsymbol{x}^-(t_1) \in \overline{\mathcal{E}}$ and the k_2 parameter of the PD controller is increased, such that $\zeta > 1$. We opt for tuning k_2 (for $t \geq t_1$) such that $\zeta > 1$ for the following reasons. Firstly, certain key characteristics of the impulsive control law, see Proposition 1, hold for $\zeta > 1$. Secondly, choosing $\zeta > 1$ (actually choosing ζ large) is desirable from a transient performance perspective. Thirdly, we will show in Section V that the proposed impulsive control law will guarantee the global uniform symptotic stability of the set-point for an arbitrarily large uncertainty in the friction coefficient by choosing ζ sufficiently large, see Assumption 3 and Remark 1.
The impulsive controller induces a velocity jump to $x_2^+(t_1)$ such that the following non-impulsive motion results in
   a) $\boldsymbol{x}^-(t_2) = \mathbf{0}$ if $\mu(t) = \underline{\mu}$, which defines the value of $x_2^+(t_1)$ and therefore the impulsive control action $k_3(x_1(t_1))$, see Section III-A,
   b) $\underline{\boldsymbol{x}}^-(t_2) \in \underline{\mathcal{E}}$ for arbitrary $\mu(t)$, which puts an additional condition on $\overline{\mu}$ and $\underline{\mu}$, see Assumption 3 in Section IV, which can, however, always be satisfied by choosing ζ large enough.

   We note that under Assumptions 1 and 3 t_2 is finite (we refer to [24] for a detailed proof).
3) The impulsive control is applied at each time-instant t_j for which $\boldsymbol{x}^-(t_j) \in \overline{\mathcal{E}}$. It holds that $\boldsymbol{x}^-(t_2) \in \underline{\mathcal{E}}$ and the control is such that $\boldsymbol{x}^-(t_j) \in \underline{\mathcal{E}}$, $j = 2, 3, \ldots$. Infinitely many impulsive actions will occur in a finite time, i.e. $t_\infty < \infty$, with $\boldsymbol{x}(t_\infty) = \mathbf{0}$, see Remark 2 after Theorem 1 in Section IV.

The resulting switching impulsive control law is now given by (3), (5) and

$$u(x_1, x_2, t) = -k_1 x_1 - k_2(t) x_2, \quad k_1, k_2 > 0,$$
$$k_2(t) = \begin{cases} k_{21} & t_0 \leq t < t_1 \\ k_{22} & t \geq t_1 \end{cases}, \tag{16}$$

such that $k_1 > 0$, $0 < \frac{k_{21}}{2\sqrt{k_1 m}} < 1$ and $\frac{k_{22}}{2\sqrt{k_1 m}} > 1$, and where t_1 is the smallest time instant $t_1 \geq t_0$ such that $\boldsymbol{x}^-(t_1) \in \overline{\mathcal{E}}$.

IV. Stability Analysis

In this section, we will show that the control design, presented in Section III, symptotically (finite-time) stabilises the set-point $\boldsymbol{x} = \mathbf{0}$. Consider the system (2) satisfying Assumption 1 and the impulsive feedback controller (3), (5), (16) with $k_3(x_1)$ satisfying (15) and $x_2^+(t_j)$ fulfilling the mixed boundary value problem (see point 2 in Proposition 1). We will call this the resulting closed-loop system and show that $\boldsymbol{x} = \mathbf{0}$ is a globally uniformly symptotically stable equilibrium point of this system. Now, let us adopt the following assumption.

Assumption 2
Solutions of the resulting closed-loop system (2), (3), (5), (16), satisfying Assumption 1, which start at $\boldsymbol{x}(t_0) \in \mathbb{R}^2$ reach the compact set $\overline{\mathcal{E}}$ in a finite time t_1 (i.e. $t_1 - t_0 < \infty$).

We now formulate two sufficient conditions for Assumption 2 in the following two propositions.

Proposition 2
Suppose the friction coefficient $\mu(x_1, x_2, t)$ satisfies Assumption 1. If the time-evolution of the friction coefficient $\mu(t) = \mu(x_1(t), x_2(t), t)$ along solutions of the closed-loop system (2), (3), (16), with $U = 0$, is piecewise constant, such that it is constant during each time-interval for which $x_2(t)$ does not change sign, and the linear part of the closed-loop system is undercritically damped (i.e. $\zeta < 1$), then the stick set $\overline{\mathcal{E}}$ is reached in finite time for any initial condition $\boldsymbol{x}(t_0) \in \mathbb{R}^2$.

Proof: In [25], Theorem 2(iii), finite-time attraction is proven for a constant value of $\mu(t)$. The proof can easily be extended to a piecewise constant $\mu(t)$ as in the proposition. ■

Proposition 3
Consider the closed-loop system (2), (3), (16), with $U = 0$. Consider a velocity-dependent friction law satisfying the decomposition $F_f(x_2) \in -mg\mu\,\mathrm{Sign}(x_2) - F_{sm}(x_2)$ instead of the friction law in (1), where μ is constant and satisfies Assumption 1, $F_{sm}(\cdot) \in \mathcal{C}^1$ and $F_f(x_2)x_2 \leq 0,\ \forall x_2$. If $k_{21} + \frac{\partial F_{sm}}{\partial x_2}(0) < 2\sqrt{mk_1}$, i.e. the linearisation of the continuous part of the closed-loop dynamics (around the origin) is undercritically damped, then the stick set $\overline{\mathcal{E}}$ is reached in finite time for any initial condition $\boldsymbol{x}(t_0) \in \mathbb{R}^2$.

Proof: Under the conditions in the proposition, Theorem 2 in [25] can be directly employed to provide the proof. ■

Given the rather generic class of friction laws considered in this paper, the conditions on the friction law in Propositions 2 and 3 can be considered to be restrictive. Note, however, that (possibly asymmetric) Coulomb friction laws with uncertain (though constant) friction coefficient form a practically relevant subclass of friction models that satisfies the conditions in Proposition 2 and that the friction law in Proposition 3 represents a general class of discontinuous, velocity-dependent friction laws (possibly including the Stribeck effect). Moreover, the formulation of less stringent conditions for the finite-time convergence to the stickset for the case of generic friction coefficients $\mu(x_1, x_2, t)$ is, to the best of the authors' knowledge, an open problem. Namely, it has been shown in [25], [26] that, even for constant μ, manifolds in state space may exist for which solutions only converge to the equilibrium set asymptotically (not in finite time). More precisely, in [25], it is shown that, under the conditions in Proposition 3 with $k_{21} + \frac{\partial F_{sm}}{\partial x_2}(0) \geq 2\sqrt{mk_1}$, solutions exist that reach the equilibrium set in infinite time. Based on Propositions 2 and 3 and the work in [25], [26], we conclude that the fact that the linearised dynamics is undercritically damped appears to be an essential condition for the finite-time attractivity of the equilibrium set. This is the reason for designing the switching controller as in (16).

We do stress here that, although more generic sufficient conditions for Assumption 2 are currently lacking, it has been widely observed in the literature (both on a model level as in experiments), see e.g. [1], [3], [4], that solutions in practice generally do converge to the stickset in finite time. In fact, this finite-time convergence to the stick set is directly related to the problems of stick-slip limit cycling and non-zero steady-state errors, which we are aiming to tackle with the control design in this paper and form the core motivation for our work. Hence, from a practical point of view, Assumption 2 is a very natural one.

Next, we adopt the following assumption.

Assumption 3
We assume that one of the following two conditions holds:

$$\underline{\mu}/\overline{\mu} > \frac{1}{2}, \tag{17}$$

or

$$\underline{\mu}/\overline{\mu} > 1 - \left(\frac{\lambda_1}{\lambda_2}\right)^{-a_1}, \text{ where } a_1 = -\frac{\lambda_1}{\lambda_2 - \lambda 1}. \tag{18}$$

Remark 1
We note that the condition (18) *in Assumption 3 can always be satisfied by choosing $\zeta = \frac{k_{22}}{2\sqrt{k_1 m}} > 1$ large enough. Namely, it holds that, firstly, the function $1 - (\lambda_1/\lambda_2)^{-a_1}$ is strictly decreasing for increasing ζ (for $\zeta > 1$) and, secondly, $\lim_{\zeta\to\infty} 1 - (\lambda_1/\lambda_2)^{-a_1} = 0$. To validate the latter statement, define $q := \lambda_1/\lambda_2$, $p := 1 - q$. We can derive that $\lim_{\zeta\to\infty} 1 - (\lambda_1/\lambda_2)^{-a_1} = \lim_{q\downarrow 0}(1 - q^{\frac{q}{1-q}}) = 1 - \lim_{q\downarrow 0} q^q = 1 - \lim_{q\downarrow 0} e^{q\ln q} = 0$. We stress here that this fact will allow us to guarantee robust stability for any uncertainty level in the friction by designing the non-impulsive part of the controller such that ζ is large enough (satisfying condition* (18)*).*

Still, we care to also provide condition (17) *($\underline{\mu}/\overline{\mu} > \frac{1}{2}$) in Assumption 3, which is independent of ζ, since this condition is less strict than condition* (18) *for ζ close to 1. Namely, $\lim_{\zeta\downarrow 1} 1 - \left(\frac{\lambda_1}{\lambda_2}\right)^{-a_1} = \lim_{q\uparrow 1}(1 - q^{\frac{q}{1-q}}) = \lim_{p\downarrow 0}(1 - (1-p)^{\frac{1-p}{p}}) = 1 - \lim_{p\downarrow 0} e^{\frac{1-p}{p}\ln(1-p)} = 1 - \frac{1}{e} \approx 0.63 > \frac{1}{2}$.*

Finally, the following theorem states the conditions under which the origin of the resulting closed-loop system is globally uniformly symptotically stable (i.e. Problem 1 is solved).

Theorem 1

Consider the resulting closed loop system (2), (3), (5), (16) satisfying Assumptions 1, 2 and 3. The origin of the resulting closed-loop system is globally uniformly symptotically stable.

Proof: For the sake of brevity we refer to [24] for a detailed proof. ∎

We note that Theorem 1 states that the proposed impulsive control law can render the set-point globally uniformly symptotically stable for a very wide class of friction models. Namely, Assumption 1 only requires the friction coefficient to be bounded from above (and below) and Assumption 3 can be satisfied for any level of uncertainty in the friction by appropriately tuning the non-impulsive part of the controller (i.e. by taking $\zeta = \frac{k_{22}}{2\sqrt{k_1 m}}$ large enough).

Remark 2

A solution of the resulting closed-loop system (2), (3), (5), (16) with a friction coefficient $\mu(x_1, x_2, t)$*, satisfying Assumption 1, and initial condition* $\boldsymbol{x}^-(t_2) \in \underline{\mathcal{E}}$ *reaches the origin in a finite time*

$$t_\infty - t_2 \leq \sqrt{\frac{2|y_2|}{g\underline{\mu}}} \frac{1}{1 - \left(1 - \frac{\underline{\mu}}{\overline{\mu}}\right)^{\frac{1}{2}}}, \qquad (19)$$

with $y_2 = x_1(t_2)$ *and* $\boldsymbol{x}(t_\infty) = \mathbf{0}$ *[24].*

V. Illustrative Example

In this section, we illustrate the effectiveness of the proposed impulsive control strategy by means of an example. Hereto, we consider a motion system as in Figure 1 with dynamics described by (2) with $m = 1$ and $g = 10$. Moreover, the friction coefficient in (2) is of the form $\mu(x_1, x_2, t) = (\mu_1 - \mu_2)/(1 + 0.5|x_2|) + \mu_2 + \mu_3 \sin(\Omega t)$, where $\mu_1 = 0.4$, $\mu_2 = 0.3$, $\mu_3 = 0.05$ and $\Omega = 4$. In this friction law one can recognise a velocity-dependency with a pronounced Stribeck effect and an explicit time-dependency. Note that this friction law satisfies Assumption 1 with $\underline{\mu} = 0.25$ and $\overline{\mu} = 0.45$, which indicates a significant possible variation on the friction coefficient and which also implies the satisfaction of Assumption 3. The possible variation of the friction coefficient is also illustrated by the dashed lines in Figure 3. Next, we employ the switching impulsive controller design proposed in Section III and described by (3), (5), (16). Herein, the control parameters are designed as $k_1 = 1$, $k_{21} = 0.5$, $k_{22} = 3$, implying that $0 < \frac{k_{21}}{2\sqrt{k_1 m}} = 0.25 < 1$ and $\frac{k_{22}}{2\sqrt{k_1 m}} = 1.5 > 1$ as proposed in Section III-B, and the impulsive control design (5) is designed using Proposition 1, see Figure 4.

We employ a numerical time-stepping scheme [23] to numerically compute solutions of the impulsive closed-loop system. Figures 5 and 6 depict a simulated response of the closed-loop system for an initial condition $x_1(0) = -4$ and $x_2(0) = -4$. Figure 5 clearly shows that the response indeed converges to the origin in finite time, while the jumps in the velocity induced by the impulsive control action are clearly visible. This figure also displays the time instants $t_1 = 3.55$ and $t_2 = 4.40$ at which the response hits, for the first time, the sets $\overline{\mathcal{E}}$ (maximal stick set) and $\underline{\mathcal{E}}$ (minimal stick set), respectively (see also Figure 6). Moreover, the response converges to the origin in the finite time $t_\infty = 4.8707$. The upper bound on t_∞ that can be computed using (19) is $t_\infty = 5.5162$. This upper bound on t_∞ is not overly conservative and can be considered to be a realistic bound on the time in which convergence to the setpoint is achieved.

We care to stress that the impulsive control design by no means exploits knowledge on the particular friction law used in this example and indeed guarantees *robust stabilisation* for any position-, velocity- and time-dependent friction coefficient satisfying the same bounds.

VI. Conclusions

In this paper, we have provided a solution to the robust set-point stabilisation problem for motion systems subject to uncertain friction. A robust stability guarantee with respect to frictional uncertainties is particularly relevant in practice, since uncertainties in the friction model are unavoidable. We propose an impulsive feedback control design, consisting of a non-impulsive state-feedback and a state-dependent impulsive feedback, that robustly stabilises the set-point for a

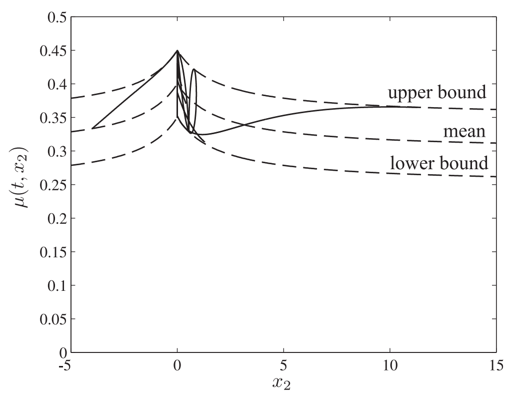

Fig. 3: Friction coefficient. Dashed lines indicate bounds on and mean of the friction coefficient and the solid line indicates the evolution of the friction coefficient along a solution of the closed-loop system.

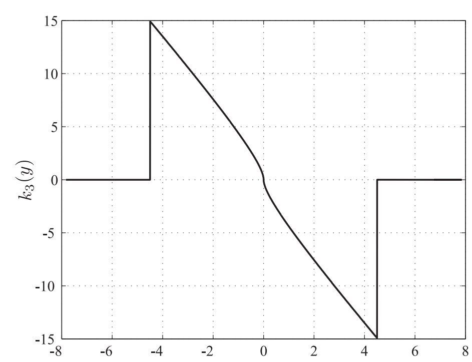

Fig. 4: Impulsive control law.

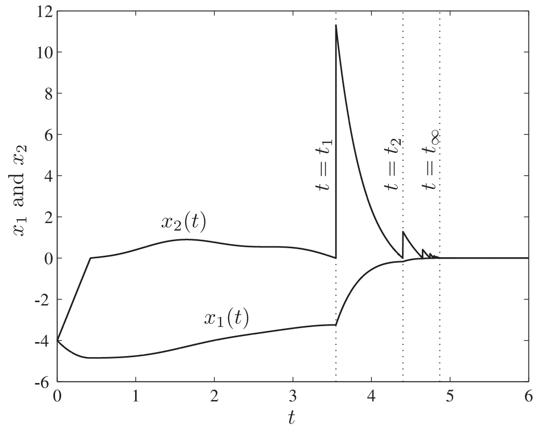

Fig. 5: Time history of the position $x_1(t)$ and the velocity $x_2(t)$ for $x_1(0) = x_2(0) = -4$.

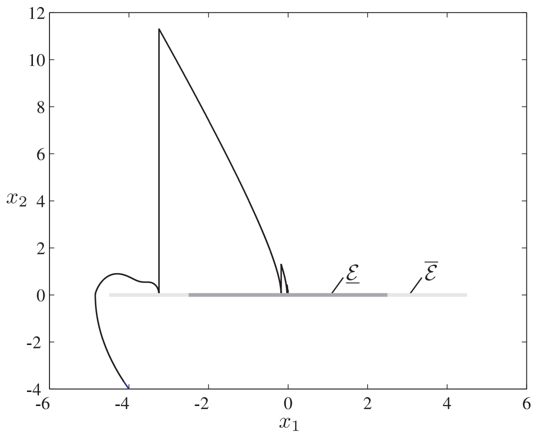

Fig. 6: Phase portrait depicting $x_2(t)$ versus $x_1(t)$ for $x_1(0) = x_2(0) = -4$.

class of position-, velocity- and time-dependent friction laws with uncertainty. Moreover, this control strategy guarantees the finite-time convergence to the set-point, thereby inducing favourable transient performance characteristics in the resulting closed loop. The results are illustrated by means of a representative motion control example.

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