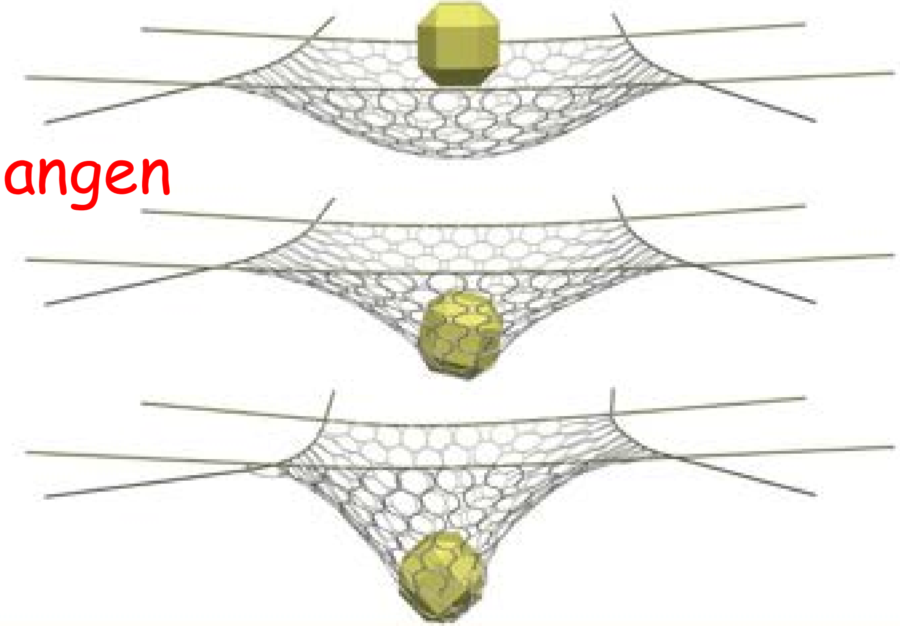
University of Stuttgart
Germany

Institute for
Nonlinear Mechanics

Nonsmooth Dynamics

a journey from convex analysis to computer code

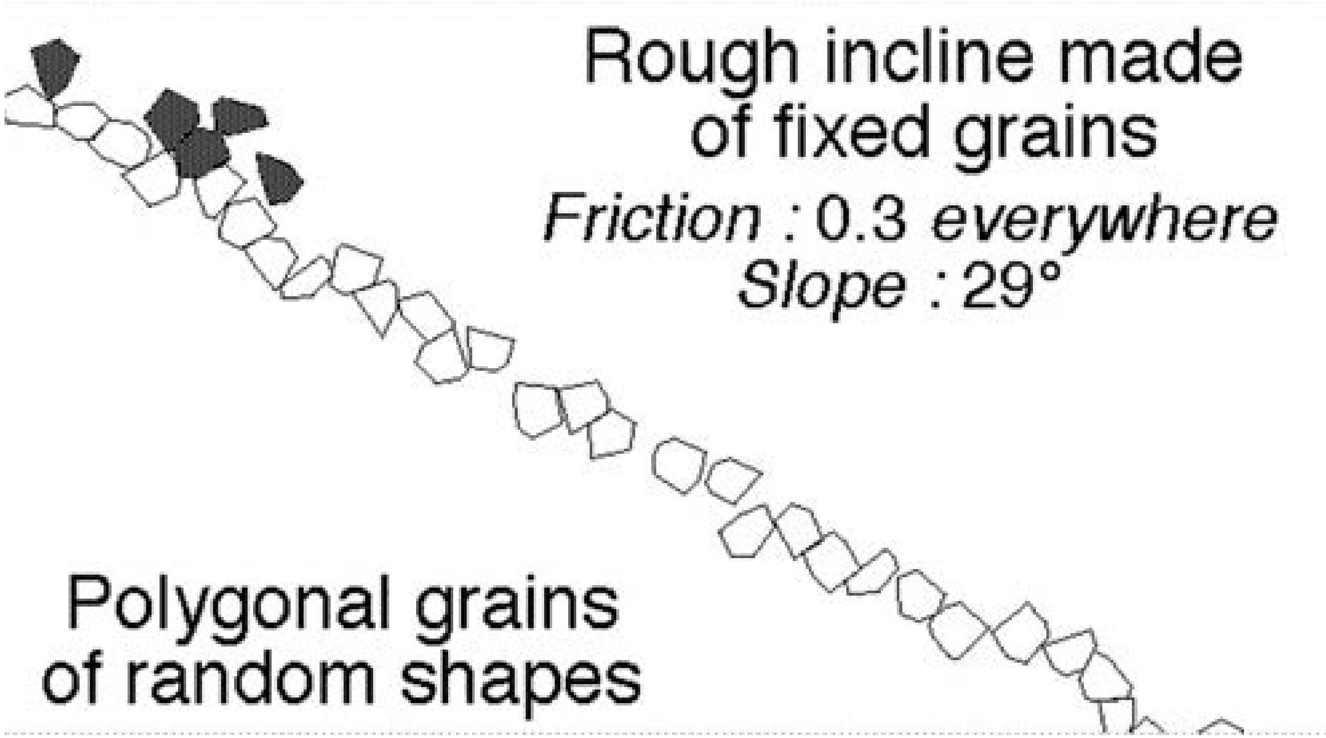
Remco Leine



Also presented at the ENNSD 2025, Erlangen

ENOC 2024
TU Delft
22 - 26 July 2024

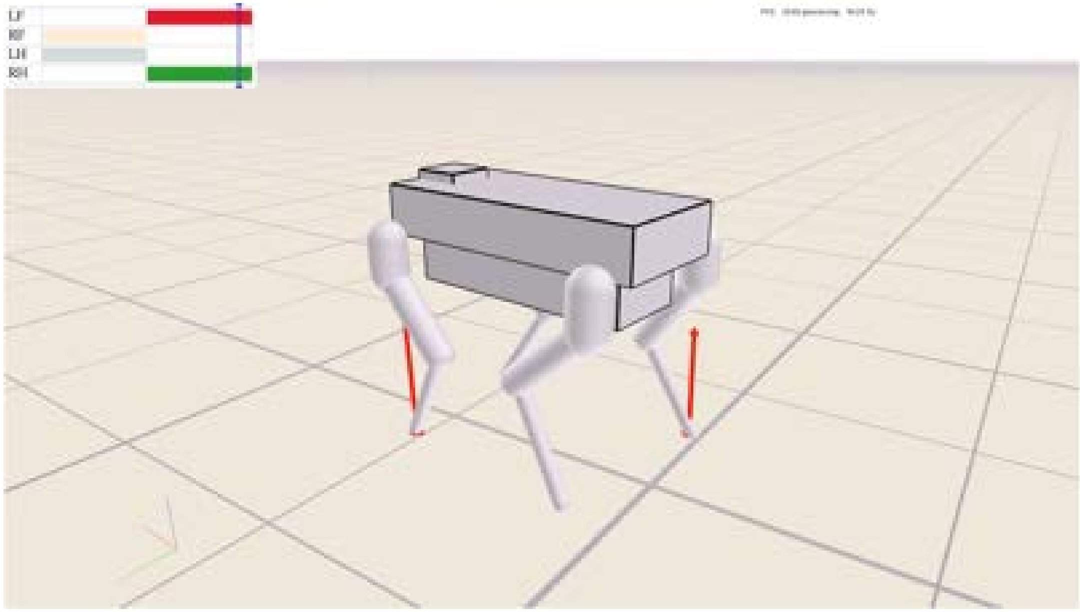
Simulation of granular media

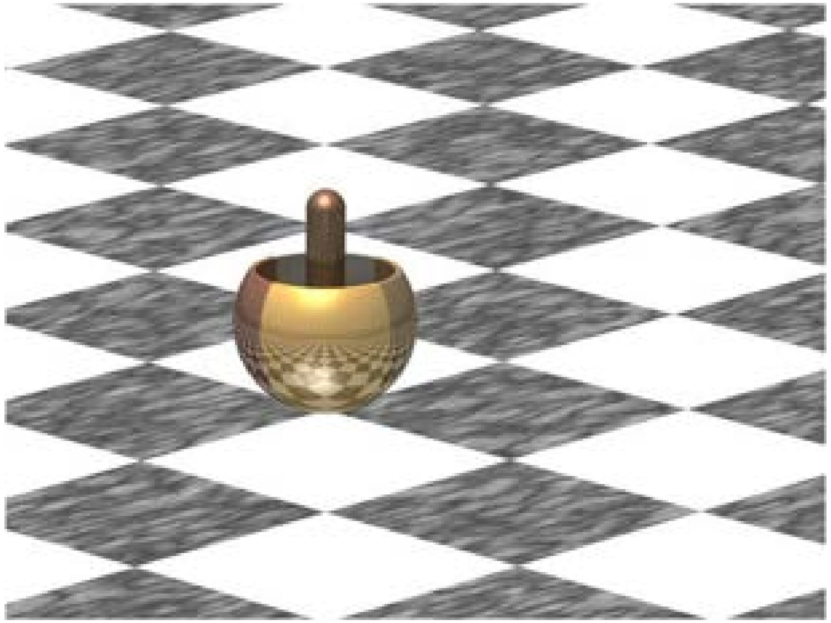


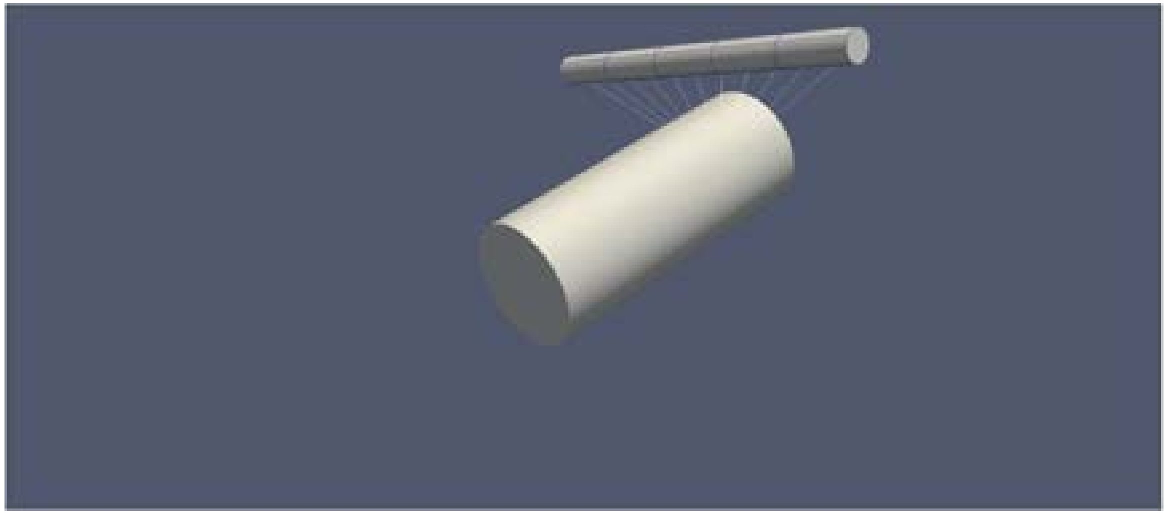
10 hours of simulation time on the high-performance computer of the University of Stuttgart?

Not really…

Application examples of Nonsmooth Dynamics

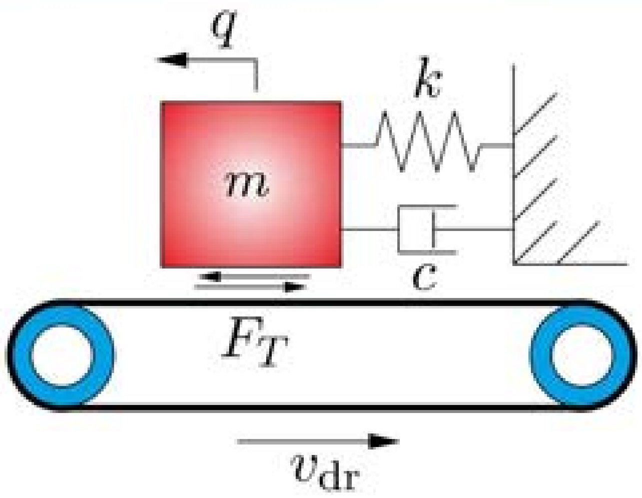




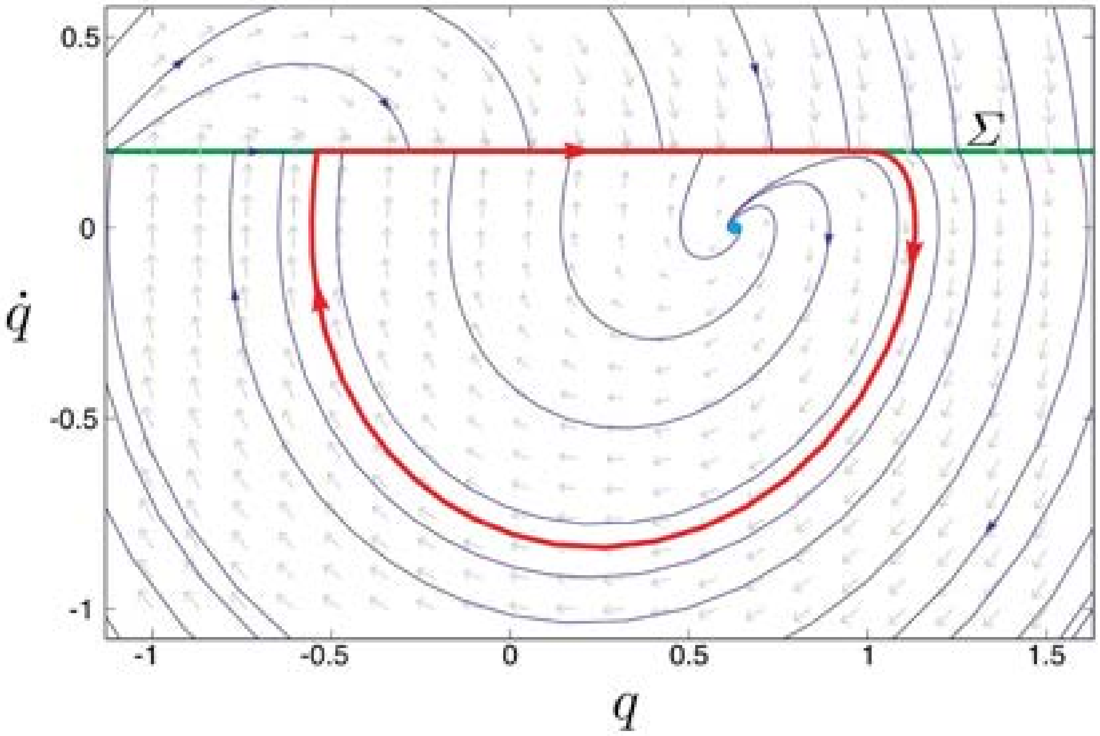


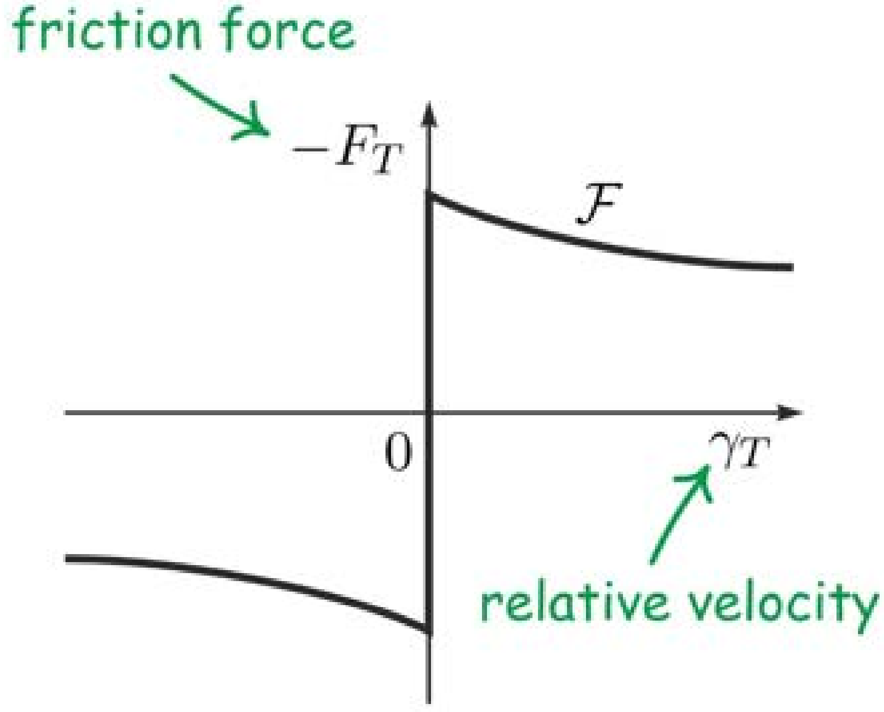
Simulations by: Giuseppe Capobianco, Simon Eugster, Jonas Breuling, R.L.

A motivating example: The block-on-belt system



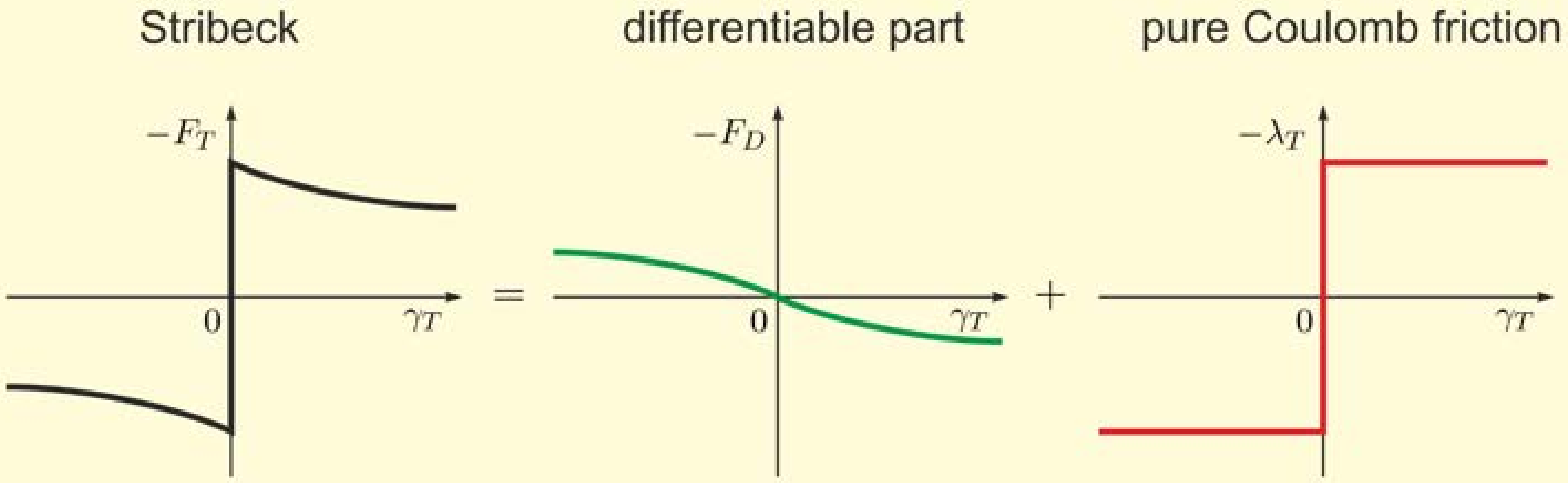
equation of motion	$m\ddot{q} + c\dot{q} + kq = F_T$
relative velocity	$\gamma_T = \dot{q} - v_{\mathrm{dr}}$
friction law	$(\gamma_T, -F_T) \in \mathrm{Graph}\,\mathcal{F}$



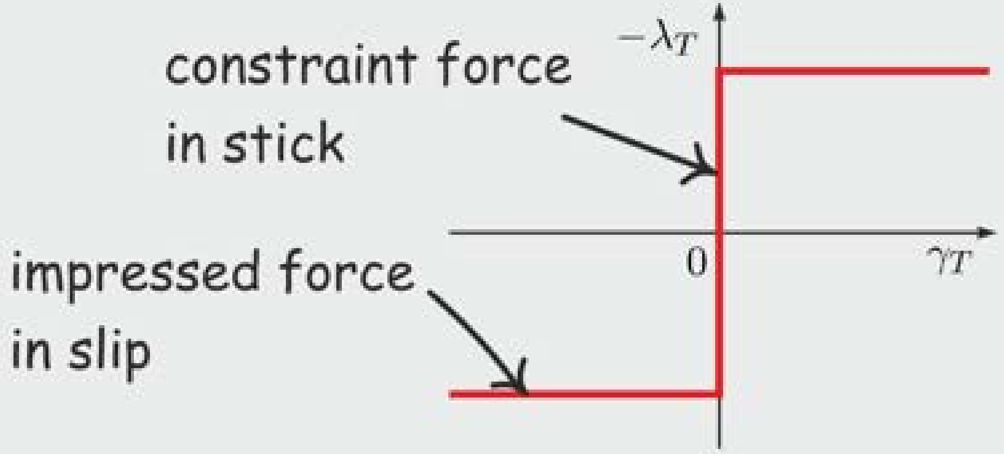


Stribeck friction law

Decomposition of the Stribeck friction law

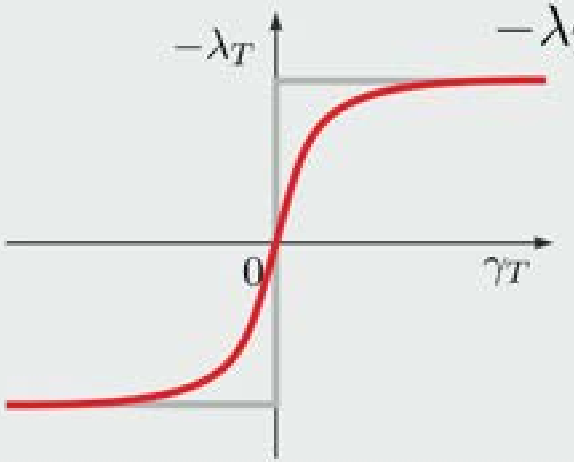


Coulomb friction is a **set-valued force law** expressing the difference in causality in the stick and slip phase



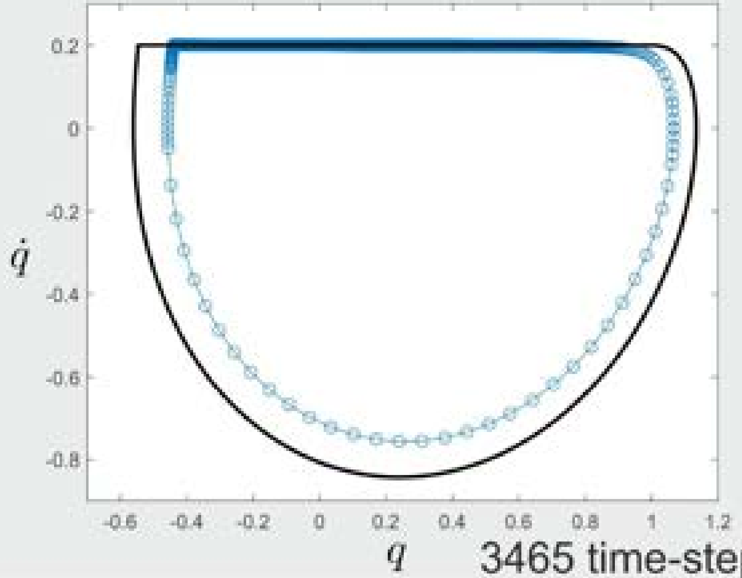
Regularization methods

Arctangent smoothing



$$-\lambda_T = \mu\lambda_N \tfrac{2}{\pi}\arctan(\varepsilon\gamma_T)$$

stiff ODE
in stick phase



3465 time-steps

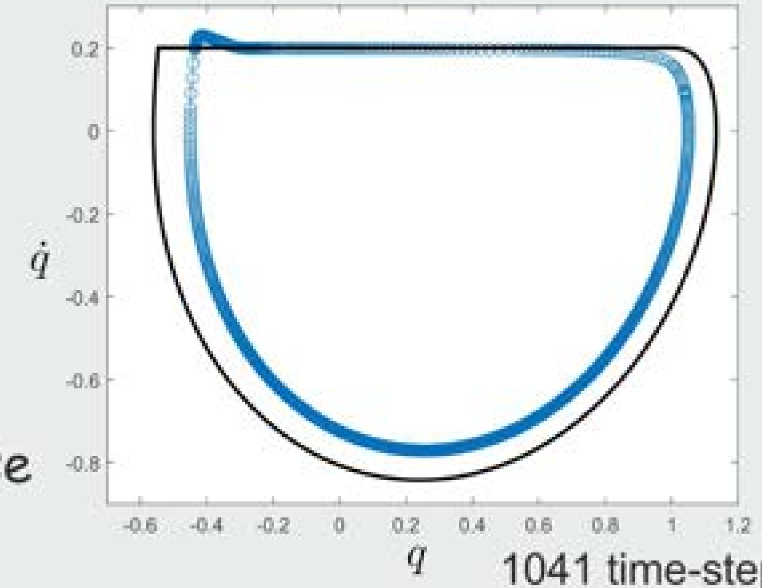
LuGre model

$$-\lambda_T = \sigma_0 z + \sigma_1 \dot z$$

$$\dot z = \gamma_T - \tfrac{\sigma_0}{\mu\lambda_N}|\gamma_T| z$$

z : bristle deflection

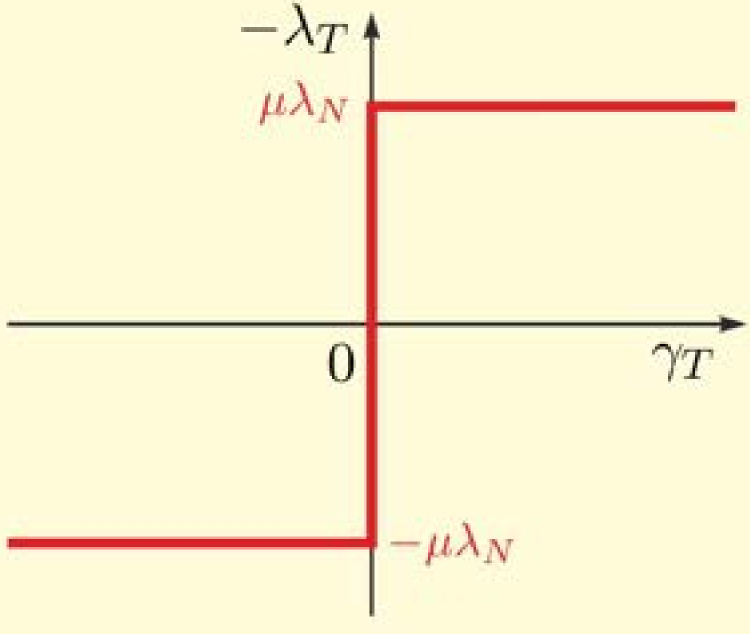
stiff ODE
in stick and slip phase



1041 time-steps

Coulomb friction as a normal cone inclusion

Primal form

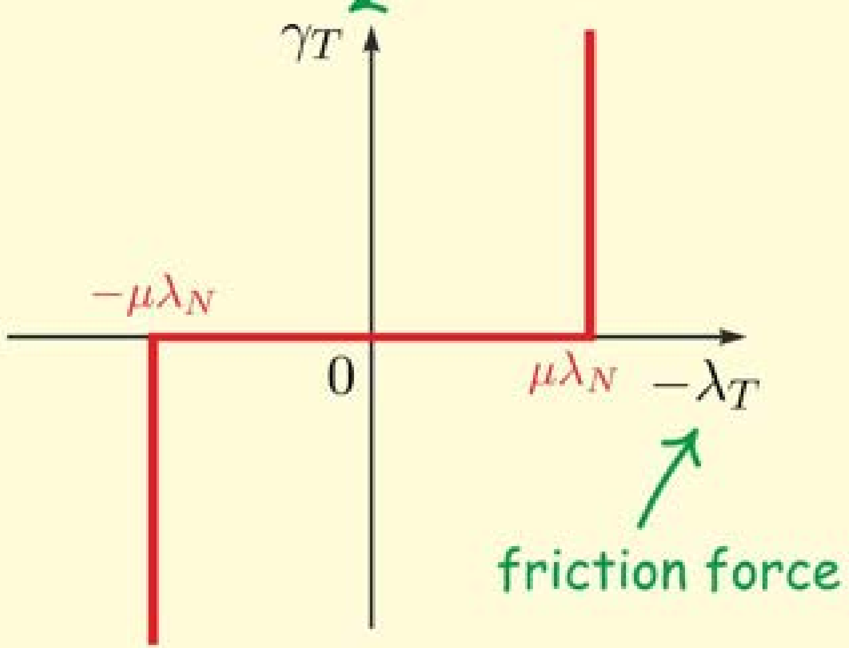


$$-\lambda_T \in \mu\lambda_N \,\mathrm{Sign}(\gamma_T)$$

set-valued
sign function

$\Longleftrightarrow$

Dual form

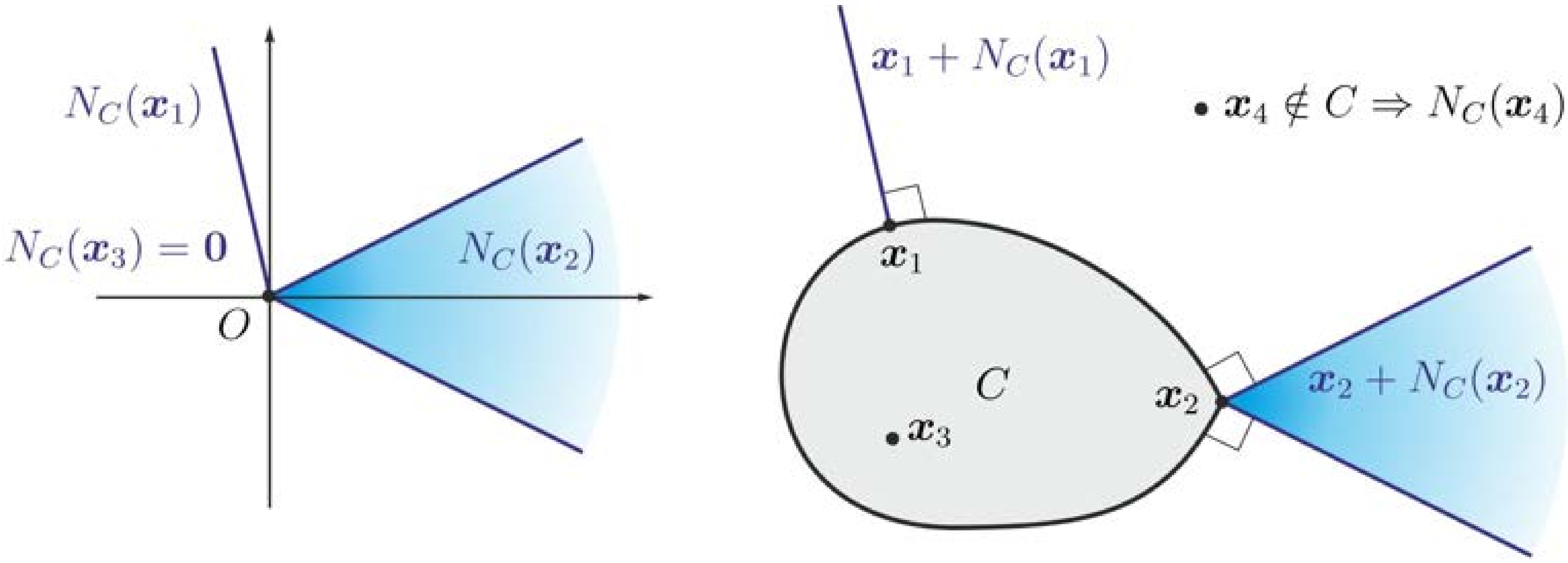


$$\gamma_T \in N_{C_T}(-\lambda_T)$$

normal cone

force reservoir: $C_T = [-\mu\lambda_N, \mu\lambda_N]$

Normal cone from convex analysis

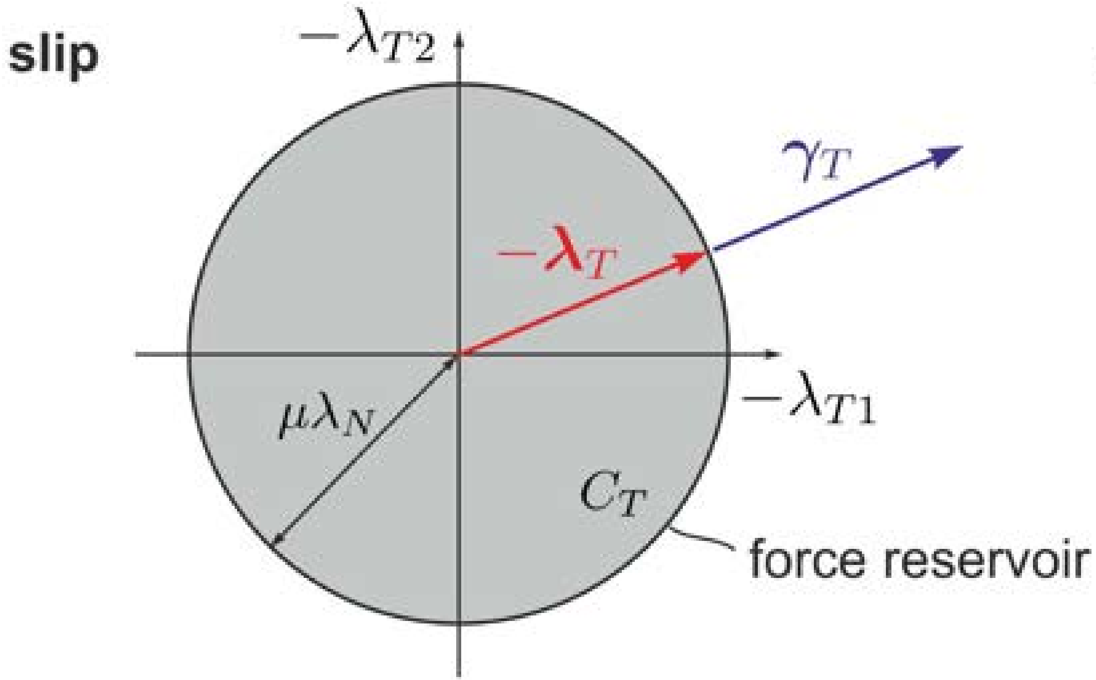


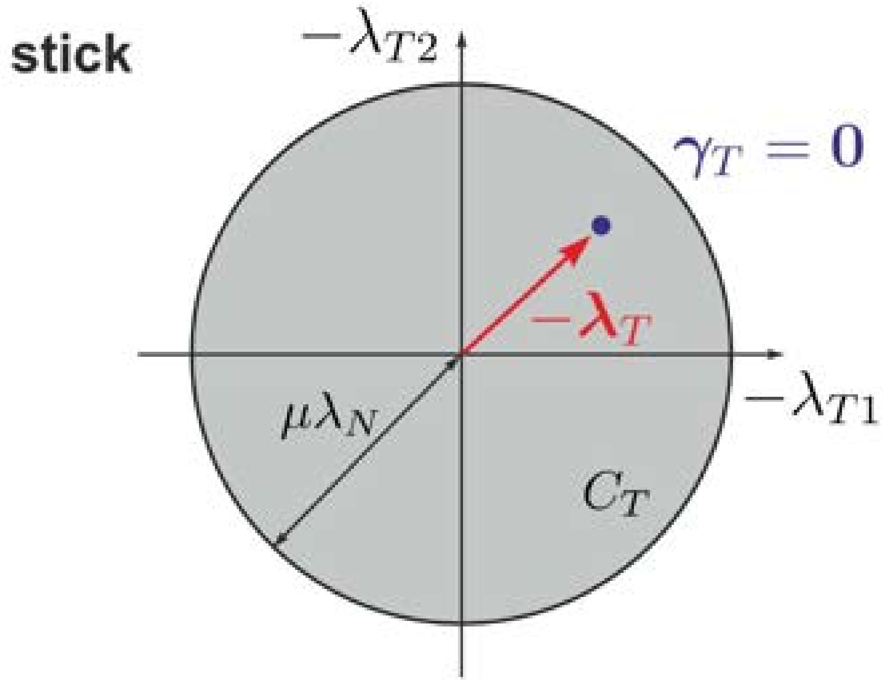
$$N_C(\boldsymbol{x}) = \{\boldsymbol{y} \mid \boldsymbol{y}^{\mathrm{T}}(\boldsymbol{x}^* - \boldsymbol{x}) \leq 0,\ \boldsymbol{x} \in C, \forall \boldsymbol{x}^* \in C\}$$

$\boldsymbol{x}$ on the boundary of C $\Rightarrow$ normal cone $N_C(\boldsymbol{x})$ consists of outward normal ray(s)

$\boldsymbol{x}$ in the interior of C $\Rightarrow$ normal cone $N_C(\boldsymbol{x})$ is zero

Coulomb's spatial friction law





$-\boldsymbol{\lambda}_T$ on the boundary of C_T

$\boldsymbol{\gamma}_T$ in outward normal ray

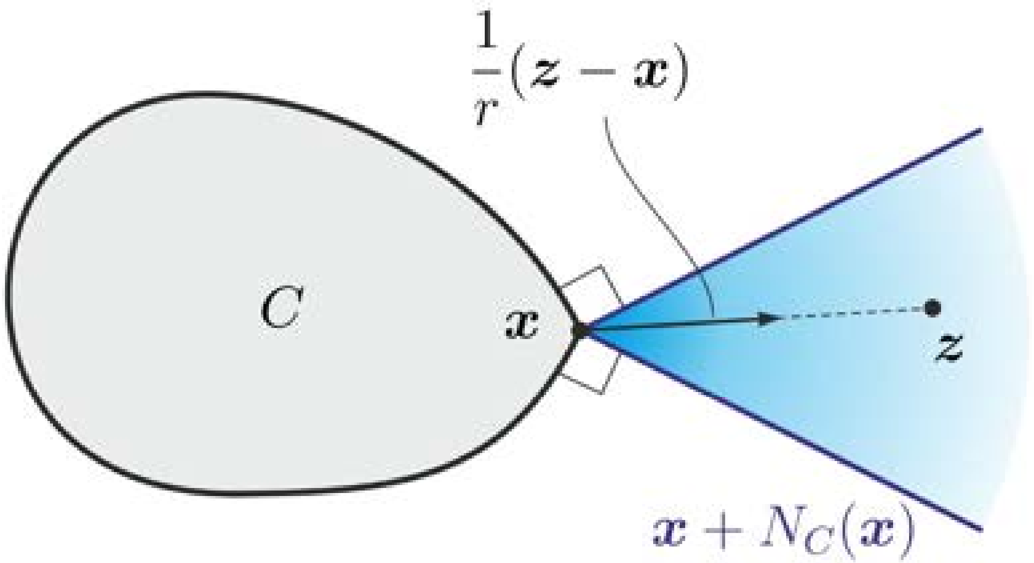
$-\boldsymbol{\lambda}_T$ in the interior of C_T

$\boldsymbol{\gamma}_T = \mathbf{0}$

$$\boldsymbol{\gamma}_T \in N_{C_T}(-\boldsymbol{\lambda}_T)$$

normal cone inclusion

Proximal point to a convex set



proximal point function:

$$\boldsymbol{x} = \mathrm{prox}_C(\boldsymbol{z})$$

$$\Updownarrow$$

$$\frac{1}{r}(\boldsymbol{z}-\boldsymbol{x}) \in N_C(\boldsymbol{x}), \quad r>0$$

substitute:

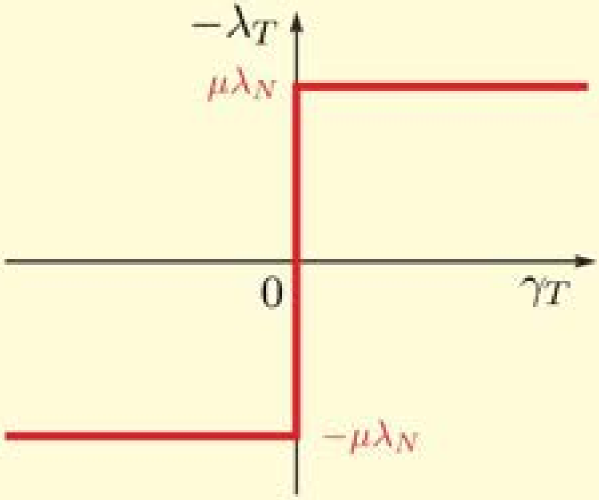
$$\left.\begin{aligned} \boldsymbol{z} &= -\boldsymbol{\lambda} + r\boldsymbol{\gamma} \\ \boldsymbol{x} &= -\boldsymbol{\lambda} \end{aligned}\right\} \Rightarrow \begin{aligned} \boldsymbol{\gamma} &= \frac{1}{r}(\boldsymbol{z}-\boldsymbol{x}) \\ \boldsymbol{\lambda} &= -\boldsymbol{x} \end{aligned}$$

$$\boldsymbol{\gamma} \in N_C(-\boldsymbol{\lambda}) \quad \Longleftrightarrow \quad -\boldsymbol{\lambda} = \mathrm{prox}_C(-\boldsymbol{\lambda} + r\boldsymbol{\gamma}) \qquad r>0$$

normal cone inclusion implicit equation

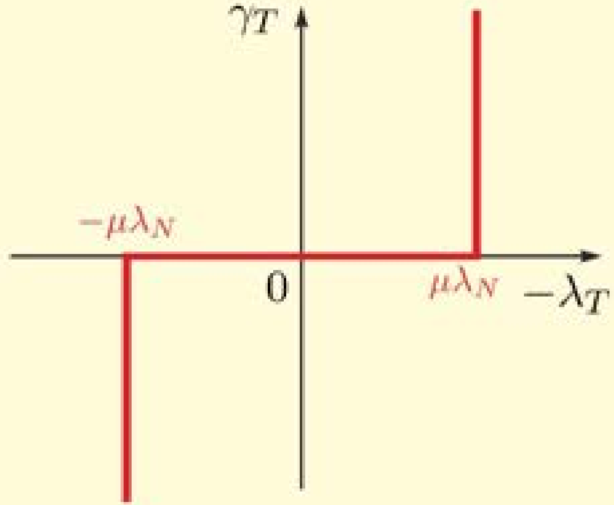
Coulomb friction as proximal point equation

Primal form $-\lambda_T \in \mu\lambda_N \,\mathrm{Sign}(\gamma_T)$



$\Longleftrightarrow$

Dual form $\gamma_T \in N_{C_T}(-\lambda_T)$

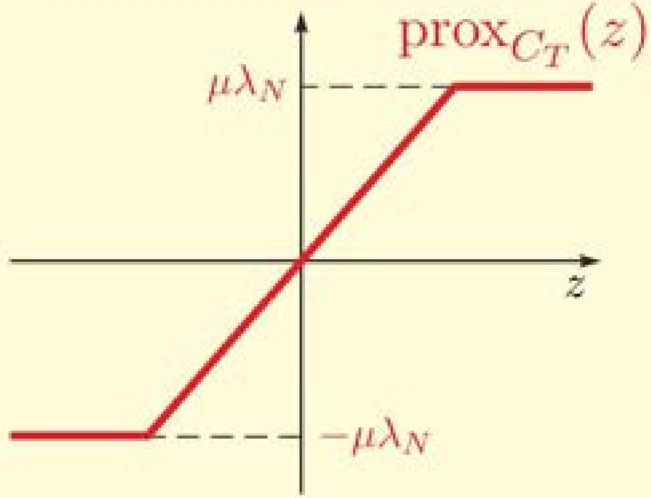


$\Updownarrow$

Proximal point equation

$$-\lambda_T = \mathrm{prox}_{C_T}(-\lambda_T + r\gamma_T)$$

force reservoir: $C_T = [-\mu\lambda_N, \mu\lambda_N]$



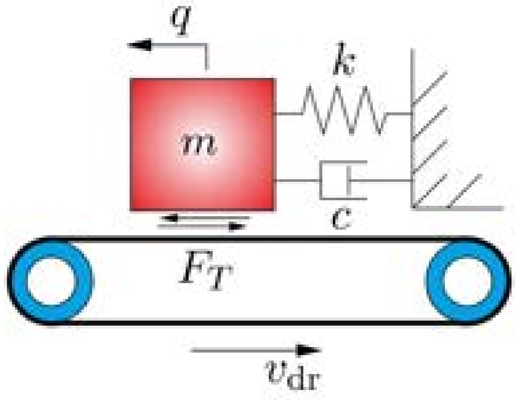
The block-on-belt system as DAE

Differential inclusion

equation of motion $m\ddot{q} + c\dot{q} + kq = F_D(\gamma_T) + \lambda_T$

relative velocity $\gamma_T = \dot{q} - v_{\mathrm{dr}}$

friction law $-\lambda_T \in \mu\lambda_N \operatorname{Sign}(\gamma_T)$



$$-\lambda_T \in \mu\lambda_N \operatorname{Sign}(\gamma_T) \iff \gamma_T \in N_{C_T}(-\lambda_T) \iff -\lambda_T = \operatorname{prox}_{C_T}(-\lambda_T + r\gamma_T)$$

Differential algebraic equation

$$\left.\begin{aligned} \dot{q} &= u \\ \dot{u} &= -\tfrac{c}{m}u - \tfrac{k}{m}q + \tfrac{1}{m}F_D(\gamma_T) + \tfrac{1}{m}\lambda_T \\ 0 &= \lambda_T + \operatorname{prox}_{C_T}(-\lambda_T + r\gamma_T) \end{aligned}\right\}$$

with $\gamma_T = u - v_{\mathrm{dr}}$

Differential algebraic equation

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \lambda)$$

$$0 = c(\boldsymbol{x}, \lambda)$$

$$\boldsymbol{x} = \begin{pmatrix} q & u \end{pmatrix}^{\mathrm{T}}, \quad \lambda = \lambda_T$$

A semi-implicit index-2 DAE solver

Differential algebraic equation

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\lambda})$$

$$\boldsymbol{0} = \boldsymbol{c}(\boldsymbol{x}, \boldsymbol{\lambda})$$

Hessenberg-2 DAE

Semi-implicit scheme

$$\boldsymbol{x}_{i+1} = \boldsymbol{x}_i + \Delta t \boldsymbol{f}(\boldsymbol{x}_i, \boldsymbol{\lambda}_{i+1})$$

$$\boldsymbol{0} = \boldsymbol{c}(\boldsymbol{x}_{i+1}, \boldsymbol{\lambda}_{i+1})$$

in each step solve $\boldsymbol{0} = \boldsymbol{c}(\boldsymbol{x}_i + \Delta t \boldsymbol{f}(\boldsymbol{x}_i, \boldsymbol{\lambda}_{i+1}), \boldsymbol{\lambda}_{i+1})$

solvable if $\det(\boldsymbol{c}_x \boldsymbol{f}_\lambda \Delta t + \boldsymbol{c}_\lambda) \neq 0$ index must be 1 or 2

Block-on-belt system with semi-impl. scheme

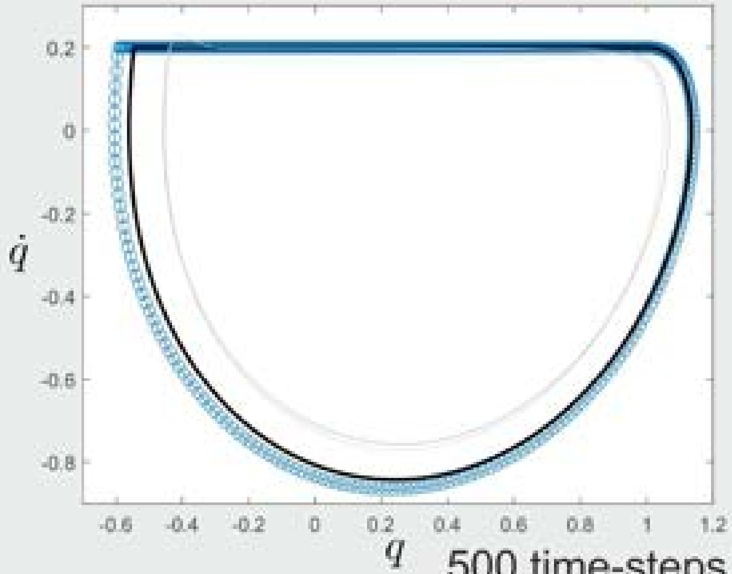
event capturing method !

$$q_{i+1} = q_i + \Delta t\, u_i$$

$$u_{i+1} = u_i + \frac{\Delta t}{m}(-cu_i - kq_i + F_D(\gamma_{T,i}) + \lambda_{T,i+1})$$

$$0 = \lambda_{T,i+1} + \mathrm{prox}_{C_T}(-\lambda_{T,i+1} + r\gamma_{T,i+1})$$

with $\gamma_{T,i} = u_i - v_{\mathrm{dr}}$



500 time-steps

Multibody system with set-valued force law

Multibody system

kinematics	$\dot{\boldsymbol{q}} = \boldsymbol{u}$
kinetics	$\boldsymbol{M}\dot{\boldsymbol{u}} - \boldsymbol{h}(\boldsymbol{q}, \boldsymbol{u}) = \boldsymbol{W}\boldsymbol{\lambda}$
rel. velocity	$\boldsymbol{\gamma} = \boldsymbol{W}^{\mathrm{T}}\boldsymbol{u} + \boldsymbol{\chi}$
force law	$\boldsymbol{\gamma} \in N_C(-\boldsymbol{\lambda})$

Semi-implicit scheme

$\boldsymbol{h}(\boldsymbol{q}_i, \boldsymbol{u}_i)$

$$\boldsymbol{q}_{i+1} = \boldsymbol{q}_i + \Delta t \boldsymbol{u}_i$$

$$\boldsymbol{M}(\boldsymbol{u}_{i+1} - \boldsymbol{u}_i) - \Delta t \boldsymbol{h}_i = \Delta t \boldsymbol{W}\boldsymbol{\lambda}_{i+1}$$

$$\boldsymbol{\gamma}_{i+1} = \boldsymbol{W}^{\mathrm{T}}\boldsymbol{u}_{i+1} + \boldsymbol{\chi}$$

$$\boldsymbol{\gamma}_{i+1} \in N_C(-\boldsymbol{\lambda}_{i+1})$$

$$\boldsymbol{\gamma}_{i+1} = \underbrace{\Delta t \boldsymbol{W}^{\mathrm{T}}\boldsymbol{M}^{-1}\boldsymbol{W}}_{\boldsymbol{A}}\boldsymbol{\lambda}_{i+1} + \underbrace{\boldsymbol{W}^{\mathrm{T}}(\boldsymbol{u}_i + \Delta t \boldsymbol{M}^{-1}\boldsymbol{h}_i) + \boldsymbol{\chi}}_{-\boldsymbol{b}}$$

$$\boldsymbol{\gamma}_{i+1} \in N_C(-\boldsymbol{\lambda}_{i+1})$$

$$\boldsymbol{y} = -\boldsymbol{\gamma}_{i+1}, \quad \boldsymbol{x} = -\boldsymbol{\lambda}_{i+1}$$

Linear normal cone inclusion problem

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$$

$$-\boldsymbol{y} \in N_C(\boldsymbol{x})$$

Overview

Constrained optimization problem

$$\min_{\boldsymbol{x}\in C} \frac{1}{2}\boldsymbol{x}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}^{\mathrm{T}}\boldsymbol{x}$$

$$\Uparrow \; \boldsymbol{A} = \boldsymbol{A}^{\mathrm{T}} > 0$$

Variational inequality

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$$

$$\forall \boldsymbol{x}^* \in C : \boldsymbol{y}^{\mathrm{T}}(\boldsymbol{x}^* - \boldsymbol{x}) \geq 0$$

$$\Leftrightarrow$$

Normal cone inclusion problem

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$$

$$-\boldsymbol{y} \in N_C(\boldsymbol{x})$$

$$\Leftrightarrow$$

Proximal point equation problem

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$$

$$\boldsymbol{x} = \mathrm{prox}_C(\boldsymbol{x} - r\boldsymbol{y})$$

$$\Downarrow \; C = \mathbb{R}_0^{n+}$$

Linear complementarity problem

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$$

$$\boldsymbol{0} \leq \boldsymbol{y} \perp \boldsymbol{x} \geq \boldsymbol{0}$$

Numerical methods

Normal cone inclusion problem

$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$

$-\boldsymbol{y} \in N_C(\boldsymbol{x})$

$\Leftrightarrow$

Proximal point equation problem

$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$

$\boldsymbol{x} = \mathrm{prox}_C(\boldsymbol{x} - r\boldsymbol{y})$

$r > 0$ can be chosen arbitrarily!

Semi-smooth Newton method

$\boldsymbol{f}(\boldsymbol{x}) := \boldsymbol{x} - \mathrm{prox}_C(\boldsymbol{x} - r(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}))$

for some initial guess $\boldsymbol{x}_0$ iterate

$$\boldsymbol{x}_{k+1} = \boldsymbol{x}_k - \boldsymbol{J}^{-1}\boldsymbol{f}(\boldsymbol{x}_k), \quad \boldsymbol{J} \in \partial_C \boldsymbol{f}(\boldsymbol{x}_k)$$

until $\|\boldsymbol{x}_{k+1} - \boldsymbol{x}_k\| < tol$

Proximal point iteration

for some initial guess $\boldsymbol{x}_0$ iterate

$$\boldsymbol{y}_k = \boldsymbol{A}\boldsymbol{x}_k + \boldsymbol{b}$$

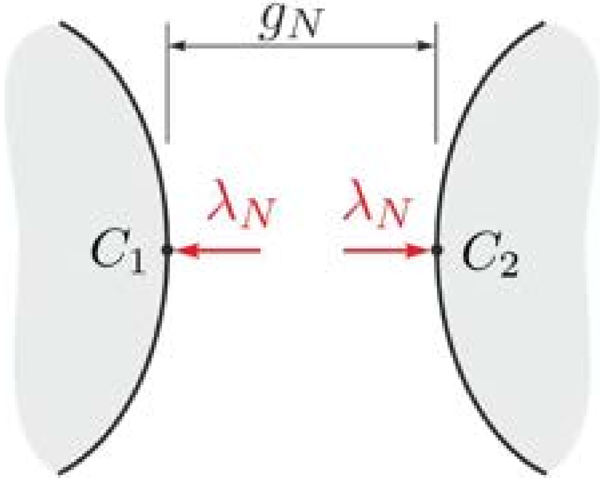
$$\boldsymbol{x}_{k+1} = \mathrm{prox}_C(\boldsymbol{x}_k - r\boldsymbol{y}_k)$$

until $\|\boldsymbol{x}_{k+1} - \boldsymbol{x}_k\| < tol$

r too small: slow convergence

r too large: divergence

Contact law in normal direction

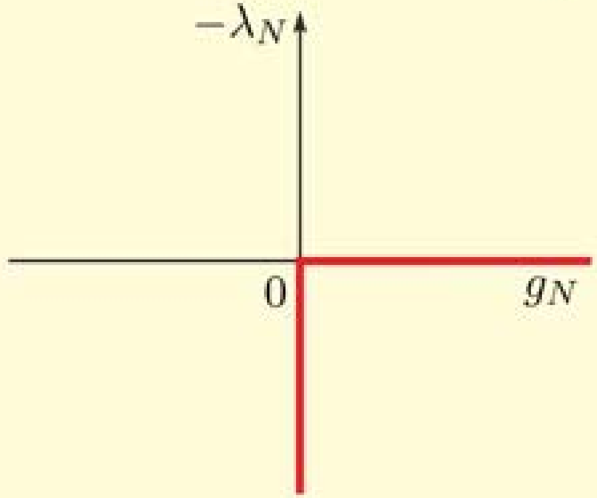


Signorini's law

no penetration	$g_N \geq 0$
no adhesion	$\lambda_N \geq 0$
no distance effects	$g_N > 0 \Longrightarrow \lambda_N = 0$

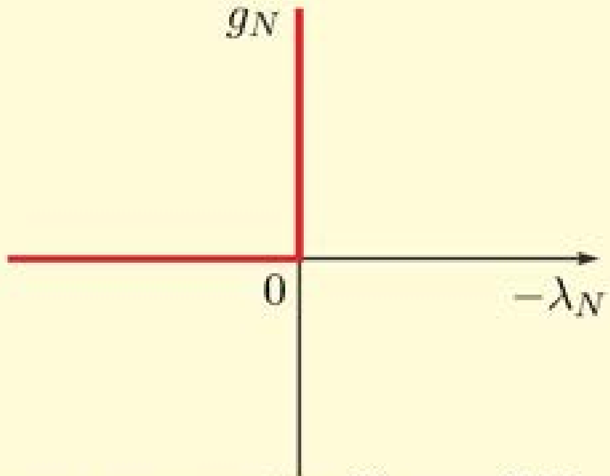
$$0 \leq g_N \perp \lambda_N \geq 0$$

Primal form $-\lambda_N \in N_{\mathbb{R}_0^+}(g_N)$



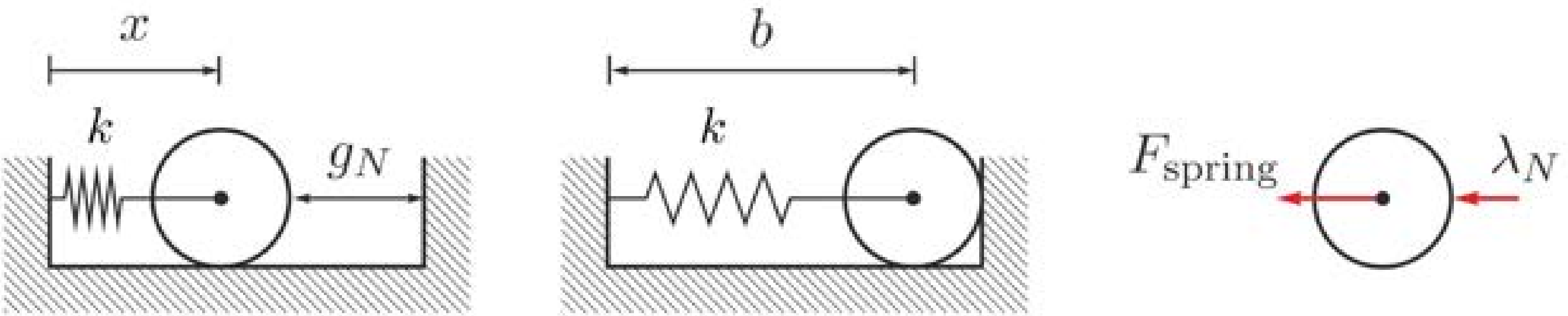
$\Longleftrightarrow$

Dual form $g_N \in N_{C_N}(-\lambda_N)$



force reservoir: $C_N = \mathbb{R}_0^-$

A Signorini contact problem in statics



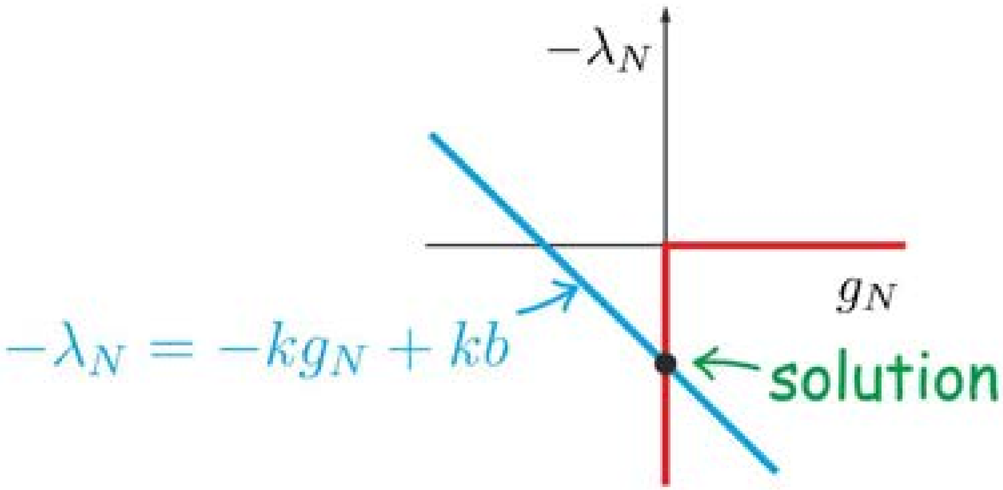
kinematics: $g_N = b - x$

equilibrium of forces: $F_{\text{spring}} + \lambda_N = 0$

constitutive equations:

spring law $F_{\text{spring}} = kx$

contact law $0 \leq g_N \perp \lambda_N \geq 0$



Contact problem

$$g_N = \frac{1}{k}\lambda_N + b$$

$$0 \leq g_N \perp \lambda_N \geq 0$$

Linear complementarity problem

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$$

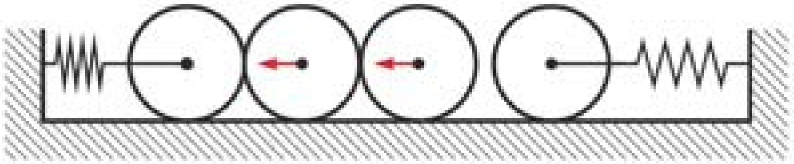
$$\boldsymbol{0} \leq \boldsymbol{y} \perp \boldsymbol{x} \geq \boldsymbol{0}$$

A multi-DOF Signorini contact problem in statics

equilibrium of forces: $\boldsymbol{K}\boldsymbol{q} = \boldsymbol{W}_N \boldsymbol{\lambda}_N + \boldsymbol{f}$

kinematics: $\boldsymbol{g}_N = \boldsymbol{W}_N^{\mathrm{T}} \boldsymbol{q} + \boldsymbol{g}_0$

constitutive equation: $\mathbf{0} \leq \boldsymbol{g}_N \perp \boldsymbol{\lambda}_N \geq \mathbf{0}$



Contact problem

$$\boldsymbol{g}_N = \underbrace{\boldsymbol{W}_N^{\mathrm{T}} \boldsymbol{K}^{-1} \boldsymbol{W}_N}_{\boldsymbol{A}} \boldsymbol{\lambda}_N + \underbrace{\boldsymbol{g}_0 + \boldsymbol{W}_N^{\mathrm{T}} \boldsymbol{K}^{-1} \boldsymbol{f}}_{\boldsymbol{b}}$$

$$\mathbf{0} \leq \boldsymbol{g}_N \perp \boldsymbol{\lambda}_N \geq \mathbf{0}$$

Linear complementarity problem

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$$

$$\mathbf{0} \leq \boldsymbol{y} \perp \boldsymbol{x} \geq \mathbf{0}$$

$\Leftrightarrow$

Proximal point equation problem

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$$

$$\boldsymbol{x} = \mathrm{prox}_C(\boldsymbol{x} - r\boldsymbol{y})$$

Dynamics: the evolutionary problem

Nonimpulsive dynamics

kinematics: $\dot{\boldsymbol{q}} = \boldsymbol{u}$

equation of motion: $\boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{u}} - \boldsymbol{h}(\boldsymbol{q}, \boldsymbol{u}) = \boldsymbol{W}_N \boldsymbol{\lambda}_N + \boldsymbol{W}_T \boldsymbol{\lambda}_T$

normal contact law: $g_{Nj} \in N_{C_N}(-\lambda_{Nj})$

friction law: $\gamma_{Tj} \in N_{C_T(\lambda_{Nj})}(-\lambda_{Tj})$

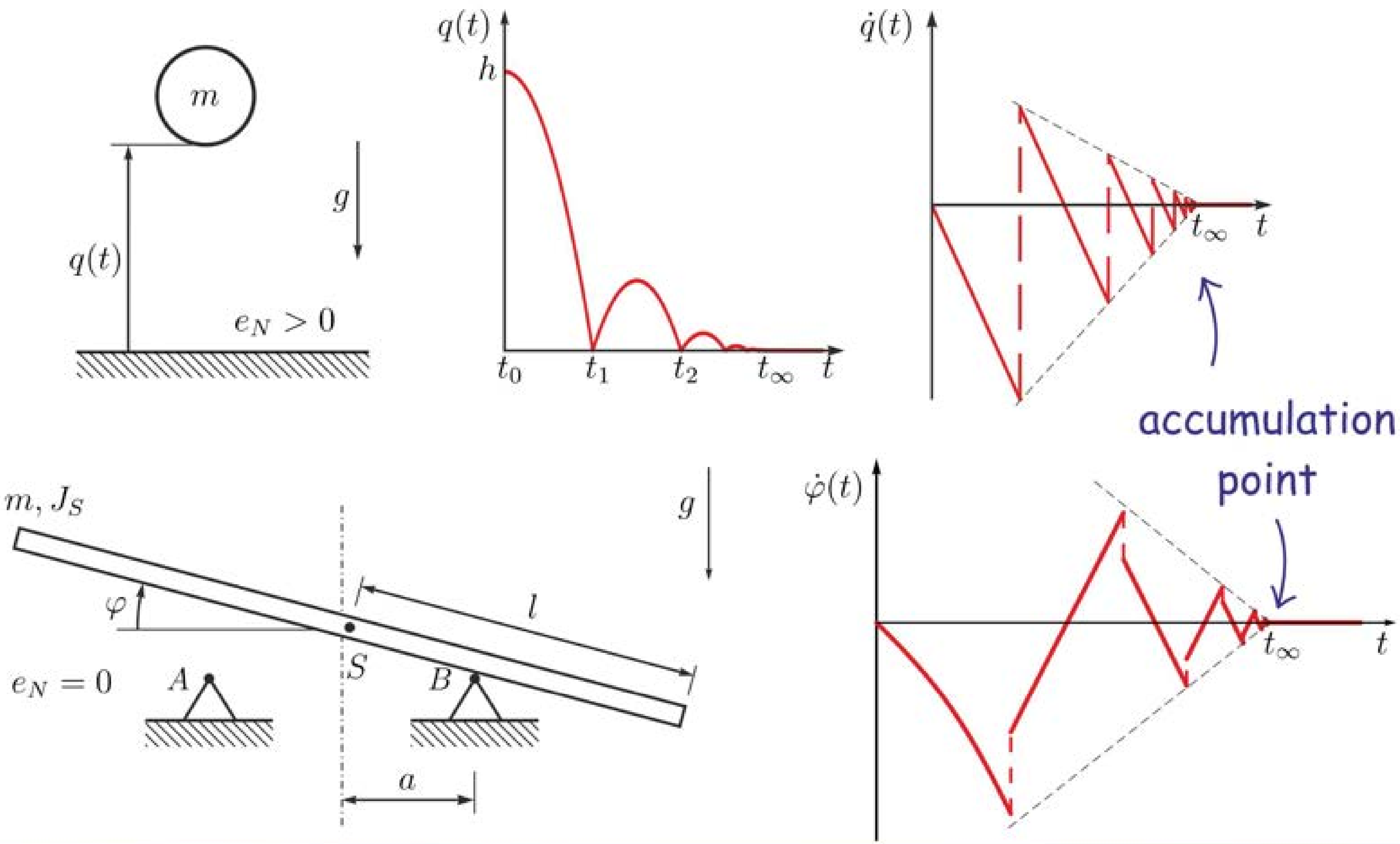
Impulsive dynamics

impact equation: $\boldsymbol{M}(\boldsymbol{q})(\boldsymbol{u}^+ - \boldsymbol{u}^-) = \boldsymbol{W}_N \boldsymbol{\Lambda}_N + \boldsymbol{W}_T \boldsymbol{\Lambda}_T$

normal impact law: $\gamma_{Nj}^+ + e_{Nj}\gamma_{Nj}^- \in N_{C_N}(-\Lambda_{Nj}), \quad j \in \mathcal{I}_N = \{j \mid g_{Nj} = 0\}$

tangential impact law: $\gamma_{Tj}^+ + e_{Tj}\gamma_{Tj}^+ \in N_{C_T(\Lambda_{Nj})}(-\Lambda_{Tj}), \quad j \in \mathcal{I}_N$

Accumulation points of collisions



The Woodpecker Toy: a simple code

only 34 lines of code...

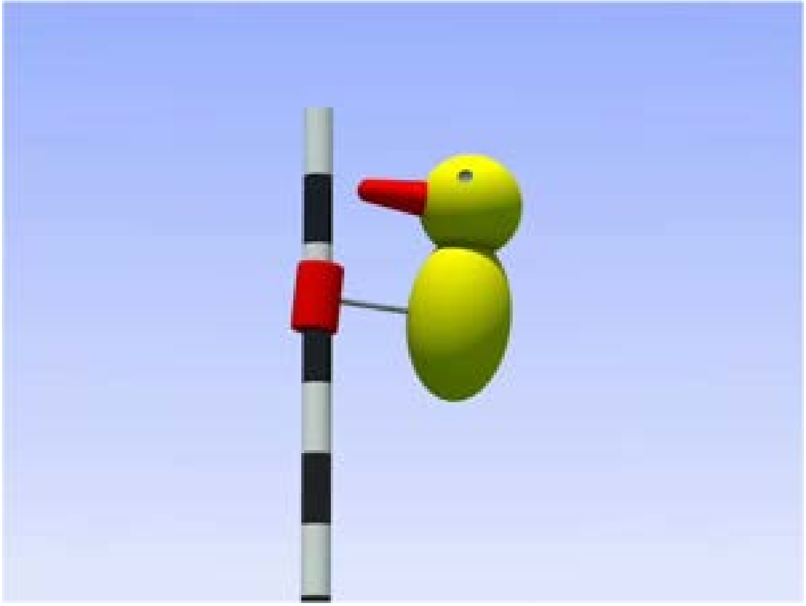
```
function [t,q,u] = woodpeckertimestepping(t0,te,q0,u0,N)
mM = 3e-4; JM = 5e-9; mS = 4.5e-3; JS = 7e-7; lM = 0.01; lG = 0.015; lS = 0.0201;   %system
rM = 0.0031; r0 = 0.0025; hM = 0.0058; hS = 0.02; grav = 9.81; cphi = 0.0056;       %parameters
mu = 0.3; eN = [0.5;0;0]; r = 1e-4; tol = 1e-8;
M =[(mS+mM) mS*lM mS*lG; mS*lM (JM+mS*lM^2) mS*lM*lG; mS*lG mS*lM*lG (JS+mS*lG^2)]; %constant
WN = [0 0 0; 0 hM -hM; -hS 0 0]; WT = [1 1 1; lM rM rM; lG-lS  0 0];              %matrices

t = linspace(t0,te,N); dt = (te-t0)/(N-1);                                          %time stamp and time step
q = [q0 zeros(3,N-1)]; u = [u0 zeros(3,N-1)]; PN = zeros(3,1); PT = zeros(3,1);     %memory allocation

for i=1:(N-1)
  tM = t(i)+dt/2; qM = q(:,i) + dt/2*u(:,i);       %midpoint
  h = [-(mS+mM)*grav; -cphi*(qM(2)-qM(3))-mS*grav*lM; -cphi*(qM(3)-qM(2))-mS*grav*lG];
  gN = [lM+lG-lS-r0-hS*qM(3);rM-r0+hM*qM(2);rM-r0-hM*qM(2)];
  I = find(gN<=0)';  %index set of closed contacts
  if ~isempty(I)
    gammaNA = WN(:,I)'*u(:,i);
    converged = false;
    while ~converged %fixed point iteration
      u(:,i+1) = u(:,i)  + M\(h*dt + WN(:,I)*PN(I) + WT(:,I)*PT(I));
      gammaNE = WN(:,I)'*u(:,i+1); gammaTE = WT(:,I)'*u(:,i+1);
      PNnew = proxCN(PN(I)-r*(gammaNE + eN(I).*gammaNA));
      PTnew = proxCT(PT(I)-r*gammaTE,mu*PN(I));
      converged = norm(PN(I)-PNnew)+norm(PT(I)-PTnew) < tol;
      PN(I) = PNnew; PT(I) = PTnew;
    end
  else  %if all contacts are open
    u(:,i+1) = u(:,i) + M\(h*dt);
  end
  q(:,i+1) = qM + u(:,i+1)*dt/2; %end timestep
end

function y = proxCN(x), y = max(0,x);             %proximal point function for CN = R_0^+
function y = proxCT(x,a), y = max(min(x,a),-a);   %proximal point function for CT = [-a,a]
```

Moreau-timestepping scheme



A semi-implicit index-2 DAE solver

Differential algebraic equation

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\lambda})$$

$$\boldsymbol{0} = \boldsymbol{c}(\boldsymbol{x}, \boldsymbol{\lambda})$$

Hessenberg-2 DAE

Semi-implicit scheme

$$\boldsymbol{x}_{i+1} = \boldsymbol{x}_i + \Delta t \boldsymbol{f}(\boldsymbol{x}_i, \boldsymbol{\lambda}_{i+1})$$

$$\boldsymbol{0} = \boldsymbol{c}(\boldsymbol{x}_{i+1}, \boldsymbol{\lambda}_{i+1})$$

in each step solve $\boldsymbol{0} = \boldsymbol{c}(\boldsymbol{x}_i + \Delta t \boldsymbol{f}(\boldsymbol{x}_i, \boldsymbol{\lambda}_{i+1}), \boldsymbol{\lambda}_{i+1})$

solvable if $\det(\boldsymbol{c}_x \boldsymbol{f}_\lambda \Delta t + \boldsymbol{c}_\lambda) \neq 0$ index must be 1 or 2

Block-on-belt system with semi-impl. scheme

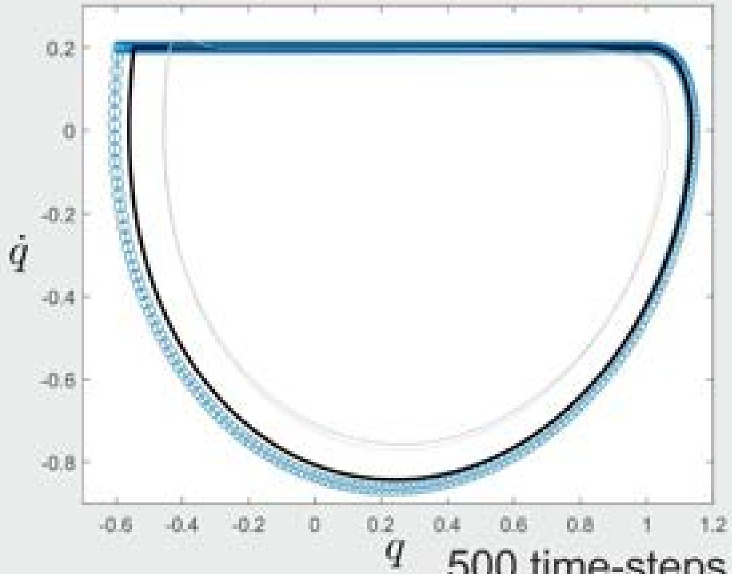
event capturing method !

$$q_{i+1} = q_i + \Delta t\, u_i$$

$$u_{i+1} = u_i + \frac{\Delta t}{m}\left(-c u_i - k q_i + F_D(\gamma_{T,i}) + \lambda_{T,i+1}\right)$$

$$0 = \lambda_{T,i+1} + \operatorname{prox}_{C_T}(-\lambda_{T,i+1} + r\gamma_{T,i+1})$$

with $\gamma_{T,i} = u_i - v_{\mathrm{dr}}$



500 time-steps

A semi-implicit index-3 DAE solver

Differential algebraic equation

$$\dot{\boldsymbol{y}} = \boldsymbol{g}(\boldsymbol{y}, \boldsymbol{z})$$

$$\dot{\boldsymbol{z}} = \boldsymbol{h}(\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{\lambda})$$

$$\boldsymbol{0} = \boldsymbol{c}(\boldsymbol{y}, \boldsymbol{z}, \boldsymbol{\lambda})$$

Hessenberg-3 DAE

Semi-implicit scheme

$$\boldsymbol{y}_{i+1} = \boldsymbol{y}_i + \Delta t \boldsymbol{g}(\boldsymbol{y}_i, \boldsymbol{z}_{i+1})$$

$$\boldsymbol{z}_{i+1} = \boldsymbol{z}_i + \Delta t \boldsymbol{h}(\boldsymbol{y}_i, \boldsymbol{z}_i, \boldsymbol{\lambda}_{i+1})$$

$$\boldsymbol{0} = \boldsymbol{c}(\boldsymbol{y}_{i+1}, \boldsymbol{z}_{i+1}, \boldsymbol{\lambda}_{i+1})$$

in each step solve

$$\boldsymbol{0} = \boldsymbol{c}(\boldsymbol{y}_i + \Delta t \boldsymbol{g}(\boldsymbol{y}_i, \boldsymbol{z}_i + \Delta t \boldsymbol{h}(\boldsymbol{y}_i, \boldsymbol{z}_i, \boldsymbol{\lambda}_{i+1}), \boldsymbol{\lambda}_{i+1})), \boldsymbol{z}_i + \Delta t \boldsymbol{h}(\boldsymbol{y}_i, \boldsymbol{z}_i, \boldsymbol{\lambda}_{i+1}), \boldsymbol{\lambda}_{i+1})$$

solvable if $\det(\boldsymbol{c}_y \boldsymbol{g}_z \boldsymbol{h}_\lambda \Delta t^2 + \boldsymbol{c}_z \boldsymbol{h}_\lambda \Delta t + \boldsymbol{c}_\lambda) \neq 0$ index must be 3 or less

Semi-implicit index-3 DAE solver in mechanics

Multibody system (frictionless)

kinematics	$\dot{\boldsymbol{q}} = \boldsymbol{u}$
kinetics	$\boldsymbol{M}\dot{\boldsymbol{u}} - \boldsymbol{h}(\boldsymbol{q}, \boldsymbol{u}) = \boldsymbol{W}_N \boldsymbol{\lambda}_N$
contact dist.	$\boldsymbol{g}_N = \boldsymbol{g}_N(\boldsymbol{q})$
Signorini's law	$\boldsymbol{g}_N \in N_{C_N}(-\boldsymbol{\lambda}_N)$

impact law is missing!!!

$$\Big\}\, \boldsymbol{0} = \boldsymbol{\lambda}_N + \mathrm{prox}_{C_N}(-\boldsymbol{\lambda}_N + r\boldsymbol{g}_N(\boldsymbol{q}))$$

Semi-implicit scheme

$$\boldsymbol{q}_{i+1} = \boldsymbol{q}_i + \Delta t \boldsymbol{u}_{i+1}$$

$$\boldsymbol{M}(\boldsymbol{u}_{i+1} - \boldsymbol{u}_i) - \Delta t \overset{\boldsymbol{h}(\boldsymbol{q}_i, \boldsymbol{u}_i)}{\boldsymbol{h}_i} = \boldsymbol{W}_N \boldsymbol{P}_{N,i+1}$$

$$\boldsymbol{g}_{N,i+1} = \boldsymbol{g}_N(\boldsymbol{q}_{i+1})$$

$$\boldsymbol{0} = \boldsymbol{P}_{N,i+1} + \mathrm{prox}_{C_N}(-\boldsymbol{P}_{N,i+1} + r\boldsymbol{g}_{N,i+1})$$

agrees for single-contact problems with the inelastic Schatzman-Paoli scheme

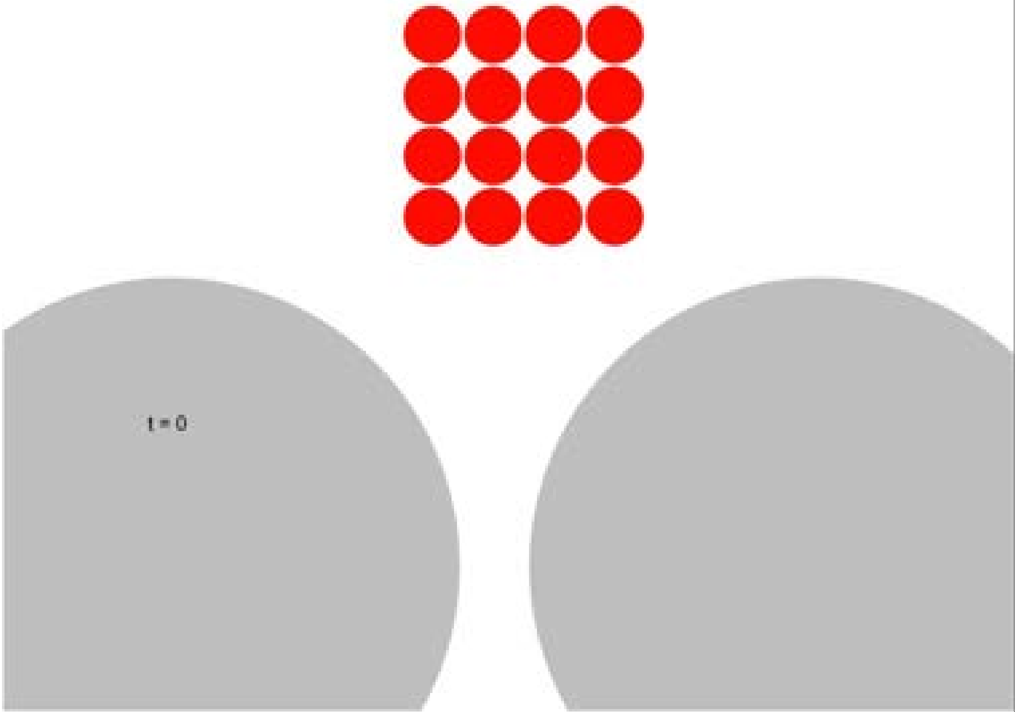
Schatzman & Paoli, 2002

Semi-implicit index-3 DAE solver in mechanics

```
function [t,q,u] = semi_implicit_scheme(sys,t0,te,q0,u0,N,tol)
% Semi-implicit scheme for frictionless inelastic impacts
t = linspace(t0,te,N); dt = (te-t0)/(N-1);          %time stamp and time step
q = zeros(sys.dim_q,N); u = zeros(sys.dim_q,N); PN = zeros(length(sys.I),1);
q(:,1) = q0; u(:,1) = u0;                            %initial condition

for i=1:(N-1)
  h = sys.h(t(i),q(:,i),u(:,i));
  M = sys.M(t(i),q(:,i));
  WN = sys.WChi(t(i),q(:,i),sys.I);
  converged = false;
  while ~converged
      u(:,i+1) = u(:,i) + M\(h*dt + WN*PN);
      q(:,i+1) = q(:,i) + u(:,i+1)*dt;
      gN = sys.gN(t(i+1),q(:,i+1),sys.I);
      PNnew = proxCN(PN-sys.r/dt*gN);
      converged = norm(PN-PNnew) < tol;
      PN = PNnew;
  end
end

function y = proxCN(x), y = max(0,x);   %proximal point function for CN = R_0^+
```



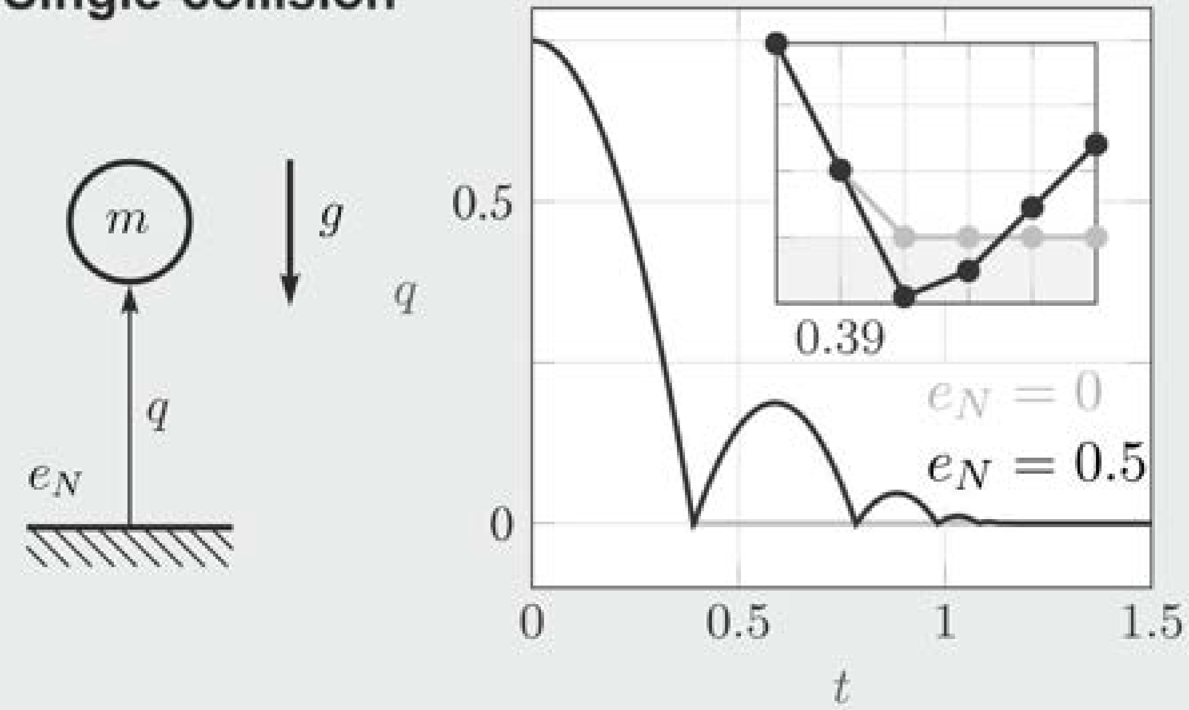
impact law is missing!!!

Schatzman-Paoli-type contact law

completely inelastic $\quad \boldsymbol{g}_{N,i+1} \in N_{C_N}(-\boldsymbol{P}_{N,i+1})$

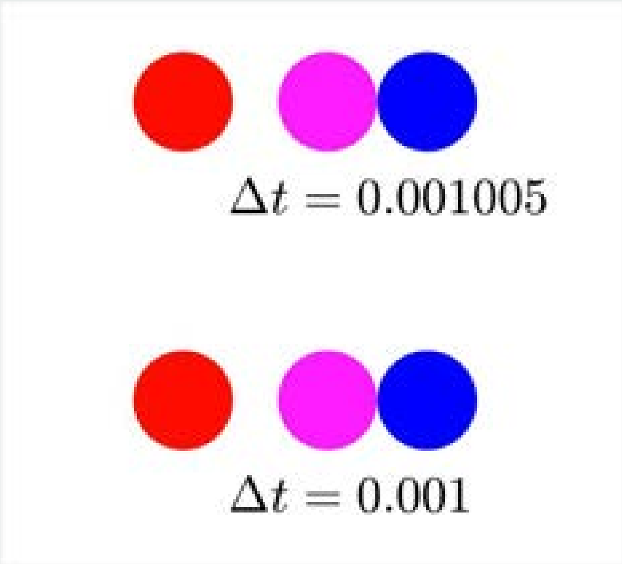
partially elastic $\quad \boldsymbol{g}_{N,i+1} + e_N \boldsymbol{g}_{N,i-1} \in N_{C_N}(-\boldsymbol{P}_{N,i+1}) \qquad 0 \leq e_N \leq 1$

Single-collision



impact is resolved over two time-steps

Multi-collision



wrong results for $e_N > 0$

Moreau timestepping

Measure differential inclusion

$$\mathrm{d}\boldsymbol{q} = \boldsymbol{u}\mathrm{d}t$$

$$\boldsymbol{M}(\boldsymbol{q})\mathrm{d}\boldsymbol{u} - \boldsymbol{h}(\boldsymbol{q},\boldsymbol{u})\mathrm{d}t = \boldsymbol{W}_N \mathrm{d}\boldsymbol{P}_N$$

$$\xi_{Nj} = \gamma_{Nj}^{+} + e_N \gamma_{Nj}^{-}$$

$$g_{Nj}(\boldsymbol{q}) = 0 : \ \xi_{Nj} \in N_{C_N}(-\mathrm{d}P_{Nj})$$

$$g_{Nj}(\boldsymbol{q}) > 0 : \ \mathrm{d}P_{Nj} = 0$$

Time stepping scheme (Moreau)

$$\boldsymbol{q}_M = \boldsymbol{q}_i + \boldsymbol{u}_i \frac{\Delta t}{2}$$

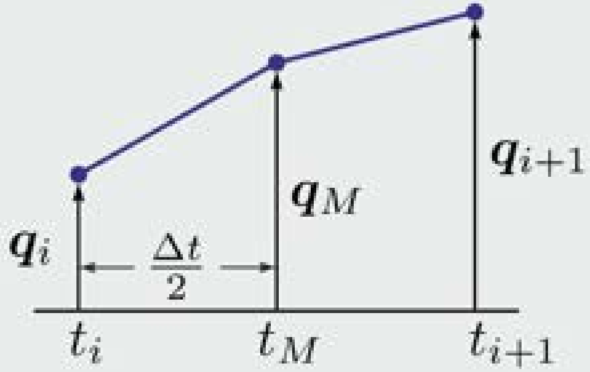
$$\boldsymbol{M}(\boldsymbol{q}_M)(\boldsymbol{u}_{i+1} - \boldsymbol{u}_i) - \boldsymbol{h}(\boldsymbol{q}_M, \boldsymbol{u}_i)\Delta t = \boldsymbol{W}_N \boldsymbol{P}_{N,i+1}$$

$$\xi_{Nj,i+1} = \gamma_{Nj,i+1} + e_N \gamma_{Nj,i}$$

$$g_{Nj}(\boldsymbol{q}_M) \le 0 : \ \xi_{Nj,i+1} \in N_{C_N}(-P_{Nj,i+1})$$

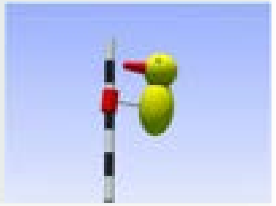
$$g_{Nj}(\boldsymbol{q}_M) > 0 : \ P_{Nj,i+1} = 0$$

$$\boldsymbol{q}_{i+1} = \boldsymbol{q}_M + \boldsymbol{u}_{i+1} \frac{\Delta t}{2}$$



Moreau, Jean, 1988

contact law on velocity level!



If we want frictional and/or elastic impacts...

Constitutive equations

friction laws
anisotropic Coulomb friction
Coulomb-Contensou friction

impact laws
gen. Newton's impact law
Poisson impact law

Glocker, Brogliato, Panagiotopoulos, Curnier, Leine, de Saxé, ...

Measure differential inclusions

Moreau, Monteiro Marques, Schatzman, Ballard, Stewart, ...

Event capturing schemes

Moreau's timestepping scheme	Moreau, Jean
Nonsmooth generalized alpha method	Acary, Brüls, Capobianco

If we w ... a ... s...

Const

frictio ... p

anis ... tic ... g ... law

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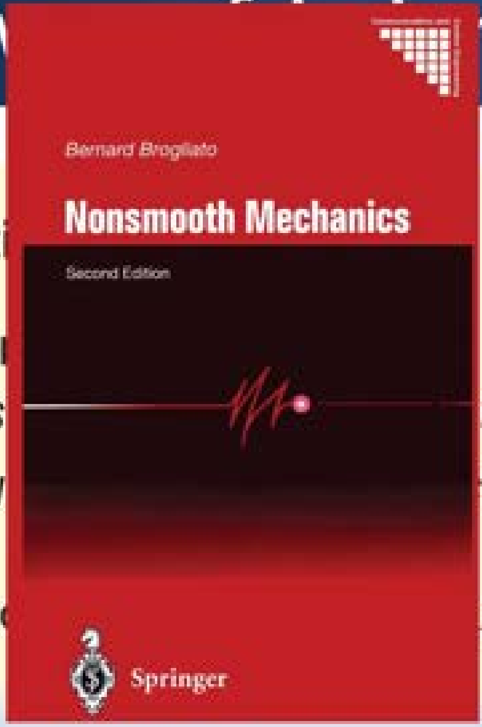
Measu ... us

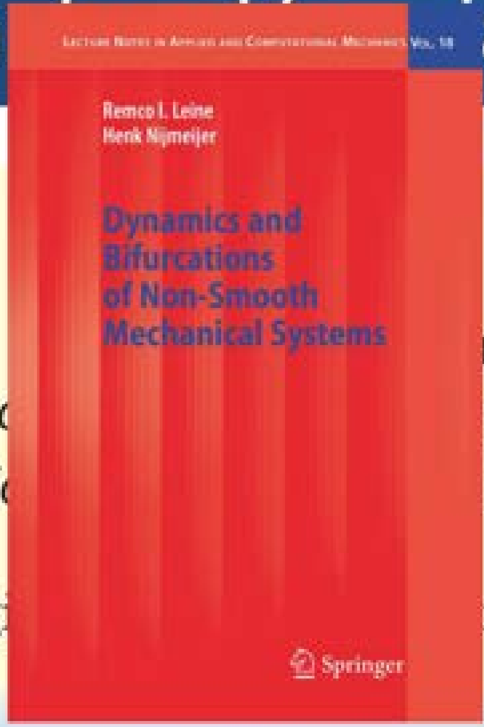
Morea ... ue ... la

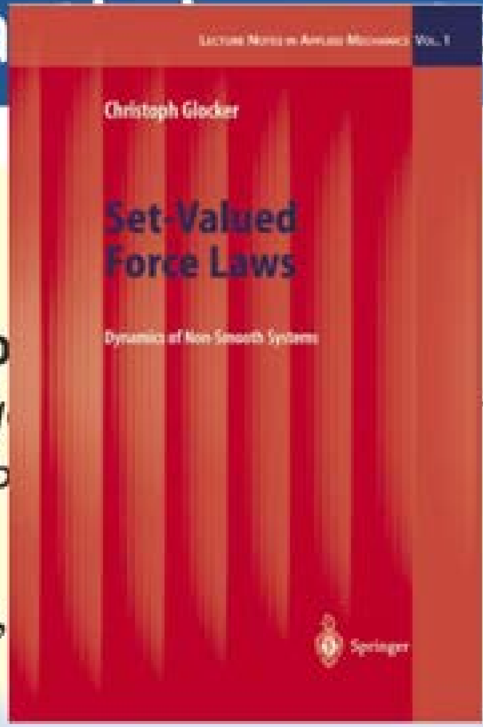
Event ... s

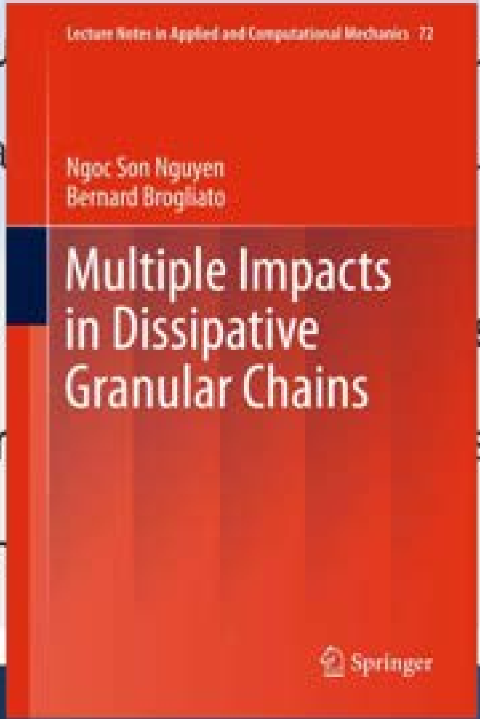
Mor ... sc

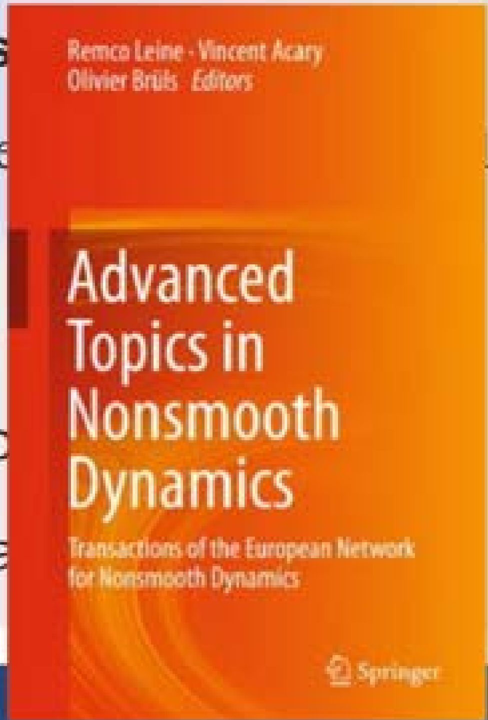
Nor ... a ... pobianco

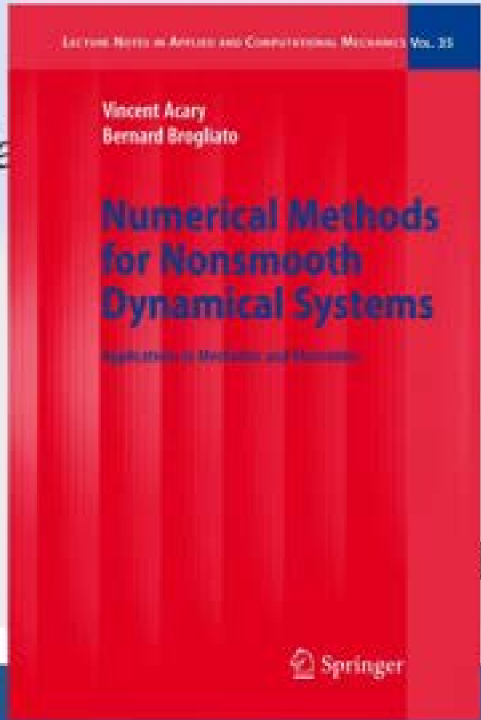












Moreau timestepping

Measure differential inclusion

$$\mathrm{d}\boldsymbol{q} = \boldsymbol{u}\mathrm{d}t$$

$$\boldsymbol{M}(\boldsymbol{q})\mathrm{d}\boldsymbol{u} - \boldsymbol{h}(\boldsymbol{q},\boldsymbol{u})\mathrm{d}t = \boldsymbol{W}_N\mathrm{d}\boldsymbol{P}_N$$

$$\xi_{Nj} = \gamma_{Nj}^{+} + e_N\gamma_{Nj}^{-}$$

$$g_{Nj}(\boldsymbol{q}) = 0: \;\; \xi_{Nj} \in N_{C_N}(-\mathrm{d}P_{Nj})$$

$$g_{Nj}(\boldsymbol{q}) > 0: \;\; \mathrm{d}P_{Nj} = 0$$

Time stepping scheme (Moreau)

$$\boldsymbol{q}_M = \boldsymbol{q}_i + \boldsymbol{u}_i\frac{\Delta t}{2}$$

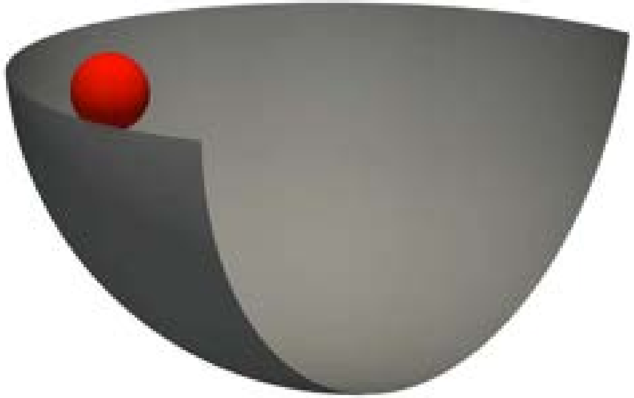
$$\boldsymbol{M}(\boldsymbol{q}_M)(\boldsymbol{u}_{i+1} - \boldsymbol{u}_i) - \boldsymbol{h}(\boldsymbol{q}_M,\boldsymbol{u}_i)\Delta t = \boldsymbol{W}_N\boldsymbol{P}_{N,i+1}$$

$$\xi_{Nj,i+1} = \gamma_{Nj,i+1} + e_N\gamma_{Nj,i}$$

$$g_{Nj}(\boldsymbol{q}_M) \leq 0: \;\; \xi_{Nj,i+1} \in N_{C_N}(-P_{Nj,i+1})$$

$$g_{Nj}(\boldsymbol{q}_M) > 0: \;\; P_{Nj,i+1} = 0$$

$$\boldsymbol{q}_{i+1} = \boldsymbol{q}_M + \boldsymbol{u}_{i+1}\frac{\Delta t}{2}$$



drift!!

contact law on velocity level!

Nonsmooth RATTLE

Stage 1

$$\boldsymbol{q}_{i+1} = \boldsymbol{q}_i + \boldsymbol{u}_{i+\frac{1}{2}}\Delta t$$

$$\boldsymbol{M}(\boldsymbol{q}_i)(\boldsymbol{u}_{i+\frac{1}{2}} - \boldsymbol{u}_i) - \boldsymbol{h}(\boldsymbol{q}_i, \boldsymbol{u}_{i+\frac{1}{2}})\frac{\Delta t}{2} = \boldsymbol{W}_N \boldsymbol{P}_{N,\mathbf{1}}$$

$$\boldsymbol{g}_N(\boldsymbol{q}_{i+1}) \in N_{C_N}(-\boldsymbol{P}_{N,\mathbf{1}})$$

contact law on position level

Stage 2

$$\boldsymbol{P}_N = \boldsymbol{P}_{N,\mathbf{1}} + \boldsymbol{P}_{N,\mathbf{2}}$$

$$\boldsymbol{M}(\boldsymbol{q}_{i+1})(\boldsymbol{u}_{i+1} - \boldsymbol{u}_{i+\frac{1}{2}}) - \boldsymbol{h}(\boldsymbol{q}_{i+1}, \boldsymbol{u}_{i+\frac{1}{2}})\frac{\Delta t}{2} = \boldsymbol{W}_N \boldsymbol{P}_{N,\mathbf{2}}$$

$$\xi_{Nj,i+1} = \gamma_{Nj,i+1} + e_N \gamma_{Nj,i}$$

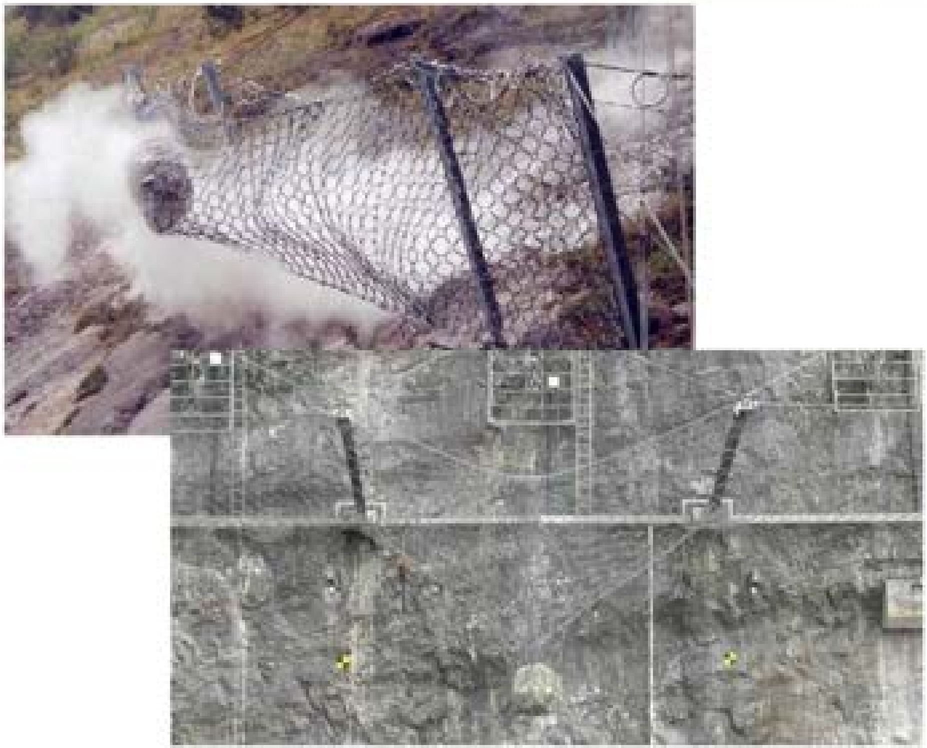
$$P_{Nj,\mathbf{1}} - r g_{Nj,i+1} \geq 0: \;\; \xi_{Nj,i+1} \in N_{C_N}(-P_{Nj})$$

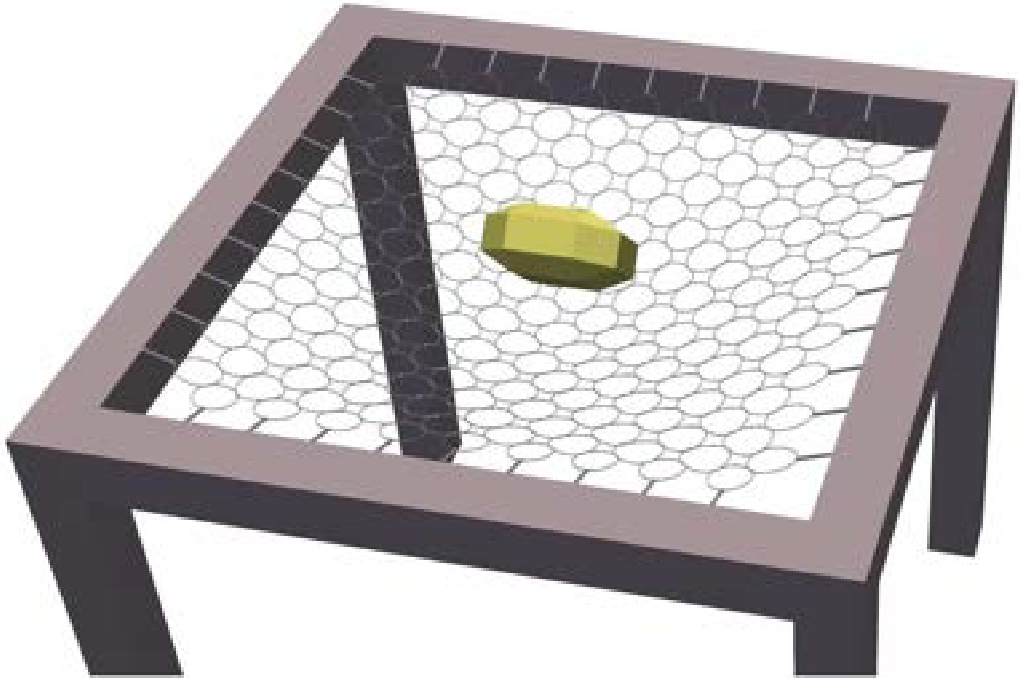
contact law on velocity level

$$P_{Nj,\mathbf{1}} - r g_{Nj,i+1} < 0: \;\; P_{Nj} = 0$$

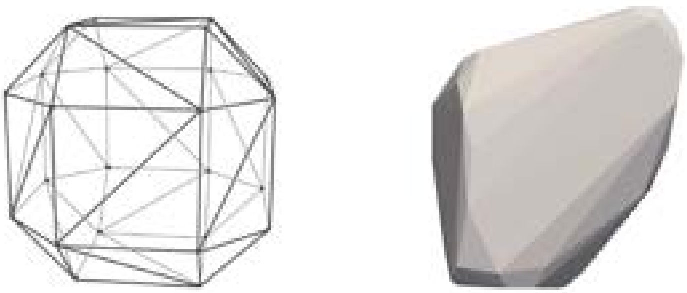
BREULING, CAPOBIANCO, EUGSTER, LEINE: A NONSMOOTH RATTLE ALGORITHM FOR MECHANICAL SYSTEMS WITH FRICTIONAL UNILATERAL CONSTRAINTS, NONLINEAR ANALYSIS: HYBRID SYSTEMS, 2024

Simulation of rockfall protection nets

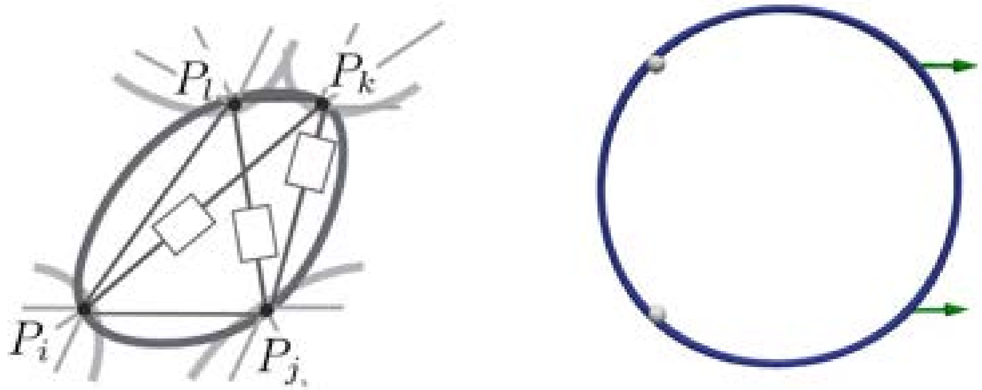




Rock model



Net model



Conclusions

- Numerical methods from nonsmooth dynamics are mature and easy to implement
- The analysis of nonsmooth systems requires the extension of Nonlinear Dynamics to nonsmooth dynamical systems:
  - stability theory
  - invariant manifolds (normal form theory, nonlinear modes)
  - bifurcation theory
  - chaos theory

MATLAB codes

With many thanks to some of my PhD students and co-workers:

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