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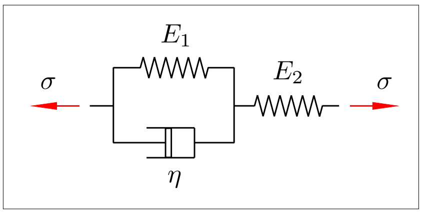
Parameter Identification and FE Implementation of a Viscoelastic Constitutive Equation Using Fractional Derivatives

Damping in viscoelastic materials can be described in several ways. In FE codes for transient calculations with direct integration usually Rayleigh-damping is provided. However, it is known that this model is not qualified to represent the damping properties of viscoelastic material over a broad range of time or frequency. Another approach uses fractional time derivatives of stresses and strains in the constitutive equations. This model requires few parameters, provides good curve fitting properties and is physically proved. In this paper a parameter identification for the fractional 3-parameter model will be carried out and its implementation into an FE code will be demonstrated.

1. Fractional 3-Parameter Model

The one-dimensional constitutive equation of the shown 3-parameter model

$$\sigma + \frac{\eta}{E_1+E_2}\,\dot{\sigma} \;=\; \frac{E_1E_2}{E_1+E_2}\,\epsilon + \eta\,\frac{E_2}{E_1+E_2}\,\dot{\epsilon} \tag{1}$$



can be generalized by fractional derivatives that can be considered as an extension of derivatives of integer order to derivatives of non-integer order (see e.g. [1]):

$$\sigma + \alpha\left(\frac{\mathrm{d}^q}{\mathrm{d}t^q}\,\sigma\right) \;=\; E\,\epsilon + \beta\left(\frac{\mathrm{d}^p}{\mathrm{d}t^p}\,\epsilon\right) \quad . \tag{2}$$

The Grünwald definition of fractional derivatives

$$\frac{\mathrm{d}^q f(t)}{\mathrm{d}t^q} = \lim_{N\to\infty}\left[\left(\frac{t}{N}\right)^{-q}\sum_{j=0}^{N-1}\frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)}\,f(t-j\,\tfrac{t}{N})\right] \quad , \qquad q \in \mathbb{R} \tag{3}$$

contains the gamma function Γ (see e.g.[2]). The Grünwald definition can be approximated by

$$\frac{\mathrm{d}^q f(t)}{\mathrm{d}t^q} \approx \left(\frac{t}{N}\right)^{-q}\sum_{j=0}^{N-1} A_{j+1}\,f(t-j\,\tfrac{t}{N}) \qquad \text{with} \qquad A_{j+1} = \frac{\Gamma(j-q)}{\Gamma(-q)\Gamma(j+1)} \tag{4}$$

as the Grünwald-coefficients.

2. Parameter Identification

A parameter identification in the time domain for the creep behavior of a viscoelastic material can be obtained from measured data by the least square fit method

$$\sum_{i=1}^{n}\left(C_{\mathrm{calc}}(t_i) - C_{\mathrm{meas}}(t_i)\right)^2 \overset{!}{=} \min. \quad , \qquad C(t) \;=\; \frac{\sigma}{\epsilon(t)} \tag{5}$$

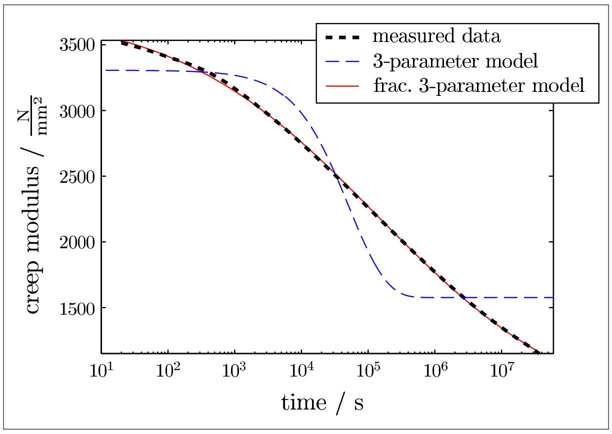
where C_{calc} and C_{meas} are the calculated and measured creep modulus, respectively. Since no general analytical solution for the strains $\epsilon(t)$ in Eq. (2) exists, the calculation of C_{calc} is carried out by a time step algorithm of the fractional differential equation (2). Inserting (4) into (2) and solving for $\epsilon(t)$ leads to

$$\epsilon(t) \;=\; \left[E + \beta\left(\frac{t}{N}\right)^{-p}\right]^{-1}\left[\sigma(t) + \alpha\left(\frac{t}{N}\right)^{-q}\sum_{j=0}^{N-1} A^{(q)}_{j+1}\sigma(t-j\,\tfrac{t}{N}) - \beta\left(\frac{t}{N}\right)^{-p}\sum_{j=1}^{N-1} A^{(p)}_{j+1}\epsilon(t-j\,\tfrac{t}{N})\right] \tag{6}$$

such that $\sigma(t)$ is a step function in time. The solution of (6) provides the required sampling points $C_{\mathrm{calc}}(t_i)$ for one set of parameters. An optimization that includes the repeated solution of (6) will then identify the material parameters.

For the polymer Delrin (Du Pont) the material parameters are identified using a time step algorithm that permits variable time steps. The optimization is performed by the MATLAB Optimization Toolbox with the additional restriction $p = q$. The identified parameters are as follows:

Parameter	Value	Dimension
E	$658,2$	$\frac{\mathrm{N}}{\mathrm{mm}^2}$
$p=q$	$0,2845$	–
α	$32,017$	s^q
β	$120\,593,0$	$\frac{\mathrm{N}}{\mathrm{mm}^2}\,\mathrm{s}^p$



3. Implementation into FE Codes

Starting point for a 3D implementation is the equation of motion in FE-formulation at time t (see [3]):

$$\int_V \underline{\underline{B}}^T\,{}^t\underline{\sigma}\,\mathrm{d}V + \underline{\underline{M}}\,{}^t\underline{\ddot{u}} = {}^t\underline{r} \tag{7}$$

The stress state ${}^t\underline{\sigma}$ is obtained from the constitutive equation (2) using (4) by resolving:

$${}^t\underline{\sigma} = \left[\underline{\underline{I}} + \underline{\underline{\alpha}}\left(\frac{t}{N}\right)^{-q}\right]^{-1}\left[\left(\underline{\underline{C}}+\underline{\underline{\beta}}\left(\frac{t}{N}\right)^{-p}\right)\underline{\underline{B}}\,{}^t\underline{u} + \underline{\underline{\beta}}\left(\frac{t}{N}\right)^{-p}\sum_{j=1}^{N-1}A^{(p)}_{j+1}\,\underline{\underline{B}}\,{}^{t-j\frac{t}{N}}\underline{u} - \underline{\underline{\alpha}}\left(\frac{t}{N}\right)^{-q}\sum_{j=1}^{N-1}A^{(q)}_{j+1}\,{}^{t-j\frac{t}{N}}\underline{\sigma}\right], \tag{8}$$

where $\underline{\underline{C}}$, $\underline{\underline{\alpha}}$ and $\underline{\underline{\beta}}$ are material dependent matrices. Inserting Equation (8) into (7) leads to

$$\underline{\underline{K}}^*\,{}^t\underline{u} + \underline{\underline{M}}\,{}^t\underline{\ddot{u}} = {}^t\underline{r} - \underline{\underline{\mu}}\left(\frac{t}{N}\right)^{-p}\sum_{j=1}^{N-1}A^{(p)}_{j+1}\,{}^{t-j\frac{t}{N}}\underline{u} + \int_V \underline{\underline{B}}^T\underline{\underline{A}}^{-1}\underline{\underline{\alpha}}\left(\frac{t}{N}\right)^{-q}\sum_{j=1}^{N-1}A^{(q)}_{j+1}\,{}^{t-j\frac{t}{N}}\underline{\sigma}\,\mathrm{d}V \tag{9}$$

containing the abbreviations

$$\underline{\underline{K}}^* = \int_V \underline{\underline{B}}^T\left[\underline{\underline{I}}+\underline{\underline{\alpha}}\left(\frac{t}{N}\right)^{-q}\right]^{-1}\left[\underline{\underline{C}}+\underline{\underline{\beta}}\right]\underline{\underline{B}}\,\mathrm{d}V \quad\text{and}\quad \underline{\underline{\mu}} = \int_V \underline{\underline{B}}^T\left[\underline{\underline{I}}+\underline{\underline{\alpha}}\left(\frac{t}{N}\right)^{-q}\right]^{-1}\underline{\underline{\beta}}\,\underline{\underline{B}}\,\mathrm{d}V \tag{10}$$

that can be calculated for each element. The structure of Eq. (9) is identical to the ordinary discretized equation of motion without damping. Therefore it can be implemented into any elastic FE-code, whereby the modified stiffness matrix $\underline{\underline{K}}^*$ has to be used instead of the stiffness $\underline{\underline{K}}$ and the external load vector $\underline{r}$ has to be modified by the additional two terms at the right hand side of Eq. (9). These terms only depend on the strain and stress history, thus they can be calculated at the beginning of each time step.

In case of a time step change the matrices $\underline{\underline{K}}^*$ and $\underline{\underline{\mu}}$ must be recalculated. In addition, the strain and stress histories that are needed in Eq. (9) at equidistant times, have to be interpolated from the known values of the past.

4. References

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