University of Stuttgart
Institute for Nonlinear Mechanics

Master's thesis

Step-size control of implicit Runge–Kutta methods for nonsmooth mechanical systems

Topic Areas:	Runge–Kutta methods, step-size control
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Prerequisites/Prior Knowledge:	nonsmooth dynamics

Implicit Runge–Kutta methods, in particular the family of Radau IIA methods [1], are widely used for the time integration of stiff ordinary differential equations (ODEs) and differential algebraic equations (DAEs). Their excellent stability properties and high order of accuracy make them highly attractive for the simulation of mechanical systems with unilateral contact, friction, and impacts [2].

A key ingredient of efficient time integration is adaptive step-size control. Classical strategies aim at achieving a prescribed accuracy with minimal computational effort by automatically adjusting the step size h. While this is well understood for smooth systems [3], the situation is fundamentally different for nonsmooth dynamics.

In such systems,

$$\dot{\boldsymbol{q}}(t) = \boldsymbol{u}(t),$$
$$\dot{\boldsymbol{u}}(t) = \boldsymbol{f}(\boldsymbol{q}(t), \boldsymbol{u}(t)) + \text{impulses},$$

the velocity $\boldsymbol{u}(t)$ locally exhibits jumps at impact times, while the position $\boldsymbol{q}(t)$ is only absolutely continuous and typically not differentiable. As a consequence, the local error is often dominated by the *incorrect resolution of impact events* rather than smooth discretization errors.

Classical embedded error estimators and Richardson extrapolation rely on Taylor expansions and therefore break down in the presence of such nonsmooth behavior. Even when monitoring only the position error, a reliable asymptotic error model is generally not available. First investigations in this direction can be found in [4].

A promising idea to overcome this difficulty is to introduce auxiliary variables by additional integration, i.e.,

$$\dot{\boldsymbol{d}}(t) = \boldsymbol{q}(t) \quad \text{and} \quad \dot{\boldsymbol{w}}(t) = \boldsymbol{d}(t).$$

These variables exhibit increased regularity, $\boldsymbol{q} \in C^0$, $\boldsymbol{d} \in C^1$ and $\boldsymbol{w} \in C^2$, even in the presence of impacts. This *regularity lifting* smooths out the nonsmooth features of the trajectory and may lead to improved asymptotic behavior of error estimators.

The idea is to construct step-size controllers based on these auxiliary variables, for instance via embedded Runge–Kutta methods or extrapolation techniques [3]. A central question to be answered in this thesis is whether such smoothed error indicators allow for robust and efficient adaptivity, or whether they fail to capture the dominant error contribution caused by impact timing.

References

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[3] E. Hairer *et al.*, *Solving Ordinary Differential Equations II*. Springer, 2002.

[4] V. Acary, "Higher order event capturing time-stepping schemes for nonsmooth multibody systems with unilateral constraints and impacts," *Applied Numerical Mathematics*, 2012.