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CONVERGENCE PROPERTIES OF MONOTONE MEASURE DIFFERENTIAL INCLUSIONS

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Abstract

In this paper, we present results providing sufficient conditions for the uniform convergence of measure differential inclusions with certain maximal monotonicity properties. The framework of measure differential inclusions allows us to describe systems with state discontinuities, such as e.g. mechanical systems with unilateral constraints. The results are illustrated by application to such a mechanical example.

Key words

Discontinuous and Impulsive Systems, Incremental Stability, Convergence.

1 Introduction

In this paper, we show that measure differential inclusions with certain maximal monotonicity conditions exhibit the convergence property. A system, which is excited by an input, is called convergent if it has a unique solution that is bounded on the whole time axis and this solution is globally asymptotically stable. Obviously, if such a solution does exist, then all other solutions converge to this solution, regardless of their initial conditions, and can be considered as a steady-state solution (Demidovich, 1967; Pavlov *et al.*, 2004).

The property of convergence can be beneficial from several points of view. Firstly, in many control problems it is required that controllers are designed in such a way that all solutions of the corresponding closed-loop system "forget" their initial conditions. Actually, one of the main tasks of feedback is to eliminate the dependency of solutions on initial conditions. In this case, all solutions converge to some steady-state solution that is determined only by the input of the closed-loop system. This input can be, for example, a command signal or a signal generated by a feedforward part of the controller or, as in the observer design problem, it can be the measured signal from the observed system. Such a convergence property of a system plays an important role in many nonlinear control problems including tracking, synchronization, observer design, and the output regulation problem, see e.g. (Pavlov *et al.*, 2005*b*) and references therein. Secondly, from a dynamics point of view, convergence is an interesting property because it excludes the possibility of different coexisting steady-state solutions: namely, a convergent system excited by a bounded (periodic) input has a *unique* bounded globally asymptotically stable (periodic) solution.

In (Demidovich, 1967), conditions for the convergence property were formulated for smooth nonlinear systems. In (Yakubovich, 1964), Lur'e-type systems, possibly with discontinuities, were considered and convergence conditions proposed. Only recently, in (Pavlov *et al.*, 2007) sufficient conditions for both continuous (though non-smooth) and discontinuous piece-wise affine (PWA) systems have been proposed. Here, we consider a class of systems described by measure differential inclusions, which includes systems with discontinuities but also allows for impulsive right-hand sides.

Systems which expose discontinuities in the state and/or vector field can be described by measure differential inclusions (Monteiro Marques, 1993; Moreau, 1988*b*; Brogliato, 1999). The differential measure of the state vector does not only consist of a part with a density with respect to the Lebesgue measure (i.e. the time-derivative of the state vector), but is also allowed to contain an atomic part. The dynamics of the system is described by an inclusion of the differential measure of the state to a state-dependent set (similar to the concept of differential inclusions). Consequently, the measure differential inclusion concept describes the continuous dynamics as well as the impulse dynamics with a single statement in terms of an inclusion and is able to describe accumulation phenomena. An advantage of this framework over other frameworks is the fact that physical interaction laws, such as friction and impact in mechanics or diode characteristics in electronics, can be formulated as set-valued force laws and be seamlessly incorporated in the formulation, see e.g. (Glocker, 2001).

Stability properties of measure differential inclusions are essential both in bifurcation analysis and the control of such systems. In (Leine and van de Wouw, 2008), results on the stability of stationary sets of measure differential inclusions (with a special focus on mechanical systems with unilateral constraints) are presented. In (Brogliato, 2004), stability properties of an equilibrium of measure differential inclusions of Lur'e-type are studied. The nonlinearities in the feedback loop are required to exhibit monotonicity properties and, if additionally passivity conditions on the linear part of the system are assured, then stability of the equilibrium can be guaranteed. Furthermore, the Lagrange-Dirichlet stability theorem is extended in (Brogliato, 2004) to measure differential inclusions describing mechanical systems with frictionless impact. Note that this work does not address the convergence property and only studies the stability of stationary solutions. However, many control problems, such as tracking control, output regulation, synchronisation and observer design require the stability analysis of time-varying solutions. The research on the stability properties of time-varying solutions of non-smooth systems is still in its infancy and the current paper should be placed in this context.

The paper is organised as follows. Section 2 provides a brief introduction to measure differential inclusions. Subsequently, we define the convergence property of dynamical systems in Section 3 and state the associated properties of convergent systems. The essential contribution of this paper lies in Section 4, in which we present sufficient conditions for the uniform convergence of measure differential inclusions with certain maximal monotonicity properties. An illustrative example of a convergent mechanical system with a unilateral constraint is discussed in detail in Section 5. Finally, Section 6 presents concluding remarks.

2 Measure Differential Inclusions

In this section, we introduce the measure differential inclusion

$$\mathrm{d}\boldsymbol{x} \in \mathrm{d}\boldsymbol{\Gamma}(t, \boldsymbol{x}(t)) \tag{1}$$

as has been proposed by Moreau (Moreau, 1988*a*). The concept of differential inclusions has been extended to measure differential inclusions in order to allow for discontinuities in $\boldsymbol{x}(t)$, see e.g. (Monteiro Marques, 1993; Moreau, 1988*b*; Brogliato, 1999).

With the differential inclusion $\dot{\boldsymbol{x}}(t) \in \boldsymbol{\mathcal{F}}(t, \boldsymbol{x}(t))$, in which $\boldsymbol{\mathcal{F}}(t, \boldsymbol{x}(t))$ is a set-valued mapping, we are able to describe a non-smooth absolutely continuous time-evolution $\boldsymbol{x}(t)$. The solution $\boldsymbol{x}(t) : \mathcal{I} \to \mathbb{R}^n$ fulfills the differential inclusion almost everywhere, because $\dot{\boldsymbol{x}}(t)$ does not exist on a Lebesgue negligible set $\mathcal{D} \subset \mathcal{I}$ of time-instances $t_i \in \mathcal{D}$ related to non-smooth state evolution. Instead of using the density $\dot{\boldsymbol{x}}(t)$, we can also write the differential inclusion using the differential measure:

$$\mathrm{d}\boldsymbol{x} \in \boldsymbol{\mathcal{F}}(t, \boldsymbol{x}(t))\,\mathrm{d}t, \tag{2}$$

which yields a measure differential inclusion. The solution $\boldsymbol{x}(t)$ fulfills the measure differential inclusion (2) *for all* $t \in I$ because of the underlying integration process being associated with measures. Moreover, writing the dynamics in terms of a measure differential inclusion allows us to study a larger class of functions $\boldsymbol{x}(t)$, as we can let $\mathrm{d}\boldsymbol{x}$ contain parts other than the Lebesgue integrable part. In order to describe a time-evolution of bounded variation which is discontinuous at isolated time-instances, we let the differential measure $\mathrm{d}\boldsymbol{x}$ also have an atomic part:

$$\mathrm{d}\boldsymbol{x} = \dot{\boldsymbol{x}}(t)\,\mathrm{d}t + (\boldsymbol{x}^+(t) - \boldsymbol{x}^-(t))\,\mathrm{d}\eta, \tag{3}$$

where $\mathrm{d}\eta$ is the atomic measure and $\boldsymbol{x}^+(t) = \lim_{\tau\downarrow 0} \boldsymbol{x}(t+\tau)$, $\boldsymbol{x}^-(t) = \lim_{\tau\uparrow 0} \boldsymbol{x}(t+\tau)$. Therefore, we extend the measure differential inclusion (2) with an atomic part as well: $\mathrm{d}\boldsymbol{x} \in \boldsymbol{\mathcal{F}}(t, \boldsymbol{x}(t))\,\mathrm{d}t + \boldsymbol{\mathcal{G}}(t, \boldsymbol{x}(t))\,\mathrm{d}\eta$. Here, $\boldsymbol{\mathcal{G}}(t, \boldsymbol{x}(t))$ is a set-valued mapping, which is in general dependent on t, $\boldsymbol{x}^-(t)$ and $\boldsymbol{x}^+(t)$. Following (Moreau, 1988*a*), we simply write $\boldsymbol{\mathcal{G}}(t, \boldsymbol{x}(t))$. More conveniently, we write the measure differential inclusion as in (1), where $\mathrm{d}\boldsymbol{\Gamma}(t, \boldsymbol{x}(t))$ is a set-valued measure function defined as

$$\mathrm{d}\boldsymbol{\Gamma}(t, \boldsymbol{x}(t)) = \boldsymbol{\mathcal{F}}(t, \boldsymbol{x}(t))\,\mathrm{d}t + \boldsymbol{\mathcal{G}}(t, \boldsymbol{x}(t))\,\mathrm{d}\eta. \tag{4}$$

The measure differential inclusion (1) has to be understood in the sense of integration and its solution $\boldsymbol{x}(t)$ is a function of locally bounded variation which fulfills $\boldsymbol{x}^+(t) = \boldsymbol{x}^-(t_0) + \int_I \boldsymbol{f}(t, \boldsymbol{x})\,\mathrm{d}t + \boldsymbol{g}(t, \boldsymbol{x})\,\mathrm{d}\eta$, for every compact interval $I = [t_0, t]$, where the functions $\boldsymbol{f}(t, \boldsymbol{x})$ and $\boldsymbol{g}(t, \boldsymbol{x})$ have to obey $\boldsymbol{f}(t, \boldsymbol{x}) \in \boldsymbol{\mathcal{F}}(t, \boldsymbol{x})$, $\boldsymbol{g}(t, \boldsymbol{x}) \in \boldsymbol{\mathcal{G}}(t, \boldsymbol{x})$. Substitution of (3) in the measure differential inclusion (1), (4) gives $\dot{\boldsymbol{x}}(t)\,\mathrm{d}t + (\boldsymbol{x}^+(t) - \boldsymbol{x}^-(t))\,\mathrm{d}\eta \in \boldsymbol{\mathcal{F}}(t, \boldsymbol{x}(t))\,\mathrm{d}t + \boldsymbol{\mathcal{G}}(t, \boldsymbol{x}(t))\,\mathrm{d}\eta$, which we can separate in the Lebesgue integrable part $\dot{\boldsymbol{x}}(t)\,\mathrm{d}t \in \boldsymbol{\mathcal{F}}(t, \boldsymbol{x}(t))\,\mathrm{d}t$, and atomic part $(\boldsymbol{x}^+(t) - \boldsymbol{x}^-(t))\,\mathrm{d}\eta \in \boldsymbol{\mathcal{G}}(t, \boldsymbol{x}(t))\,\mathrm{d}\eta$ from which we can retrieve $\dot{\boldsymbol{x}}(t) \in \boldsymbol{\mathcal{F}}(t, \boldsymbol{x}(t))$ and the jump condition $\boldsymbol{x}^+(t) - \boldsymbol{x}^-(t) \in \boldsymbol{\mathcal{G}}(t, \boldsymbol{x}(t))$. It should be noted that the state $\boldsymbol{x}$ of (1) may be confined to a so-called admissible set, which we denote by $\mathcal{X}$. Here, we will assume that the measure differential inclusions under study are consistent, where the consistency property implies that if the initial condition is taken in the admissible set, i.e. $\boldsymbol{x}_0 = \boldsymbol{x}(t_0)$ is such that $\boldsymbol{x}_0 \in \mathcal{X}$, then there exist a solution in forward time that resides in the admissible domain, i.e. $\boldsymbol{x}(t) \in \mathcal{X}$ for almost all $t \geq t_0$ (Leine and van de Wouw, 2008).

3 Convergent Systems

In this section, we briefly discuss the definition of convergence and certain properties of convergent systems. In the definition of convergence, the Lyapunov stability of solutions of (1) plays a central role. For definitions of (uniform) stability and attractivity of measure differential inclusions we refer to (Leine and van de Wouw, 2008). The definitions of convergence properties presented here extend the definition given in (Demidovich, 1967) (see also (Pavlov *et al.*, 2005*a*)).

We consider systems of the form

$$\mathrm{d}\boldsymbol{x} \in \mathrm{d}\boldsymbol{\Gamma}(\boldsymbol{x}, \boldsymbol{w}(t)), \tag{5}$$

with state $\boldsymbol{x} \in \mathbb{R}^n$ and input $\boldsymbol{w} \in \mathbb{R}^d$. The right-hand side of (5) is assumed to be continuous in $\boldsymbol{w}$. In the following, we will consider the class $\overline{\mathbb{PC}}_d$ of piecewise continuous inputs $\boldsymbol{w}(t) : \mathbb{R} \to \mathbb{R}^d$ which are bounded on $\mathbb{R}$.

Let us formally define the property of convergence.

Definition 1
System (5) is said to be

- convergent *if there exists a solution $\bar{\boldsymbol{x}}_w(t)$ satisfying the following conditions, for every input $\boldsymbol{w} \in \overline{\mathbb{PC}}_d$:*

  (i) $\bar{\boldsymbol{x}}_w(t)$ *is defined and bounded for all* $t \in \mathbb{R}$,
  (ii) $\bar{\boldsymbol{x}}_w(t)$ *is globally attractively stable.*

- uniformly convergent *if it is convergent and $\bar{\boldsymbol{x}}_w(t)$ is globally uniformly attractively stable, for every input $\boldsymbol{w} \in \overline{\mathbb{PC}}_d$.*
- exponentially convergent *if it is convergent and $\bar{\boldsymbol{x}}_w(t)$ is globally exponentially stable, for every input $\boldsymbol{w} \in \overline{\mathbb{PC}}_d$.*

The wording 'attractively stable' has been used instead of the usual term 'asymptotically stable', because attractivity of solutions in (measure) differential inclusions can be asymptotic or symptotic (finite-time attractivity) (Leine and van de Wouw, 2008).

The solution $\bar{\boldsymbol{x}}_w(t)$ is called a *steady-state solution*. As follows from the definition of convergence, any solution of a convergent system "forgets" its initial condition and converges to some steady-state solution. In general, the steady-state solution $\bar{\boldsymbol{x}}_w(t)$ may be non-unique (where the subscript emphasizes the fact that the steady-state solution depends on $\boldsymbol{w}(t)$). But for any two steady-state solutions $\bar{\boldsymbol{x}}_{w,1}(t)$ and $\bar{\boldsymbol{x}}_{w,2}(t)$ it holds that $\|\bar{\boldsymbol{x}}_{w,1}(t) - \bar{\boldsymbol{x}}_{w,2}(t)\| \to 0$ as $t \to +\infty$. At the same time, for *uniformly* convergent systems the steady-state solution is unique, as formulated below.

Property 1 ((Pavlov *et al.*, 2005*b*; Pavlov *et al.*, 2005*a*))
If system (5) is uniformly convergent, then, for any input $\boldsymbol{w}(t) \in \overline{\mathbb{PC}}_d$, the steady-state solution $\bar{\boldsymbol{x}}_w(t)$ is the only solution defined and bounded for all $t \in \mathbb{R}$.

Uniformly convergent systems excited by periodic or constant inputs exhibit the following property, that is particularly useful in, for example, bifurcation analyses of periodically perturbed systems.

Property 2 ((Demidovich, 1967; Pavlov *et al.*, 2005*b*))
Suppose system (5) with a given input $\boldsymbol{w}(t)$ is uniformly convergent. If the input $\boldsymbol{w}(t)$ is constant, the corresponding steady-state solution $\bar{\boldsymbol{x}}_w(t)$ is also constant; if the input $\boldsymbol{w}(t)$ is periodic with period T, then the corresponding steady-state solution $\bar{\boldsymbol{x}}_w(t)$ is also periodic with the same period T.

4 Convergence of Maximal Monotone Systems

4.1 Maximal Monotonicity

Let us first define maximal monotone set-valued functions.

Definition 2 (Maximal Monotone Set-valued Function)

- *A set-valued function $\boldsymbol{\mathcal{F}}(\boldsymbol{x}) : \mathbb{R}^n \to \mathbb{R}^n$ is called monotone if its graph is monotone in the sense that for all $(\boldsymbol{x}, \boldsymbol{y}) \in \mathrm{Graph}(\boldsymbol{\mathcal{F}})$ and for all $(\boldsymbol{x}^*, \boldsymbol{y}^*) \in \mathrm{Graph}(\boldsymbol{\mathcal{F}})$ it holds that $(\boldsymbol{y} - \boldsymbol{y}^*)^{\mathrm{T}}(\boldsymbol{x} - \boldsymbol{x}^*) \geq 0$. In addition, if $(\boldsymbol{y} - \boldsymbol{y}^*)^{\mathrm{T}}(\boldsymbol{x} - \boldsymbol{x}^*) \geq \alpha \|\boldsymbol{x} - \boldsymbol{x}^*\|^2$ for some $\alpha > 0$, then the set-valued map is strictly monotone.*
- *A monotone set-valued function $\boldsymbol{\mathcal{F}}(\boldsymbol{x})$ is called maximal monotone if there exists no other monotone set-valued function whose graph strictly contains the graph of $\boldsymbol{\mathcal{F}}$. If $\boldsymbol{\mathcal{F}}$ is strictly monotone and maximal, then it is called strictly maximal monotone.*

In this section we will consider the dynamics of measure differential inclusions (5) with certain maximal monotonicity conditions on $\boldsymbol{\Gamma}(\boldsymbol{x}, \boldsymbol{w}(t))$. In particular, we study systems for which $\boldsymbol{\Gamma}(\boldsymbol{x}, \boldsymbol{w}(t))$ can be split in a state-dependent part and an input-dependent part. The state-dependent part is, with the help of a maximal monotonicity requirement, assumed to be strictly passive with respect to the Lebesgue measure and passive with respect to the atomic measure. Such kind of systems will be simply referred to as 'maximal monotone systems' in the following.

We first formalise maximal monotone systems in Section 4.2, subsequently give sufficient conditions for the existence of a compact positively invariant set in Section 4.3 (which plays an important role in the proof for convergence) and finally give sufficient conditions for convergence in Section 4.4.

4.2 Maximal monotone systems

Let $\boldsymbol{x} \in \mathbb{R}^n$ be the state-vector of the system and $\boldsymbol{w} \in \mathbb{R}^m$ be the input vector. Consider the time-evolution of $\boldsymbol{x}$ to be governed by a measure differential equation of the form

$$\mathrm{d}\boldsymbol{x} = -\,\mathrm{d}\boldsymbol{a} - \boldsymbol{c}(\boldsymbol{x})\,\mathrm{d}t + \mathrm{d}\boldsymbol{b}(\boldsymbol{w}), \tag{6}$$

where $\boldsymbol{c} : \mathbb{R}^n \to \mathbb{R}^n$ is a single-valued function and $\mathrm{d}\boldsymbol{a}$ and $\mathrm{d}\boldsymbol{b}(\boldsymbol{w})$ are differential measures with densities with respect to $\mathrm{d}t$ and $\mathrm{d}\eta$, i.e. $\mathrm{d}\boldsymbol{a} = \boldsymbol{a}'_t\,\mathrm{d}t + \boldsymbol{a}'_\eta\,\mathrm{d}\eta$, and $\mathrm{d}\boldsymbol{b}(\boldsymbol{w}) = \boldsymbol{b}'_t(\boldsymbol{w})\,\mathrm{d}t + \boldsymbol{b}'_\eta(\boldsymbol{w})\,\mathrm{d}\eta$. We assume $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{b}'_\eta(\boldsymbol{w})$ to be bounded from above by a constant β. Basically, this gives an upper-bound on the energy input of the impulsive inputs. Such an assumption makes sense from the physical point of view, see the example in Section 5. The quantities $\boldsymbol{a}'_t$ and $\boldsymbol{a}'_\eta$, which are functions of time along solutions of (6), obey the set-valued laws

$$\boldsymbol{a}'_t \in \boldsymbol{\mathcal{A}}(\boldsymbol{x}), \tag{7}$$

$$\boldsymbol{a}'_\eta \in \boldsymbol{\mathcal{A}}(\boldsymbol{x}^+), \tag{8}$$

where $\boldsymbol{\mathcal{A}}$ is a set-valued mapping. The dynamics can be decomposed in a Lebesgue measurable part and an atomic part. The Lebesgue measurable part gives the differential equation $\dot{\boldsymbol{x}}(t) := \boldsymbol{x}'_t = -\boldsymbol{a}'_t(\boldsymbol{x}(t)) - \boldsymbol{c}(\boldsymbol{x}(t)) + \boldsymbol{b}'_t(\boldsymbol{w}(t))$, which forms with the set-valued law (7) a differential inclusion $\dot{\boldsymbol{x}} \in -\boldsymbol{\mathcal{A}}(\boldsymbol{x}) - \boldsymbol{c}(\boldsymbol{x}) + \boldsymbol{b}'_t(\boldsymbol{w})$ a.e. The atomic part gives the state-reset rule $\boldsymbol{x}^+ - \boldsymbol{x}^- := \boldsymbol{x}'_\eta = -\boldsymbol{a}'_\eta + \boldsymbol{b}'_\eta(\boldsymbol{w})$. In mechanics, the state-reset rule is called the impact equation. The above impact law (8), for which $\boldsymbol{\mathcal{A}}$ is only a function of $\boldsymbol{x}^+$, corresponds to a completely inelastic impact equation. Because of the similarity between the laws (7) and (8), we can combine these laws into the measure law

$$\mathrm{d}\boldsymbol{a} \in \mathrm{d}\boldsymbol{\mathcal{A}}(\boldsymbol{x}^+) = \boldsymbol{\mathcal{A}}(\boldsymbol{x}^+)(\,\mathrm{d}t + \mathrm{d}\eta). \tag{9}$$

The equality of measures (6) together with the measure law (9) constitutes a measure differential inclusion

$$\mathrm{d}\boldsymbol{x} \in -\,\mathrm{d}\boldsymbol{\mathcal{A}}(\boldsymbol{x}^+) - \boldsymbol{c}(\boldsymbol{x})\,\mathrm{d}t + \mathrm{d}\boldsymbol{b}(\boldsymbol{w}) := \mathrm{d}\boldsymbol{\Gamma}(\boldsymbol{x},\boldsymbol{w}). \tag{10}$$

The set-valued operator $\boldsymbol{\mathcal{A}}(\boldsymbol{x})$ models the non-smooth dissipative elements in the system. We assume that $\boldsymbol{\mathcal{A}}(\boldsymbol{x})$ is a maximal monotone set-valued mapping, see Definition 2. Moreover, we assume that $\boldsymbol{0} \in \boldsymbol{\mathcal{A}}(\boldsymbol{0})$. This last assumption together with the monotonicity assumption implies the condition $\boldsymbol{x}^{\mathrm{T}}\boldsymbol{a} \geq \boldsymbol{0}$ for all $\boldsymbol{a} \in \boldsymbol{\mathcal{A}}(\boldsymbol{x})$ and for any $\boldsymbol{x} \in \mathcal{X}$, i.e. the action of $\boldsymbol{\mathcal{A}}$ is passive. Furthermore, we assume that $\boldsymbol{\mathcal{A}}(\boldsymbol{x}) + \boldsymbol{c}(\boldsymbol{x})$ is a strictly maximal monotone set-valued mapping, i.e. there exists an $\alpha > 0$ such that

$$\begin{aligned}(\boldsymbol{x}_2 - \boldsymbol{x}_1)^{\mathrm{T}}\,(\boldsymbol{a}_2 + \,\boldsymbol{c}(\boldsymbol{x}_2) - \boldsymbol{a}_1 - \boldsymbol{c}(\boldsymbol{x}_1))\\ \geq \alpha\|\boldsymbol{x}_2 - \boldsymbol{x}_1\|^2,\end{aligned} \tag{11}$$

for all $\boldsymbol{a}_1 \in \boldsymbol{\mathcal{A}}(\boldsymbol{x}_1)$, $\boldsymbol{a}_2 \in \boldsymbol{\mathcal{A}}(\boldsymbol{x}_2)$ and for any two states $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathcal{X}$.

4.3 Existence of a compact positively invariant set

The existence of a compact positively invariant set plays an important role in the proof of convergence as will become clear in Section 4.4. Clearly, if the impulsive inputs are passive in the sense that $(\boldsymbol{x}^+)^{\mathrm{T}}\boldsymbol{b}'_\eta(\boldsymbol{w}(t)) \leq 0$ for all t, then the system is dissipative for large $\|\boldsymbol{x}\|$ and all solutions must be bounded. In the following theorem, we give a less stringent sufficient condition for the existence of a compact positively invariant set of (10) based on a dwell-time condition (Hespanha and Morse, 1999) on the occurrence of the impulsive inputs.

Theorem 1

A consistent measure differential inclusion of the form (10) has a compact positively invariant set if

1. *$\boldsymbol{\mathcal{A}}(\boldsymbol{x})$ is a maximal monotone set-valued mapping with $\boldsymbol{0} \in \boldsymbol{\mathcal{A}}(\boldsymbol{0})$,*
2. *$\boldsymbol{\mathcal{A}}(\boldsymbol{x}) + \boldsymbol{c}(\boldsymbol{x})$ is a strictly maximal monotone set-valued mapping, i.e. there exists an $\alpha > 0$ such that (11) is satisfied,*
3. *there exists a $\beta \in \mathbb{R}$ such that $(\boldsymbol{x}^+)^{\mathrm{T}}\boldsymbol{b}'_\eta(\boldsymbol{w}) \leq \beta$ for all $\boldsymbol{x} \in \mathcal{X}$, i.e. the energy input of the impulsive inputs is bounded from above,*
4. *the time-instances t_i for which the input is impulsive are separated by the dwell-time $\tau \leq t_{i+1} - t_i$, with $\tau = \frac{\delta}{2(\delta-1)\alpha}\ln(1 + \frac{2\beta}{\delta^2\gamma^2})$ and $\gamma := \max(0, \sup_{t\in\mathbb{R},\boldsymbol{a}'_t(\boldsymbol{0})\in\boldsymbol{\mathcal{A}}(\boldsymbol{0})} \frac{-\boldsymbol{a}'_t(\boldsymbol{0}) - \boldsymbol{c}(\boldsymbol{0}) + \boldsymbol{b}'_t(\boldsymbol{w}(t))}{\alpha})$ for some $\delta > 1$.*

Proof. Consider the Lyapunov candidate function $W = \frac{1}{2}\boldsymbol{x}^T\boldsymbol{x}$. The differential measure of W has a density $\dot{W}$ with respect to the Lebesgue measure $\mathrm{d}t$ and a density $W^+ - W^-$ with respect to the atomic measure $\mathrm{d}\eta$, i.e. $\mathrm{d}W = \dot{W}\,\mathrm{d}t + (W^+ - W^-)\,\mathrm{d}\eta$. We first evaluate the density $\dot{W}$: $\dot{W} = \boldsymbol{x}^{\mathrm{T}}(-\boldsymbol{a}'_t - \boldsymbol{c}(\boldsymbol{x}) + \boldsymbol{b}'_t(\boldsymbol{w})) = \boldsymbol{x}^{\mathrm{T}}(-\boldsymbol{a}'_t - \boldsymbol{c}(\boldsymbol{x}) + \boldsymbol{a}'_t(\boldsymbol{0}) + \boldsymbol{c}(\boldsymbol{0})) + \boldsymbol{x}^{\mathrm{T}}(-\boldsymbol{a}'_t(\boldsymbol{0}) - \boldsymbol{c}(\boldsymbol{0}) + \boldsymbol{b}'_t(\boldsymbol{w}))$, with $\boldsymbol{a}'_t \in \boldsymbol{\mathcal{A}}(\boldsymbol{x})$ and $\boldsymbol{a}'_t(\boldsymbol{0}) \in \boldsymbol{\mathcal{A}}(\boldsymbol{0})$. Due to strict monotonicity of $\boldsymbol{\mathcal{A}}(\boldsymbol{x}) + \boldsymbol{c}(\boldsymbol{x})$, there exists a constant $\alpha > 0$ such that

$$\begin{aligned}\dot{W} &\leq -\alpha\|\boldsymbol{x}\|^2 + \boldsymbol{x}^{\mathrm{T}}(-\boldsymbol{a}'_t(\boldsymbol{0}) - \boldsymbol{c}(\boldsymbol{0}) + \boldsymbol{b}'_t(\boldsymbol{w})),\\ &\leq -\|\boldsymbol{x}\|\Big(\alpha\|\boldsymbol{x}\|\\ &\quad - \sup_{t\in\mathbb{R},\boldsymbol{a}'_t(\boldsymbol{0})\in\boldsymbol{\mathcal{A}}(\boldsymbol{0})}\left\{-\boldsymbol{a}'_t(\boldsymbol{0}) - \boldsymbol{c}(\boldsymbol{0}) + \boldsymbol{b}'_t(\boldsymbol{w}(t))\right\}\Big).\end{aligned} \tag{12}$$

Note that $\dot{W} < 0$ for $\boldsymbol{x}$ satisfying

$$\|\boldsymbol{x}\| > \sup_{t\in\mathbb{R},\boldsymbol{a}'_t(\boldsymbol{0})\in\boldsymbol{\mathcal{A}}(\boldsymbol{0})} \frac{-\boldsymbol{a}'_t(\boldsymbol{0}) - \boldsymbol{c}(\boldsymbol{0}) + \boldsymbol{b}'_t(\boldsymbol{w}(t))}{\alpha}. \tag{13}$$

For $\|\boldsymbol{x}\| > \gamma$, with γ as defined in the theorem, we can prove an exponential decay of W (in between state jumps at $t = t_i$). Note that the function $f(x) = -(1 - \frac{1}{\delta})\alpha x^2$ is greater than $g(x) = -\alpha x^2 + \gamma\alpha x$ for $x > \delta\gamma$, where $\delta > 1$ is an arbitrary constant and $\gamma > 0$. It therefore holds that $\dot{W} \leq -(1 - \frac{1}{\delta})\alpha\|\boldsymbol{x}\|^2$ for $\|\boldsymbol{x}\| \geq \delta\gamma$, i.e.

$$\dot{W} \leq -2\left(1 - \frac{1}{\delta}\right)\alpha W \quad \text{for } \|\boldsymbol{x}\| \geq \delta\gamma. \tag{14}$$

Subsequently, we consider the jump $W^+ - W^-$ of W: $W^+ - W^- = \frac{1}{2}(\boldsymbol{x}^+ + \boldsymbol{x}^-)^{\mathrm{T}}(\boldsymbol{x}^+ - \boldsymbol{x}^-)$, with $\boldsymbol{x}^+ - \boldsymbol{x}^- = -\boldsymbol{a}'_\eta + \boldsymbol{b}'_\eta(\boldsymbol{w})$ and $\boldsymbol{a}'_\eta \in \boldsymbol{\mathcal{A}}(\boldsymbol{x}^+)$. Elimination of $\boldsymbol{x}^-$ and exploiting the monotonicity of $\boldsymbol{\mathcal{A}}(\boldsymbol{x})$ gives

$$\begin{aligned}W^+ - W^- &= \frac{1}{2}(2\boldsymbol{x}^+ + \boldsymbol{a}'_\eta - \boldsymbol{b}'_\eta(\boldsymbol{w}))^{\mathrm{T}}\left(-\boldsymbol{a}'_\eta + \boldsymbol{b}'_\eta(\boldsymbol{w})\right)\\ &= (\boldsymbol{x}^+)^{\mathrm{T}}\left(-\boldsymbol{a}'_\eta + \boldsymbol{b}'_\eta(\boldsymbol{w})\right) - \frac{1}{2}\left\|\boldsymbol{a}'_\eta - \boldsymbol{b}'_\eta(\boldsymbol{w})\right\|^2 \leq \beta,\end{aligned} \tag{15}$$

in which we used the assumption that the energy input of the impulsive inputs $\boldsymbol{b}'_\eta(\boldsymbol{w})$ is bounded from above by β (see condition 3 in the theorem) and the monotonicity and passivity of $\boldsymbol{\mathcal{A}}$. Then, due to (14), for the non-impulsive part of the motion it holds that if $\|\boldsymbol{x}(t_0)\| \leq \gamma$ then $\|\boldsymbol{x}(t)\| \leq \gamma$ for all $t \in [t_0, t^*]$ (if no state resets occur in this time interval). Moreover, as far as the state resets are concerned, (15) shows that a state reset from a state $\boldsymbol{x}^-(t_i) \in \mathcal{V}$ with $\mathcal{V} = \{\boldsymbol{x} \in \mathcal{X} \mid \|\boldsymbol{x}\| \leq \delta\gamma\}$ can only occur to $\boldsymbol{x}^+(t_i)$ such that $W(\boldsymbol{x}^+(t_i)) := \frac{1}{2}\|\boldsymbol{x}^+(t_i)\|^2 \leq$

$W(\boldsymbol{x}^-(t_i)) + \beta \leq \frac{1}{2}\delta^2\gamma^2 + \beta$ (note hereto the specific form of $W = \frac{1}{2}\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}$). During the following open time-interval (t_i, t_{i+1}) for which $\boldsymbol{b}'_\eta(\boldsymbol{w}(t)) = \mathbf{0}$, the function W evolves as $W(\boldsymbol{x}^-(t_{i+1})) = W(\boldsymbol{x}^+(t_i)) + \int_{(t_i,t_{i+1})} \mathrm{d}W$, which may involve impulsive motion due to dissipative impulses $\boldsymbol{a}'_\eta$. Let $t_\mathcal{V} \in (t_i, t_{i+1})$ be the time-instance for which $\|\boldsymbol{x}^-(t_\mathcal{V})\| = \delta\gamma$. The function W will necessarily decrease during the time-interval $(t_i, t_\mathcal{V})$ due to (14) and $W^+ - W^- = -(\boldsymbol{x}^+)^{\mathrm{T}}\boldsymbol{a}'_\eta - \frac{1}{2}\left\|\boldsymbol{a}'_\eta\right\|^2 \leq 0$ (the state-dependent impulses are passive). It therefore holds that

$$W(\boldsymbol{x}^-(t_\mathcal{V})) \leq e^{-2(1-\frac{1}{\delta})\alpha(t_\mathcal{V}-t_i)}W(\boldsymbol{x}^+(t_i)), \qquad (16)$$

because $\mathrm{d}W \leq -2(1-\frac{1}{\delta})\alpha W\,\mathrm{d}t + (W^+ - W^-)\,\mathrm{d}\eta \leq -2(1-\frac{1}{\delta})\alpha W\,\mathrm{d}t$ for positive measures. Using $W(\boldsymbol{x}^-(t_\mathcal{V})) = \frac{1}{2}\delta^2\gamma^2$ and $W(\boldsymbol{x}^+(t_i)) \leq \frac{1}{2}\delta^2\gamma^2 + \beta$ in the exponential decrease (16) gives $\frac{1}{2}\delta^2\gamma^2 \leq e^{-2(1-\frac{1}{\delta})\alpha(t_\mathcal{V}-t_i)}(\frac{1}{2}\delta^2\gamma^2 + \beta)$ or $t_\mathcal{V} - t_i \leq \frac{\delta}{2(\delta-1)\alpha}\ln(1 + \frac{2\beta}{\delta^2\gamma^2})$. Consequently, if the next impulsive time-instance t_{i+1} of the input is after $t_\mathcal{V}$, then the solution $\boldsymbol{x}(t)$ has enough time to reach $\mathcal{V}$. Hence, if the impulsive time-instance of the input are separated by the dwell-time τ, i.e. $t_{i+1} - t_i \geq \tau$, with

$$\tau = \frac{\delta}{2(\delta-1)\alpha}\ln(1 + \frac{2\beta}{\delta^2\gamma^2}), \qquad (17)$$

then the set

$$\mathcal{W} = \left\{\boldsymbol{x} \in \mathcal{X} \mid \frac{1}{2}\|\boldsymbol{x}\|^2 \leq \frac{1}{2}\delta^2\gamma^2 + \beta\right\} \qquad (18)$$

is a compact positively invariant set. □

Typically, we would like the invariant set $\mathcal{W}$ to be as small as possible, as it gives an upper-bound for the trajectories of the system. On the other hand, we also want the dwell-time to be as small as possible. The constant $\delta > 1$ plays in interesting role in the above theorem. By increasing δ, we allow the invariant set $\mathcal{W}$ to be larger, thereby decreasing the dwell-time τ. So, there is a kind of pay-off between the size of the invariant set and the dwell-time. Any *finite* value of δ is sufficient to prove the existence of a *compact* positively invariant set. We therefore can take the dwell-time τ to be an arbitrary small value, but not infinitely small. This brings us to the following corollary:

Corollary 1
If the size of the compact positively invariant set is not of interest, then Condition 4 in Theorem 1 can be replaced by an arbitrary small dwell-time $\tau > 0$.

Proof. Taking the limit of $\delta \to \infty$ gives the condition $\tau > 0$ for arbitrary γ and β. □

It therefore suffices to assume that the impulsive inputs are separated in time (which is not a strange assumption from a physical point of view) and simply put τ equal to the (unknown) minimal time-lapse between the impulsive inputs. Then, we calculate the corresponding value of δ and obtain the size of the compact positively invariant set.

In this section, we have presented a sufficient condition for the existence of a compact positively invariant set, but the attractivity of solutions outside $\mathcal{W}$ to $\mathcal{W}$ is not guaranteed. If in addition the system is incrementally attractively stable, for which we will give a sufficient condition in Section 4.4, then it is also assured that all solutions outside $\mathcal{W}$ converge to $\mathcal{W}$.

4.4 Conditions for convergence

In the following theorem, it is stated that strictly maximal monotone measure differential inclusions are uniformly convergent.

Theorem 2
A consistent measure differential inclusion of the form (10) is exponentially convergent if

1. *$\boldsymbol{\mathcal{A}}(\boldsymbol{x})$ is a maximal monotone set-valued mapping, with $\mathbf{0} \in \boldsymbol{\mathcal{A}}(\mathbf{0})$,*
2. *$\boldsymbol{\mathcal{A}}(\boldsymbol{x}) + \boldsymbol{c}(\boldsymbol{x})$ is a strictly maximal monotone set-valued mapping,*
3. *system (10) exhibits a compact positively invariant set.*

Proof. Let us first show that system (10) is incrementally attractively stable, i.e. all solutions of (10) converge to each other for positive time. Consider hereto two solutions $\boldsymbol{x}_1(t)$ and $\boldsymbol{x}_2(t)$ of (10) and a Lyapunov candidate function $V = \frac{1}{2}\|\boldsymbol{x}_2 - \boldsymbol{x}_1\|^2$. Consequently, the differential measure of V satisfies: $\mathrm{d}V = \frac{1}{2}(\boldsymbol{x}_2^+ + \boldsymbol{x}_2^- - \boldsymbol{x}_1^+ - \boldsymbol{x}_1^-)^{\mathrm{T}}(\mathrm{d}\boldsymbol{x}_2 - \mathrm{d}\boldsymbol{x}_1)$, with $\mathrm{d}\boldsymbol{x}_1 = -\mathrm{d}\boldsymbol{a}_1 - \boldsymbol{c}(\boldsymbol{x}_1)\,\mathrm{d}t + \mathrm{d}\boldsymbol{b}(\boldsymbol{w})$, $\mathrm{d}\boldsymbol{x}_2 = -\mathrm{d}\boldsymbol{a}_2 - \boldsymbol{c}(\boldsymbol{x}_2)\,\mathrm{d}t + \mathrm{d}\boldsymbol{b}(\boldsymbol{w})$, where $\mathrm{d}\boldsymbol{a}_1 \in \boldsymbol{\mathcal{A}}(\boldsymbol{x}_1^+)$ and $\mathrm{d}\boldsymbol{a}_2 \in \boldsymbol{\mathcal{A}}(\boldsymbol{x}_2^+)$. The differential measure of V has a density $\dot{V}$ with respect to the Lebesgue measure $\mathrm{d}t$ and a density $V^+ - V^-$ with respect to the atomic measure $\mathrm{d}\eta$, i.e. $\mathrm{d}V = \dot{V}\,\mathrm{d}t + (V^+ - V^-)\,\mathrm{d}\eta$. We first evaluate the density $\dot{V}$:

$$\begin{aligned}\dot{V} &= -(\boldsymbol{x}_2 - \boldsymbol{x}_1)^{\mathrm{T}}(\boldsymbol{a}'_t(\boldsymbol{x}_2) + \boldsymbol{c}(\boldsymbol{x}_2) - \boldsymbol{b}'_t(\boldsymbol{w}) \\ &\qquad - \boldsymbol{a}'_t(\boldsymbol{x}_1) - \boldsymbol{c}(\boldsymbol{x}_1) + \boldsymbol{b}'_t(\boldsymbol{w})) \\ &= -(\boldsymbol{x}_2 - \boldsymbol{x}_1)^{\mathrm{T}}(\boldsymbol{a}'_t(\boldsymbol{x}_2) + \boldsymbol{c}(\boldsymbol{x}_2) - \boldsymbol{a}'_t(\boldsymbol{x}_1) - \boldsymbol{c}(\boldsymbol{x}_1)),\end{aligned} \qquad (19)$$

where $\boldsymbol{a}'_t(\boldsymbol{x}_1) \in \boldsymbol{\mathcal{A}}(\boldsymbol{x}_1)$ and $\boldsymbol{a}'_t(\boldsymbol{x}_2) \in \boldsymbol{\mathcal{A}}(\boldsymbol{x}_2)$, since both solutions $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ correspond to the same perturbation $\boldsymbol{w}$. Due to strict monotonicity of $\boldsymbol{\mathcal{A}}(\boldsymbol{x}) + \boldsymbol{c}(\boldsymbol{x})$, there exists a constant $\alpha > 0$ such that $\dot{V} \leq -\alpha\|\boldsymbol{x}_2 - \boldsymbol{x}_1\|^2$. Subsequently, we consider the jump $V^+ - V^-$ of V: $V^+ - V^- = \frac{1}{2}(\boldsymbol{x}_2^+ + \boldsymbol{x}_2^- - \boldsymbol{x}_1^+ - \boldsymbol{x}_1^-)^{\mathrm{T}}\left(\boldsymbol{x}_2^+ - \boldsymbol{x}_2^- - \boldsymbol{x}_1^+ + \boldsymbol{x}_1^-\right)$, with $\boldsymbol{x}_1^+ - \boldsymbol{x}_1^- = -\boldsymbol{a}'_\eta(\boldsymbol{x}_1) + \boldsymbol{b}'_\eta(\boldsymbol{w})$, for $\boldsymbol{a}'_\eta(\boldsymbol{x}_1) \in \boldsymbol{\mathcal{A}}(\boldsymbol{x}_1^+)$, and $\boldsymbol{x}_2^+ - \boldsymbol{x}_2^- = -\boldsymbol{a}'_\eta(\boldsymbol{x}_2) + \boldsymbol{b}'_\eta(\boldsymbol{w})$, for $\boldsymbol{a}'_\eta(\boldsymbol{x}_2) \in \boldsymbol{\mathcal{A}}(\boldsymbol{x}_2^+)$. Elimination of $\boldsymbol{x}_1^-$ and $\boldsymbol{x}_2^-$ and exploiting the monotonicity of $\boldsymbol{\mathcal{A}}(\boldsymbol{x})$ gives

$$
\begin{aligned}
V^+ - V^- &= \frac{1}{2}(2\boldsymbol{x}_2^+ + \boldsymbol{a}'_\eta(\boldsymbol{x}_2) - 2\boldsymbol{x}_1^+ - \boldsymbol{a}'_\eta(\boldsymbol{x}_1))^{\mathrm{T}} \\
&\qquad \left(-\boldsymbol{a}'_\eta(\boldsymbol{x}_2) + \boldsymbol{a}'_\eta(\boldsymbol{x}_1)\right) \\
&= -(\boldsymbol{x}_2^+ - \boldsymbol{x}_1^+)^{\mathrm{T}} \left(\boldsymbol{a}'_\eta(\boldsymbol{x}_2) - \boldsymbol{a}'_\eta(\boldsymbol{x}_1)\right) \\
&\quad - \frac{1}{2}\left\|\boldsymbol{a}'_\eta(\boldsymbol{x}_2) - \boldsymbol{a}'_\eta(\boldsymbol{x}_1)\right\|^2 \\
&\leq 0.
\end{aligned} \tag{20}
$$

It therefore holds that V strictly decreases over every non-empty compact time-interval as long as $\boldsymbol{x}_2 \neq \boldsymbol{x}_1$. In turn, this implies that all solutions of (10) converge to each other exponentially (and therefore uniformly).

Finally we use Lemma 2 in (Yakubovich, 1964), which formulates that if a system exhibits a compact positively invariant set, then the existence of a solution that is bounded for $t \in \mathbb{R}$ is guaranteed. We will denote this 'steady-state' solution by $\bar{\boldsymbol{x}}_w(t)$. The original lemma is formulated for differential equations (possibly with discontinuities, therewith including differential inclusions, with bounded right-hand sides). Here, we use this lemma for measure differential inclusions and would like to note that the proof of the lemma allows for such extensions if we only require continuous dependence on initial conditions. The latter is guaranteed for monotone measure differential inclusions, because incremental stability implies a continuous dependence on initial conditions.

Since all solutions of (10) are globally exponentially stable, also $\bar{\boldsymbol{x}}_w(t)$ is a globally exponentially stable solution. This concludes the proof that the measure differential inclusion (10) is exponentially convergent. □

5 Illustrative Example: a One-way clutch

In this section, we present an example of a mechanical system with a set-valued force law (modelling a one-way clutch) that illustrates the power of the result in Theorem 2.

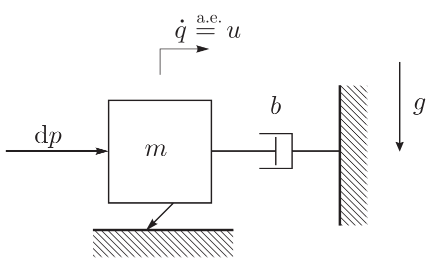

Figure 1. Mass with one-way clutch and impulsive actuation.

The time-evolution of the velocity of a mass m subjected to a one-way clutch, a dashpot $b > 0$ and an external input (considering both bounded and impulsive contributions), see Figure 1, can be described by the equality of measures

$$
m\,\mathrm{d}u = \mathrm{d}p + \mathrm{d}s - bu\,\mathrm{d}t. \tag{21}
$$

We can decompose the differential measure $\mathrm{d}s$ of the one-way clutch in $\mathrm{d}s = \lambda\,\mathrm{d}t + \Lambda\,\mathrm{d}\eta$, where $\lambda := s'_t$ is the contact force and $\Lambda = s'_\eta$ is the contact impulse. The differential impulse measure $\mathrm{d}s$ of the one-way clutch obeys the set-valued force law $-\mathrm{d}s \in \mathrm{Upr}(u^+)$. The set-valued function $\mathrm{Upr}(x)$ is the unilateral primitive (Glocker, 2001):

$$
\begin{aligned}
-y \in \mathrm{Upr}(x) &\iff 0 \leq x \perp y \geq 0 \\
&\iff x \geq 0,\, y \geq 0,\, xy = 0,
\end{aligned} \tag{22}
$$

being a maximal monotone operator.

The input consists of a bounded force f and an impulse F: $\mathrm{d}p = f\,\mathrm{d}t + F\,\mathrm{d}\eta$. We assume that an impulsive input $F > 0$ is transmitted by firing bullets with mass m_0 and constant speed $v \leq v_{\max}$ on the left side of the mass m. We assume a completely inelastic impact. If $u \geq v$, then the bullet is not able to hit the mass m and then the impulse F equals zero. If $u^+ < v$, then the impulse F equals the mass of the bullet multiplied with its velocity jump: $F = m_0(v - u^+)$. Similarly, we assume that an impulsive input $F < 0$ is transmitted by firing on the right side of the mass m with a speed $v < 0$, bounded by $|v| \leq v_{\max}$. The energy input $u^+F = m_0u^+(v - u^+)$ of the impulse F is maximal when $u^+ = \frac{1}{2}v$ and is therefore bounded from above by $\beta := \frac{1}{4}m_0v_{\max}^2 \geq |u^+F|$.

We first prove the existence of a compact positively invariant set with Theorem 1. Theorem 1 uses the Lyapunov function $W(u) = \frac{1}{2}u^2$, which we recognise to be the kinetic energy divided by the mass m. The time-derivative $\dot{W}$ gives, using $u\lambda = 0$, $\dot{W} \leq -\frac{b}{m}u^2 + u\sup_{t\in\mathbb{R}}(f(t))$, and it therefore holds that $\alpha = \frac{b}{m}$ and $\gamma = \frac{1}{b}\sup_{t\in\mathbb{R}}(f(t))$ with α and γ defined in Theorem 1. Theorem 1 states that if the time-instances t_i of the impulses F are separated by the dwell-time $\tau = \frac{\delta}{2(\delta-1)\alpha}\ln(1 + \frac{2\beta}{\delta^2\gamma^2})$, then the set $\mathcal{W} = \left\{u \in \mathbb{R}^+ \mid \frac{1}{2}u^2 \leq \frac{1}{2}\delta^2\gamma^2 + \beta\right\}$ is a compact positively invariant set for arbitrary $\delta > 1$. Following Corollary 1, we conclude that the dwell-time can be made arbitrary small by increasing δ. We therefore take τ to be smaller than the minimal time-lapse between two succeeding impulsive time-instances, which gives a lower bound for δ.

Just as in the proof of Theorem 2, we prove incremental stability using the Lyapunov function $V = \frac{1}{2}(u_2 - u_1)^2$. First, we consider the time-derivative $\dot{V}$:

$$
\begin{aligned}
\dot{V} &= (u_2 - u_1)(\dot{u}_2 - \dot{u}_1) \\
&= (u_2 - u_1)\frac{1}{m}(\lambda_2 - bu_2 - \lambda_1 + bu_1) \\
&= (u_2 - u_1)\frac{1}{m}(\lambda_2 - \lambda_1) - \frac{b}{m}(u_2 - u_1)^2, \\
&\qquad \text{with } -\lambda_1 \in \mathrm{Upr}(u_1),\ -\lambda_2 \in \mathrm{Upr}(u_2) \\
&\leq -\frac{b}{m}(u_2 - u_1)^2.
\end{aligned} \tag{23}
$$

Subsequently, we consider a jump in V:

$$\begin{aligned} V^+ - V^- =& V(u_1^+, u_2^+) - V(u_1^-, u_2^-) \\ =& \frac{1}{2}(u_2^+ - u_1^+)^2 - \frac{1}{2}(u_2^- - u_1^-)^2 \\ =& \frac{1}{2}(u_2^+ + u_2^- - u_1^+ - u_1^-) \\ & \qquad (u_2^+ - u_2^- - u_1^+ + u_1^-). \end{aligned} \tag{24}$$

Following the proof of Theorem 2, we eliminate u_1^- and u_2^- by substituting the impact equation $m(u_j^+ - u_j^-) = \Lambda_j + F$, $j = 1, 2$:

$$\begin{aligned} V^+ - V^- &= \frac{1}{2}(2u_2^+ - \frac{1}{m}\Lambda_2 - 2u_1^+ + \frac{1}{m}\Lambda_1)\frac{1}{m}(\Lambda_2 - \Lambda_1) \\ &= (u_2^+ - u_1^+)\frac{1}{m}(\Lambda_2 - \Lambda_1) - \frac{1}{2m^2}(\Lambda_2 - \Lambda_1)^2 \leq 0. \end{aligned} \tag{25}$$

Hence, it holds for the differential measure $\mathrm{d}V$ that $\mathrm{d}V = \dot{V}\,\mathrm{d}t + (V^+ - V^-)\,\mathrm{d}\eta \leq -\alpha(u_2 - u_1)^2\,\mathrm{d}t$, with $\alpha = \frac{b}{m}$. Integration of $\mathrm{d}V$ over a non-empty time-interval therefore leads to a strict decrease of the function V as long as $u_2 \neq u_1$. This proves incremental stability. Consequently, the system is exponentially convergent (see Theorem 2). This property implies, see Definition 1, that for any input, with measure $\mathrm{d}p$, with a bounded Lebesgue measurable part $f\,\mathrm{d}t$ and the impulsive part relating to bounded energy, the system exhibits a unique bounded steady-state solution to which all other solutions converge exponentially.

6 Conclusions

In this paper, we have presented sufficient conditions for the convergence property of a class of measure differential inclusions with certain maximal monotonicity properties. The results are illustrated by application to a mechanical system with a unilateral constraint (a mass-damper system with a one-way clutch and impulsive inputs). Future work involves the exploitation of such convergence properties to design tracking controllers for mechanical systems with unilateral constraints.

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