Chapter 20
Towards Finite Element Model Updating Based on Nonlinear Normal Modes

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Abstract Local nonlinearities typically occur due to large deformation in certain parts of a structure or due to the presence of nonlinear coupling elements. Often the dynamic behavior of such elements is a priori unknown and has to be investigated experimentally before they can be included in numerical calculations. In this contribution an integrated method for estimation of linear as well as nonlinear system parameters based on the nonlinear normal modes (NNMs) of the structure is proposed. The characteristics of the nonlinear and linear parts of an assembly both contribute to its NNMs. Assuming that the functional form of the nonlinearity is known or can be estimated through non-parametric identification techniques, this feature can be exploited for the purpose of model updating. For the updating process the measured and calculated NNMs of a system are compared and their difference is minimized. In this context the numerical calculation of NNMs is performed using the Harmonic Balance Method (HBM). The properties of the proposed method are demonstrated on the numerical example of a 4DOF oscillator with a cubic nonlinearity. Furthermore, the effectiveness of the method is shown by updating the FE-model of a beam with cubic nonlinearity based on experimental data.

Keywords Nonlinear normal modes • FE-model updating • Nonlinear identification • Harmonic balance method

20.1 Introduction

In linear structures updating of FE-parameters is one of the standard tools for the adjustment of analytical models based on experimental data [1, 2]. There exist various methods and different criteria e.g. based on Frequency Response Functions or modal data. However, in reality most systems behave in a nonlinear way as for example local nonlinearities are caused by joints or geometric nonlinearities due to large displacements. Additionally, nonlinear material properties may be relevant. All of these nonlinearities can remarkably affect the systems overall behavior and therefore cause linear model updating techniques to fail. In these cases the nonlinearities must be identified and included in the analytical model.

For identification the of nonlinear systems there exist numerous techniques [3, 4], yet many of them are limited to SDOF systems or weak nonlinearity and some are difficult to relate to a clear physical meaning [5]. Others treat the linear parts of the structure separated from the nonlinear parts [6] which is on the one hand more complicated as it requires a combination of several measurements and analytical methods for the same structure and on the other hand it is sometimes difficult to isolate the effects of nonlinearities in coupled structures [7].

This contribution addresses some of these issues by using the concept of Nonlinear Normal Modes for a structural model updating. This concept was pioneered by Rosenberg [8] and provides a clear mathematical and physical framework as well as a conceptual relation to linear modes [9]. Therefore, it recently received increasing attention in the structural dynamics community [10, 11]. Rosenberg defines NNMs as synchronous periodic motion of a conservative system. This definition is restrictive in a certain sense, as it refers to conservative systems only. Still, for lightly damped systems as considered in this paper, the NNMs of the underlying conservative system are closely related to the characteristics of the damped system. It should be mentioned at this point that the numerical methods can be extended to non-conservative systems [12–14] and there are also approaches for the extension of experimental methods [15].

The idea of using the NNMs of a system to identify parameters and to update experimental models is based on the fact that NNMs contain both: information about the underlying linear system as well as information about the nonlinear behavior.

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G. Kerschen (ed.), *Nonlinear Dynamics, Volume 1*, Conference Proceedings of the Society for Experimental Mechanics Series, DOI 10.1007/978-3-319-15221-9_20

In this context, the article focuses on linear systems with local nonlinearities that may strongly affect the overall behavior of the system. Particularly, the influence of a local cubic nonlinearity is considered in the following, but the extension to any other conservative nonlinearity is straightforward.

The general strategy of the method is outlined in Sect. 20.2. In Sect. 20.3 the calculation method for the simulated NNMs which is based on the Harmonic Balance Method including higher harmonics is described in some detail. Subsequently, in Sect. 20.4, some properties of the proposed method are examined on a 4DOF lumped mass system. Afterwards, the functionality of the method is demonstrated on a real experimental setup consisting of a beam with attached cubic spring element in Sect. 20.5. The paper closes with a conclusion and some aspects of future work in Sect. 20.6.

20.2 Model Updating Methodology

In this section an integrated methodology for updating of linear and nonlinear parameters of a system is described. The method is based on the comparison of experimental and analytical backbone curves (BBCs) associated to the NNMs. Hence, the fundamental property of NNMs which will be exploited for the update, is the frequency-energy dependence or frequency-amplitude dependence respectively. Figure 20.1 schematically shows the method used for the update which consists of two main steps: one is the numerical calculation of the NNM, which is described in the next section, and the other one is the experimental extraction of the NNM's frequency-energy dependence, which will be briefly addressed in Sect. 20.5.

Based on this, the idea is to minimize the difference between experimental BBCs (BBC_{ei}) and analytical BBCs (BBC_{ai}) leading to the objective function of the updating process

$$\Delta = \sum_{i}^{l} \|(BBC_{ai} - BBC_{ei})\| \rightarrow min. \tag{20.1}$$

In this objective function l denotes the total number of NNMs considered for the update. The difference between the measured and simulated NNMs can be expressed in terms of the difference of the associated backbone curves which can be minimized in a least squares sense

$$\|BBC_{ai} - BBC_{ei}\| = \sum_{j}^{n} (BBC_{ai}(\chi_j, \boldsymbol{\alpha}) - BBC_{ei}(\chi_j))^2, \tag{20.2}$$

where χ_j are the points where the BBCs are analytically evaluated and measured respectively. For minimization of the objective function a Nelder-Mead simplex algorithm is used and generally all unknown system parameters which are included in the parameter vector $\boldsymbol{\alpha}$ can be updated simultaneously to obtain a best fit of the analytical model to the experimental results. The vector $\boldsymbol{\alpha}$ can contain linear system parameters, nonlinear system parameters or both.

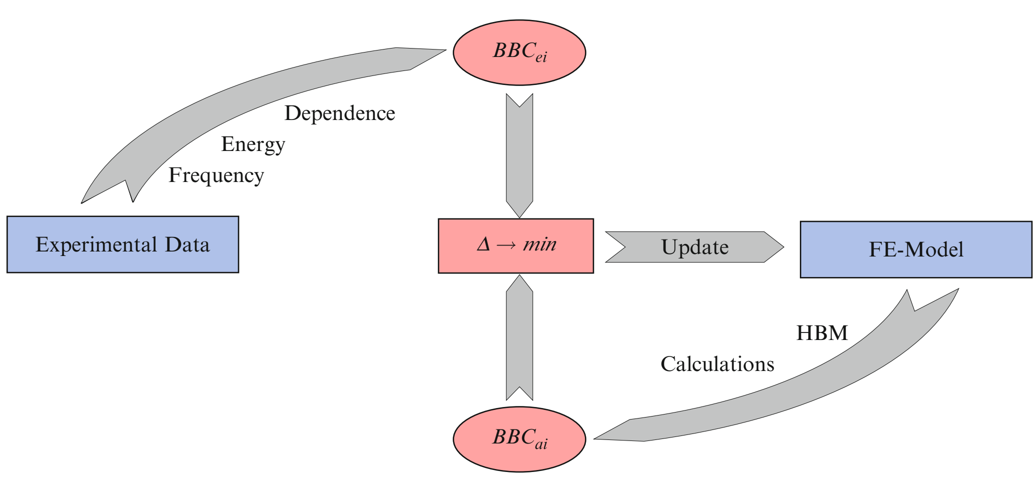

Fig. 20.1 Scheme for FE-model update based on nonlinear normal modes

20.3 Harmonic Balance Method for NNM Calculation

There are several methods for the numerical calculation of NNMs like for example shooting methods [11, 16] and also different versions of HBM [14, 17] based methods. In this paper the method of choice is based on the HBM in combination with analytical formulations for nonlinear forces due to three main reasons:

1. The high computational performance also for large scale structures.
2. The filtering property of the HBM in connection with NNM calculations.
3. The direct comparability of calculated results with experimental data.

The significance of the first point is obvious as an updating process may require numerous function evaluations to reach the best parameters. The second point will be described in more detail at the end of this section and the last point in the context of the experimental example in Sect. 20.5.

In general an autonomous, undamped mechanical system can be represented by the equation of motion

$$\boldsymbol{M}\ddot{\boldsymbol{x}} + \boldsymbol{K}\boldsymbol{x} + \boldsymbol{F}_{\mathrm{nl}}(\boldsymbol{x}, t) = \boldsymbol{0}, \tag{20.3}$$

where $\boldsymbol{M}$ denotes the mass matrix, $\boldsymbol{K}$ the linear stiffness matrix and $\boldsymbol{F}_{\mathrm{nl}}(\boldsymbol{x}, t)$ represents a vector of nonlinear, conservative forces. In this particular case of a conservative system a simplified HBM ansatz of the form

$$\boldsymbol{x}(t) = \hat{\boldsymbol{x}}_0 + \sum_{\nu=1}^{\nu_{\mathrm{h}}} \hat{\boldsymbol{x}}_\nu \cos(\nu\omega t) \tag{20.4}$$

can be made, taking into account a number of ν_{h} higher harmonics. This ansatz contrasts to the general HBM ansatz in the way that only the cosine part (or sine part) of the Fourier series has to be taken into account. This is due to the autonomous and conservative nature of the system which makes the phase arbitrary and constant. With this ansatz for the calculation of the solution for the displacement amplitudes $\hat{\boldsymbol{x}}_\nu$ in Eq. (20.4) the nonlinear forces $\boldsymbol{F}_{\mathrm{nl}}(\boldsymbol{x}, t)$ can be developed in a Fourier series dependent on the displacement $\boldsymbol{x}$ leading to

$$\boldsymbol{F}_{\mathrm{nl}}(\boldsymbol{x}, t) = \hat{\boldsymbol{F}}_{\mathrm{nl},0} + \sum_{\nu=1}^{\nu_{\mathrm{h}}} \hat{\boldsymbol{F}}_{\mathrm{nl},\nu} \cos(\nu\omega t). \tag{20.5}$$

Using these Fourier series the equation of motion can be transformed into a matrix form in the frequency domain yielding

$$\underbrace{\begin{bmatrix} \boldsymbol{K} & \boldsymbol{0} & \boldsymbol{0} \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{H}_{\mathrm{lin},1} & \boldsymbol{0} \cdots & \boldsymbol{0} \\ \vdots & \vdots & \vdots \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \cdots & \boldsymbol{H}_{\mathrm{lin},\nu_{\mathrm{h}}} \end{bmatrix}}_{\boldsymbol{H}_{\mathrm{lin}}} \underbrace{\begin{bmatrix} \hat{\boldsymbol{x}}_0 \\ \hat{\boldsymbol{x}}_1 \\ \vdots \\ \hat{\boldsymbol{x}}_{\nu_{\mathrm{h}}} \end{bmatrix}}_{\hat{\boldsymbol{x}}} + \underbrace{\begin{bmatrix} \hat{\boldsymbol{F}}_{\mathrm{nl},0}(\hat{\boldsymbol{x}}_0, \hat{\boldsymbol{x}}_1, \ldots, \hat{\boldsymbol{x}}_{\nu_{\mathrm{h}}}) \\ \hat{\boldsymbol{F}}_{\mathrm{nl},1}(\hat{\boldsymbol{x}}_0, \hat{\boldsymbol{x}}_1, \ldots, \hat{\boldsymbol{x}}_{\nu_{\mathrm{h}}}) \\ \vdots \\ \hat{\boldsymbol{F}}_{\mathrm{nl},\nu_{\mathrm{h}}}(\hat{\boldsymbol{x}}_0, \hat{\boldsymbol{x}}_1, \ldots, \hat{\boldsymbol{x}}_{\nu_{\mathrm{h}}}) \end{bmatrix}}_{\hat{\boldsymbol{f}}_{\mathrm{nl}}\left(\hat{\boldsymbol{x}},\omega\right)} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{0} \end{bmatrix}, \tag{20.6}$$

$$\text{with} \quad \boldsymbol{H}_{\mathrm{lin},\nu} = \boldsymbol{K} - (\nu\omega)^2 \boldsymbol{M}. \tag{20.7}$$

As the nonlinear forces depend on the amplitudes and this relation is not known explicitly in the frequency domain the equation has to be solved iteratively. For this purpose a Newton-like method is used along with a continuation method. The system of equations in Eq. (20.6) consists of $N + 1$ equations and $N + 2$ unknowns as the frequency is unknown as well as all entries of the displacement vector $\hat{\boldsymbol{x}}$. To circumvent the problem of an underdetermined system a normalization is performed. This normalization can be made with respect to the total energy in the system or to the amplitude of a specific DOF. In an experimental setup it is difficult to measure the total energy of the system, hence in this case the normalization will be made with respect to a chosen DOF. For the solution with the Newton method a Jacobian matrix of the system has to be calculated. This calculation is the key factor for the computational performance of the solution algorithm, but the details are beyond the scope of this paper. For a detailed description of the analytical approach for the nonlinear force and Jacobian matrix calculation applied to the cubic nonlinearity and the description of the solution method the reader is referred to [18].

As already mentioned an interesting property of the HBM in context with NNM calculation is the filtering characteristic, which means that depending on the ansatz function in Eq. (20.4) certain internal resonances can be neglected and still the global trend of the NNM is represented correctly [17]. This is shown in Fig. 20.2 through the example of the first mode of a beam with cubic nonlinearity similar to the one depicted in Fig. 20.7. It can be observed that the calculation with five harmonics shows two internal resonances: one of type 1:3 and one of type 1:5 which are both caused by modal interactions of the first mode with the second mode. In contrast, the calculation with three harmonics filters out the 1:5 internal resonance and the calculation with one harmonic also the 1:3 internal resonance. This is apparent as the solution can only contain motions governed by frequencies included in the ansatz in Eq. (20.4). However, in regions apart from the internal resonances the frequency-energy dependency as well as the mode shapes which are depicted in Fig. 20.2 on the right are captured quite well even with the rough single harmonic approximation.

This is particularly interesting, when it is a priori known which harmonics have a major effect or even lead to internal resonances and hence should be included in the calculations. This is the case in the application of model updating, when experimental data exists prior to the numerical calculation of the NNMs.

20.4 Parametric Study of an Example System

Before some practical aspects of the proposed method are demonstrated, it has to be stated that this is by no means a complete study of all relevant factors that can influence the performance of the updating process. The following investigations of a four degree of freedom (4DOF) lumped mass system with cubic nonlinearity is rather an exemplary selection of some factors which might be of interest for the subsequent experimental demonstration. The 4DOF system considered as numerical example is displayed in Fig. 20.3.

For this system a reference calculation of the NNMs is made, which will be put in place of the experimental data in Fig. 20.1. This provides the advantage that the system parameters are actually known and can be varied for testing purpose.

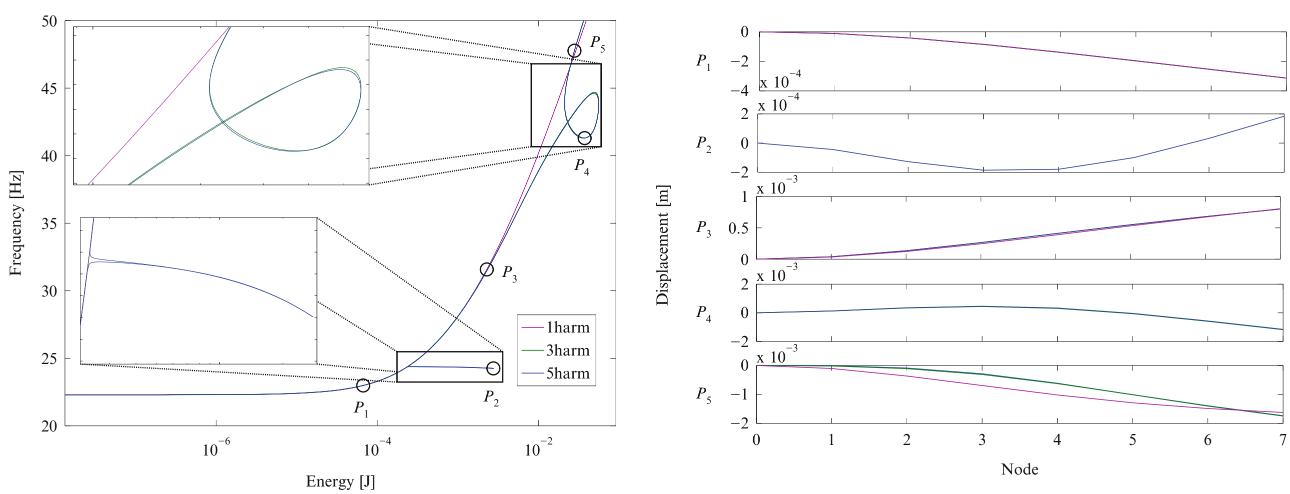

Fig. 20.2 *Left*: Filtering of HBM with different number of harmonics for the first mode of a beam with cubic nonlinearity. *Right*: Comparison of the mode shapes for the points $P1$ to $P5$ with different numbers of harmonics

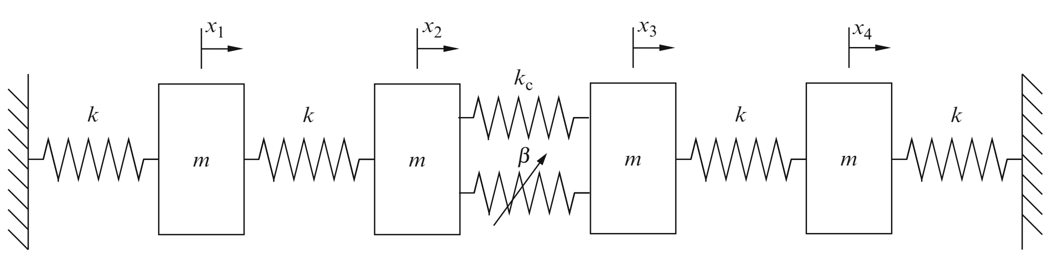

Fig. 20.3 4DOF system with cubic coupling element

Table 20.1 Parameters of reference system and parameter range for start values

Parameter	Value	Unit	Start parameter range (%)
m	1	kg	± 20
k	1	N/m	± 20
k_c	1	N/m	± 30
β	0.5	N/m^3	± 30

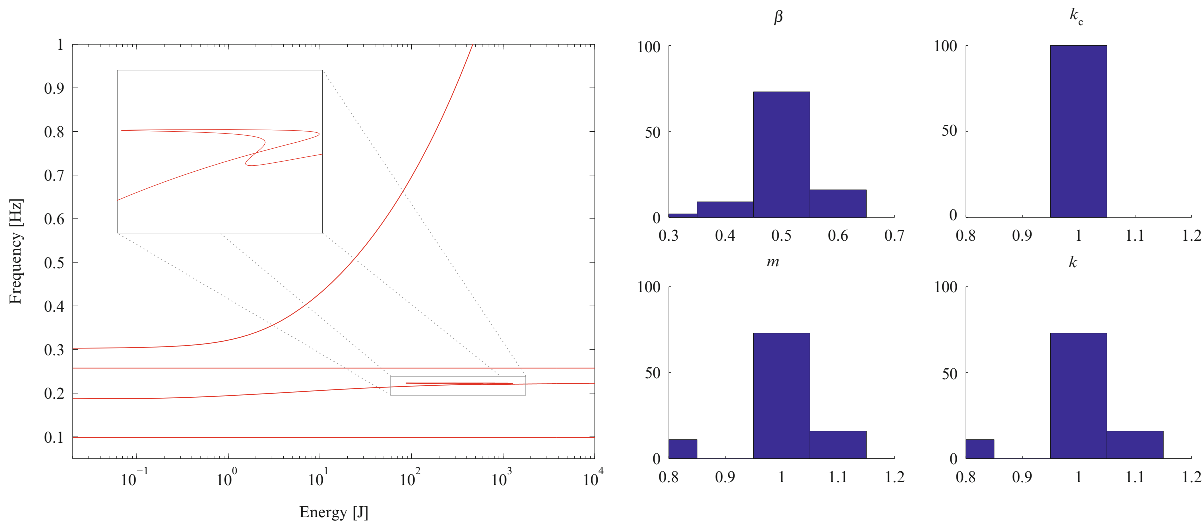

Fig. 20.4 *Left*: Frequency energy plot for reference system (three harmonics). *Right*: Results of parameter update for maximum energy $E_{max} = 10^1$ J

The parameters of the reference system are listed in Table 20.1 and the Frequency Energy Plot (FEP) is shown in Fig. 20.4. For each of the following parametric studies a number of 100 different start parameter sets is used. Based on the reference solution the start setup with randomly distributed parameters in a certain range around the actual parameters of the system is chosen and the performance of the updating process is investigated. The range for the start parameters is listed in Table 20.1.

The first factor having an influence on the performance of the algorithm is the selected energy range for the update. For the investigation of this factor only a single harmonic approximation is used to avoid internal resonances distorting the significance of the results. For very low energy the system behaves approximately linear. Therefore, it is hard to find a good parameter estimation of the nonlinear parameter β if the regarded energy range is too low. This can be seen in Fig. 20.4 (right), where the low energy range leads to problems in estimating especially the nonlinear parameter. On the other hand the probability that the updating algorithm reaches local minima and thus fails, increases with very high energy ranges, as can be observed in Fig. 20.5 (right). Furthermore, in a real experimental setup it is difficult to measure at high energy levels as the system is then subject to high loads which can cause damage. This will indeed be the limiting factor in the experimental investigation in the next section. Also the computational time for the analytic model naturally increases with increasing energy range. Hence, it is desired to select a trade-off energy range for the update, such as in Fig. 20.5 (left), which means that the amplitudes are high enough that the effect of the nonlinearity is significant but as low as possible to prevent damage of the experimental setup, avoid local minima and to increase the computational performance. In any case, for the most start parameters the updated parameters are significantly closer to the real parameters and provide a good model for further investigations.

A second point which obviously influences the updating process is the number of harmonics taken into account. One consequence of the consideration of higher harmonics is that the system in Eq. (20.6) apparently increases in size, which makes the calculations more expensive. Another effect is the occurrence of internal resonances. This feature of NNMs leads to multi-valued curves in the FEP, like in Fig. 20.4 (left), and causes problems in the updating with the basic least squares objective function. A conceivable workaround is the change of the normalization to a specific DOF or even a higher harmonic of a DOF instead of the total energy. However, a detailed study of the effect of internal resonances on the model updating is still an open question for future research. For the following experimental demonstration internal resonances do not play a role and if one stays away from the internal resonances the consideration of the higher harmonics does not have a significant effect on the performance of the updating algorithm, as can be observed in Fig. 20.6 (left).

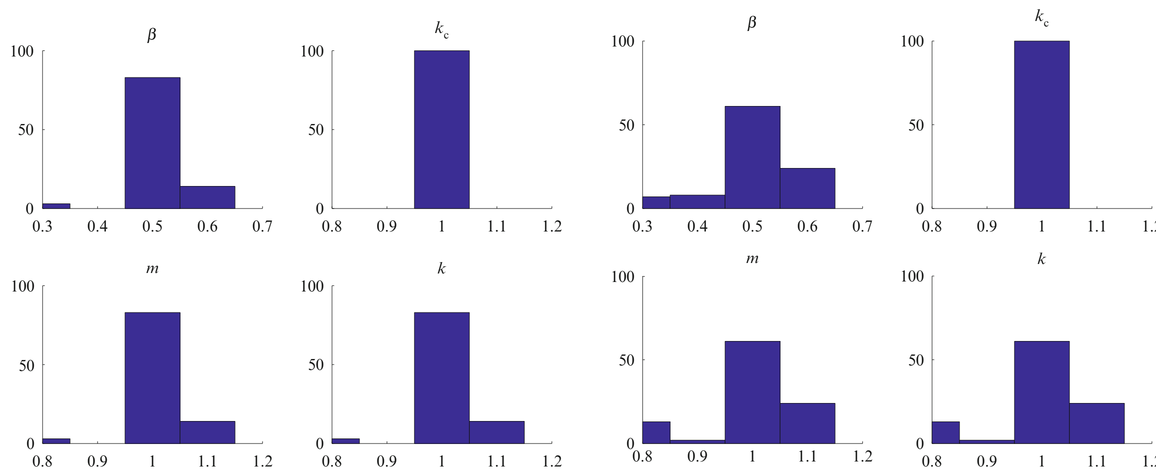

Fig. 20.5 *Left*: Results of parameter update for maximum energy $E_{max} = 10^2$ J. *Right*: Results of parameter update for maximum energy $E_{max} = 10^4$ J

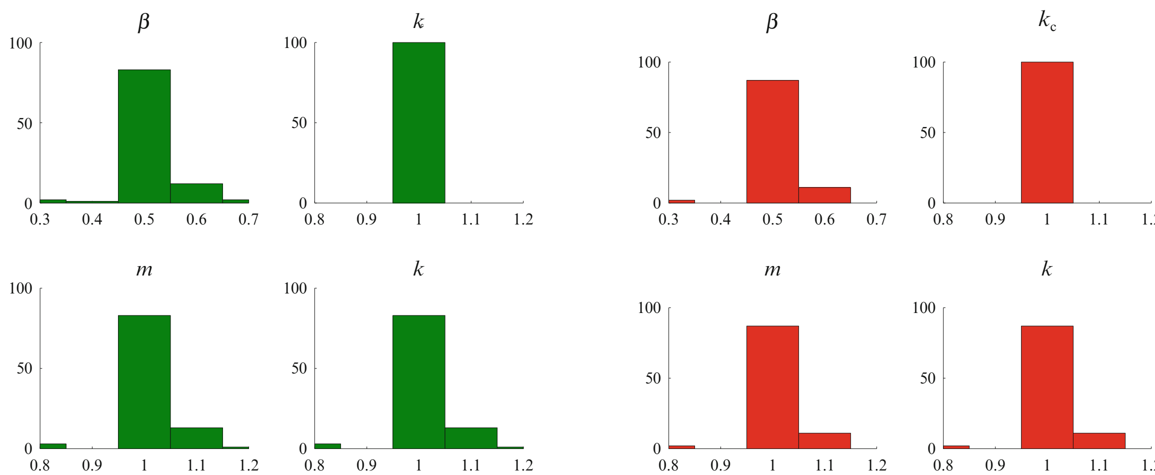

Fig. 20.6 *Left*: Results of parameter update for maximum energy $E_{max} = 1e2$ J and three harmonics. *Right*: Results of parameter update for maximum energy $E_{max} = 1e2$ J and modes 1, 2 and 4

The third aspect which will be regarded is that in an experiment it is sometimes difficult to obtain complete and accurate measurements of all NNMs. In Fig. 20.6 (right) the results for the update with only three of the NNMs of the system is run. All other settings are the same like in Fig. 20.5 (left). Comparison of both results shows that it is possible to get a similar quality of the results with an incomplete set of modes. However, one has to be careful as for example in the regarded system not all modes are affected by the nonlinearity and the choice of the wrong modes might cause the update, especially of the nonlinear parameter, to fail. So for an experimental setup one should generally have some idea about how the modes look like before choosing the NNMs considered for the updating.

In summary, the parametric study shows that even for poorly estimated start parameters the updating mostly converges to reasonable solutions and one has some freedom in choosing the amplitude range for the update, the number of harmonics and the number of modes taken into account.

20.5 Model Updating Results for Beam with Cubic Nonlinearities

In this section the proposed method is applied to a real experimental setup. The system consists of a clamped beam with a small beam at the tip and is similar to the ECL benchmark beam [19]. The small beam will encounter large displacements and therefore show cubic behavior as already reported in several publications before [20, 21]. The main beam is modeled by

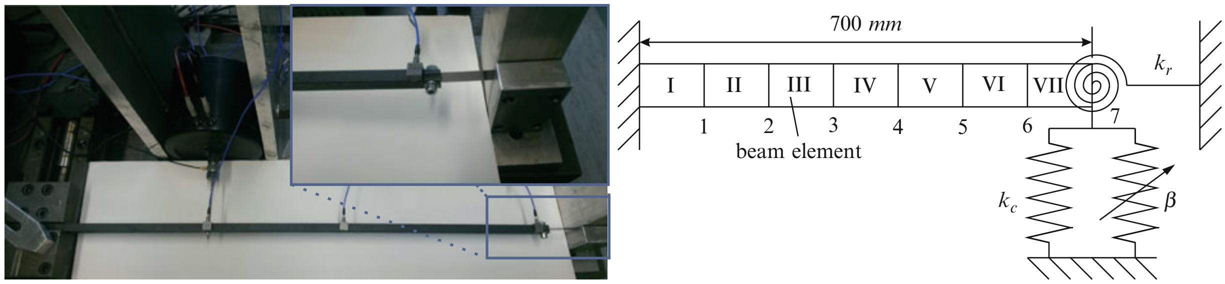

Fig. 20.7 *Left*: Photo of the experimental setup. *Right*: Schematic sketch of the numerical model

Table 20.2 Start parameters of model update for the beam structure and updated parameters

Parameter	Value before update	Value after update	Unit
E	165	186	GPa
k_c	8,000	5,525	N/m
k_r	300	217	Nm/rad
β	300×10^6	203×10^6	N/m^3

a total number of seven beam elements and 14DOFs. In the analytical model the small beam at the tip is replaced by linear translational and rotational springs with spring constants k_c and k_r respectively as well as a cubic spring with constant β. A photo of the experimental setup and a sketch of the numerical model is shown in Fig. 20.7.

The experimental extraction of the NNMs follows the approach of Peeters [20], who proposed a method which consists of three steps:

1. Tuning of the excitation to a single mode based on a phase quadrature criterion.
2. Turning off the excitation and measuring the free decay response.
3. Extracting the ridge of the Wavelet Transform (WT) of the free decay response.

With this method it is possible to extract the time-frequency dependence of a single NNM. In combination with calculation of the ridge of the WT the frequency-amplitude dependence can be reconstructed as well as the evolution of the NNM shape over amplitude with respect to the measured points. An advantage of the WT in combination with the HBM is that the WT decomposes the signal into its frequency components. Hence, the result of the WT of the time signal can be directly related to the respective harmonic coefficients of the HBM based calculations of the NNM.

For the measurements the beam is excited on node 2 with an electrodynamic shaker and tuned to a single mode. Then the excitation is switched off and the free decay is recorded. In this experimental setup the acceleration is measured with three accelerometers located at nodes $2, 4$ and 7. For the updating only the first two NNMs are taken into account as it is difficult and time consuming to accurately measure all NNMs. Still, the first two modes together with the three measured points provide a sufficient data base to start a parameter update for the parameters of the small coupling beam which will be expressed in terms of k_c, k_r, β and the Young's modulus E of the main beam.

The first five harmonics were included in the calculations as these were found to be significant in the experimental results. Yet, in the objective function only the first harmonic component was compared to the respective component of the measured results. Due to the nature of the wavelet transform yielding frequency-decomposed amplitudes for all times of the free decay it is generally possible to extract frequency-amplitude dependencies of all harmonics of the measurement. Based on this an arbitrary harmonic or even multiple harmonics can be included in the objective function for the updating. Typically the first harmonic governs the time signal and therefore is the natural choice as it provides a reasonable signal-to-noise ratio. In some cases, especially in the presence of internal resonances, it might be interesting to chose another harmonic component or multiple harmonics for the objective function. The use of the frequency-amplitude dependence of three measured points simultaneously implies an update based on frequency-amplitude dependence of each point and the relation of their amplitudes dependent on the instantaneous frequency. Thus, some additional information about the amplitude dependent NNM shape evaluated at these three points is included as well.

The parameters of an exemplary start setup before the update are listed in Table 20.2 and compared to the updated parameters. Even for such a poor start parameter estimation the updating procedure converges to a reasonable solution. For comparison the Young's modulus of the main beam was also estimated by a linear modal analysis as 185 GPa which is a deviation less than 1 % compared to the Young's modulus estimated by the nonlinear model update. Generally, the method

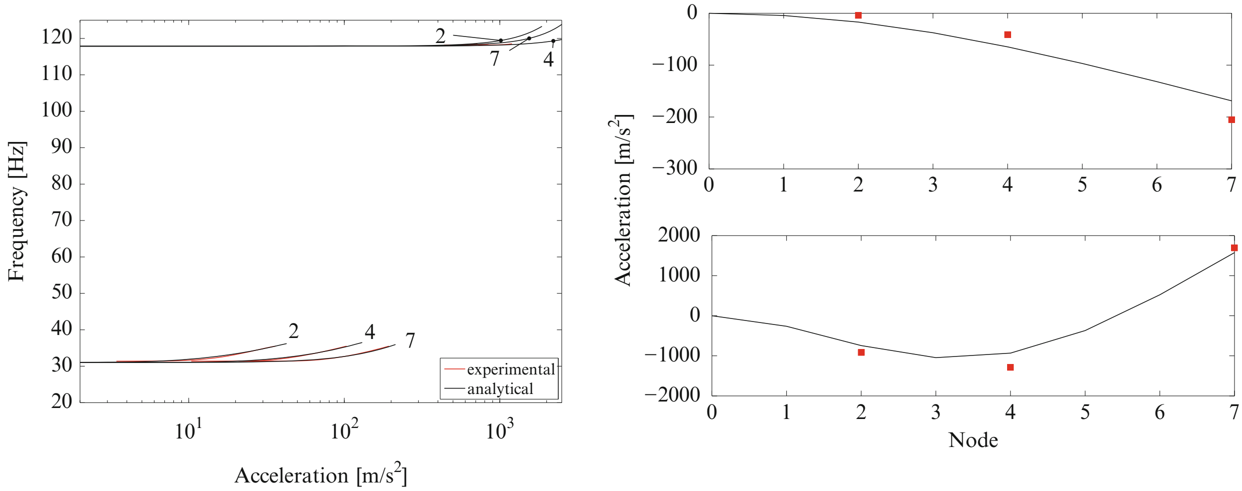

Fig. 20.8 *Left*: Comparison of first harmonic of frequency-amplitude dependence of analytical and experimental. *Right*: Comparison of the mode shapes for the measured nodes 2, 4, 7 with the analytical mode shape (*top*: mode 1 at 34.5 Hz, *bottom*: mode 2 at 118.5 Hz)

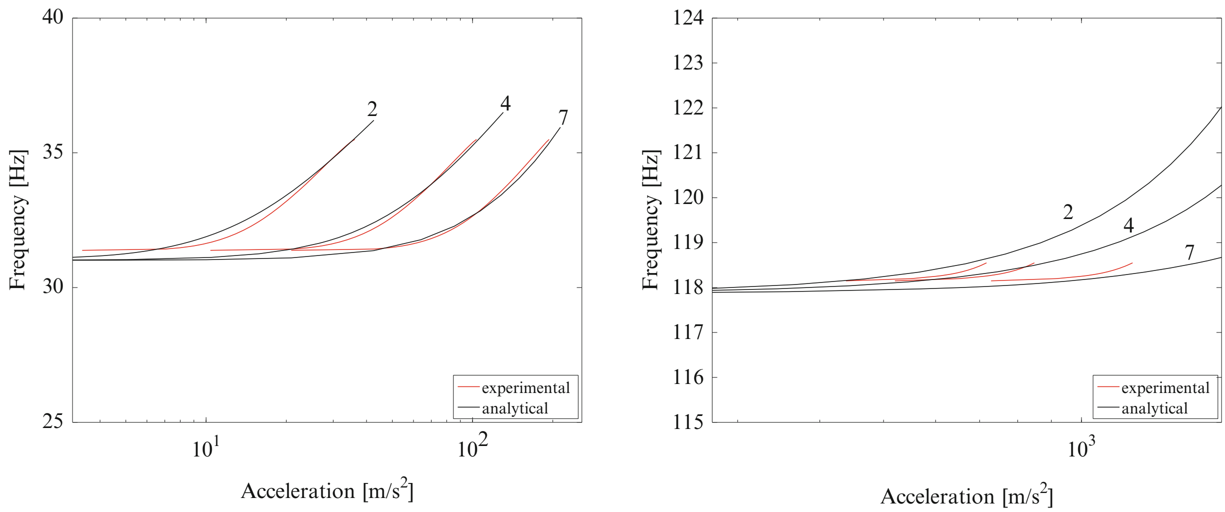

Fig. 20.9 *Left*: Zoom on the frequency-amplitude dependence curve of first NNM for points 2, 4 and 7. *Right*: Zoom on the frequency-amplitude dependence curve of second NNM for points 2, 4 and 7

seems not very sensitive on the start parameters and provides very similar updating results for different start parameters within a wide range. Also the computational time which is for the update with five harmonics around 90 s on a 3.2 GHz Intel i5 desktop computer seems acceptable.

Figure 20.8 shows a comparison of the results of the updated analytical model and the measured NNMs for the first two modes and all three measured points (nodes 2, 4, 7) on the beam. On the right the frequency-amplitude dependence is depicted for both, analytical and experimental results. Especially the first NNM shows very good agreement over a large amplitude range. For the second NNM it was difficult to obtain an accurate measurement for a wide amplitude range. However, in the range which could be measured the agreement is also satisfying. The mode shapes for the first two NNMs of the beam at a medium amplitude level, with corresponding frequencies of 34.5 Hz for the first and 118.5 Hz for the second NNM, are compared on the right of Fig. 20.8 and show good agreement as well.

Figure 20.9 shows a zoom on the frequency-amplitude dependence curves of the first (left) and the second (right) NNM for the experimental and the numerical results for the considered nodes.

20.6 Conclusion and Future Work

This paper presents a method for the integration of local nonlinearities in a structural model updating procedure based on the NNMs of the structure. With this method linear and nonlinear system parameters have been identified simultaneously. For the updating a least squares minimization of the difference between the measured BBCs and the analytical BBCs is used. The analytical calculations are made with a HBM ansatz taking into account higher harmonics. Experimentally the NNMs were extracted as the ridges of the WTs of free decay measurements in resonance. The updating algorithm was tested on a numerical test case of a lumped mass system and some features of the method were illustrated. Subsequently, the updating was successfully applied to a real experimental setup consisting of a beam structure with cubic nonlinearity. It has been shown that the application of the HBM is reasonable in this context as the computational algorithm provides fast calculations which is required for an updating with numerous function evaluations. Also the filtering of internal resonances and the direct comparability of HBM results with measured WTs of the free decay responses are interesting features of the method. For the experimental setup it turns out that a critical point is the accurate measurement of the NNMs over a broad amplitude range and for all NNMs. The free decay measurement method which was applied is also time consuming as it requires to measure each mode separately. Nevertheless, even experimental results for smaller amplitude ranges, for few measured points and an incomplete set of considered NNMs were sufficient to successfully identify parameters of the beam structure.

In future research the method will be applied to more complex structures. In this context the consideration of other nonlinearities including non-smooth and non-conservative ones will be a challenge as well as the improvement of the methods for the measurements and the updating algorithm. Another interesting point which has to be analyzed in more detail is the effect of internal resonances in the context of the model updating.

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