Chapter 9
A Variational Formulation of Classical Nonlinear Beam Theories

Simon R. Eugster & Jonas Harsch

Abstract This article intends to present a concise theory of spatial nonlinear classical beams followed by a special treatment of the planar case. Hereby the considered classical beams are understood as generalized one-dimensional continua that model the mechanical behavior of three-dimensional beam-like objects. While a one-dimensional continuum corresponds to a deformable curve in space, parametrized by a single material coordinate and time, a generalized continuum is augmented by further kinematical quantities depending on the very same parameters. We introduce the following three nonlinear spatial beams: The Timoshenko beam, the Euler–Bernoulli beam and the inextensible Euler–Bernoulli beam. In the spatial theory, the Euler–Bernoulli beam and its inextensible companion are presented as constrained theories. In the planar case, both constrained theories are additionally described using an alternative kinematics that intrinsically satisfies the defining constraints of these theories.

9.1 Introduction

One particular reason for confusion in beam theory is the lack of a consistent naming in literature. Hence, whenever talking and writing about beam theory, it is crucial to clarify this ambiguity by defining the kinematics of the discussed theory. In this article, a *Timoshenko beam* is considered as a generalized one-dimensional continuum described by a spatial curve, its centerline, augmented in each point of the curve by an orthonormal director triad. For beam-like three-dimensional elastic bodies the director triads model the cross sections, which remain plane and rigid for all configurations. Alternative names given in literature are “special Cosserat rod”, see Antman (2005), “Simo–Reissner beam” referring to Simo (1985) and Reissner

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(1981), "geometrically exact beam", see (Betsch and Steinmann, 2003; Eugster et al, 2014), or "poutres naturelle" ("natural beam") in French literature as for instance in Ballard and Millard (2009). If the director triad is constrained such that one of the directors always algins with the centerline's tangent, we call the beam an *Euler–Bernoulli beam* (Eugster, 2015; Eugster and Steigmann, 2020). Also here alternative names are around, i.e., "Kirchhoff–Clebsch rod" (Steigmann and Faulkner, 1993), "Kirchhoff–Love rod" (Greco and Cuomo, 2013), "Kirchhoff rod" (Meier et al, 2014), or "Navier–Bernoulli beam" (Ballard and Millard, 2009). We call the beam an *inextensible Euler–Bernoulli beam* if in addition the norm of the centerline's tangent remains unchanged throughout the motion.

Since the beginning of continuum mechanics, beam theory has been an omnipresent research field that has newly received its attention not only in soft robotics (Deutschmann et al, 2018; Eugster and Deutschmann, 2018; Till et al, 2019) but also in the field of mechanical metamaterials (Barchiesi et al, 2019b). Many metamaterials are composed of networks of beams such as the class of pantographic materials (dell'Isola et al, 2019a,b, 2020b), which were analyzed in various forms, see among others (Alibert et al, 2003; Andreaus et al, 2018; Barchiesi et al, 2019a, 2020; Boutin et al, 2017; Capobianco et al, 2018; dell'Isola et al, 2016a,c; dell'Isola and Steigmann, 2015; Giorgio et al, 2017; Maurin et al, 2019; Rahali et al, 2015; Shirani et al, 2019; Steigmann and dell'Isola, 2015). The presented beam theories are formulated in a variational setting, where the principle of virtual work plays the role of the fundamental postulate in mechanics, see (Eugster and Glocker, 2017; dell'Isola et al, 2020a; dell'Isola and Placidi, 2011; dell'Isola and Seppecher, 1995; Eugster and dell'Isola, 2017, 2018). The principle states that the sum of all virtual work contributions of the modeled mechanical effects must vanish for all virtual displacements. For static problems, the total virtual work is composed of internal and external virtual work contributions, which model, respectively, mechanical interactions of material points of the beam among themselves as well as mechanical interactions of material points of the beam and the environment. For dynamic problems, virtual work contributions incorporating inertial effects of the beam have to be added. Moreover, the principle of virtual work must then hold for all time instants. The main task in the development of a beam theory is the definition of each virtual work contribution. This task is by no means unique and can be considered as the modeling procedure in mechanics. Since the (inextensible) Euler–Bernoulli beam can be considered as a constrained Timoshenko beam, the virtual work contributions thereof are formulated first in Sections 9.2–9.5. We pursue the following strategy for the internal and external virtual work contributions. The internal virtual work contributions are related to the variation of a strain energy function that depends on the kinematical quantities describing the beam (Section 9.2). The set of strain energy functions is reduced in Section 9.3 by the requirement of an invariance principle. More precisely, we postulate the invariance under superimposed rigid body motions of the strain energy function. Similar invariance conditions are obtained when advocating for a change of observer as discussed in Steigmann (2017). This leads us not only to the most general strain energy function that guarantees the invariance principle, but also to the internal virtual work of the Timoshenko beam.

The suitable external virtual work contributions are subsequently obtained by an integration by parts procedure. Accordingly, the form of the internal virtual work defines the external force effects that the beam can resist. These are for the classical theories, distributed forces and couples as well as point forces and couples at both ends of the beam. The virtual work contributions of the inertial effects are derived in the sense of an induced theory in which the beam is considered as a constrained three-dimensional body, see (Antman, 2005; Eugster, 2015). The total virtual work of the (inextensible) Euler–Bernoulli beam is obtained in Section 9.6 by augmenting the strain energy function in the sense of a Lagrange multiplier method (Bersani et al, 2019; dell'Isola et al, 2016b). In Section 9.7, the motion of the beams are restricted to be planar. For the Timoshenko beam, the parametrization of the required rotation fields becomes trivial. For both the Euler–Bernoulli beam and the inextensible Euler–Bernoulli beam, a minimal set of kinematical descriptors can be found. In these minimal formulations, the virtual work contributions of the constraints vanish.

9.2 Notation and Kinematics

We regard tensors as linear transformations from a three-dimensional vector space $\mathbb{E}^3$ to itself and use standard notation such as $\mathbf{A}^\mathrm{T}$, $\mathbf{A}^{-1}$, $\det(\mathbf{A})$. These are, respectively, the transpose, the inverse, and the determinant of a tensor $\mathbf{A}$. The set of tensors is denoted by $\mathrm{L}(\mathbb{E}^3;\mathbb{E}^3)$. The tensor $\mathbf{1}$ stands for the identity tensor, which leaves every vector $\mathbf{a} \in \mathbb{E}^3$ unchanged, i.e. $\mathbf{a} = \mathbf{1a}$. We use Skw to denote the linear subspace of skew tensors and $Orth^+ = \{\mathbf{A} \in \mathrm{L}(\mathbb{E}^3;\mathbb{E}^3) | \mathbf{A}^\mathrm{T}\mathbf{A} = \mathbf{A}\mathbf{A}^\mathrm{T} = \mathbf{1} \wedge \det(\mathbf{A}) = +1\}$ to identify the group of rotation tensors. The tensor product of three-vectors is indicated by interposing the symbol $\otimes$. Latin and Greek indices take values in $\{1,2,3\}$ and $\{2,3\}$, respectively, and, when repeated, are summed over their ranges. Furthermore, we abbreviate the arguments in functions depending on the three components (a_1, a_2, a_3) or merely on the last two components (a_2, a_3) of a vector $\mathbf{a} \in \mathbb{E}^3$ by (a_i) or (a_α), respectively. Derivatives of functions $f = f(s,t)$ with respect to s and t are denoted by a prime $f' = \partial f/\partial s$ and a dot $\dot{f} = \partial f/\partial t$, respectively. The variation of a function $f = f(s,t)$, denoted by a delta, is the derivative with respect to the parameter ε of a one-parameter family $\hat{f} = \hat{f}(s,t;\varepsilon)$ evaluated at $\varepsilon = 0$, i.e. $\delta f(s,t) = \partial \hat{f}/\partial\varepsilon(s,t;0)$. The one-parameter family satisfies $f(s,t) = \hat{f}(s,t;0)$.

Next, we introduce the required kinematical quantities for the spatial nonlinear Timoshenko beam theory. The motion of the *centerline* is the mapping $\mathbf{r}\colon I \times \mathbb{R} \to \mathbb{E}^3, (s,t) \mapsto \mathbf{r}(s,t)$, where, for each instant of time $t \in \mathbb{R}$, the closed interval $I = [l_1, l_2] \subset \mathbb{R}$ parametrizes the set of beam points. We make the convenient choice to use as material coordinate the arc length parameter s of the *reference centerline* $\mathbf{r}_0\colon I \to \mathbb{E}^3$. To capture cross-sectional orientations of beam-like bodies, the kinematics of the centerline is augmented by the motion of positively oriented director triads $\mathbf{d}_i : I \times \mathbb{R} \to \mathbb{E}^3$. The directors $\mathbf{d}_\alpha(s,t)$ span the plane and rigid cross section of the beam for the material coordinate s at time t. The positively oriented director triads in the reference configuration are given by the mappings $\mathbf{D}_i : I \to \mathbb{E}^3$. While $\mathbf{D}_1$ is identified with the unit tangent to the reference centerline $\mathbf{r}_0$, i.e., $\mathbf{D}_1 = \mathbf{r}_0'$, the vectors $\mathbf{D}_2(s)$ and $\mathbf{D}_3(s)$ are identified with the geometric principal axes of the cross sections.

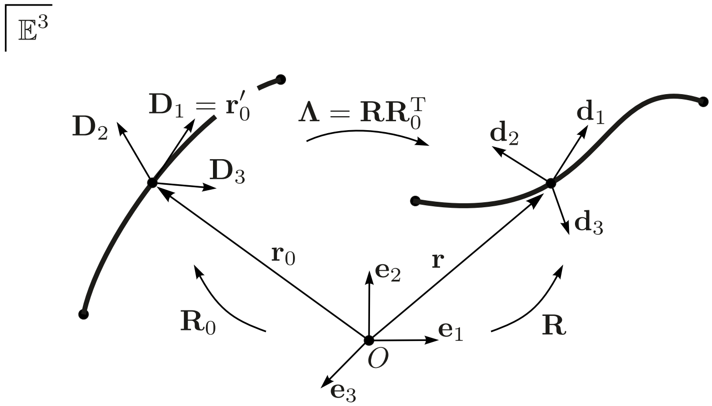

Fig. 9.1 Kinematics of a spatial Timoshenko beam.

With the reference and current rotation fields $\mathbf{R}_0 \colon I \to Orth^+$ and $\mathbf{R} \colon I \times \mathbb{R} \to Orth^+$, respectively, the reference and current director triads are related to a fixed right-handed inertial frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ by

$$\mathbf{D}_i(s) = \mathbf{R}_0(s)\mathbf{e}_i \ , \quad \mathbf{d}_i(s,t) = \mathbf{R}(s,t)\mathbf{e}_i \ . \tag{9.1}$$

Using the identity tensor in the form $\mathbf{1} = \mathbf{e}_i \otimes \mathbf{e}_i$ together with the relations (9.1), the current and reference rotation fields can be expressed as

$$\mathbf{R}_0 = \mathbf{R}_0\mathbf{1} = \mathbf{R}_0(\mathbf{e}_i \otimes \mathbf{e}_i) = (\mathbf{R}_0\mathbf{e}_i) \otimes \mathbf{e}_i = \mathbf{D}_i \otimes \mathbf{e}_i \ , \tag{9.2}$$

$$\mathbf{R} = \mathbf{R}\mathbf{1} = \mathbf{R}(\mathbf{e}_i \otimes \mathbf{e}_i) = (\mathbf{R}\mathbf{e}_i) \otimes \mathbf{e}_i = \mathbf{d}_i \otimes \mathbf{e}_i \ . \tag{9.3}$$

With the inverse relations of (9.1) at hand and exploiting the equivalence of the inverse and the transpose for rotations, we can relate all bases by

$$\mathbf{e}_i = \mathbf{R}_0^{\mathrm{T}}(s)\mathbf{D}_i(s) = \mathbf{R}^{\mathrm{T}}(s,t)\mathbf{d}_i(s,t) \ . \tag{9.4}$$

To capture the deformation between the reference and the current configuration, we introduce the rotation field $\mathbf{\Lambda} : I \times \mathbb{R} \to Orth^+, (s,t) \mapsto \mathbf{\Lambda}(s,t) = \mathbf{R}(s,t)\mathbf{R}_0^{\mathrm{T}}(s)$, which rotates the reference director triads to the current director triads, i.e.,

$$\mathbf{d}_i(s,t) = \mathbf{\Lambda}(s,t)\mathbf{D}_i(s) \ . \tag{9.5}$$

Using the identity tensor in the form $\mathbf{1} = \mathbf{D}_i \otimes \mathbf{D}_i$ and repeating the steps as in (9.3), we can represent the rotation between reference and current configuration as

$$\mathbf{\Lambda} = \mathbf{\Lambda 1} = \mathbf{\Lambda}(\mathbf{D}_i \otimes \mathbf{D}_i) = (\mathbf{\Lambda D}_i) \otimes \mathbf{D}_i \overset{(9.5)}{=} \mathbf{d}_i \otimes \mathbf{D}_i \ . \tag{9.6}$$

Using (9.1) and (9.4), the rate of change of the reference director triad with respect to the arc length s is expressed as

$$\mathbf{D}_i'(s) = (\mathbf{R}_0(s)\mathbf{e}_i)' = \mathbf{R}_0'(s)\mathbf{e}_i \overset{(9.4)}{=} \mathbf{R}_0'(s)\mathbf{R}_0^{\mathrm{T}}(s)\mathbf{D}_i(s) = \tilde{\boldsymbol{\kappa}}_0(s)\mathbf{D}_i(s) \ , \tag{9.7}$$

where we have introduced the *reference curvature* $\tilde{\boldsymbol{\kappa}}_0(s) = \mathbf{R}_0'(s)\mathbf{R}_0^{\mathrm{T}}(s)$. Taking the derivative with respect to s of (9.2), the reference curvature can be expressed with respect to the $\mathbf{D}_i \otimes \mathbf{D}_j$-basis as

$$\tilde{\boldsymbol{\kappa}}_0 = \tilde{\kappa}_{ij}^0 \mathbf{D}_i \otimes \mathbf{D}_j = (\mathbf{D}_i \otimes \mathbf{e}_i)'(\mathbf{D}_j \otimes \mathbf{e}_j)^{\mathrm{T}} = \mathbf{D}_i' \otimes \mathbf{D}_i = (\mathbf{D}_i \cdot \mathbf{D}_j')\mathbf{D}_i \otimes \mathbf{D}_j \ . \tag{9.8}$$

The reference curvature $\tilde{\boldsymbol{\kappa}}_0$ is skew-symmetric, i.e., $\tilde{\boldsymbol{\kappa}}_0^{\mathrm{T}} = -\tilde{\boldsymbol{\kappa}}_0$. This follows straightforwardly from

$$\mathbf{0} = (\mathbf{1})' = (\mathbf{R}_0\mathbf{R}_0^{\mathrm{T}})' = \mathbf{R}_0'\mathbf{R}_0^{\mathrm{T}} + \mathbf{R}_0{\mathbf{R}_0'}^{\mathrm{T}} = \mathbf{R}_0'\mathbf{R}_0^{\mathrm{T}} + (\mathbf{R}_0'\mathbf{R}_0^{\mathrm{T}})^{\mathrm{T}} \ . \tag{9.9}$$

Thus $\tilde{\boldsymbol{\kappa}}_0(s)$ has an associated axial vector $\mathrm{ax}(\tilde{\boldsymbol{\kappa}}_0(s)) = \boldsymbol{\kappa}_0(s) \in \mathbb{E}^3$ defined by the relation $\tilde{\boldsymbol{\kappa}}_0(s)\mathbf{a} = \boldsymbol{\kappa}_0(s) \times \mathbf{a} \ \forall \mathbf{a} \in \mathbb{E}^3$. The reference curvature can thus also be expressed by the vector valued function

$$\boldsymbol{\kappa}_0 = \kappa_i^0 \mathbf{D}_i = \mathrm{ax}(\tilde{\boldsymbol{\kappa}}_0) = \frac{1}{2}\varepsilon_{ijk}\kappa_{kj}^0 \mathbf{D}_i = \frac{1}{2}\varepsilon_{ijk}(\mathbf{D}_k \cdot \mathbf{D}_j')\mathbf{D}_i \ , \tag{9.10}$$

where ε_{ijk} denotes the Levi-Civita permutation symbol, which is ± 1 for even and odd permutations of $\{1, 2, 3\}$, respectively, and zero otherwise.

The *current curvature* $\tilde{\mathbf{w}}(s,t) = \mathbf{R}'(s,t)\mathbf{R}^{\mathrm{T}}(s,t) \in Skw$ and its axial representation $\mathbf{w}(s,t) \in \mathbb{E}^3$ capture the rate of change of the current director triad with respect to the arc length parameter s and emerge in the relation

$$\mathbf{d}_i' = (\mathbf{R}\mathbf{e}_i)' = \mathbf{R}'\mathbf{e}_i \overset{(9.4)}{=} \mathbf{R}'\mathbf{R}^{\mathrm{T}}\mathbf{d}_i = \tilde{\mathbf{w}}\mathbf{d}_i = \mathbf{w} \times \mathbf{d}_i \ . \tag{9.11}$$

The skew symmetry of $\tilde{\mathbf{w}}(s,t)$ follows from the analogous computations as carried out in (9.9). The current curvature can be represented with respect to the $\mathbf{d}_i \otimes \mathbf{d}_j$-basis as

$$\tilde{\mathbf{w}} = \tilde{w}_{ij}\mathbf{d}_i \otimes \mathbf{d}_j = (\mathbf{d}_i \otimes \mathbf{e}_i)'(\mathbf{d}_j \otimes \mathbf{e}_j)^{\mathrm{T}} = \mathbf{d}_i' \otimes \mathbf{d}_i = (\mathbf{d}_i \cdot \mathbf{d}_j')\mathbf{d}_i \otimes \mathbf{d}_j \ , \tag{9.12}$$

or as vector-valued function

$$\mathbf{w} = w_i \mathbf{d}_i = \mathrm{ax}(\tilde{\mathbf{w}}) = \frac{1}{2}\varepsilon_{ijk}\tilde{w}_{kj}\mathbf{d}_i = \frac{1}{2}\varepsilon_{ijk}(\mathbf{d}_k \cdot \mathbf{d}_j')\mathbf{d}_i \ . \tag{9.13}$$

The rate of change of the current directors with respect to time t is described by the *angular velocity* $\tilde{\boldsymbol{\omega}}(s,t) = \dot{\mathbf{R}}(s,t)\mathbf{R}^{\mathrm{T}}(s,t) = \dot{\mathbf{\Lambda}}(s,t)\mathbf{\Lambda}^{\mathrm{T}}(s,t) \in Skw$, which appears together with the corresponding axial vector $\boldsymbol{\omega}(s,t) \in \mathbb{E}^3$ in

$$\dot{\mathbf{d}}_i = (\mathbf{R}\mathbf{e}_i)^{\cdot} = \dot{\mathbf{R}}\mathbf{e}_i \overset{(9.4)}{=} \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}}\mathbf{d}_i = \tilde{\boldsymbol{\omega}}\mathbf{d}_i = \boldsymbol{\omega} \times \mathbf{d}_i \ . \tag{9.14}$$

The angular velocity can thus be represented as

$$\tilde{\boldsymbol{\omega}} = \tilde{\omega}_{ij}\mathbf{d}_i \otimes \mathbf{d}_j = (\mathbf{d}_i \otimes \mathbf{e}_i)^{\cdot}(\mathbf{d}_j \otimes \mathbf{e}_j)^{\mathrm{T}} = \dot{\mathbf{d}}_i \otimes \mathbf{d}_i = (\mathbf{d}_i \cdot \dot{\mathbf{d}}_j)\mathbf{d}_i \otimes \mathbf{d}_j \ , \tag{9.15}$$

or as vector-valued function in the form

$$\boldsymbol{\omega} = \omega_i \mathbf{d}_i = \mathrm{ax}(\tilde{\boldsymbol{\omega}}) = \frac{1}{2}\varepsilon_{ijk}\tilde{\omega}_{kj}\mathbf{d}_i = \frac{1}{2}\varepsilon_{ijk}(\mathbf{d}_k \cdot \dot{\mathbf{d}}_j)\mathbf{d}_i \ . \tag{9.16}$$

The skew-symmetry of $\tilde{\boldsymbol{\omega}}$ can be verified similar to (9.9).

The rate of change of the current directors under a variation of the current configuration is captured by the skew symmetric *virtual rotation* $\delta\tilde{\boldsymbol{\phi}} = \delta\mathbf{R}\mathbf{R}^{\mathrm{T}} = \delta\boldsymbol{\Lambda}\boldsymbol{\Lambda}^{\mathrm{T}}$ with its axial vector $\delta\boldsymbol{\phi}(s,t) \in \mathbb{E}^3$. Both representations can be recognized in

$$\delta\mathbf{d}_i = \delta(\mathbf{R}\mathbf{e}_i) = \delta\mathbf{R}\mathbf{e}_i = \delta\mathbf{R}\mathbf{R}^{\mathrm{T}}\mathbf{d}_i = \delta\tilde{\boldsymbol{\phi}}\mathbf{d}_i = \delta\boldsymbol{\phi} \times \mathbf{d}_i \ . \tag{9.17}$$

As before, the virtual rotation can be represented either as the tensor function

$$\delta\tilde{\boldsymbol{\phi}} = \delta\tilde{\phi}_{ij}\mathbf{d}_i \otimes \mathbf{d}_j = \delta(\mathbf{d}_i \otimes \mathbf{e}_i)(\mathbf{d}_j \otimes \mathbf{e}_j)^{\mathrm{T}} = \delta\mathbf{d}_i \otimes \mathbf{d}_i = (\mathbf{d}_i \cdot \delta\mathbf{d}_j)\mathbf{d}_i \otimes \mathbf{d}_j \ , \tag{9.18}$$

or as the vector-valued function

$$\delta\boldsymbol{\phi} = \delta\phi_i \mathbf{d}_i = \mathrm{ax}(\delta\tilde{\boldsymbol{\phi}}) = \frac{1}{2}\varepsilon_{ijk}\delta\tilde{\phi}_{kj}\mathbf{d}_i = \frac{1}{2}\varepsilon_{ijk}(\mathbf{d}_k \cdot \delta\mathbf{d}_j)\mathbf{d}_i \ . \tag{9.19}$$

Due to the symmetry of second derivatives, the partial derivative with respect to s and the variation δ commute, i.e., $\delta(\mathbf{d}_i') = (\delta\mathbf{d}_i)' = \delta\mathbf{d}_i'$. This identity can be reformulated by using (9.11), (9.17) and subtracting the left-hand side from the right-hand side, yielding $\delta(\mathbf{w} \times \mathbf{d}_i) - (\delta\boldsymbol{\phi} \times \mathbf{d}_i)' = 0$. Application of the product rule, applying once again (9.11) and (9.17) as well as making use of the skew-symmetry of the cross product, i.e., $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} \ , \forall \mathbf{a}, \mathbf{b} \in \mathbb{E}^3$, we get

$$\delta\mathbf{w} \times \mathbf{d}_i + \mathbf{w} \times (\delta\boldsymbol{\phi} \times \mathbf{d}_i) - \delta\boldsymbol{\phi}' \times \mathbf{d}_i + \delta\boldsymbol{\phi} \times (\mathbf{d}_i \times \mathbf{w}) = 0 \ . \tag{9.20}$$

Using the Jacobi identity $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) = -\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) \ \forall \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{E}^3$ and applying twice the skew-symmetry of the cross-product, the above equation reduces to

$$(\delta\mathbf{w} - \delta\boldsymbol{\phi}' - \delta\boldsymbol{\phi} \times \mathbf{w}) \times \mathbf{d}_i = 0 \ . \tag{9.21}$$

Since (9.21) must hold for arbitrary $\mathbf{d}_i \in \mathbb{E}^3$, we can conclude that

$$\delta\boldsymbol{\phi}' = \delta\mathbf{w} - \delta\boldsymbol{\phi} \times \mathbf{w} \ . \tag{9.22}$$

9.3 Strain Energy Functional

Following Shirani et al (2019), we assume the strain energy E stored in a beam segment $[s_1, s_2] \subset I$ to be expressed as

$$E = \int_{s_1}^{s_2} U \, \mathrm{d}s \, , \tag{9.23}$$

where U, the strain energy function per unit reference arc length s, is a function of the list $\{\mathbf{r}, \mathbf{r}', \boldsymbol{\Lambda}, \boldsymbol{\Lambda}'\}$ and possibly depends explicitly on s, i.e.,

$$U = U(\mathbf{r}, \mathbf{r}', \boldsymbol{\Lambda}, \boldsymbol{\Lambda}'; s) \, . \tag{9.24}$$

The explicit s-dependence may arise from an initial curvature of the beam, or from nonuniform material properties.

By the requirement that strain energy functions must ensure invariance under superimposed rigid body motions, we can reduce the set of possible strain energy functions. Thus, for $\mathbf{Q}(t) \in Orth^+$ and $\mathbf{c}(t) \in \mathbb{E}^3$, the strain energy function U must be invariant under the transformations

$$\begin{aligned} \mathbf{r} &\mapsto \mathbf{r}^+ = \mathbf{Q}\mathbf{r} + \mathbf{c} \, , & \mathbf{r}' &\mapsto (\mathbf{r}^+)' = \mathbf{Q}\mathbf{r}' \, , \\ \boldsymbol{\Lambda} &\mapsto \boldsymbol{\Lambda}^+ = \mathbf{Q}\boldsymbol{\Lambda} \, , & \boldsymbol{\Lambda}' &\mapsto (\boldsymbol{\Lambda}^+)' = \mathbf{Q}\boldsymbol{\Lambda}' \, . \end{aligned} \tag{9.25}$$

By choosing $\mathbf{Q}(t) = \mathbf{1}$ and $\mathbf{c}(t) \in \mathbb{E}^3$ arbitrary, we get the condition

$$U(\mathbf{r}, \mathbf{r}', \boldsymbol{\Lambda}, \boldsymbol{\Lambda}'; s) = U(\mathbf{r}^+, (\mathbf{r}^+)', \boldsymbol{\Lambda}^+, (\boldsymbol{\Lambda}^+)'; s) = U(\mathbf{r} + \mathbf{c}, \mathbf{r}', \boldsymbol{\Lambda}, \boldsymbol{\Lambda}'; s) \, , \tag{9.26}$$

from which we conclude that $U(\mathbf{r}, \mathbf{r}', \boldsymbol{\Lambda}, \boldsymbol{\Lambda}'; s) = \tilde{U}(\mathbf{r}', \boldsymbol{\Lambda}, \boldsymbol{\Lambda}'; s)$ has to be independent of the centerline $\mathbf{r}$. For a particular material coordinate $s \in I$, we choose the rotation $\mathbf{Q}(t) = \boldsymbol{\Lambda}^{\mathrm{T}}(s, t)$ together with a vanishing displacement $\mathbf{c}(t) = 0$, which yields the condition

$$\tilde{U}(\mathbf{r}', \boldsymbol{\Lambda}, \boldsymbol{\Lambda}'; s) = \tilde{U}(\boldsymbol{\Lambda}^{\mathrm{T}}\mathbf{r}', \boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{\Lambda}, \boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{\Lambda}'; s) = \tilde{U}(\boldsymbol{\Lambda}^{\mathrm{T}}\mathbf{r}', \mathbf{1}, \boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{\Lambda}'; s) \, . \tag{9.27}$$

Due to $\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{\Lambda} = \mathbf{1}$, the strain energy per unit arc length $\tilde{U}$ may not depend on the argument $\boldsymbol{\Lambda}$ either. Hence, we conclude that if the strain energy function U is invariant under superimposed rigid body motions then there is a strain energy function $\tilde{W}$ which is related to U by

$$U(\mathbf{r}, \mathbf{r}', \boldsymbol{\Lambda}, \boldsymbol{\Lambda}'; s) = \tilde{W}(\boldsymbol{\Lambda}^{\mathrm{T}}\mathbf{r}', \boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{\Lambda}'; s) \, . \tag{9.28}$$

The reverse direction, which implies from condition (9.28) the invariance of U, is obtained immediately since the kinematic quantities $\boldsymbol{\Lambda}^{\mathrm{T}}\mathbf{r}' = (\boldsymbol{\Lambda}^+)^{\mathrm{T}}(\mathbf{r}^+)'$ and $\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{\Lambda}' = (\boldsymbol{\Lambda}^+)^{\mathrm{T}}(\boldsymbol{\Lambda}^+)'$ are invariant under the transformations (9.25). For that reason, we take these kinematic quantities as generalized strain measures of the beam, where

$$\boldsymbol{\Gamma}(s,t) = \Gamma_i(s,t)\mathbf{D}_i(s) = \boldsymbol{\Lambda}^{\mathrm{T}}(s,t)\mathbf{r}'(s,t) = (\mathbf{r}'(s,t)\cdot\mathbf{d}_i(s,t))\mathbf{D}_i(s) \qquad (9.29)$$

incorporates in its $\mathbf{D}_i$-components the projection of the centerline's tangent $\mathbf{r}'$ onto the current directors $\mathbf{d}_i$. Identifying $\mathbf{R}' = \tilde{\mathbf{w}}\mathbf{R}$ in (9.11), the second generalized strain measure $\boldsymbol{\Lambda}^{\mathrm{T}}(s,t)\boldsymbol{\Lambda}'(s,t) \in Skw$ can be related to the reference and current curvature by

$$\begin{aligned}\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{\Lambda}' &= (\mathbf{R}\mathbf{R}_0^{\mathrm{T}})^{\mathrm{T}}(\mathbf{R}\mathbf{R}_0^{\mathrm{T}})' = \mathbf{R}_0\mathbf{R}^{\mathrm{T}}\mathbf{R}'\mathbf{R}_0^{\mathrm{T}} + \mathbf{R}_0\mathbf{R}_0'^{\mathrm{T}} \\ &= \boldsymbol{\Lambda}^{\mathrm{T}}\tilde{\mathbf{w}}\boldsymbol{\Lambda} - \tilde{\boldsymbol{\kappa}}_0 = \tilde{\boldsymbol{\kappa}} - \tilde{\boldsymbol{\kappa}}_0\ ,\end{aligned} \qquad (9.30)$$

where we have introduced the current curvature pulled-back to the reference configuration $\tilde{\boldsymbol{\kappa}} = \boldsymbol{\Lambda}^{\mathrm{T}}\tilde{\mathbf{w}}\boldsymbol{\Lambda}$. Making use of (9.6) and the definition of $\tilde{\mathbf{w}}$ given in (9.12), $\tilde{\boldsymbol{\kappa}}$ can be represented as

$$\tilde{\boldsymbol{\kappa}} = \tilde{\kappa}_{ij}\mathbf{D}_i\otimes\mathbf{D}_j = \boldsymbol{\Lambda}^{\mathrm{T}}\tilde{\mathbf{w}}\boldsymbol{\Lambda} = \tilde{w}_{ij}\boldsymbol{\Lambda}^{\mathrm{T}}(\mathbf{d}_i\otimes\mathbf{d}_j)\boldsymbol{\Lambda} = \tilde{w}_{ij}\mathbf{D}_i\otimes\mathbf{D}_j\,, \qquad (9.31)$$

which shows that the components of $\tilde{\boldsymbol{\kappa}}$ in the $\mathbf{D}_i\otimes\mathbf{D}_j$-basis and $\tilde{\mathbf{w}}$ in the $\mathbf{d}_i\otimes\mathbf{d}_j$-basis coincide, i.e., $\tilde{\kappa}_{ij} = \tilde{w}_{ij}$. The skew symmetry of $\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{\Lambda}'$ allows us to write the strain measure also as the vector-valued function

$$\boldsymbol{\kappa} - \boldsymbol{\kappa}_0 = (\kappa_i - \kappa_i^0)\mathbf{D}_i = \mathrm{ax}(\tilde{\boldsymbol{\kappa}}) - \mathrm{ax}(\tilde{\boldsymbol{\kappa}}_0) \qquad (9.32)$$

with their corresponding components in the $\mathbf{D}_i$-basis given by

$$\kappa_i = \frac{1}{2}\varepsilon_{ijk}\tilde{\kappa}_{kj} = \frac{1}{2}\varepsilon_{ijk}(\mathbf{d}_k\cdot\mathbf{d}_j')\ ,\quad \kappa_i^0 = \frac{1}{2}\varepsilon_{ijk}\tilde{\kappa}_{kj}^0 = \frac{1}{2}\varepsilon_{ijk}(\mathbf{D}_k\cdot\mathbf{D}_j')\ . \qquad (9.33)$$

Replacing $\boldsymbol{\Lambda}^{\mathrm{T}}\boldsymbol{\Lambda}'$ with its axial vector (9.32) and putting the dependence of the strain energy function on a pre-curved reference configuration indicated by a non-vanishing $\boldsymbol{\kappa}_0$ into the explicit s-dependence, we obtain the final form of the objective strain energy function per unit arc length

$$U(\mathbf{r},\mathbf{r}',\boldsymbol{\Lambda},\boldsymbol{\Lambda}';s) = W(\boldsymbol{\Gamma},\boldsymbol{\kappa};s) \qquad (9.34)$$

which solely depends on (9.29) and the current parts of (9.32).

Whenever an explicit strain energy function is required in the following, we choose the quadratic strain energy function of the form

$$W(\boldsymbol{\Gamma},\boldsymbol{\kappa};s) = \sum_{i=1}^{3}\left\{\frac{1}{2}E_i(\Gamma_i - \delta_{i1})^2 + \frac{1}{2}F_i(\kappa_i - \kappa_i^0)^2\right\}\ . \qquad (9.35)$$

Note that there is a possible s-dependence in the axial stiffness E_1, the shear stiffnesses E_2 and E_3, the torsional stiffness F_1 as well as the flexural stiffnesses F_2 and F_3.

If the beam models a body that could also be described as a three-dimensional continous body with isotropic material, the stiffnesses are often related to the material properties and the geometry of the body. Then, the axial stiffness E_1 is given by the

Young's modulus times the cross-sectional area, the shear stiffnesses E_2 and E_3 are shear modulus times the cross-sectional area multiplied with an appropriate shear correction factor (Timoshenko and Goodier, 1951; Cowper, 1966), the torsional stiffness F_1 is shear modulus times the polar moment of the cross section and the flexural stiffnesses F_2 and F_3 are Young's modulus times the appropriate second moment of area of the cross section.

9.4 Virtual Work Contributions

With the objective strain energy function (9.34), we define the internal virtual work contributions of the beam using the first variation of the beam's total strain energy E in accordance with

$$\delta W^{\mathrm{int}} = -\delta E = -\int_{l_1}^{l_2} \delta W \,\mathrm{d}s = -\int_{l_1}^{l_2} \left\{ \frac{\partial W}{\partial \Gamma_i}\delta\Gamma_i + \frac{\partial W}{\partial \kappa_i}\delta\kappa_i \right\} \mathrm{d}s \,. \tag{9.36}$$

In the following, the variations of the generalized strain measures are derived. The variations of the components of (9.29) can be computed using (9.17) together with the invariance of the triple product with respect to even permutations, i.e., $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})\ \forall \mathbf{a},\mathbf{b},\mathbf{c}\in\mathbb{E}^3$, and the skew-symmetry of the cross product

$$\delta\Gamma_i = \delta(\mathbf{r}'\cdot\mathbf{d}_i) = (\delta\mathbf{r}' - \delta\boldsymbol{\phi}\times\mathbf{r}')\cdot\mathbf{d}_i \,. \tag{9.37}$$

Again using the properties of the triple product, the relation of the permutation symbol and the Kronecker delta $\varepsilon_{ijk}\varepsilon_{jkl} = \varepsilon_{ijk}\varepsilon_{ljk} = 2\delta_{il}$ as well as (9.17), the variation of the first equation in (9.33) is given as

$$\begin{aligned}
\delta\kappa_i &= \frac{1}{2}\varepsilon_{ijk}\delta(\mathbf{d}_k\cdot\mathbf{d}_j') \overset{(9.17)}{=} \frac{1}{2}\varepsilon_{ijk}[(\delta\boldsymbol{\phi}\times\mathbf{d}_k)\cdot\mathbf{d}_j' + \mathbf{d}_k\cdot(\delta\boldsymbol{\phi}\times\mathbf{d}_j)'] \\
&= \frac{1}{2}\varepsilon_{ijk}[(\delta\boldsymbol{\phi}\times\mathbf{d}_k)\cdot\mathbf{d}_j' + \mathbf{d}_k\cdot(\delta\boldsymbol{\phi}'\times\mathbf{d}_j + \delta\boldsymbol{\phi}\times\mathbf{d}_j')] \\
&= \frac{1}{2}\varepsilon_{ijk}\mathbf{d}_k\cdot(\delta\boldsymbol{\phi}'\times\mathbf{d}_j) = \frac{1}{2}\varepsilon_{ijk}\delta\boldsymbol{\phi}'\cdot(\mathbf{d}_j\times\mathbf{d}_k) \\
&= \frac{1}{2}\varepsilon_{ijk}\varepsilon_{jkl}(\delta\boldsymbol{\phi}'\cdot\mathbf{d}_l) = \delta\boldsymbol{\phi}'\cdot\mathbf{d}_i \,.
\end{aligned} \tag{9.38}$$

Substituting (9.22) and the variation of the two generalized strain measures (9.37) and (9.38) into the internal virtual work (9.36), their final form is given by

$$
\begin{aligned}
\delta W^{\mathrm{int}} &= -\int_{l_1}^{l_2} \left\{ \frac{\partial W}{\partial \Gamma_i} \delta \Gamma_i + \frac{\partial W}{\partial \kappa_i} \delta \kappa_i \right\} \mathrm{d}s \\
&= -\int_{l_1}^{l_2} \left\{ (\delta \mathbf{r}' - \delta \boldsymbol{\phi} \times \mathbf{r}') \cdot \mathbf{n} + \delta \boldsymbol{\phi}' \cdot \mathbf{m} \right\} \mathrm{d}s \\
&= -\int_{l_1}^{l_2} \left\{ (\delta \mathbf{r}' - \delta \boldsymbol{\phi} \times \mathbf{r}') \cdot \mathbf{n} + (\delta \mathbf{w} - \delta \boldsymbol{\phi} \times \mathbf{w}) \cdot \mathbf{m} \right\} \mathrm{d}s \; ,
\end{aligned} \tag{9.39}
$$

where the generalized internal forces $\mathbf{n} = \frac{\partial W}{\partial \Gamma_i} \mathbf{d}_i$ and $\mathbf{m} = \frac{\partial W}{\partial \kappa_i} \mathbf{d}_i$ have been introduced. The interpretation of these generalized forces is postponed to the next section. However, note that in the case of inelastic behavior, where no strain energy function might be available, the internal virtual work could be defined by (9.39) in which the generalized forces $\mathbf{n}$ and $\mathbf{m}$ follow different constitutive laws than here. According to Germain (1973) or Eugster and Glocker (2017), the internal virtual work must vanish for all rigid virtual displacements, i.e., for $\delta \mathbf{r}(s,t) = \delta \mathbf{c}(t) + \delta \boldsymbol{\varphi}(t) \times \mathbf{r}(s,t)$ and $\delta \boldsymbol{\phi}(s,t) = \delta \boldsymbol{\varphi}(t)$, where $\delta \mathbf{c}(t), \delta \boldsymbol{\varphi}(t) \in \mathbb{E}^3$. Independent of the constitutive assumption, this is granted when using the internal virtual work (9.39).

Using integration by parts in the second line of (9.39), we can rewrite the internal virtual work as

$$
\begin{aligned}
\delta W^{\mathrm{int}} = \int_{l_1}^{l_2} &\left\{ \delta \mathbf{r} \cdot \mathbf{n}' + \delta \boldsymbol{\phi} \cdot (\mathbf{m}' + \mathbf{r}' \times \mathbf{n}) \right\} \, \mathrm{d}s \\
&- \sum_{i=1}^{2} (-1)^i \left\{ \delta \mathbf{r} \cdot \mathbf{n} + \delta \boldsymbol{\phi} \cdot \mathbf{m} \right\} |_{s=l_i} \; .
\end{aligned} \tag{9.40}
$$

For the static case, where only the internal virtual work contributions equilibrate the external virtual work contributions, (9.40) gives us the form of the external forces that we allow in our beam theory. These are *distributed external forces* $\overline{\mathbf{n}} : I \times \mathbb{R} \to \mathbb{E}^3$ and *distributed external couples* $\overline{\mathbf{m}} : I \times \mathbb{R} \to \mathbb{E}^3$. In addition we allow point-wise defined *external forces* $\overline{\mathbf{n}}_1, \overline{\mathbf{n}}_2 : \mathbb{R} \to \mathbb{E}^3$ and *couples* $\overline{\mathbf{m}}_1, \overline{\mathbf{m}}_2 : \mathbb{R} \to \mathbb{E}^3$ to be applied on the boundaries l_1 and l_2 of the beam. This leads to the virtual work contributions of the external forces of the form

$$
\delta W^{\mathrm{ext}} = \int_{l_1}^{l_2} \left\{ \delta \mathbf{r} \cdot \overline{\mathbf{n}} + \delta \boldsymbol{\phi} \cdot \overline{\mathbf{m}} \right\} \, \mathrm{d}s + \sum_{i=1}^{2} \left\{ \delta \mathbf{r} \cdot \overline{\mathbf{n}}_i + \delta \boldsymbol{\phi} \cdot \overline{\mathbf{m}}_i \right\} |_{s=l_i} \; . \tag{9.41}
$$

If we want to allow countable many point forces inside the beam, the open set (l_1, l_2) has to be divided into further open sets, where the point forces are applied on the corresponding boundaries.

In order to formulate the virtual work contributions of inertial effects, we proceed differently as before. For a meanwhile, we assume that the beam is a three-dimensional continuous body whose points in the reference configuration occupy the positions

$$\mathbf{X}(s,\theta_\alpha) = \mathbf{r}_0(s) + \theta_\alpha \mathbf{D}_\alpha(s) \; . \tag{9.42}$$

Hence, every material point in the reference configuration is addressed by the coordinates $(s,\theta_2,\theta_3) \in B \subset \mathbb{R}^3$. We assume that the cross sections of the beam are spanned by the reference directors $\mathbf{D}_2$ and $\mathbf{D}_3$ such that θ_2 and θ_3 are the cross section coordinates. In the sense of an induced theory, we assume the beam to be a constrained three-dimensional continuum whose current configuration is restricted to

$$\mathbf{x}(s,\theta_\alpha,t) = \mathbf{r}(s,t) + \theta_\alpha \mathbf{d}_\alpha(s,t) \; . \tag{9.43}$$

In fact the kinematical ansatz (9.43) restricts the motion of the cross sections to remain plane and rigid for any motion. The virtual work of the inertial forces of a three-dimensional continuum is commonly defined as

$$\delta W^{\mathrm{dyn}} = -\int_B \delta\mathbf{x}\cdot\ddot{\mathbf{x}}\,\mathrm{d}m = -\int_B \delta\mathbf{x}\cdot\ddot{\mathbf{x}}\,\rho_0\,\mathrm{d}A\mathrm{d}s \; , \tag{9.44}$$

where $\rho_0 : B \to \mathbb{R}$ is the beam's *mass density* per unit volume in its reference configuration and $\mathrm{d}A$ is the cross-sectional surface element in the beam's reference configuration. For convenience, we introduce $\boldsymbol{\rho}(s,\theta_\alpha,t) = \theta_\alpha \mathbf{d}_\alpha(s,t)$ with its corresponding skew-symmetric tensor $\tilde{\boldsymbol{\rho}}(s,\theta_\alpha,t) \in Skw$. Using (9.17), the virtual displacements admissible with respect to the position field (9.43) are obtained as

$$\delta\mathbf{x} = \delta\mathbf{r} + \delta\boldsymbol{\phi}\times\theta_\alpha\mathbf{d}_\alpha = \delta\mathbf{r} + \delta\boldsymbol{\phi}\times\boldsymbol{\rho} = \delta\mathbf{r} - \tilde{\boldsymbol{\rho}}\delta\boldsymbol{\phi} \; . \tag{9.45}$$

Making use of the kinematic relation (9.14) and by taking the first and second time derivative of the position field (9.43), the *velocity* and *acceleration fields* are

$$\begin{aligned} \dot{\mathbf{x}} &= \dot{\mathbf{r}} + \boldsymbol{\omega}\times\theta_\alpha\mathbf{d}_\alpha = \dot{\mathbf{r}} + \boldsymbol{\omega}\times\boldsymbol{\rho} \; , \\ \ddot{\mathbf{x}} &= \ddot{\mathbf{r}} + \dot{\boldsymbol{\omega}}\times\boldsymbol{\rho} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\boldsymbol{\rho}) = \ddot{\mathbf{r}} - \tilde{\boldsymbol{\rho}}\dot{\boldsymbol{\omega}} + \tilde{\boldsymbol{\omega}}\tilde{\boldsymbol{\omega}}\boldsymbol{\rho} \; . \end{aligned} \tag{9.46}$$

For the next steps, we introduce some abbreviations for integral expressions that will appear in the upcoming derivation and which are related to the zeroth, first and second moment with respect to the mass density ρ_0. The *cross section mass density* per unit of s is defined as

$$A_{\rho_0}(s) := \int_{A(s)} \rho_0\,\mathrm{d}A \; , \tag{9.47}$$

where $A(s) = \{(\theta_2,\theta_3) \in \mathbb{R}^2 | (s,\theta_2,\theta_3) \in B\}$. In case that the centerline does not coincide with the *line of centroids* $\mathbf{r}_{\mathrm{c}} : I\times\mathbb{R} \to \mathbb{E}^3, (s,t) \mapsto \mathbf{r}_{\mathrm{c}}(s,t)$ a *coupling term*

$$\mathbf{c} = A_{\rho_0}(\mathbf{r}_{\mathrm{c}} - \mathbf{r}) = \int_{A(s)} \boldsymbol{\rho}\rho_0\,\mathrm{d}A = \int_{A(s)} \theta_\alpha\mathbf{d}_\alpha\rho_0\,\mathrm{d}A \tag{9.48}$$

will remain. Using (9.14) and (9.48), the second time derivative of the coupling term is

$$\ddot{\mathbf{c}} = (\boldsymbol{\omega}\times\mathbf{c})^{\dot{}} = \dot{\boldsymbol{\omega}}\times\mathbf{c} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{c}) = -\tilde{\mathbf{c}}\dot{\boldsymbol{\omega}} + \tilde{\boldsymbol{\omega}}\tilde{\boldsymbol{\omega}}\mathbf{c} \; . \tag{9.49}$$

Note that we denoted the skew-symmetric tensor corresponding to $\mathbf{c}$ by $\tilde{\mathbf{c}}$. The last required integrated quantity is the *cross section inertia density* defined as

$$\mathbf{I}_{\rho_0} = \int_{A(s)} \tilde{\boldsymbol{\rho}}\tilde{\boldsymbol{\rho}}^{\mathrm{T}} \rho_0 \,\mathrm{d}A \,. \tag{9.50}$$

Furthermore, the time derivative of $\mathbf{h} = \mathbf{I}_{\rho_0}\boldsymbol{\omega}$, i.e., the product of the cross section inertia density $\mathbf{I}_{\rho_0}$ and the angular velocity $\boldsymbol{\omega}$ is

$$\dot{\mathbf{h}}(s,t) = (I_{ij}^{\rho_0}\omega_j \mathbf{d}_i)^{\cdot} = I_{ij}^{\rho_0}(\dot{\omega}_j \mathbf{d}_i + \omega_j \dot{\mathbf{d}}_i) \,. \tag{9.51}$$

Note that the components of cross section inertia density $\mathbf{I}_{\rho_0} = I_{ij}^{\rho_0}\mathbf{d}_i \otimes \mathbf{d}_j$ in the $\mathbf{d}_i \otimes \mathbf{d}_j$-basis are constant with respect to time t. Since the cross product of two collinear vectors vanishes, i.e., $0 = \boldsymbol{\omega} \times \boldsymbol{\omega} = \boldsymbol{\omega} \times \omega_k \mathbf{d}_k$, we can extend the above equation, which leads together with (9.14) to the compact form

$$\dot{\mathbf{h}} = I_{ij}^{\rho_0}\mathbf{d}_i \otimes \mathbf{d}_j(\dot{\omega}_k \mathbf{d}_k + \boldsymbol{\omega} \times \omega_k \mathbf{d}_k) + \boldsymbol{\omega} \times I_{ij}^{\rho_0}\omega_j \mathbf{d}_i = \mathbf{I}_{\rho_0}\dot{\boldsymbol{\omega}} + \tilde{\boldsymbol{\omega}}\mathbf{I}_{\rho_0}\boldsymbol{\omega} \,. \tag{9.52}$$

With the acceleration vector (9.46) and the virtual displacements (9.45), the contributions of the inertial forces to the virtual work are given by

$$\delta W^{\mathrm{dyn}} = -\int_B \delta\mathbf{x}\cdot\ddot{\mathbf{x}} \,\mathrm{d}m = -\int_B (\delta\mathbf{r} - \tilde{\boldsymbol{\rho}}\delta\boldsymbol{\phi})\cdot(\ddot{\mathbf{r}} - \tilde{\boldsymbol{\rho}}\dot{\boldsymbol{\omega}} + \tilde{\boldsymbol{\omega}}\tilde{\boldsymbol{\omega}}\boldsymbol{\rho})\rho_0 \,\mathrm{d}A\mathrm{d}s \,. \tag{9.53}$$

The integration over the body B can be split in an integration over the cross section $A(s)$ and an integration along the arc length s. Together with the integrated quantities defined above, the properties of the triple product, the skew symmetry $\tilde{\boldsymbol{\rho}} = -\tilde{\boldsymbol{\rho}}^{\mathrm{T}}$, as well as the relation $\tilde{\mathbf{a}}\tilde{\mathbf{b}}\tilde{\mathbf{b}}\mathbf{a} = -\tilde{\mathbf{b}}\tilde{\mathbf{a}}\tilde{\mathbf{a}}\mathbf{b}$ for $\tilde{\mathbf{a}}, \tilde{\mathbf{b}} \in Skw$, where $\mathbf{a} = \mathrm{ax}(\tilde{\mathbf{a}})$, $\mathbf{b} = \mathrm{ax}(\tilde{\mathbf{b}})$, we obtain

$$\begin{aligned}\delta W^{\mathrm{dyn}} = -\int_{l_1}^{l_2} \Big\{ \delta\mathbf{r}\cdot\Big(\ddot{\mathbf{r}}\int_{A(s)}\rho_0\mathrm{d}A - \int_{A(s)}\tilde{\boldsymbol{\rho}}\rho_0\mathrm{d}A\dot{\boldsymbol{\omega}} + \tilde{\boldsymbol{\omega}}\tilde{\boldsymbol{\omega}}\int_{A(s)}\boldsymbol{\rho}\rho_0\mathrm{d}A\Big) \\ + \delta\boldsymbol{\phi}\cdot\Big(\int_{A(s)}\tilde{\boldsymbol{\rho}}\rho_0\mathrm{d}A\ddot{\mathbf{r}} + \int_{A(s)}\tilde{\boldsymbol{\rho}}\tilde{\boldsymbol{\rho}}^{\mathrm{T}}\rho_0\mathrm{d}A\dot{\boldsymbol{\omega}} + \tilde{\boldsymbol{\omega}}\int_{A(s)}\tilde{\boldsymbol{\rho}}\tilde{\boldsymbol{\rho}}^{\mathrm{T}}\rho_0\mathrm{d}A\boldsymbol{\omega}\Big)\Big\}\mathrm{d}s \\ = -\int_{l_1}^{l_2}\{\delta\mathbf{r}\cdot(A_{\rho_0}\ddot{\mathbf{r}} - \tilde{\mathbf{c}}\dot{\boldsymbol{\omega}} + \tilde{\boldsymbol{\omega}}\tilde{\boldsymbol{\omega}}\mathbf{c}) + \delta\boldsymbol{\phi}\cdot(\tilde{\mathbf{c}}\ddot{\mathbf{r}} + \mathbf{I}_{\rho_0}\dot{\boldsymbol{\omega}} + \tilde{\boldsymbol{\omega}}\mathbf{I}_{\rho_0}\boldsymbol{\omega})\}\mathrm{d}s.\end{aligned} \tag{9.54}$$

Using (9.49) and (9.52), the virtual work contributions of the inertial terms can be written in the compact form

$$\delta W^{\mathrm{dyn}} = -\int_{l_1}^{l_2} \Big\{\delta\mathbf{r}\cdot(A_{\rho_0}\ddot{\mathbf{r}} + \ddot{\mathbf{c}}) + \delta\boldsymbol{\phi}\cdot(\mathbf{c}\times\ddot{\mathbf{r}} + \dot{\mathbf{h}})\Big\} \,\mathrm{d}s \,. \tag{9.55}$$

9.5 Principle of Virtual Work and Equations of Motion

The principle of virtual work can be stated as following. For all admissible virtual displacements and for any instant of time t, the total virtual work of the beam must vanish, i.e.,

$$\delta W^{\mathrm{tot}} = \delta W^{\mathrm{dyn}} + \delta W^{\mathrm{int}} + \delta W^{\mathrm{ext}} = 0 \quad \forall \delta \mathbf{r}, \delta \boldsymbol{\phi}, t \, . \tag{9.56}$$

Inserting the individual contributions of the virtual work (9.39), (9.41) and (9.55) into the principle of virtual work (9.56) leads to the *weak variational formulation* of the spatial nonlinear Timoshenko beam

$$\begin{aligned} \delta W^{\mathrm{tot}} = \int_{l_1}^{l_2} \{ & \delta \mathbf{r} \cdot (\overline{\mathbf{n}} - A_{\rho_0} \ddot{\mathbf{r}} - \ddot{\mathbf{c}}) + \delta \boldsymbol{\phi} \cdot (\overline{\mathbf{m}} - \mathbf{c} \times \ddot{\mathbf{r}} - \dot{\mathbf{h}}) - (\delta \mathbf{r}' - \delta \boldsymbol{\phi} \times \mathbf{r}') \cdot \mathbf{n} \\ & - \delta \boldsymbol{\phi}' \cdot \mathbf{m} \} \, \mathrm{d}s + \sum_{i=1}^{2} \{ \delta \mathbf{r} \cdot \overline{\mathbf{n}}_i + \delta \boldsymbol{\phi} \cdot \overline{\mathbf{m}}_i \} |_{s=l_i} = 0 \quad \forall \delta \mathbf{r}, \, \delta \boldsymbol{\phi}, \, t \, . \end{aligned} \tag{9.57}$$

Using the internal virtual work in the form (9.40), i.e., after integration by parts, we end up with the *strong variational formulation* of the spatial nonlinear Timoshenko beam

$$\begin{aligned} \delta W^{\mathrm{tot}} = \int_{l_1}^{l_2} & \{ \delta \mathbf{r} \cdot (\mathbf{n}' + \overline{\mathbf{n}} - A_{\rho_0} \ddot{\mathbf{r}} - \ddot{\mathbf{c}}) \\ & + \delta \boldsymbol{\phi} \cdot (\mathbf{m}' + \mathbf{r}' \times \mathbf{n} + \overline{\mathbf{m}} - \mathbf{c} \times \ddot{\mathbf{r}} - \dot{\mathbf{h}}) \} \, \mathrm{d}s \\ & + \sum_{i=1}^{2} \{ \delta \mathbf{r} \cdot (\overline{\mathbf{n}}_i - (-1)^i \mathbf{n}) + \delta \boldsymbol{\phi} \cdot (\overline{\mathbf{m}}_i - (-1)^i \mathbf{m}) \} |_{s=l_i} = 0, \; \forall \delta \mathbf{r}, \, \delta \boldsymbol{\phi}, \, t. \end{aligned} \tag{9.58}$$

By the fundamental lemma of calculus of variations, (9.58) can only be fulfilled if the equations of motion of the Timoshenko beam

$$\begin{aligned} \mathbf{n}' + \overline{\mathbf{n}} &= A_{\rho_0} \ddot{\mathbf{r}} + \ddot{\mathbf{c}} \\ \mathbf{m}' + \mathbf{r}' \times \mathbf{n} + \overline{\mathbf{m}} &= \mathbf{c} \times \ddot{\mathbf{r}} + \dot{\mathbf{h}} \end{aligned} \tag{9.59}$$

are satisfied in the interior of the beam, $s \in (l_1, l_2)$, together with the boundary conditions $\mathbf{n}(l_1, t) = -\overline{\mathbf{n}}_1(t)$, $\mathbf{m}(l_1, t) = -\overline{\mathbf{m}}_1(t)$ and $\mathbf{n}(l_2, t) = \overline{\mathbf{n}}_2(t)$, $\mathbf{m}(l_2, t) = \overline{\mathbf{m}}_2(t)$, see also (Antman, 2005; Dill, 1992). Certainly appropriate initial conditions in time have to be stated.

For getting an interpretation of the generalized internal forces $\mathbf{n}$ and $\mathbf{m}$ introduced in the internal virtual work contributions (9.39), we consider the static problem of a cantilever beam with reference length L, i.e., $I = [0, L]$. The beam is clamped at $s = 0$ and subjected to distributed forces $\overline{\mathbf{n}}$ and couples $\overline{\mathbf{m}}$ as well as a point force and a couple $\overline{\mathbf{n}}_2$, $\overline{\mathbf{m}}_2$ at $s = L$. For the clamped end, the boundary conditions and the admissible virtual displacements and rotations read

$$\mathbf{r}(0) = 0 \;, \quad \delta\mathbf{r}(0) = 0 \;, \quad \boldsymbol{\phi}(0) = 0 \;, \quad \delta\boldsymbol{\phi}(0) = 0 \;. \tag{9.60}$$

Disregarding inertial effects and applying the admissible virtual displacements at the boundary (9.60), the equations of motion (9.59) turn into the equilibrium conditions

$$\begin{aligned} \mathbf{n}' + \overline{\mathbf{n}} &= 0 \\ \mathbf{m}' + \mathbf{r}' \times \mathbf{n} + \overline{\mathbf{m}} &= 0 \;, \end{aligned} \tag{9.61}$$

together with the force boundary conditions $\mathbf{n}(L) = \overline{\mathbf{n}}_2$ and $\mathbf{m}(L) = \overline{\mathbf{m}}_2$ at the end of the beam. The boundary condition $\mathbf{n}(L) = \overline{\mathbf{n}}_2$ identifies $\mathbf{n}(L)$ as the force applied at the end $s = L$; the same holds for the couple.

For $\overline{s} \in I$, the fundamental theorem of calculus allows us to write $\int_{\overline{s}}^{L} \mathbf{n}' \mathrm{d}s = \mathbf{n}(L) - \mathbf{n}(\overline{s})$. Using the first equilibrium equation in (9.61) together with the boundary condition $\mathbf{n}(L) = \overline{\mathbf{n}}_2$, we get

$$\mathbf{n}(\overline{s}) = \mathbf{n}(L) - \int_{\overline{s}}^{L} \mathbf{n}' \,\mathrm{d}s = \overline{\mathbf{n}}_2 + \int_{\overline{s}}^{L} \overline{\mathbf{n}} \,\mathrm{d}s \;. \tag{9.62}$$

Accordingly, $\mathbf{n}(\overline{s})$ corresponds to the force exerted by the segment $(\overline{s}, L]$ on the part $[0, \overline{s}]$. We consequently identify $\mathbf{n} = N\mathbf{d}_1 + Q_2\mathbf{d}_2 + Q_3\mathbf{d}_3$ as the *resultant contact force*, where $N = \partial W/\partial \Gamma_1$ and $Q_\alpha = \partial W/\partial \Gamma_\alpha$ correspond to the axial force and the shear forces, respectively.

Using the relation $\int_{\overline{s}}^{L} \mathbf{m}' \mathrm{d}s = \mathbf{m}(L) - \mathbf{m}(\overline{s})$ together with the second equilibrium equation in (9.61), the boundary condition $\mathbf{m}(L) = \overline{\mathbf{m}}_2$ and subsequent integration by parts, we obtain

$$\begin{aligned} \mathbf{m}(\overline{s}) &= \mathbf{m}(L) + \int_{\overline{s}}^{L} \{\mathbf{r}' \times \mathbf{n} + \overline{\mathbf{m}}\} \,\mathrm{d}s \\ &= \overline{\mathbf{m}}_2 + \mathbf{r}(L) \times \mathbf{n}(L) - \mathbf{r}(\overline{s}) \times \mathbf{n}(\overline{s}) + \int_{\overline{s}}^{L} \{\overline{\mathbf{m}} - \mathbf{r} \times \mathbf{n}'\} \,\mathrm{d}s \;. \end{aligned} \tag{9.63}$$

For a fixed $\overline{s} \in I$, let $\Delta\mathbf{r}(s) = \mathbf{r}(s) - \mathbf{r}(\overline{s})$ be the vector connecting the point $\mathbf{r}(\overline{s})$ with $\mathbf{r}(s)$ for an arbitrary $s \in I$. Substituting $\mathbf{r}(s) = \mathbf{r}(\overline{s}) + \Delta\mathbf{r}(s)$ in the above equation, using the first equilibrium equation in (9.61) and the identity (9.62), the terms with $\mathbf{r}(\overline{s})$ cancel and we get

$$\mathbf{m}(\overline{s}) = \overline{\mathbf{m}}_2 + \Delta\mathbf{r}(L) \times \overline{\mathbf{n}}_2 + \int_{\overline{s}}^{L} \{\overline{\mathbf{m}} + \Delta\mathbf{r} \times \overline{\mathbf{n}}\} \,\mathrm{d}s \;. \tag{9.64}$$

From (9.64), it becomes apparent that $\mathbf{m}(\overline{s})$ is the couple exerted by the segment $(\overline{s}, L]$ on the part $[0, \overline{s}]$. We identify $\mathbf{m} = T\mathbf{d}_1 + M_2\mathbf{d}_2 + M_3\mathbf{d}_3$ as the *resultant contact couple*, where $T = \partial W/\partial \kappa_1$ and $M_\alpha = \partial W/\partial \kappa_\alpha$ correspond to the twisting couple and the bending couples, respectively.

9.6 Constrained Beam Theories

In the preceding sections, we have established the governing equations for a spatial nonlinear Timoshenko beam. To end up with an Euler–Bernoulli or an inextensible Euler–Bernoulli beam, we have to prescribe further constraints. For the Euler–Bernoulli beam, the current director $\mathbf{d}_1$ must align with the centerline's tangent $\mathbf{r}'$. For an inextensible Euler–Bernoulli beam, also the norm of the centerline's tangent must remain constant, i.e., $\|\mathbf{r}'\| = \|\mathbf{r}_0'\| = 1$. All required constraint conditions can be formulated in the form $g(s,t) = 0$. To incorporate such a constraint into the variational formulation, i.e., in the principle of virtual work (9.56), we augment the strain energy functional of the whole beam (9.23) in accordance with

$$E^* = E + \overline{E} \, , \quad \overline{E} = -\int_{l_1}^{l_2} g(s,t)\sigma(s,t) \, \mathrm{d}s \, , \tag{9.65}$$

where the Lagrange multiplier field $\sigma : I \times \mathbb{R} \to \mathbb{R}$ has been introduced. Again, the internal virtual work is obtained by the variation of the strain energy functional, which in the constrained case reads as

$$\delta W^{\mathrm{int},*} = -\delta E^* = -\delta E - \delta\overline{E} = \delta W^{\mathrm{int}} + \delta W_{\mathrm{c}}^{\mathrm{int}} \, , \; \delta W_{\mathrm{c}}^{\mathrm{int}} = \delta W_{\mathrm{c},1}^{\mathrm{int}} + \delta W_{\mathrm{c},2}^{\mathrm{int}} \, ,$$
$$\delta W_{\mathrm{c},1}^{\mathrm{int}} = \int_{l_1}^{l_2} g(s,t)\delta\sigma(s,t) \, \mathrm{d}s \, , \quad \delta W_{\mathrm{c},2}^{\mathrm{int}} = \int_{l_1}^{l_2} \delta g(s,t)\sigma(s,t) \, \mathrm{d}s \, . \tag{9.66}$$

The first additional internal virtual work contribution $\delta W_{\mathrm{c},1}^{\mathrm{int}}$ corresponds to the weak form of the constraint condition, which is important for a later numerical treatment. The second contribution $\delta W_{\mathrm{c},2}^{\mathrm{int}}$ represents the virtual work of the constraint forces. For multiple constraints, the contributions are just summed up.

9.6.1 Nonlinear Euler–Bernoulli Beam

If the current director $\mathbf{d}_1$ aligns with the centerline's tangent $\mathbf{r}'$, then the cross sections spanned by the directors $\mathbf{d}_\alpha$ remain orthogonal to $\mathbf{r}'$. This restriction can be formulated by the two conditions

$$g_\alpha(s,t) = \Gamma_\alpha(s,t) = \mathbf{d}_\alpha(s,t) \cdot \mathbf{r}'(s,t) = 0 \, . \tag{9.67}$$

For this case, the constraint conditions coincide with vanishing shear deformations, see (9.29). Identifying $\delta g_\alpha = \delta\Gamma_\alpha = \mathbf{d}_\alpha \cdot (\delta\mathbf{r}' - \delta\boldsymbol{\phi} \times \mathbf{r}')$, together with its definition given in (9.37), the virtual work contributions of the constraints are

$$\delta W^{\text{int}}_{\text{c},1} = \int_{l_1}^{l_2} \delta\sigma_\alpha g_\alpha \, \mathrm{d}s = \int_{l_1}^{l_2} \delta\sigma_\alpha (\mathbf{d}_\alpha \cdot \mathbf{r}') \, \mathrm{d}s \; ,$$
$$\delta W^{\text{int}}_{\text{c},2} = \int_{l_1}^{l_2} \sigma_\alpha \delta g_\alpha \, \mathrm{d}s = \int_{l_1}^{l_2} (\delta\mathbf{r}' - \delta\boldsymbol{\phi} \times \mathbf{r}') \cdot \mathbf{n}_C \, \mathrm{d}s \; , \tag{9.68}$$

where we have introduced $\mathbf{n}_C = \sigma_\alpha \mathbf{d}_\alpha$. Hence, the two Lagrange multiplier fields σ_α act as shear constraint forces to enforce the vanishing shear deformations Γ_α in the Euler–Bernoulli beam.

9.6.2 Nonlinear Inextensible Euler–Bernoulli Beam

If, in addition to (9.67), we further prescribe an inextensibility constraint, the constraint condition

$$g_1(s,t) = \Gamma_1(s,t) - 1 = \mathbf{d}_1(s,t) \cdot \mathbf{r}'(s,t) - 1 = 0 \tag{9.69}$$

must hold. For the physically reasonable situation that $\mathbf{d}_1 \cdot \mathbf{r}' > 0$ and since $\mathbf{d}_\alpha \cdot \mathbf{r}' = 0$, the constraint condition (9.69) coincides with the condition that the centerline's tangent has unit length. Indeed $0 = \|\mathbf{r}'\| - 1 = [(\mathbf{d}_i \cdot \mathbf{r}')(\mathbf{d}_i \cdot \mathbf{r}')]^{1/2} - 1 \overset{(9.67)}{=} [(\mathbf{d}_1 \cdot \mathbf{r}')(\mathbf{d}_1 \cdot \mathbf{r}')]^{1/2} - 1 = \mathbf{d}_1 \cdot \mathbf{r}' - 1$. Identifying $\delta g_1 = \delta\Gamma_1$, together with its definition given in (9.37), the virtual work contributions of the constraints (9.67) and (9.69) are

$$\delta W^{\text{int}}_{\text{c},1} = \int_{l_1}^{l_2} \delta\sigma_i g_i \, \mathrm{d}s = \int_{l_1}^{l_2} \delta\sigma_i (\mathbf{d}_i \cdot \mathbf{r}' - \delta_{i1}) \, \mathrm{d}s \; ,$$
$$\delta W^{\text{int}}_{\text{c},2} = \int_{l_1}^{l_2} \sigma_i \delta g_i \, \mathrm{d}s = \int_{l_1}^{l_2} (\delta\mathbf{r}' - \delta\boldsymbol{\phi} \times \mathbf{r}') \cdot \mathbf{n}_C \, \mathrm{d}s \; , \tag{9.70}$$

with $\mathbf{n}_C = \sigma_i \mathbf{d}_i$. For the inextensible Euler–Bernoulli beam, the resultant contact forces are pure reaction forces that guarantee unshearability and inextensiblity of the beam.

9.7 Constrained and Unconstrained Planar Beam Theories

In the previous section, we have shown how to augment the principle of virtual work to also treat the Euler–Bernoulli beam theory and its inextensible version as a constrained theory in a variational setting. In this section, we will restrict the motion of the beams to be planar. Furthermore, we will work out the virtual work contributions of the constrained theories in detail for a possible finite element analysis as it is presented in Harsch and Eugster (2020). For the (inextensible)

Euler–Bernoulli beam, we will also choose kinematical descriptions that satisfy the constraint conditions intrinsically. These formulations are then called minimal formulations referring to the terminology of minimal coordinates in finite degree of freedom mechanics.

9.7.1 Timoshenko Beam

For the planar case, the kinematics of the beam's centerline is restricted to the $\mathbf{e}_1$-$\mathbf{e}_2$-plane. The position vector of the centerline, the tangent vector and its derivative with respect to the arc length parameter s for the reference and current configurations are given by

$$
\begin{aligned}
\mathbf{r}_0(s) &= X(s)\mathbf{e}_1 + Y(s)\mathbf{e}_2 \ , & \mathbf{r}(s,t) &= x(s,t)\mathbf{e}_1 + y(s,t)\mathbf{e}_2 \ , \\
\mathbf{r}_0'(s) &= X'(s)\mathbf{e}_1 + Y'(s)\mathbf{e}_2 \ , & \mathbf{r}'(s,t) &= x'(s,t)\mathbf{e}_1 + y'(s,t)\mathbf{e}_2 \ , \\
\mathbf{r}_0''(s) &= X''(s)\mathbf{e}_1 + Y''(s)\mathbf{e}_2 \ , & \mathbf{r}''(s,t) &= x''(s,t)\mathbf{e}_1 + y''(s,t)\mathbf{e}_2 \ ,
\end{aligned}
\tag{9.71}
$$

using the coordinate functions $X, Y \colon I \to \mathbb{R}$ and $x, y \colon I \times \mathbb{R} \to \mathbb{R}$. Planar rotations are given by rotations around $\mathbf{D}_3 = \mathbf{d}_3 = \mathbf{e}_3$. As depicted in Fig. 9.2, the reference

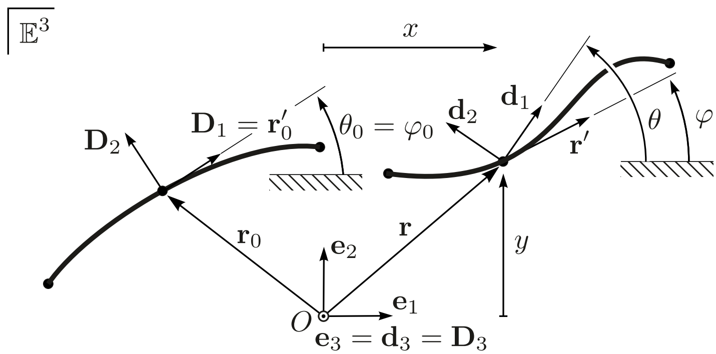

Fig. 9.2 Graphic illustration of planar position and rotation fields.

and current director triads are

$$
\begin{aligned}
\mathbf{D}_1 &= \cos\theta_0(s)\mathbf{e}_1 + \sin\theta_0(s)\mathbf{e}_2 \ , & \mathbf{D}_2 &= -\sin\theta_0(s)\mathbf{e}_1 + \cos\theta_0(s)\mathbf{e}_2 \ , \\
\mathbf{d}_1 &= \cos\theta(s,t)\mathbf{e}_1 + \sin\theta(s,t)\mathbf{e}_2 \ , & \mathbf{d}_2 &= -\sin\theta(s,t)\mathbf{e}_1 + \cos\theta(s,t)\mathbf{e}_2 \ ,
\end{aligned}
\tag{9.72}
$$

where $\theta_0 : I \to \mathbb{R}$ parameterizes the absolute angle of the reference director $\mathbf{D}_1$ with respect to the vector $\mathbf{e}_1$ and $\theta : I \times \mathbb{R} \to \mathbb{R}$ the absolute angle of the current director $\mathbf{d}_1$.

The rotation of the reference configuration is given by $\mathbf{R}_0 = R^0_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$, with the components $R^0_{ij} = \mathbf{e}_i \cdot \mathbf{D}_j$. The current rotation field is given analogously by $\mathbf{R} = R_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$ with $R_{ij} = \mathbf{e}_i \cdot \mathbf{d}_j$. Both components can be written in matrix notation as

$$[R^0_{ij}] = \begin{pmatrix} \cos\theta_0 & -\sin\theta_0 & 0 \\ \sin\theta_0 & \cos\theta_0 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \quad [R_{ij}] = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{9.73}$$

The rotation field $\boldsymbol{\Lambda} = \mathbf{R}\mathbf{R}_0^{\mathrm{T}} = R_{ik}R^0_{jk}\mathbf{e}_i \otimes \mathbf{e}_j = \Lambda_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$ has the components in matrix form given by

$$[\Lambda_{ij}] = [R_{ik}][R^0_{kj}]^{\mathrm{T}} = \begin{pmatrix} \cos(\theta-\theta_0) & -\sin(\theta-\theta_0) & 0 \\ \sin(\theta-\theta_0) & \cos(\theta-\theta_0) & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{9.74}$$

Computing the derivatives of the directors (9.72) with respect to s, we get

$$\mathbf{D}_1' = \theta_0'\mathbf{D}_2,\ \mathbf{D}_2' = -\theta_0'\mathbf{D}_1,\ \mathbf{d}_1' = \theta'\mathbf{d}_1,\ \mathbf{d}_2' = -\theta'\mathbf{d}_1,\ \mathbf{D}_3' = \mathbf{d}_3' = 0 . \tag{9.75}$$

The curvatures $\tilde{\boldsymbol{\kappa}} - \tilde{\boldsymbol{\kappa}}_0 = (\tilde{\kappa}_{ij} - \tilde{\kappa}^0_{ij})\mathbf{D}_i \otimes \mathbf{D}_j$ and their associated axial vectors $\boldsymbol{\kappa} - \boldsymbol{\kappa}_0 = (\kappa_i - \kappa^0_i)\mathbf{d}_i$, together with their components are given in (9.33). The components can be obtained by inserting the directors (9.72) and their partial derivatives (9.75). This yields the very simple planar curvatures

$$\begin{aligned} [\tilde{\kappa}_{ij}] &= \begin{pmatrix} 0 & -\theta' & 0 \\ \theta' & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \qquad [\tilde{\kappa}^0_{ij}] = \begin{pmatrix} 0 & -\theta_0' & 0 \\ \theta_0' & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \\ \boldsymbol{\kappa} &= \kappa_3\mathbf{e}_3 = \theta'\mathbf{e}_3 , \qquad \boldsymbol{\kappa}_0 = \kappa^0_3\mathbf{e}_3 = \theta_0'\mathbf{e}_3 . \end{aligned} \tag{9.76}$$

The generalized strain measure $\boldsymbol{\Gamma}$ from (9.29) is computed using (9.71) together with (9.72) and reads

$$\boldsymbol{\Gamma} = \Gamma_1\mathbf{D}_1 + \Gamma_2\mathbf{D}_2 = (\mathbf{r}'\cdot\mathbf{d}_1)\mathbf{D}_1 + (\mathbf{r}'\cdot\mathbf{d}_2)\mathbf{D}_2 . \tag{9.77}$$

The third component Γ_3 of the strain measure vanishes, because the current tangential vector has no component in $\mathbf{d}_3$-direction.

The velocities and accelerations of the beam's centerline are easily computed from (9.71) as

$$\dot{\mathbf{r}} = \dot{x}\mathbf{e}_1 + \dot{y}\mathbf{e}_2 , \quad \ddot{\mathbf{r}} = \ddot{x}\mathbf{e}_1 + \ddot{y}\mathbf{e}_2 . \tag{9.78}$$

The rate of change of the director triad $\dot{\mathbf{d}}_i$ is obtained by replacing the derivative with respect to s in (9.75) by the time derivative. The angular velocity (9.15) can be easily computed in the $\mathbf{e}_i \otimes \mathbf{e}_j$-basis as

$$\tilde{\boldsymbol{\omega}} = \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}} = \tilde{\omega}_{ij}\mathbf{e}_i \otimes \mathbf{e}_j \ , \quad [\tilde{\omega}_{ij}] = [\dot{R}_{ik}][R_{kj}]^{\mathrm{T}} = \begin{pmatrix} 0 & -\dot{\theta} & 0 \\ \dot{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ . \tag{9.79}$$

Since $\mathbf{d}_3 = \mathbf{e}_3$, the associated axial vector $\boldsymbol{\omega}$ can be represented as

$$\boldsymbol{\omega} = \mathrm{ax}(\tilde{\boldsymbol{\omega}}) = \dot{\theta}\mathbf{e}_3 \ . \tag{9.80}$$

The variation of the beam's centerline and the variation of the tangent vector are

$$\delta\mathbf{r} = \delta x\mathbf{e}_1 + \delta y\mathbf{e}_2 \ , \quad \delta\mathbf{r}' = \delta x'\mathbf{e}_1 + \delta y'\mathbf{e}_2 \ . \tag{9.81}$$

Also for the virtual rotation $\delta\tilde{\boldsymbol{\phi}}$ given in (9.18), it is easiest to compute its components in the $\mathbf{e}_i \otimes \mathbf{e}_j$-basis as

$$\delta\tilde{\boldsymbol{\phi}} = \delta\boldsymbol{\Lambda}\boldsymbol{\Lambda}^{\mathrm{T}} = \delta\tilde{\phi}_{ij}\mathbf{e}_i \otimes \mathbf{e}_j, \quad [\delta\tilde{\phi}_{ij}] = [\delta\Lambda_{ik}][\Lambda_{kj}]^{\mathrm{T}} = \begin{pmatrix} 0 & -\delta\theta & 0 \\ \delta\theta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \ . \tag{9.82}$$

The associated axial vector and its partial derivative with respect to the arc length parameter s read

$$\delta\boldsymbol{\phi} = \mathrm{ax}(\delta\tilde{\boldsymbol{\phi}}) = \delta\theta\mathbf{e}_3 \ , \quad \delta\boldsymbol{\phi}' = \delta\theta'\mathbf{e}_3 \ . \tag{9.83}$$

Dropping the index in the shear force Q_2 and the bending couple M_3, the planar form of the resultant contact forces and couples defined in (9.39) are

$$\begin{aligned} \mathbf{n} &= N\mathbf{d}_1 + Q\mathbf{d}_2 = (N\cos\theta - Q\sin\theta)\,\mathbf{e}_1 + (N\sin\theta + Q\cos\theta)\mathbf{e}_2 \\ &= n_1\mathbf{e}_1 + n_2\mathbf{e}_2 \ , \\ \mathbf{m} &= M\mathbf{d}_3 = M\mathbf{e}_3 \ . \end{aligned} \tag{9.84}$$

Note the just introduced abbreviations $n_1 = N\cos\theta - Q\sin\theta$ and $n_2 = N\sin\theta + Q\cos\theta$.

For the sake of compact notation, we define the mapping

$$^{\perp} : \mathbb{E}^3 \to \mathbb{E}^3 \ , \quad \mathbf{a} \mapsto \mathbf{a}^{\perp} = \mathbf{A}\mathbf{a} \ , \quad \mathbf{A} = \mathbf{e}_2 \otimes \mathbf{e}_1 - \mathbf{e}_1 \otimes \mathbf{e}_2 \ , \tag{9.85}$$

which rotates a vector in the $\mathbf{e}_1$-$\mathbf{e}_2$ plane around the $\mathbf{e}_3$-axis by $\pi/2$ in the mathematically positive sense. Accordingly, we can write

$$\delta\mathbf{r}' - \delta\boldsymbol{\phi} \times \mathbf{r}' = \delta x'\mathbf{e}_1 + \delta y'\mathbf{e}_2 + \delta\theta y'\mathbf{e}_1 - \delta\theta x'\mathbf{e}_2 = \delta\mathbf{r}' - \delta\theta\mathbf{r}'^{\perp} \ . \tag{9.86}$$

Inserting the variation of the tangent vector (9.81), the above computed expression, the variation of the virtual rotation (9.83) and the planar contact forces and couples given in (9.84) into the internal virtual work contributions (9.39), we obtain its planar form

$$\delta W^{\text{int}} = -\int_{l_1}^{l_2} \{(\delta \mathbf{r}' - \delta\theta \mathbf{r}'^{\perp}) \cdot \mathbf{n} + \delta\theta' M\} \, \mathrm{d}s \, . \tag{9.87}$$

The external forces $\overline{\mathbf{n}} = \overline{n}_1 \mathbf{e}_1 + \overline{n}_2 \mathbf{e}_2$ only act in the $\mathbf{e}_1$-$\mathbf{e}_2$-plane and external couples are of the form $\overline{\mathbf{m}} = \overline{M}\mathbf{e}_3$. Same holds for the external forces $\overline{\mathbf{n}}_i = \overline{n}_1^i \mathbf{e}_1 + \overline{n}_2^i \mathbf{e}_2$ and couples $\overline{\mathbf{m}}_i = \overline{M}_i \mathbf{e}_3$ at the boundaries of the beam. The virtual work contributions of the external forces are straightforwardly obtained by inserting (9.81) and (9.83) into (9.41), i.e.,

$$\delta W^{\text{ext}} = \int_{l_1}^{l_2} \left\{ \delta \mathbf{r} \cdot \overline{\mathbf{n}} + \delta\theta \overline{M} \right\} \, \mathrm{d}s + \sum_{i=1}^{2} \left\{ \delta \mathbf{r} \cdot \overline{\mathbf{n}}_i + \delta\theta \overline{M}_i \right\} |_{s=l_i} \, . \tag{9.88}$$

For the sake of brevity, we assume that the centerline $\mathbf{r}$ corresponds with the line of centroids $\mathbf{r}_{\mathrm{c}}$. Hence, the coupling term $\mathbf{c} = A_{\rho_0}(\mathbf{r}_{\mathrm{c}} - \mathbf{r})$ and its time derivatives vanish. Moreover, we assume a homogeneous mass distribution in the cross section such that the directors $\mathbf{d}_\alpha$ coincide with the geometric principal axes of the beam's cross section. Thus, the components of the cross section inertia density $\mathbf{I}_{\rho_0} = I_{ij}^{\rho_0} \mathbf{d}_i \otimes \mathbf{d}_j$ can be arranged in matrix form given by the diagonal matrix

$$[I_{ij}^{\rho_0}] = \mathrm{Diag}[I_1, I_2, I_3] = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \, . \tag{9.89}$$

The product of the cross section inertia density with the angular velocity and its time derivative (9.52) are given by

$$\mathbf{h} = \mathbf{I}_{\rho_0} \boldsymbol{\omega} = I_3 \dot{\theta} \mathbf{e}_3 \, , \quad \dot{\mathbf{h}} = \mathbf{I}_{\rho_0} \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{I}_{\rho_0} \boldsymbol{\omega} = I_3 \ddot{\theta} \mathbf{e}_3 \, . \tag{9.90}$$

Using the above derived simplifications, together with the variation and acceleration of the centerline given in (9.81) and (9.78), respectively, the virtual work contributions of the inertial forces (9.55) reduce to

$$\delta W^{\text{dyn}} = -\int_{l_1}^{l_2} \left\{ \delta \mathbf{r} \cdot A_{\rho_0} \ddot{\mathbf{r}} + \delta\theta I_3 \ddot{\theta} \right\} \, \mathrm{d}s \, . \tag{9.91}$$

The total virtual work of the planar Timoshenko beam is given by assembling the individual contributions from (9.87), (9.88) and (9.91) which yields

$$\begin{aligned} \delta W^{\text{tot}} = \int_{l_1}^{l_2} & \{ \delta \mathbf{r} \cdot (\overline{\mathbf{n}} - A_{\rho_0} \ddot{\mathbf{r}}) + \delta\theta (\overline{M} - I_3 \ddot{\theta} + \mathbf{r}'^{\perp} \cdot \mathbf{n}) - \delta \mathbf{r}' \cdot \mathbf{n} - \delta\theta' M \} \mathrm{d}s \\ & + \sum_{i=1}^{2} \left\{ \delta \mathbf{r} \cdot \overline{\mathbf{n}}_i + \delta\theta \overline{M}_i \right\} |_{s=l_i} \, . \end{aligned} \tag{9.92}$$

By identifying the first set of constraint conditions (9.67) with the quantities given in (9.77), we get

$$g_2(s,t) = \Gamma_2 = \mathbf{r}' \cdot \mathbf{d}_2 = 0 \;, \quad g_3(s,t) = \Gamma_3 = \mathbf{r}' \cdot \mathbf{d}_3 = 0 \;, \tag{9.93}$$

where the constraint g_3 is trivially fulfilled. With $\delta g_2 = \delta\Gamma_2 = \delta\mathbf{r}' \cdot \mathbf{d}_2 - \delta\theta\mathbf{r}' \cdot \mathbf{d}_1$, the spatial virtual work contributions given in (9.68) reduces to

$$\delta W^{\text{int}}_{\text{c},1} = \int_{l_1}^{l_2} \delta\sigma_2(\mathbf{r}' \cdot \mathbf{d}_2) \, \mathrm{d}s \;, \quad \delta W^{\text{int}}_{\text{c},2} = \int_{l_1}^{l_2} \sigma_2(\delta\mathbf{r}' \cdot \mathbf{d}_2 - \delta\theta\mathbf{r}' \cdot \mathbf{d}_1) \, \mathrm{d}s \;. \tag{9.94}$$

Adding the virtual work contributions above to the unconstrained total virtual work of the Timoshenko beam (9.92), the Euler–Bernoulli beam model is obtained.

In addition to (9.93), the inextensibility condition (9.69) can be met in the form

$$g_1(s,t) = \Gamma_1(s,t) - 1 = \mathbf{d}_1 \cdot \mathbf{r}' - 1 = \|\mathbf{r}'\| - 1 = g - 1 = 0 \;, \tag{9.95}$$

where we have introduced the abbreviation $g = ||\mathbf{r}'||$. The variation of the current stretch is given by

$$\delta\Gamma_1 = \delta g = \frac{\delta\mathbf{r}' \cdot \mathbf{r}'}{g} \;, \tag{9.96}$$

which leads for the inextensible Euler–Bernoulli beam to the additional virtual work contributions

$$\delta W^{\text{int}}_{\text{c},1} = \int_{l_1}^{l_2} \delta\sigma_1(g-1) \, \mathrm{d}s \;, \quad \delta W^{\text{int}}_{\text{c},2} = \int_{l_1}^{l_2} \sigma_1 \frac{\delta\mathbf{r}' \cdot \mathbf{r}'}{g} \, \mathrm{d}s \;. \tag{9.97}$$

Adding the virtual work contributions above, together with (9.94) to the unconstrained total virtual work of the planar Timoshenko beam (9.92), the inextensible planar Euler–Bernoulli beam model is obtained.

9.7.2 Euler–Bernoulli Beam

In this section, we show how to formulate the planar Euler–Bernoulli beam theory with coordinates that meet the required constraint conditions and for which the constraint forces become obsolete. By inserting the planar versions for the tangent vector (9.71) and the second director (9.72) into the first equality given in (9.93), we can express the absolute angle of the current cross section as

$$\theta = \arctan\left(\frac{y'}{x'}\right) \;. \tag{9.98}$$

Recapitulating the abbreviation $g = \|\mathbf{r}'\| = [(x')^2 + (y')^2]^{1/2}$, the variation and the partial derivative with respect to s of (9.98) are given by

$$\delta\theta = \frac{x'\delta y' - y'\delta x'}{g^2} = \frac{\mathbf{r}'^{\perp} \cdot \delta\mathbf{r}'}{g^2} \;, \quad \theta' = \frac{x'y'' - y'x''}{g^2} = \frac{\mathbf{r}'^{\perp} \cdot \mathbf{r}''}{g^2} \;, \tag{9.99}$$

where we have used $\frac{\mathrm{d}}{\mathrm{d}x}\arctan(x) = (1+x^2)^{-1}$ and the chain rule of differential calculus. With the same arguments, the first and second time derivative of (9.98) are given by

$$\dot{\theta} = \frac{\mathbf{r}'^{\perp}\cdot\dot{\mathbf{r}}'}{g^2}\ , \quad \ddot{\theta} = \frac{\mathbf{r}'^{\perp}\cdot\ddot{\mathbf{r}}'}{g^2} - \frac{2\dot{\theta}\mathbf{r}'\cdot\dot{\mathbf{r}}'}{g^2}\ , \tag{9.100}$$

where, for the second identity, we have used the property $\mathbf{a}^{\perp}\cdot\mathbf{a} = 0\ \forall\mathbf{a}\in\mathbb{E}^3$. Using the skew symmetry of the rotation operation (9.85)[1] and the linearity of the dot product, the variation of θ' is computed straightforwardly as

$$\delta\theta' = \frac{1}{g^2}\left(\delta\mathbf{r}''\cdot\mathbf{r}'^{\perp} - \delta\mathbf{r}'\cdot[2\theta'\mathbf{r}' + \mathbf{r}''^{\perp}]\right)\ . \tag{9.101}$$

With the above derived relations at hand, we are able to replace all quantities depending on θ in the virtual work of the Euler–Bernoulli beam. After minor rearrangements, this leads to the compact internal virtual work contributions of the Euler–Bernoulli beam

$$\begin{aligned}\delta W^{\mathrm{int}} &= -\int_{l_1}^{l_2}\{\delta\varGamma_1\,N + \delta\theta' M\}\,\mathrm{d}s\\ &= -\int_{l_1}^{l_2}\left\{\frac{1}{g}\delta\mathbf{r}'\cdot\left(\mathbf{r}'N - \frac{M}{g}[2\theta'\mathbf{r}' + \mathbf{r}''^{\perp}]\right) + \delta\mathbf{r}''\cdot\mathbf{r}'^{\perp}\frac{M}{g^2}\right\}\mathrm{d}s\ .\end{aligned} \tag{9.102}$$

Note, that the integral of the virtual work has to exist, thus we require the beam's centerline to be at least C^1-continuous. This has to be kept in mind for a later discretization.

Inserting the first identity of (9.99) into the planar virtual work contributions of the external forces, given in (9.88), we get

$$\begin{aligned}\delta W^{\mathrm{ext}} = &\int_{l_1}^{l_2}\left\{\delta\mathbf{r}\cdot\overline{\mathbf{n}} + \frac{\mathbf{r}'^{\perp}\cdot\delta\mathbf{r}'}{g^2}\overline{M}\right\}\mathrm{d}s\\ &+ \sum_{i=1}^{2}\left\{\delta\mathbf{r}\cdot\overline{\mathbf{n}}_i + \frac{\mathbf{r}'^{\perp}\cdot\delta\mathbf{r}'}{g^2}\overline{M}_i\right\}\Bigg|_{s=l_i}\ .\end{aligned} \tag{9.103}$$

Using the relations given in (9.99) and (9.100), the virtual work contributions of the inertial forces (9.91) are given by

$$\delta W^{\mathrm{dyn}} = -\int_{l_1}^{l_2}\left\{\delta\mathbf{r}\cdot A_{\rho_0}\ddot{\mathbf{r}} + I_3\frac{\mathbf{r}'^{\perp}\cdot\delta\mathbf{r}'}{g^4}\left(\mathbf{r}'^{\perp}\cdot\ddot{\mathbf{r}}' - 2\dot{\theta}\mathbf{r}'\cdot\dot{\mathbf{r}}'\right)\right\}\mathrm{d}s\ . \tag{9.104}$$

Note that very often the cumbersome contribution containing I_3 is omitted, see Elishakoff et al (2015) for a discussion about that issue.

---

[1] Using the property that $\mathbf{A}$ in (9.85) is skew symmetric, i.e., $\mathbf{A}^{\mathrm{T}} = -\mathbf{A}$, we get $\mathbf{a}\cdot\mathbf{b}^{\perp} = \mathbf{a}\cdot\mathbf{A}\mathbf{b} = (\mathbf{A}^{\mathrm{T}}\mathbf{a})\cdot\mathbf{b} = -(\mathbf{A}\mathbf{a})\cdot\mathbf{b} = -\mathbf{a}^{\perp}\cdot\mathbf{b}$ and thus the variation of the rotated tangential vector and the centerline's second derivative can be swapped by a sign change, i.e., $\delta\mathbf{r}'^{\perp}\cdot\mathbf{r}'' = -\delta\mathbf{r}'\cdot\mathbf{r}''^{\perp}$.

Additionally, the inextensibility (9.96) can be enforced by adding the planar version of the constraint virtual work contributions (9.97) to the total planar virtual work, which leads to the mixed formulation (minimal formulation and inextensibility constraint) of the planar inextensible Euler–Bernoulli beam.

9.7.3 Inextensible Euler–Bernoulli Beam

The third constraint condition (9.69) can also be satisfied by choosing a new set of coordinates. Computing the components of the current tangent vector in the $\mathbf{d}_i$-basis yields

$$\mathbf{r}' = (\mathbf{r}' \cdot \mathbf{d}_i)\mathbf{d}_i \overset{(9.93)}{=} (\mathbf{r}' \cdot \mathbf{d}_1)\mathbf{d}_1 = \mathit{\Gamma}_1 \mathbf{d}_1 \overset{(9.95,9.72)}{=} \cos\theta \mathbf{e}_1 + \sin\theta \mathbf{e}_2 \ , \quad (9.105)$$

which necessarily fulfills (9.69). In what follows, $\mathbf{r}' = \mathbf{r}'(\theta)$ is considered as function of θ. The position vector at (s, t) is obtained as the integrated quantity

$$\mathbf{r}(s,t) = \int_{l_1}^{s} \mathbf{r}'(\theta(\bar{s},t)) \, \mathrm{d}\bar{s} + \bar{\mathbf{r}}(t) \ , \quad (9.106)$$

where $\bar{\mathbf{r}}(t)$ is the time dependent reference point at $s = l_1$. Computing the time derivative of the above equation, we get the velocity vector

$$\dot{\mathbf{r}}(s,t) = \int_{l_1}^{s} \frac{\partial \mathbf{r}'}{\partial \theta}(\theta(\bar{s},t))\dot{\theta}(\bar{s},t) \, \mathrm{d}\bar{s} + \dot{\bar{\mathbf{r}}}(t) \ . \quad (9.107)$$

Accordingly, the acceleration vector is obtained as

$$\ddot{\mathbf{r}}(s,t) = \int_{l_1}^{s} \left\{ \frac{\partial^2 \mathbf{r}'}{\partial \theta^2}(\theta(\bar{s},t))\dot{\theta}^2(\bar{s},t) + \frac{\partial \mathbf{r}'}{\partial \theta}(\theta(\bar{s},t))\ddot{\theta}(\bar{s},t) \right\} \mathrm{d}\bar{s} + \ddot{\bar{\mathbf{r}}}(t) \ . \quad (9.108)$$

The variation of the position vector is given as

$$\delta\mathbf{r}(s,t) = \int_{l_1}^{s} \frac{\partial \mathbf{r}'}{\partial \theta}(\theta(\bar{s},t))\delta\theta(\bar{s},t) \, \mathrm{d}\bar{s} + \delta\bar{\mathbf{r}}(t) \ . \quad (9.109)$$

Using the last identity of (9.95) and inserting its variation $\delta\mathit{\Gamma}_1 = 0$ into the internal virtual work of the Euler–Bernoulli beam given in (9.102), we get

$$\delta W^{\mathrm{int}} = -\int_{l_1}^{l_2} \delta\theta' M \, \mathrm{d}s \ . \quad (9.110)$$

The external virtual work contributions of the inextensible Euler–Bernoulli beam in minimal formulation are given as

$$\delta W^{\text{ext}} = \int_{l_1}^{l_2} \left\{ \left(\int_{l_1}^{s} \frac{\partial \mathbf{r}'}{\partial \theta} \delta\theta \, \mathrm{d}\bar{s} + \delta\bar{\mathbf{r}} \right) \cdot \overline{\mathbf{n}} + \delta\theta \overline{M} \right\} \, \mathrm{d}s + \sum_{i=1}^{2} \left\{ \left(\int_{l_1}^{s} \frac{\partial \mathbf{r}'}{\partial \theta} \delta\theta \, \mathrm{d}\bar{s} + \delta\bar{\mathbf{r}} \right) \cdot \overline{\mathbf{n}}_i + \delta\theta \overline{M}_i \right\} \bigg|_{s=l_i} . \tag{9.111}$$

Inserting (9.108) and (9.109) into the virtual work contributions of the inertial forces given in (9.55) we obtain the cumbersome relation

$$\begin{aligned} \delta W^{\text{dyn}} &= -\int_{l_1}^{l_2} \left\{ \delta\mathbf{r} \cdot A_{\rho_0} \ddot{\mathbf{r}} + \delta\theta I_3 \ddot{\theta} \right\} \, \mathrm{d}s \\ &= -\int_{l_1}^{l_2} A_{\rho_0} \left\{ \int_{l_1}^{s} \delta\theta \frac{\partial \mathbf{r}'}{\partial \theta} \cdot \frac{\partial^2 \mathbf{r}'}{\partial \theta^2} \dot{\theta}^2 \, \mathrm{d}\tilde{s} + \int_{l_1}^{s} \delta\theta \frac{\partial \mathbf{r}'}{\partial \theta} \, \mathrm{d}\tilde{s} \cdot \ddot{\bar{\mathbf{r}}} \right. \\ &\quad + \int_{l_1}^{s} \delta\theta \frac{\partial \mathbf{r}'}{\partial \theta} \cdot \frac{\partial \mathbf{r}'}{\partial \theta} \ddot{\theta} \, \mathrm{d}\tilde{s} + \delta\bar{\mathbf{r}} \cdot \int_{l_1}^{s} \frac{\partial^2 \mathbf{r}'}{\partial \theta^2} \dot{\theta}^2 \, \mathrm{d}\tilde{s} \\ &\quad \left. + \delta\bar{\mathbf{r}} \cdot \int_{l_1}^{s} \frac{\partial \mathbf{r}'}{\partial \theta} \ddot{\theta} \, \mathrm{d}\tilde{s} + \delta\bar{\mathbf{r}} \cdot \ddot{\bar{\mathbf{r}}} + \delta\theta \frac{I_3}{A_{\rho_0}} \ddot{\theta} \right\} \mathrm{d}s \, . \end{aligned} \tag{9.112}$$

If we apply this theory of the planar inextensible Euler–Bernoulli beam to the static consideration of a clamped straight cantilever subjected to a force or a couple at the end, the principle of virtual work leads us directly to the equations known from the elastica theory.

Even though this minimal formulation of the inextensible Euler–Bernoulli beam would be suitable for a subsequent finite element analysis, we will not pursue this any further in Harsch and Eugster (2020). The double integral expressions that appear in the virtual work expressions of distributed forces and couples (9.111) as well as in the intertia terms (9.112) make a numerical treatment extremely cumbersome and not to strive for. Just think about the numerical error of the position (9.106) that cumulates with increasing beam length.

9.8 Conclusion

In this article we presented the derivation of the equations of motion describing the three classical beams, i.e., the Timoshenko beam, the Euler–Bernoulli beam as well as its inextensible companion. The governing equations for the beams were obtained within the variational framework of the principle of virtual work. The applied variational formulation is beneficial not only to add constraints in the sense of the Lagrange multiplier method but also for a subsequent finite element formulation as shown in Harsch and Eugster (2020) for the planar theories. Therefore we additionally elaborated all virtual work contributions of the classical planar theories both as constrained and as unconstrained theories. The corresponding virtual work contributions are ready for a Bubnov–Galerkin discretization.

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