On the Eigensolutions of Circular Plate with Viscoelastic Filling Media using Fractional Derivatives

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Abstract: Slots with viscoelastic filling media increase the local damping of circular saw blade, which stabilizes the cutting process. In order to model the damping behaviour more accurately, a fractional viscoelastic model is introduced. Applying viscoelastic reciprocal theorem and eigenfunction expansion theorem leads to exact eigensolutions of the system with viscoelastic filling media. The method is applied to a circular plate with viscoelastic filling media. The comparison with a finite element simulation shows good agreement.

Keywords: fractional viscoelasticity, Green's function, vibration of plates, eigensolution.

0. NOMENCLATURE

$\boldsymbol{r}$: placement vector

$\boldsymbol{\theta}$: angle vector

$\boldsymbol{u}$: displacement vector

m: mass

F: external force

α: fractional order

*J(**r**):* unit step jump function

ρ: mass density

λ: Lamé first parameter

μ: shear Modulus

λ^*: complex damping modulus

μ^*: loss factor

σ_{ij}: stress tensor

ε_{ij}: strain tensor

E: Young's modulus

η: loss factor

$\overline{K}$: complex modulus

f: complex eigenvalue

$\boldsymbol{\psi}(\boldsymbol{r})$: complex eigenvector of the slotted plate

$\boldsymbol{\phi}(\boldsymbol{r})$: eigenvector of perfect plate

D_0: flexural rigidity

σ_0: Poisson's ratio

h: thickness of the plate

1. INTRODUCTION

In order to improve the surface quality of woodcuts by rotating sawblades, the steel blades are modified by including thin slots which are filled by viscoelastic media (Fig.1.). As the cutting process excites bending modes of the rotating blade, the surface quality of the cuts is deteriorated. An improved surface quality is achieved by filling optimal shaped thin slots by viscoelastic material which provides high damping in a broad frequency range when undergoing shear deformation due to the relative blade deflection within the viscoelastic layers.

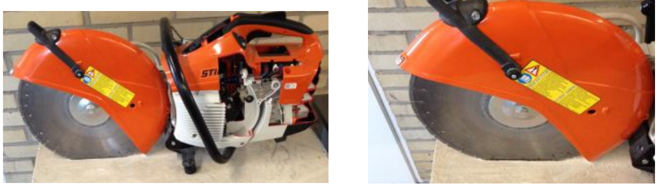

Fig. 1. a circular saw blade with slots and a cutting piece

The purpose of deliberately included slots in circular saw blades with or without filling media has been investigated by many scientists and engineers for a long time. Nishio, et al. showed by experiments and finite element simulation in a detailed way what the advantages and disadvantages the slots cause the performance of circular saw blades. Essentially, the number and the length of the slots will decrease the stiffness of the blade, which cause the decrease in the critical rotational speed. However, with a clever configuration of the slots as well as the filling media, additional local damping is introduced, so as to stabilize the cutting process. Gudmundson also studied how the cracks influence the eigenfrequencies of thin plates, proposing a method of calculating the disturbed eigenfrequencies based on the "cut-out energy". Shen developed a rigorous mathematical model based on Betti's reciprocal theorem and Green's function. In his work, an analytical formulation of the eigensolutions due to the filling media in the slots is deduced. Perturbation method is applied accordingly, in order to solve the eigensolutions. Both Shen and Nishio found out, that the configurations as well as the number of slots play a significant role in influencing the eigensolutions. Shen also showed that the Kelvin-Voigt viscoelastic model can be applied combined with the aforementioned method to model the damping properties of filling media in the slots of circular saw blade.

However, it is well known that the Kelvin-Voigt viscoelastic model indicates the linear frequency dependence of damping, which violates the physics. Viscoelastic model based on fractional derivatives provide a much weaker frequency dependence of damping. Therefore, this paper is contributed to the improvement of modelling the damping behaviour of the filling media by means of a fractional Kelvin-Voigt viscoelastic model (Gaul., Schmidt et al.).

2. THEORETICAL BASIS

2.1 Boundary Value Problem Formulation

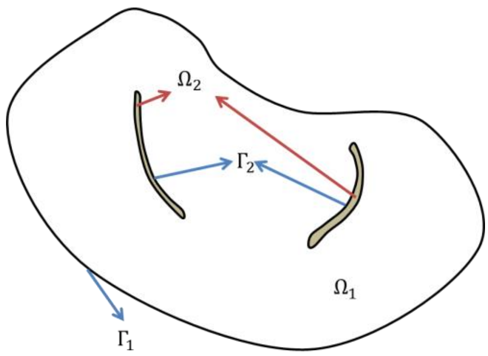

Fig.2. an elastic structure containing damping slots

As is shown in Fig. 2., Γ_1 and Γ_2 are the boundaries of elastic structure and damping slots respectively while Ω_1 and Ω_2 are the domains of these two materials.

In order to combine the filling material in the slots with the steel, we define a unit step function *J(**r**)* as follows:

$$J(\boldsymbol{r}) = \begin{cases} 1, & \boldsymbol{r} \in \Omega_{\boldsymbol{1}} \\ 0, & \boldsymbol{r} \in \Omega_{\boldsymbol{0}} \end{cases} \tag{1}$$

where $\Omega_{\boldsymbol{1}}$ is the domain of the slots and Ω_0 for the steel.

With $J(\boldsymbol{r})$, all the material parameters are functions of the field position.

$$\lambda(\boldsymbol{r}) = \lambda_0 - \lambda_1 J(\boldsymbol{r}), \tag{2a}$$

$$\mu(\boldsymbol{r}) = \mu_0 - \mu_1 J(\boldsymbol{r}), \tag{2b}$$

$$\rho(\boldsymbol{r}) = \rho_0 - \rho_1 J(\boldsymbol{r}), \tag{2c}$$

$$\lambda^*(\boldsymbol{r}) = \lambda_0^* J(\boldsymbol{r}), \tag{2d}$$

$$\mu^*(\boldsymbol{r}) = \mu_0^* J(\boldsymbol{r}), \tag{2e}$$

The field equation with example boundary conditions is:

$$\sigma_{ij,j}(\boldsymbol{u}(\boldsymbol{r},t); \lambda, \mu) = \rho(\boldsymbol{r}) \ddot{u}_i(\boldsymbol{r},t) \tag{3a}$$

The Dirichlet boundary condition is:

$$\boldsymbol{u}(\boldsymbol{r}) = \boldsymbol{0} \ \text{ on } \Gamma_1, \quad u_1 = u_2 \ \text{ on } \ \Gamma_2 \tag{3b}$$

which means there is no relative displacement between the steel and the damping media.

The Neumann boundary condition describes a traction free boundary on Γ_1 :

$$\sigma_{ij,j}(\boldsymbol{u}(\boldsymbol{r},t); \lambda, \mu) n_j = 0 \ \text{ on } \Gamma_1 \tag{3c}$$

2.2 Fractional Viscoelastic Formulation

The general definition of a fractional derivative by Riemann-Liouville is:

$$D^\alpha\big(\boldsymbol{u}(t)\big) = \frac{d^\alpha \boldsymbol{u}(t)}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{\boldsymbol{u}(\tau)}{(t-\tau)^\alpha} d\tau\,, \tag{4}$$

where the fractional order α satisfies $0 < \alpha < 1$ and the Euler-Gamma function Γ is defined as

$$\Gamma(1-\alpha) = \int_0^\infty x^{-\alpha} e^{-x} dx\,. \tag{5}$$

As the modal analysis is based on a harmonic Ansatz, we can transform the displacement field $\boldsymbol{u}(t)$ from the time domain to the frequency domain by means of the Fourier transform which is in the case of fractional derivative order.

$$D^\alpha\big(\boldsymbol{u}(t)\big) = (i\omega)^\alpha U e^{i\omega t} \tag{6}$$

The constitutive equation of a fractional Kelvin-Voigt model is then:

$$\sigma_{ij}(\boldsymbol{u};\lambda,\mu,\lambda^*,\mu^*,\alpha)=\lambda\delta_{ij}\varepsilon_{kk}(\boldsymbol{u})+2\mu\varepsilon_{ij}(\boldsymbol{u})$$

$$+\lambda^*\delta_{ij}\frac{d^\alpha\varepsilon_{kk}(\boldsymbol{u})}{dt^\alpha}+2\mu^*\frac{d^\alpha\varepsilon_{ij}(\boldsymbol{u})}{dt^\alpha} \quad (7)$$

Assuming the system is geometrically linear, the strain and displacement field has the following relationship:

$$\varepsilon_{ij}(\boldsymbol{u})=\tfrac{1}{2}(u_{i,j}+u_{j,i}). \quad (8)$$

$$\frac{d^\alpha\varepsilon_{ij}(\boldsymbol{u})}{dt^\alpha}=\frac{1}{2}\left[\frac{d^\alpha u_{i,j}(\boldsymbol{u})}{dt^\alpha}+\frac{d^\alpha u_{j,i}(\boldsymbol{u})}{dt^\alpha}\right]. \quad (8)$$

Inserting (8) and (9) into (7), we have the constitutive equation in frequency domain:

$$\sigma_{ij}(\boldsymbol{u},f;\lambda,\mu,\lambda^*,\mu^*,\alpha)=\lambda\delta_{ij}\varepsilon_{kk}(\boldsymbol{u})+2\mu\varepsilon_{ij}(\boldsymbol{u})$$

$$+f^\alpha\left[\lambda^*\delta_{ij}\varepsilon_{kk}(\boldsymbol{u})+2\mu^*\varepsilon_{ij}(\boldsymbol{u})\right], \quad (10)$$

The complex modulus, which contains loss factor η is:

$$\bar{K}=E[1+(i\eta\omega)^\alpha]. \quad (11)$$

According to the following viscoelastic reciprocal theorem (Christensen),
an isotropic viscoelastic body, when subjected to two different states of loading with corresponding body forces, surface stresses, and displacements, $F_i,\ \sigma_i, u_i, F_i', \sigma_i', u_i'$, *respectively, has a field variable solution which satisfies the relationship*

$$\int_B [\boldsymbol{\sigma_i}*\boldsymbol{du_i'}]\boldsymbol{da}+\int_V [\boldsymbol{F_i}*\boldsymbol{du_i'}]\boldsymbol{dv}$$
$$=\int_B [\boldsymbol{\sigma_i'}*\boldsymbol{du_i}]\boldsymbol{da}+\int_V [\boldsymbol{F_i'}*\boldsymbol{du_i}]\boldsymbol{dv}. \quad (12)$$

Together with the divergence theorem, we derive the following equation in polar coordinate:

$$\int_\Omega \sigma_{ij}(\boldsymbol{u},f;\lambda,\mu,\lambda^*,\mu^*,\alpha)\varepsilon_{ij}(\boldsymbol{u}')dr^2$$
$$=\int_\Omega \sigma_{ij}(\boldsymbol{u}',f;\lambda,\mu,\lambda^*,\mu^*,\alpha)\varepsilon_{ij}(\boldsymbol{u})dr^2$$
$$=\int_\Omega I(\boldsymbol{u},\boldsymbol{u}';\lambda,\mu)\ dr^2+f^\alpha\int_\Omega I(\boldsymbol{u},\boldsymbol{u}';\lambda^*,\mu^*)\ dr^2$$

where

$$I(\boldsymbol{u},\boldsymbol{u}';\lambda,\mu)=\lambda\varepsilon_{kk}(\boldsymbol{u})\varepsilon_{kk}(\boldsymbol{u}')+2\mu\varepsilon_{ij}(\boldsymbol{u})\varepsilon_{ij}(\boldsymbol{u}') \quad (13)$$

2.3 Exact Eigensolutions

The complex-valued Green's function under a concentrated force $\delta(\boldsymbol{r}-\boldsymbol{r}_0)e^{ft}$ within the unslotted system satisfies also the following equation:

$$\frac{d}{dx_j}\left[\sigma_{ij}\left(\boldsymbol{G^k}(\boldsymbol{r}|\boldsymbol{r}_0),f;\lambda_0,\mu_0,0,0\right)\right]=f^2\rho_0(\boldsymbol{r})\boldsymbol{G_i^k}(\boldsymbol{r}|\boldsymbol{r}_0)$$
$$+\delta_{ik}\delta(\boldsymbol{r}-\boldsymbol{r}_0) \quad (14)$$

Combining (12), (13), (14), one can build the following integral equation.

$$\boldsymbol{\psi}^k(\boldsymbol{r}_0)=f^2\int_\Omega \rho_1J(\boldsymbol{r})\boldsymbol{\psi}(\boldsymbol{r})\cdot\boldsymbol{G^k}(\boldsymbol{r}|\boldsymbol{r}_0)dr^2$$
$$-f^\alpha\int_\Omega \boldsymbol{I}(\boldsymbol{G^k}(\boldsymbol{r}|\boldsymbol{r}_0),\boldsymbol{\psi}(\boldsymbol{r});\lambda_0^*J(\boldsymbol{r}),\mu_0^*J(\boldsymbol{r}))dr^2$$
$$+\int_\Omega \boldsymbol{I}(\boldsymbol{G^k}(\boldsymbol{r}|\boldsymbol{r}_0),\boldsymbol{\psi}(\boldsymbol{r});\lambda_1J(\boldsymbol{r}),\mu_1J(\boldsymbol{r}))dr^2,$$
$$k=1,2,3 \quad (15)$$

By means of an orthonormal eigenfunction expansion theorem, the exact eigensolutions can be derived as follows for the example boundary conditions:

$$\boldsymbol{\psi}(\boldsymbol{r})=\boldsymbol{\phi_n}(\boldsymbol{r})-\sum_{\substack{m=1\\m\neq n}}^{\infty}\frac{U\left(\psi,\phi_{m;}f\right)}{f^2+\omega_m^2}\boldsymbol{\phi_m}(\boldsymbol{r}). \quad (16)$$

$$f^2=-\omega_m^2-U\left(\boldsymbol{\psi},\boldsymbol{\phi_{m;}}f\right). \quad (17)$$

3. APPLICATION TO CIRCULAR PLATES

Consider the transverse vibration of a thin slotted plate, we have found from experiment that geometrical nonlinearity is negligible, and therefore it suffices to use the Kirchhoff plate model. The eigenvalue problem is formulated in (18), $\nabla^4$is a biharmonic operator, which can be decomposed into two second-order equations (Szabo).

$$\nabla^4\boldsymbol{\phi}(\boldsymbol{r},\boldsymbol{\theta})-\beta^4\boldsymbol{\phi}(\boldsymbol{r},\boldsymbol{\theta})=0,\ \beta=\frac{\omega^2m}{D_0} \quad (18)$$

The corresponding eigenfunction is a linear combination of Bessel functions (Meirovitch).The eigenvalues $\omega_{m,\pm n}$ and eigenfunctions can be classified by the number of the nodal diameters m and nodal circles n. The eigenfunctions must satisfy the orthonormal conditions.

$$\int_A \rho_0h\Phi_{mn}(\boldsymbol{r})\,\Phi_{pq}(\boldsymbol{r})dA=\delta_{mp}\delta_{nq}$$
$$m,p=0,1,2,\ldots,\infty$$
$$n,q=\pm0,\pm1,\pm2,\ldots,\pm\infty. \quad (19)$$

$$\int_A I(\Phi_{mn}(r),\Phi_{pq}(r);D_0,\sigma_0)\,dA=\omega_{mn}^2\delta_{mp}\delta_{nq}$$
$$m,p=0,1,2,\ldots,\infty$$
$$n,q=\pm0,\pm1,\pm2,\ldots,\pm\infty. \quad (20)$$

with the bilinear operator

$$
\begin{aligned}
&I(u, v; D_0, \sigma_0) \\
&= D_0 \Bigg[\nabla^2 u \nabla^2 v \\
&+ 2(1-\nu_0) \left\{ \left(\frac{1}{r}\frac{\partial^2 u}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial u}{\partial \theta} \right) \left(\frac{1}{r}\frac{\partial^2 v}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial v}{\partial \theta} \right) \right. \\
&\left. - \frac{1}{2}\left[\left(\frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} \right)\frac{\partial^2 v}{\partial r^2} + \left(\frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} \right)\frac{\partial^2 u}{\partial r^2} \right] \right\} \Bigg]
\end{aligned}
\tag{21}
$$

The elastic eigenfunctions corresponding to each eigenmode are computed and plotted from Fig. 3. to Fig. 6.

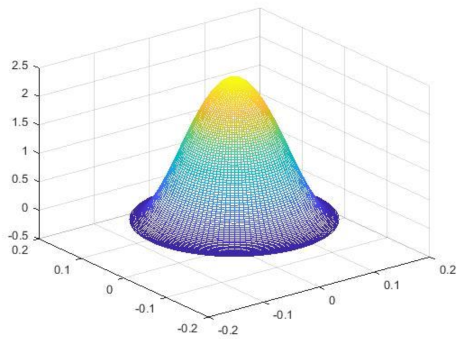

Fig. 3. elastic eigenfunction Φ_{01}

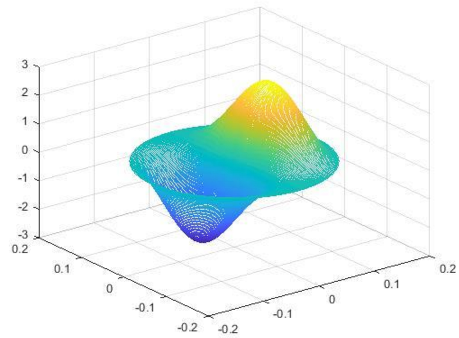

Fig. 4. elastic eigenfunction Φ_{11}

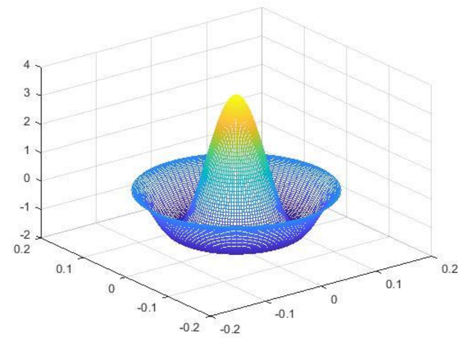

Fig. 5. elastic eigenfunction Φ_{02}

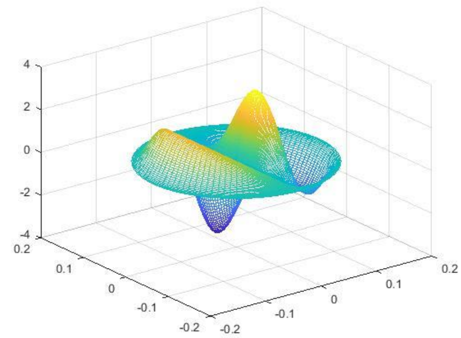

Fig. 6. elastic eigenfunction Φ_{12}

Inserting the eigenvalues and eigenfunctions of unslotted circular plates into the formulae (15), (16) and considering the geometry of the slots as the perturbation parameter, we can build up the following first-order perturbation solution in order to calculate the eigenvalues and eigenvectors of the circular plates with viscoelastic filling media.

$$
\boldsymbol{\psi_n}(\boldsymbol{r}) = \boldsymbol{\phi_n}(\boldsymbol{r}) - \sum_{\substack{m=1 \\ m \neq n}}^{\infty} \frac{U(\psi, \phi_{m;} f_n)}{f_n^2 + \omega_m^2} \boldsymbol{\phi_m}(\boldsymbol{r}) + O(\Omega_c^2)
\tag{21}
$$

$$
f_n^2 = -\omega_m^2 - U(\boldsymbol{\psi}, \boldsymbol{\phi_{m;}} f) + O(\Omega_c^2)
\tag{22}
$$

4. VALIDATION BY FINITE ELEMENT SIMULATION

A numerical modal analysis is carried out based on finite element method in order to validate the mathematical model.

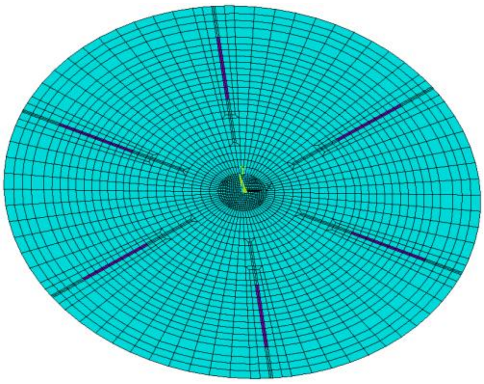

Fig. 7. FE model of the circular plate with viscoelasic media filled in rectangular slots

As is shown in Fig. 7., a circular plate containing six equally distributed rectangular slots filled by viscoelastic media is built using ANSYS v18. The radius is 0.15m and the thickness of the plate is 0.0016m. The length of each slot is 0.05m whereas the width is 0.002m. The slots start from the inner radius of 0.075m from the plate centre till the outer radius of 0.125m from the plate centre. The shell element 181 using linear shape function is applied in the calculation.

Table 1. Material parameters of the FE model

Material	Mass density (kg/m^3)	Young's modulus (kg/m^2)	Poisson's ratio	Loss factor	Fractional order
steel	$7.85e^3$	2.1e11	0.3		
Viscoelastic media	$3.9e^3$	1.05e11	0.3	0.05	0.4

In the mathematical model, modal damping is calculated for each mode with the input of the fractional order regarding the viscoelastic material property. Such modal dampings are used as the inputs for the FE model, so that we can compare the eigenfrequencies among two methods and unslotted Kirchhoff plate model.

Table 2. Eigenfrequencies of the mathematical model, FE model and the unslotted Kirchhoff plate model

Mode	FE Model (Hz)	Mathematical model (Hz)	Unslotted Kirchhoff plate model(Hz)
(0,1)	180.88	181.86	181.55
(1,1)	377.41	378.44	377.81
(0,2)	706.76	701.21	706.77
(1,2)	1084.5	1079.3	1081

The relative differences between two models and their difference with the Kirchhoff plate model are below 1%, which shows good agreement.

5. CONCLUSIONS AND DISCUSSION

It has been proven that fractional viscoelastic model has the advantages over those classical models, such as Kelvin-Voigt viscoelastic model, Maxwell model and three parameter model. In the Kelvin-Voigt model, as the frequency increases, the dependence of damping with respect to frequence is linear, which violates the physics. By means of fractional viscoelastic model, such a problem can be remedied by a weaker frequency dependency of damping.

Moreover, with possibly fewer parameters, fractional viscoelastic model shows a better fitting to the experiment, which is another advantage.

In this presented paper, the fractional viscoelastic model is applied to model the damping property due to the filling media in the slots of circular saw blades. It is shown that fractional viscoelastic model can be applied to model such a coupling problem.

Although the fractional viscoelastic model is easy to implement in the frequency domain, the complexity of the presented mathematical model itself restrict the application of a more generalized fractional viscoelastic model.

It would be of great interest if a new and simpler mathematical model, which combines the primary material and the filling media, can be developed. And therefore, finer fractional viscoelastic model can be put into practice.

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