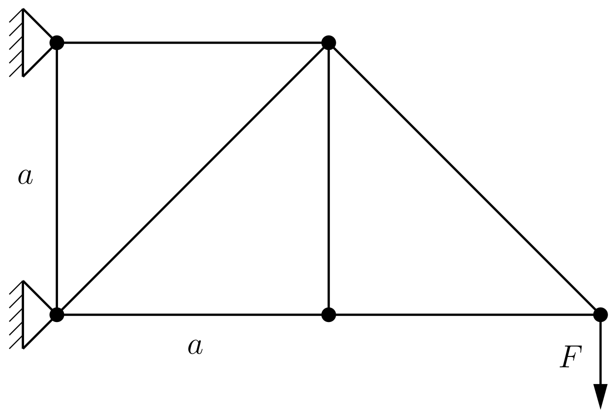
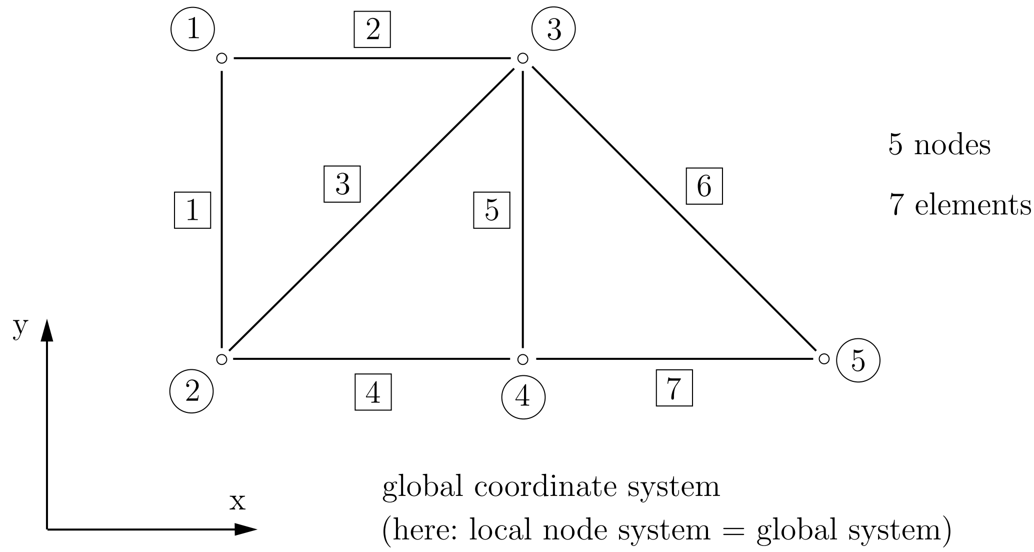
Static calculation of a planar truss structure:



Six steps:

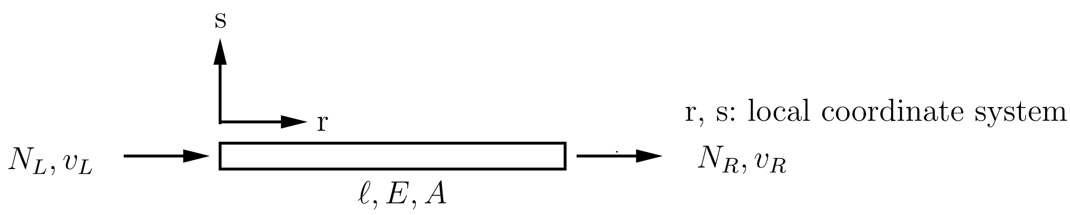
1. Discretization
2. Element matrices
3. Transformation
4. Assembly
5. Boundary conditions
6. Solution

1. step: Discretization/Decomposition into nodes and element



2. step: Element matrices

rod element (truss structure: only tensile and compressive loads, no bending moments)



- matrix of element displacements (nodal displacements)

$$\underline{\tilde{v}} = \begin{bmatrix} v_L \\ v_R \end{bmatrix}$$

- matrix of nodal forces

$$\underline{\tilde{S}} = \begin{bmatrix} N_L \\ N_R \end{bmatrix}$$

- element stiffness matrix (see Exercis # 4)

$$\underline{\tilde{S}} = \underline{\tilde{K}}\,\underline{\tilde{v}}$$

$$\begin{bmatrix} N_L \\ N_R \end{bmatrix} = \frac{EA}{\ell} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} v_L \\ v_R \end{bmatrix}$$

$$\underline{\tilde{K}} = \frac{EA}{\ell} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

! ℓ may be different for different elements !

$\ell_1 = a$

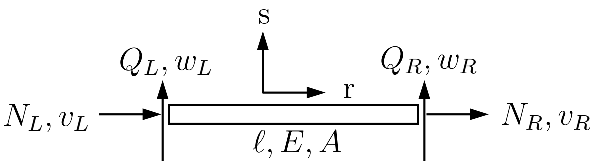
$\ell_2 = \sqrt{2}\,a$

3. step: Transformation

here: 2 global degrees of freedom u_x, u_y for each node

(no rotational dof, truss structure!)

$\rightsquigarrow$ extension of the element stiffness matrix

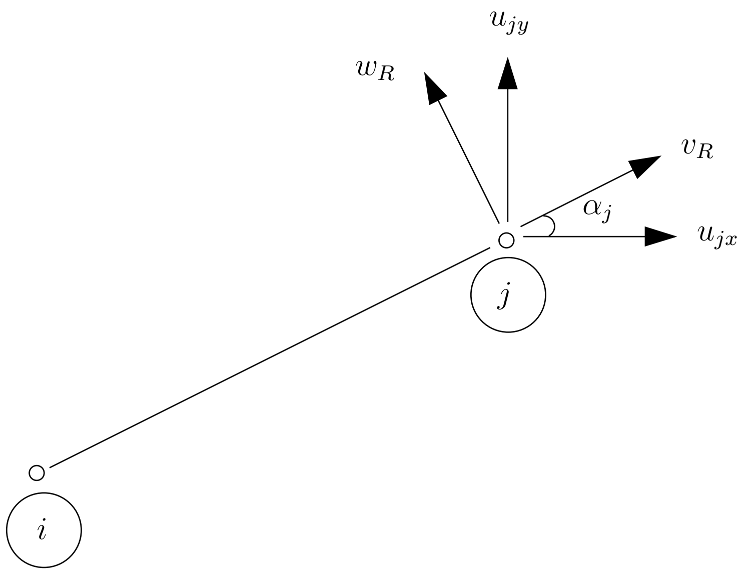


$$\underbrace{\begin{bmatrix} N_L \\ Q_L \\ N_R \\ Q_R \end{bmatrix}}_{\underline{S}} = \underbrace{\frac{EA}{\ell}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{\underline{K}} \underbrace{\begin{bmatrix} v_L \\ w_L \\ v_R \\ w_R \end{bmatrix}}_{\underline{v}}$$

Q_R, Q_L are always equal to zero because of truss structure.

The additional dof w_R, w_L are needed for the transformation.

<u>Transformation</u>



$$\begin{aligned} u_{jx} &= v_R \cos\alpha_j - w_R \sin\alpha_j \\ u_{jy} &= v_R \sin\alpha_j + w_R \cos\alpha_j \end{aligned}$$

$$\underbrace{\begin{bmatrix} u_{ix} \\ u_{iy} \\ u_{jx} \\ u_{jy} \end{bmatrix}}_{\underline{u}} = \underbrace{\begin{bmatrix} \cos\alpha_i & -\sin\alpha_i & 0 & 0 \\ \sin\alpha_i & \cos\alpha_i & 0 & 0 \\ 0 & 0 & \cos\alpha_i & -\sin\alpha_i \\ 0 & 0 & \sin\alpha_i & \cos\alpha_i \end{bmatrix}}_{\underline{T}^T} \underbrace{\begin{bmatrix} v_L \\ w_L \\ v_R \\ w_R \end{bmatrix}}_{\underline{v}}$$

$$\rightarrow \underline{u} = \underline{T}^T \underline{v} \quad \left(\underline{T}^T \text{ is orthogonal, i.e.} \, \underline{T}^{-1} = \underline{T}^T\,!\right)$$

same for nodal forces

$$\rightarrow \underline{S} = \underline{T}^T \underline{S}$$

Transformed stiffness relation:

$$\underline{S} = \underline{K}\ \underline{v} \quad \text{(a)} \qquad \underline{\hat{u}} = \underline{T}^T \underline{v} = \underline{T}^{-1} \underline{v} \ \rightsquigarrow\ \underline{v} = \underline{T}\ \underline{\hat{u}} \quad \text{(b)}$$

from (a) and (b):

$$\begin{aligned} \underline{S} &= \underline{K}\ \underline{T}\ \underline{\hat{u}} \\ \underline{T}^T\ \underline{S} &= \underline{T}^T \underline{K}\ \underline{T}\ \underline{\hat{u}} \\ \underline{\hat{S}} &= \underline{T}^T \underline{K}\ \underline{T}\ \underline{\hat{u}} \qquad \leftarrow\ \underline{\hat{S}} = \underline{T}^T \underline{S} \end{aligned}$$

$$\Rightarrow\ \underline{\hat{S}} = \underline{\hat{K}}\ \underline{\hat{u}} \quad \text{with} \quad \underline{\hat{K}} = \underline{T}^T \underline{K}\ \underline{T}$$

$$\Rightarrow \text{ form of } \underline{\hat{S}}:\ \underline{\hat{S}} = \begin{bmatrix} S_{ix} \\ S_{iy} \\ S_{jx} \\ S_{jy} \end{bmatrix}$$

We have now

$$\underline{u} = \begin{bmatrix} u_{ix} \\ u_{iy} \\ u_{jx} \\ u_{jy} \end{bmatrix} = \begin{bmatrix} \underline{u}_i \\ \underline{u}_j \end{bmatrix} \quad \text{displacements of nodes } i \text{ and } j \text{ in global coordinates}$$

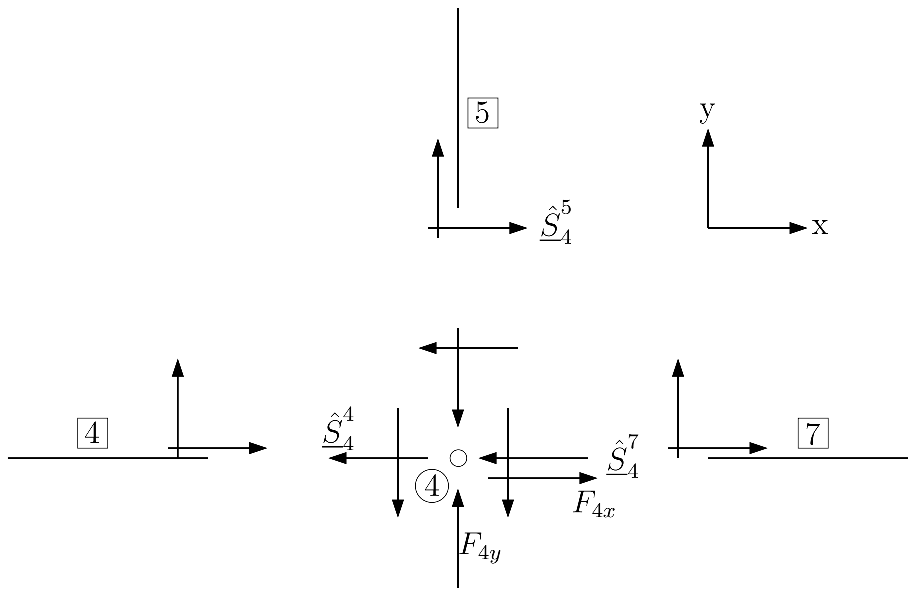
and for each element

$$\underline{S}^e = \begin{bmatrix} S^e_{ix} \\ S^e_{iy} \\ S^e_{jx} \\ S^e_{jy} \end{bmatrix} = \begin{bmatrix} \underline{S}^e_i \\ \underline{S}^e_j \end{bmatrix} \quad \text{nodal forces at element e (between nodes } i \text{ and } j)$$

$$\underline{S}^e = \underline{K}^e\ \underline{u}\ \Rightarrow\ \underbrace{\begin{bmatrix} \underline{S}^e_i \\ \underline{S}^e_j \end{bmatrix}}_{\underline{S}^e} = \underbrace{\begin{bmatrix} \underline{K}^e_{ii} & \underline{K}^e_{ij} \\ \underline{K}^e_{ji} & \underline{K}^e_{jj} \end{bmatrix}}_{\underline{K}^e} \underbrace{\begin{bmatrix} \underline{u}_i \\ \underline{u}_j \end{bmatrix}}_{\underline{u}} \qquad (*)$$

4. step: Assembly of Global System

$\rightsquigarrow$ balances of forces at each node



e.g. node ④:

here: $F_{4x} = F_{4y} = 0$

$$\sum F_x = 0 : F_{4x} - S^4_{4x} - S^5_{4x} - S^7_{4x} = 0$$

$$\sum F_y = 0 : F_{4y} - S^4_{4y} - S^5_{4y} - S^7_{4y} = 0$$

$$\Rightarrow \hat{\underline{S}}_4^4 + \hat{\underline{S}}_4^5 + \hat{\underline{S}}_4^7 = \hat{\underline{f}}_4 \qquad (**)$$

general:

$$\hat{\underline{f}}_i = \sum_e \hat{\underline{S}}_i^e$$

Expressing nodal forces through displacements by using (*)

e.g. element $\boxed{4}$ between node ② and ④

$$\begin{bmatrix} \hat{\underline{S}}_2^4 \\ \hline \hat{\underline{S}}_4^4 \end{bmatrix} = \left[\begin{array}{c|c} \hat{\underline{K}}_{22}^4 & \hat{\underline{K}}_{24}^4 \\ \hline \hat{\underline{K}}_{42}^4 & \hat{\underline{K}}_{44}^4 \end{array}\right] \begin{bmatrix} \hat{\underline{u}}_2 \\ \hline \hat{\underline{u}}_4 \end{bmatrix} \quad \text{partitioning}$$

or more explicit

$$\begin{bmatrix} \hat{\underline{S}}_{2x}^4 \\ \hat{\underline{S}}_{2y}^4 \\ \hline \hat{\underline{S}}_{4x}^4 \\ \hat{\underline{S}}_{4y}^4 \end{bmatrix} = \frac{EA}{a} \left[\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} \hat{\underline{u}}_{2x} \\ \hat{\underline{u}}_{2y} \\ \hline \hat{\underline{u}}_{4x} \\ \hat{\underline{u}}_{4y} \end{bmatrix}$$

$$\Rightarrow \underline{\underline{\hat{\underline{S}}_4^4 = \hat{\underline{K}}_{42}^4 \, \hat{\underline{u}}_2 + \hat{\underline{K}}_{44}^4 \, \hat{\underline{u}}_4}} \quad \text{similarly for the other nodal forces}$$

in (**):

$$\hat{\underline{K}}_{42}^4 \, \hat{\underline{u}}_2 + \hat{\underline{K}}_{44}^4 \, \hat{\underline{u}}_4 + \hat{\underline{K}}_{43}^5 \, \hat{\underline{u}}_3 + \hat{\underline{K}}_{44}^5 \, \hat{\underline{u}}_4 + \hat{\underline{K}}_{44}^7 \, \hat{\underline{u}}_4 + \hat{\underline{K}}_{45}^7 \, \hat{\underline{u}}_5 = \hat{\underline{f}}_4$$

$$\hat{\underline{K}}_{42}^4 \, \hat{\underline{u}}_2 + \hat{\underline{K}}_{43}^5 \, \hat{\underline{u}}_3 + \left(\hat{\underline{K}}_{44}^4 + \hat{\underline{K}}_{44}^5 + \hat{\underline{K}}_{44}^7\right) \hat{\underline{u}}_4 + \hat{\underline{K}}_{45}^7 \, \hat{\underline{u}}_5 = \hat{\underline{f}}_4$$

similarly for all nodes.

In matrix form for the 5 nodes:

$$
\left[\begin{array}{c|c|c|c|c}
\underline{\hat{K}}^1_{11} + \underline{\hat{K}}^2_{11} & \underline{\hat{K}}^1_{12} & \underline{\hat{K}}^2_{13} & & \\
\hline
\underline{\hat{K}}^1_{21} & \underline{\hat{K}}^1_{22} + \underline{\hat{K}}^3_{22} + \underline{\hat{K}}^4_{22} & \underline{\hat{K}}^3_{23} & \underline{\hat{K}}^4_{24} & \\
\hline
\underline{\hat{K}}^2_{31} & \underline{\hat{K}}^2_{32} & \underline{\hat{K}}^2_{33} + \underline{\hat{K}}^3_{33} + \underline{\hat{K}}^5_{33} + \underline{\hat{K}}^6_{33} & \underline{\hat{K}}^5_{34} & \underline{\hat{K}}^6_{35} \\
\hline
 & \underline{\hat{K}}^4_{42} & \underline{\hat{K}}^5_{43} & \underline{\hat{K}}^4_{44} + \underline{\hat{K}}^5_{44} + \underline{\hat{K}}^7_{44} & \underline{\hat{K}}^7_{45} \\
\hline
 & & \underline{\hat{K}}^6_{53} & \underline{\hat{K}}^7_{54} & \underline{\hat{K}}^6_{55} + \underline{\hat{K}}^7_{55}
\end{array}\right]
\left[\begin{array}{c}
\underline{\hat{u}}_1 \\ \hline \underline{\hat{u}}_2 \\ \hline \underline{\hat{u}}_3 \\ \hline \underline{\hat{u}}_4 \\ \hline \underline{\hat{u}}_5
\end{array}\right]
=
\left[\begin{array}{c}
\underline{\hat{f}}_1 \\ \hline \underline{\hat{f}}_2 \\ \hline \underline{\hat{f}}_3 \\ \hline \underline{\hat{f}}_4 \\ \hline \underline{\hat{f}}_5
\end{array}\right]
$$

⇒ placement of the submatrices corresponds to the indices of the nodes.

⇒ assembly of the global stiffness matrix can be carried out quite easily.

1) Partitioning of the element stiffness matrix according to the nodes

   e.g. element $\boxed{1}$

$$
\underline{\hat{K}}^1 = \frac{EA}{a}\left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ \hline 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right] = \left[\begin{array}{cc} \underline{\hat{K}}^1_{11} & \underline{\hat{K}}^1_{12} \\ \underline{\hat{K}}^1_{21} & \underline{\hat{K}}^1_{22} \end{array}\right]
$$

2) Adding the submatrices onto the corresponding position of the global stiffness matrix

   $\underline{\hat{K}}^1_{11}$ onto $K_{(1,1)}$, $\underline{\hat{K}}^1_{12}$ onto $K_{(1,2)}$ etc.

<u>5. step:</u> Boundary conditions

system of equations

$$\underline{K} \begin{bmatrix} u_{1x} \\ u_{1y} \\ u_{2x} \\ u_{2y} \\ u_{3x} \\ u_{3y} \\ u_{4x} \\ u_{4y} \\ u_{5x} \\ u_{5y} \end{bmatrix} = \begin{bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \\ F_{3x} \\ F_{3y} \\ F_{4x} \\ F_{4y} \\ F_{5x} \\ F_{5y} \end{bmatrix}$$

boundary coonditon's: $u_{1x} = u_{1y} = u_{2x} = u_{2y} = 0$

$\rightsquigarrow$ $F_{1x}, F_{1y}, F_{2x}, F_{2y}$: unknown forces

loads: $F_{3x} = F_{3y} = F_{4x} = F_{4y} = F_{5x} = 0, \quad F_{5y} = -F$

$\rightsquigarrow$ $u_{3x}, u_{3y}, u_{4x}, u_{4y}, u_{5x}, u_{5y}$: unknown displacements

Either the displacements or the forces are given, the corresponding other variable is unknown.

General form:

$$\underbrace{\begin{bmatrix} \underline{K}_{uu} & \underline{K}_{uF} \\ \underline{K}_{Fu} & \underline{K}_{FF} \end{bmatrix}}_{=\underline{K}} \begin{bmatrix} \underline{u} \\ \underline{\bar{u}} \end{bmatrix} = \begin{bmatrix} \underline{\bar{f}} \\ \underline{f} \end{bmatrix}$$

$\underline{\bar{u}}$, $\underline{\bar{f}}$: given (known) displacements/forces

$\underline{u}$, $\underline{f}$: unknown variables

6. step: Solution

1. Determination of the unknown displacements

   $$\underline{K}_{uu}\,\underline{u} = \underline{\bar{f}} - \underline{K}_{uF}\,\underline{\bar{u}}$$

   for homogeneous boundary conditions ($\underline{\bar{u}} = \underline{0}$)

   $$\rightsquigarrow \underline{K}_{uu}\,\underline{u} = \underline{\bar{f}}$$

   In most cases, only the displacements are of particular interest, that's why the other rows and columns are usually cancelled or not formulated from the beginning.

$$\left[\begin{array}{c|c|c} & & \\ & \underline{K}_{uu} & \\ & & \end{array}\right] \left[\begin{array}{c} u_{3x} \\ u_{3y} \\ \hline u_{4x} \\ u_{4y} \\ \hline u_{5x} \\ u_{5y} \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ \hline 0 \\ 0 \\ \hline 0 \\ -F \end{array}\right]$$

If the unknown forces are to be determined:

$$\rightsquigarrow \underline{K}_{Fu}\,\underline{u} + \underline{K}_{FF}\,\underline{\bar{u}} = \underline{f} \quad \text{with}\,\underline{u}\,\text{from solution above}$$

for homogeneous boundary conditions

$$\rightsquigarrow \underline{K}_{Fu}\,\underline{u} = \underline{f}$$

Building up the (reduced) stiffness matrix of the system by use of the index table

1. Numbering of the degrees of freedom (dof) and the r̈eal d̈of

$$\left[\begin{array}{ccc|c} u_{1x} & & 1 & 0 \\ u_{1y} & & 2 & 0 \\ u_{2x} & & 3 & 0 \\ u_{2y} & & 4 & 0 \\ u_{3x} & & 5 & 1 \\ u_{3y} & & 6 & 2 \\ u_{4x} & & 7 & 3 \\ u_{4y} & & 8 & 4 \\ u_{5x} & & 9 & 5 \\ u_{5y} & & 10 & 6 \end{array}\right]$$
$$\uparrow \quad \uparrow$$

system degrees of freedom real degrees of freedom
Originally, the system has 10 dof (5 nodes with 2 displacements each). But 4 dof are blocked by the boundary conditions, so that 6 r̈eal d̈of remain. This is also the size of the reduced system matrix.

2. Formulation of the <u>element index tables</u> (each element has 4 local dof)

element 1

$$\left[\begin{array}{c|cc|c} 1 & u_{1x} & 1 & 0 \\ 2 & u_{1y} & 2 & 0 \\ 3 & u_{2x} & 3 & 0 \\ 4 & u_{2y} & 4 & 0 \end{array}\right]$$

element 2

$$\left[\begin{array}{c|cc|c} 1 & u_{1x} & 1 & 0 \\ 2 & u_{1y} & 2 & 0 \\ 3 & u_{3x} & 5 & 1 \\ 4 & u_{3y} & 6 & 2 \end{array}\right]$$

element 3

$$\left[\begin{array}{c|cc|c} 1 & u_{2x} & 3 & 0 \\ 2 & u_{2y} & 4 & 0 \\ 3 & u_{3x} & 5 & 1 \\ 4 & u_{3y} & 6 & 2 \end{array}\right]$$

element 4

$$\left[\begin{array}{c|cc|c} 1 & u_{2x} & 3 & 0 \\ 2 & u_{2y} & 4 & 0 \\ 3 & u_{4x} & 7 & 3 \\ 4 & u_{4y} & 8 & 4 \end{array}\right]$$

element 5

$$\left[\begin{array}{c|cc|c} 1 & u_{3x} & 5 & 1 \\ 2 & u_{3y} & 6 & 2 \\ 3 & u_{4x} & 7 & 3 \\ 4 & u_{4y} & 8 & 4 \end{array}\right]$$

$\vdots$

The element stiffness matrix is of the form:

$$\underline{\hat{K}}^e = \begin{bmatrix} \hat{K}^e(1,1) & \hat{K}^e(1,2) & \cdots & \hat{K}^e(1,4) \\ \hat{K}^e(2,1) & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \hat{K}^e(4,1) & \cdots & \cdots & \hat{K}^e(4,4) \end{bmatrix}$$

The elements of the element stiffness matrices are now added onto the system stiffness matrix according to the index tables

$$
\begin{bmatrix}
\underline{\hat{K}}_{11}^{1}+\underline{\hat{K}}_{11}^{2} & \underline{\hat{K}}_{12}^{1} & \underline{\hat{K}}_{13}^{2} & & \\
\underline{\hat{K}}_{21}^{1} & \underline{\hat{K}}_{22}^{1}+\underline{\hat{K}}_{22}^{3}+\underline{\hat{K}}_{22}^{4} & \underline{\hat{K}}_{23}^{3} & \underline{\hat{K}}_{24}^{4} & \\
\underline{\hat{K}}_{31}^{2} & \underline{\hat{K}}_{32}^{2} & \underline{\hat{K}}_{33}^{2}+\underline{\hat{K}}_{33}^{3}+\underline{\hat{K}}_{33}^{5}+\underline{\hat{K}}_{33}^{6} & \underline{\hat{K}}_{34}^{5} & \underline{\hat{K}}_{35}^{6} \\
 & \underline{\hat{K}}_{42}^{4} & \underline{\hat{K}}_{43}^{5} & \underline{\hat{K}}_{44}^{4}+\underline{\hat{K}}_{44}^{5}+\underline{\hat{K}}_{44}^{7} & \underline{\hat{K}}_{45}^{7} \\
 & & \underline{\hat{K}}_{53}^{6} & \underline{\hat{K}}_{54}^{7} & \underline{\hat{K}}_{55}^{6}+\underline{\hat{K}}_{55}^{7}
\end{bmatrix}
\begin{bmatrix}
\underline{\hat{u}}_1 \\ \underline{\hat{u}}_2 \\ \underline{\hat{u}}_3 \\ \underline{\hat{u}}_4 \\ \underline{\hat{u}}_5
\end{bmatrix}
=
\begin{bmatrix}
\underline{\hat{f}}_1 \\ \underline{\hat{f}}_2 \\ \underline{\hat{f}}_3 \\ \underline{\hat{f}}_4 \\ \underline{\hat{f}}_5
\end{bmatrix}
$$