Numerical Simulation of Fractionally Damped Mechanical Systems Using Infinite State Representation

Matthias Hinze, André Schmidt, Remco I. Leine

Institute for Nonlinear Mechanics, University of Stuttgart, Pfaffenwaldring 9, 70569 Stuttgart , Germany
E-mail: hinze@inm.uni-stuttgart.de

Abstract: In this paper we propose an approach for the numerical simulation of fractionally damped mechanical systems which may be adapted to the solution of fractional differential equations. The method is based on the infinite state representation of the Caputo fractional derivative. This representation usually contains an integral with a weakly singular kernel, which leads to problems in numerical computations. In this contribution we introduce a reformulation with a kernel that decays to zero at both ends of the integration interval. We show that the accuracy of numerical simulations significantly improves by using the proposed reformulation.

Key Words: fractional differential equation, fractional damping, springpot, infinite state representation, numerical approximation

1 Introduction

This paper presents a numerical method for the simulation of mechanical systems with fractional damping and fractional differential equations by exploiting a reformulation of the infinite state representation, which has originally been developed within the context of Lyapunov stability theory.

Fractional calculus is a mathematical theory dealing with derivatives and integrals of arbitrary (non-integer) order [1, 2] with a variety of applications. Particularly in mechanics, fractional damping may arise through the modelling of mechanical systems with viscoelastic components. Complex rheological models for viscoelastic materials are often described through an array of classical Kelvin or Maxwell elements, inevitably resulting in a model with a large number of parameters. It has been shown that the viscoelastic behaviour of complex materials is in many applications well represented by fractional order elements, so called 'springpots' with only a few parameters [3, 4]. Actually, a springpot may be interpreted as an arrangement of infinitely many Kelvin or Maxwell elements [5–8]. This can be realized using the infinite state or diffusive representation of fractional derivatives and integrals which has been introduced in [9, 10]. Furthermore, this idea leads to a potential energy representation for a springpot [5, 11] which in turn can be used to prove stability of equilibria for fractionally damped mechanical systems. In this context, we developed in [12] a reformulation of the infinite state representation of fractional derivatives which extracts both the energy preserving and dissipating components of a springpot. This naturally leads to a Lyapunov functional for the rigorous proof of Lyapunov stability of equilibria for a class of mechanical systems with fractional damping.

The main idea of the present contribution is to use this reformulation to come to a novel numerical scheme. The infinite state representation is widely spread in numerical methods dealing with fractional derivative and integral, e.g. in the famous articles [13, 14] and [15], as this approach leads to systems of ordinary instead of fractional differential equations. In contrast to direct approaches to solve fractional differential equations [16–18], these methods do not require the consideration of the solution history which leads to notably less computational and memory costs. The accuracy of the approximation based on the infinite state representation depends on the integration scheme (Gauss-Laguerre [13], composite Gauss-Legendre [19], composite Gauss-Jacobi quadrature [20, 21]), the way of handling the weakly singular integration kernel and the order of differentiation [19–21]. Furthermore, the prediction of the asymptotic behaviour of solutions may be incorrect [22]. In this contribution, we propose to use the reformulation of the fractional derivative introduced in [12] to approximate fractionally damped mechanical systems and fractional differential equations by systems of ordinary differential equations using a composite Gauss-Legendre quadrature and find solutions using an ODE-solver. The proposed scheme has the advantage of a non-singular integration kernel that asymptotically decays to zero, such that the truncation error and the quadrature error of the integral are small. The current paper studies the performance of the scheme in comparison to other numerical methods using the infinite state representation.

In Sec. 2 we set notation and derive the reformulation of the infinite state representation as introduced in [12]. Based on this reformulation, we develop the numerical scheme for the simulation of fractionally damped mechanical systems in Sec. 3. Subsequently, in Sec. 4 we generalize the approach for the solution of fractional differential equations. Both schemes are tested on benchmark problems and compared to existing numerical methods based on the infinite state representation. Finally, in Sec. 5 we draw conclusions and state some generalizations of the proposed method.

2 Fractional derivative and reformulated infinite state representation

We consider the Caputo derivative of a continuously differentiable function $q(t)$ with $q(t) = \dot{q}(t) = 0$ for $t \leq t_0$ if $t_0 \in (-\infty, 0]$ or $q(t)$ and $\dot{q}(t)$ bounded for $t \leq 0$ if $t_0 = -\infty$

This work is supported by the Federal Ministry of Education and Research of Germany under Grant No. 01IS17096.

and differentiation order $\alpha \in (0,1)$ given by

$$ {}^{C}D^{\alpha}_{t_0+}q(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} (t-\tau)^{-\alpha}\dot{q}(\tau)\mathrm{d}\tau. \tag{1} $$

Thereby, we think of $t = 0$ as initial time and the interval $(t_0, 0]$ as the non-zero or significant history of q, i.e. we consider an initialized fractional derivative of q. The infinite state representation of (1), see [15], can be written as

$$ {}^{C}D^{\alpha}_{t_0+}q(t) = \int_0^{\infty} \mu_{1-\alpha}(\omega)y(\omega,t)\mathrm{d}\omega, \tag{2} $$

where the function μ_α is defined as

$$ \mu_\alpha(\eta) := \frac{\sin(\alpha\pi)}{\pi}\eta^{-\alpha} \tag{3} $$

and the infinite states $y(\eta,t),\ \eta \in (0,\infty)$ solve the initial value problem

$$ \dot{y}(\eta,t) = -\eta y(\eta,t) + \dot{q}(t), \quad y(\eta,t_0) = 0. \tag{4} $$

Additionally, we propose to consider a second kind of infinite states $Y(\eta,t),\ \eta \in (0,\infty)$ that fulfil

$$ \dot{Y}(\eta,t) = -\eta Y(\eta,t) + q(t), \quad Y(\eta,t_0) = 0 \tag{5} $$

and $\dot{Y} = y$. In the following, we will introduce an expansion of the integration kernel in (2) by the term $\omega^2 + r^2$ for a real number $r > 0$. The following properties will prove to be useful in the proposed reformulation.

Proposition 1:

$$ \int_0^{\infty} \frac{\mu_\alpha(\omega)}{\omega + s}\mathrm{d}\omega = s^{-\alpha}, \quad s \in \mathbb{C}\backslash\mathbb{R}^-,\ \alpha \in (0,1). \tag{6} $$

Proof. Due to the relation for the Laplace transform of $\mathrm{e}^{-\omega t}$

$$ \begin{aligned} \mathcal{L}\{\mathrm{e}^{-\omega t}\}(s) &= \int_0^{\infty} \mathrm{e}^{-\omega t}\mathrm{e}^{-st}\mathrm{d}t = \int_0^{\infty} \mathrm{e}^{-(\omega+s)t}\mathrm{d}t \\ &= \left[-\frac{1}{\omega+s}\mathrm{e}^{-(\omega+s)t}\right]_0^{\infty} = \frac{1}{\omega+s} \end{aligned} $$

we obtain Eq. (6) using the property of the Gamma function

$$ \Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin(\alpha\pi)} $$

and Fubini's theorem as

$$ \begin{aligned} \int_0^{\infty} \frac{\mu_\alpha(\omega)}{\omega+s}\mathrm{d}\omega &= \frac{\sin(\alpha\pi)}{\pi} \int_0^{\infty} \omega^{-\alpha} \int_0^{\infty} \mathrm{e}^{-(\omega+s)t}\mathrm{d}t\mathrm{d}\omega \\ &= \frac{\sin(\alpha\pi)}{\pi} \int_0^{\infty} \mathrm{e}^{-st} \int_0^{\infty} \omega^{-\alpha}\mathrm{e}^{-\omega t}\mathrm{d}\omega\mathrm{d}t \\ &= \frac{\sin(\alpha\pi)}{\pi} \int_0^{\infty} \mathrm{e}^{-st}\Gamma(1-\alpha)t^{\alpha-1}\mathrm{d}t \\ &= \frac{\sin(\alpha\pi)}{\pi}\Gamma(1-\alpha)\Gamma(\alpha)s^{-\alpha} = s^{-\alpha}. \end{aligned} $$

□

Remark 2:
The result of Prop. 1 is also used in [15] (without proof). A similar proof as the one given here may be found in [23].

Proposition 3:
For $\alpha \in (0,1)$ and $r > 0$, the identities

$$ \int_0^{\infty} \frac{\mu_{1-\alpha}(\omega)}{\omega^2+r^2}\mathrm{d}\omega = \cos\left(\frac{\alpha\pi}{2}\right) r^{\alpha-2}, \tag{7} $$

$$ \int_0^{\infty} \frac{\mu_{1-\alpha}(\omega)\omega}{\omega^2+r^2}\mathrm{d}\omega = \sin\left(\frac{\alpha\pi}{2}\right) r^{\alpha-1} \tag{8} $$

hold.

Proof. Substitute $\eta = \omega^2$ and $\mathrm{d}\eta = 2\omega\mathrm{d}\omega$ in the integral and obtain

$$ \begin{aligned} \int_0^{\infty} \frac{\mu_{1-\alpha}(\omega)}{\omega^2+r^2}\mathrm{d}\omega &= \frac{\sin(\alpha\pi)}{\pi} \int_0^{\infty} \frac{\omega^{\alpha-1}}{\omega^2+r^2}\mathrm{d}\omega \\ &= \frac{\sin(\alpha\pi)}{2\pi} \int_0^{\infty} \frac{\eta^{\frac{\alpha}{2}-1}}{\eta+r^2}\mathrm{d}\eta \\ &= \frac{\sin(\alpha\pi)}{2\sin\left(\frac{\alpha\pi}{2}\right)} \int_0^{\infty} \frac{\mu_{1-\frac{\alpha}{2}}(\eta)}{\eta+r^2}\mathrm{d}\eta. \end{aligned} $$

Using the sine-double-angle formula and (6), we directly obtain (7). The proof of (8) is analogous. □

Expansion of (2) by the term $\omega^2 + r^2$ for a fixed $r > 0$ leads together with (4), (5), (7) and (8) to

$$ \begin{aligned} &{}^{C}D^{\alpha}_{t_0+}q(t) \\ &= \int_0^{\infty} \frac{\mu_{1-\alpha}(\omega)\omega}{\omega^2+r^2}\omega y(\omega,t)\mathrm{d}\omega \\ &\quad + r^2 \int_0^{\infty} \frac{\mu_{1-\alpha}(\omega)}{\omega^2+r^2} y(\omega,t)\mathrm{d}\omega \\ &= \int_0^{\infty} \frac{\mu_{1-\alpha}(\omega)\omega}{\omega^2+r^2}\mathrm{d}\omega\,\dot{q}(t) - \int_0^{\infty} \frac{\mu_{1-\alpha}(\omega)\omega}{\omega^2+r^2}\dot{y}(\omega,t)\mathrm{d}\omega \\ &\quad + r^2 \int_0^{\infty} \frac{\mu_{1-\alpha}(\omega)}{\omega^2+r^2}\mathrm{d}\omega\, q(t) \\ &\quad - r^2 \int_0^{\infty} \frac{\mu_{1-\alpha}(\omega)\omega}{\omega^2+r^2} Y(\omega,t)\mathrm{d}\omega \\ &= \sin\left(\frac{\alpha\pi}{2}\right) r^{\alpha-1}\dot{q}(t) - \int_0^{\infty} \frac{\mu_{1-\alpha}(\omega)\omega}{\omega^2+r^2}\dot{y}(\omega,t)\mathrm{d}\omega \\ &\quad + \cos\left(\frac{\alpha\pi}{2}\right) r^{\alpha}q(t) - r^2 \int_0^{\infty} \frac{\mu_{1-\alpha}(\omega)\omega}{\omega^2+r^2} Y(\omega,t)\mathrm{d}\omega. \end{aligned} \tag{9} $$

The advantage of this reformulation of the fractional derivative is the new kernel

$$ K(\alpha,\omega) = \frac{\mu_{1-\alpha}(\omega)\omega}{\omega^2+r^2} = \frac{\sin(\alpha\pi)}{\pi}\frac{\omega^{\alpha}}{\omega^2+r^2}, \tag{10} $$

that is integrable on $\mathbb{R}^+_0$ and fulfils

$$ \lim_{\omega\to 0} K(\alpha,\omega) = \lim_{\omega\to\infty} K(\alpha,\omega) = 0, \quad \alpha \in (0,1). \tag{11} $$

Furthermore, for a fixed value of α the function $K(\alpha,\omega)$ has a maximum at

$$ \omega_{\max} = \sqrt{\frac{\alpha}{2-\alpha}}r. \tag{12} $$

Hence, the position of $\omega_{\max}$ may be adjusted by the magnitude of r. The graphs of $K(\alpha,\omega)$ for different values of

$\alpha \in (0,1)$ and $\omega_{\max} = 1$ (i.e. $r^2 = \frac{2-\alpha}{\alpha}$) are displayed in Fig. 1.

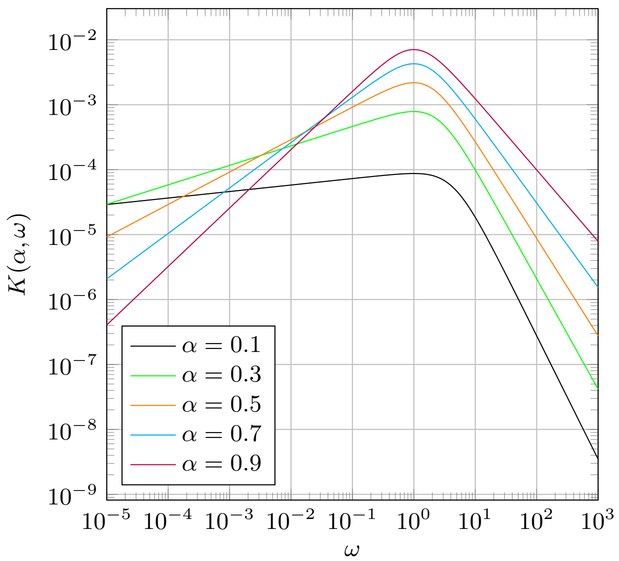

Fig. 1: $K(\alpha, \omega)$ for different values of $\alpha \in (0, 1)$.

3 Application to fractionally damped mechanical systems

3.1 Derivation of the method

In the following, we consider a (nonlinear) mechanical system with differentiable initial function $q_0(t)$

$$\begin{cases} m\ddot{q}(t) + c\,{}^{C}D^{\alpha}_{t_0+}q(t) = h(t, q(t), \dot{q}(t)), \\ \qquad\qquad q(t) = q_0(t), \quad t_0 \le t \le 0 \end{cases} \tag{13}$$

with parameters $m, c > 0$. Using (9) and (10), we can interpret (13) as an infinite-dimensional first-order system

$$\begin{cases} \dot{q}(t) = v(t), \\ m\dot{v}(t) - c\displaystyle\int_0^\infty K(\alpha,\omega)\dot{y}(\omega,t)\mathrm{d}\omega \\ \qquad = h(t,q(t),v(t)) - c\cos\left(\frac{\alpha\pi}{2}\right) r^\alpha q(t) \\ \qquad\quad - c\sin\left(\frac{\alpha\pi}{2}\right) r^{\alpha-1} v(t) \\ \qquad\quad + cr^2 \displaystyle\int_0^\infty K(\alpha,\omega)Y(\omega,t)\mathrm{d}\omega, \\ \dot{Y}(\eta,t) = y(\eta,t),\ \eta \in (0,\infty), \\ \dot{y}(\eta,t) = v(t) - \eta y(\eta,t),\ \eta \in (0,\infty), \end{cases} \tag{14}$$

with initial conditions

$$\begin{cases} q(0) = q_0(0),\ v(0) = \dot{q}_0(0), \\ Y(\eta,0) = \displaystyle\int_{t_0}^0 \mathrm{e}^{\eta\tau} q_0(\tau)\mathrm{d}\tau, \\ y(\eta,0) = \displaystyle\int_{t_0}^0 \mathrm{e}^{\eta\tau} \dot{q}_0(\tau)\mathrm{d}\tau,\ \eta \in (0,\infty). \end{cases} \tag{15}$$

In a numerical approach, the integration of the infinite states is discretized by N sampling points $\eta_j, j = 1,\dots,N$, neglecting the interval (η_N, ∞). Thereby, we approximate (14) and (15) by a $(2N+2)$-dimensional initial value problem

$$\begin{cases} A\dot{x}(t) = f(t, x(t)), \\ x(0) = x_0 \end{cases} \tag{16}$$

with $x = (q, v, Y_1, \dots, Y_N, y_1, \dots, y_N)^{\mathrm{T}}$ and

$$A = \begin{pmatrix} 1 & 0 & 0\cdots0 & 0\cdots\cdots\cdots0 \\ 0 & m & 0\cdots0 & (-cK(\alpha,\eta_j)w(\eta_j))_j \\ 0 & 0 & & \\ \vdots & \vdots & I_{N\times N} & 0_{N\times N} \\ 0 & 0 & & \\ 0 & 0 & & \\ \vdots & \vdots & 0_{N\times N} & I_{N\times N} \\ 0 & 0 & & \end{pmatrix}, \tag{17}$$

as well as $f = (f_1, \dots, f_{2N+2})^{\mathrm{T}}$ with

$$\begin{aligned} f_1(t,x(t)) &= v(t), \\ f_2(t,x(t)) &= h(t,q(t),v(t)) - c\cos\left(\frac{\alpha\pi}{2}\right) r^\alpha q(t) \\ &\quad - c\sin\left(\frac{\alpha\pi}{2}\right) r^{\alpha-1} v(t) \\ &\quad + cr^2 \sum_{i=1}^{N} K(\alpha,\eta_i)\, Y_i(t)\, w(\eta_i), \\ f_{2+j}(t,x(t)) &= y_j(t), \quad j = 1,\dots,N, \\ f_{2+N+j}(t,x(t)) &= v(t) - \eta_j y_j(t), \quad j = 1,\dots,N. \end{aligned} \tag{18}$$

Here, $\eta_j, w(\eta_j),\ j = 1,\dots,N$ are the sampling points and weights of the quadrature used, respectively. The initial values x_0 are chosen according to (15).

3.2 Benchmark problem 1

Consider the mechanical system with initial function

$$\begin{cases} \ddot{q}(t) + {}^{C}D^{\alpha}_{-\infty+}q(t) + q(t) = t^2 + \dfrac{2}{\Gamma(3-\alpha)}t^{2-\alpha} + 2, \\ \qquad\qquad q(t) = 0, \quad t \le 0 \end{cases} \tag{19}$$

with solution $q(t) = t^2$. The procedure described above has been realised for different values of $\alpha \in (0,1)$ with $2N = 500$ internal states Y_j and y_j, $j = 1,\dots,N$. We have chosen $r^2 = \frac{2-\alpha}{\alpha}$ and a logarithmically dense grid in $[10^{-8}, 10^{8}]$ (centred around $\omega_{\max} = 1$) of $K = 25$ points η_k, $k = 1,\dots,K$. The integral over each interval $(0,\eta_1), (\eta_1,\eta_2), \dots, (\eta_{K-1},\eta_K)$ has been approximated by a Gauss-Legendre quadrature with $J = 10$ sampling points, such that $N = K \cdot J$. Finally, the system of the form (16) has been solved in MATLAB with `ode15s.m`. In Fig. 2 the relative error

$$\Delta^{(1)}(t) = \left|\frac{q(t) - \tilde{q}(t)}{q(t)}\right| + \left|\frac{\dot{q}(t) - \dot{\tilde{q}}(t)}{\dot{q}(t)}\right| \tag{20}$$

between the exact $(q, \dot{q})$ and the numerical solution $(\tilde{q}, \dot{\tilde{q}})$ as function of time t is shown for various values of $\alpha \in (0,1)$. Furthermore, we have used an existing numerical scheme based on [13] to compare the proposed methods. Therefore, we have considered the infinite state representation of (19) without reformulation (9) and the approximation

$$\int_0^\infty \mu_{1-\alpha}(\omega)y(\omega,t)\mathrm{d}\omega \approx \sum_{i=1}^{N} \mu_{1-\alpha}(\eta_i)y_i(t)w(\eta_i) \quad (21)$$

using a composite Gauss-Jacobi quadrature with the same parameters as above (instead of Gauss-Laguerre quadrature as in [13]). We have obtained (19) in first-order form

$$\begin{cases} \dot q(t) = v(t), \\ \dot v(t) = t^2 + \dfrac{2}{\Gamma(3-\alpha)} t^{2-\alpha} + 2 - q(t) \\ \qquad - \displaystyle\sum_{i=1}^{N} \mu_{1-\alpha}(\eta_i) y_i(t) w(\eta_i), \\ \dot y_j(t) = v(t) - \eta_j y_j(t), \quad j = 1,\dots,N \end{cases} \quad (22)$$

and have used `ode15s.m` again to solve (22) and show the relative errors in Fig. 2 as well.

4 Application to fractional differential equations

4.1 Derivation of the method

The method described above can be adapted to the case $m = 0$, i.e. for a fractional differential equation with initial function of the form

$$\begin{cases} {}^C D^\alpha_{t_0+} q(t) = \tilde h(t, q(t)), \\ \qquad q(t) = q_0(t), \quad t_0 \le t \le 0. \end{cases} \quad (23)$$

This leads to a $(2N+1)$-dimensional first-order initial-value problem

$$\begin{cases} B\dot z(t) = g(t, z(t)), \\ \quad z(0) = z_0 \end{cases} \quad (24)$$

with $z = (q, Y_1, \dots, Y_N, y_1, \dots, y_N)^{\mathrm{T}}$,

$$B = \begin{pmatrix} \sin\left(\frac{\alpha\pi}{2}\right) r^{\alpha-1} & 0\cdots 0 & (-K(\alpha,\eta_j)w(\eta_j))_j \\ 0 & & \\ \vdots & I_{N\times N} & 0_{N\times N} \\ 0 & & \\ -1 & & \\ \vdots & 0_{N\times N} & I_{N\times N} \\ -1 & & \end{pmatrix} \quad (25)$$

and $g = (g_1, \dots, g_{2N+1})^{\mathrm{T}}$ with

$$\begin{aligned} g_1(t, z(t)) &= \tilde h(t, q(t)) - \cos\left(\frac{\alpha\pi}{2}\right) r^\alpha q(t) \\ &\quad + r^2 \sum_{i=1}^{N} K(\alpha, \eta_i)\, Y_i(t)\, w(\eta_i), \end{aligned} \quad (26)$$

$$\begin{aligned} g_{1+j}(t, z(t)) &= y_j(t), \quad j = 1,\dots,N, \\ g_{1+N+j}(t, z(t)) &= -\eta_j y_j(t), \quad j = 1,\dots,N. \end{aligned}$$

In this case, the initial condition has the form

$$z_0 = \left(q_0(0), \left(\int_{t_0}^0 \mathrm{e}^{\eta_j\tau} q_0(\tau)\mathrm{d}\tau\right)_j, \left(\int_{t_0}^0 \mathrm{e}^{\eta_j\tau} \dot q_0(\tau)\mathrm{d}\tau\right)_j\right)^{\mathrm{T}}. \quad (27)$$

4.2 Benchmark problems 2 and 3

Consider the following fractional differential equations with initial function

$$\begin{cases} {}^C D^\alpha_{-\infty+} q(t) = \tilde h_i(t, q(t)), \\ \qquad q(t) = q_{0,i}(t), \quad t \le 0, \end{cases} \quad i = 1,2 \quad (28)$$

with

$$\tilde h_1(t, q(t)) = 1 - q(t), \quad q_{0,1}(t) = 0, \quad (29)$$

$$\tilde h_2(t, q(t)) = -q(t), \quad q_{0,2}(t) = 1 \quad (30)$$

and closed form solutions $q(t) = 1 - E_\alpha(-t^\alpha)$ and $q(t) = E_\alpha(-t^\alpha)$, respectively, where E_α is the one-parameter Mittag-Leffler function. The problem in (30), in contrast to the examples before leads to non-zero initial conditions (27) in the form

$$z_{0,2} = \left(1, \frac{1}{\eta_1}, \dots, \frac{1}{\eta_N}, 0, \dots, 0\right). \quad (31)$$

A numerical solution of (28) with (29) and (30) has been obtained using the method described above where the integration of internal states has again been approximated by a composite Gauss-Legendre quadrature with parameters as in Sec. 3.2 and the system in the form (24) has again been solved using MATLAB's solver `ode15s.m`. The values of the Mittag-Leffler function have been computed with the help of [24]. The time dependent absolute error

$$\Delta^{(0)}(t) = |q(t) - \tilde q(t)| \quad (32)$$

between the exact solution $q(t)$ and the numerical solution $\tilde q(t)$ for various values of $\alpha \in (0,1)$ is shown in Figs. 3 and 4. As in Sec. 3.2, we compare the proposed method to a scheme, which also uses the infinite state representation without reformulation (9). Therefore, the system (28) is transformed into the equivalent fractional integral equation

$$q(t) = \lim_{\tau\to-\infty} q_{0,i}(\tau) + I^\alpha_{-\infty+} \tilde h_i(t, q(t)), \quad i = 1,2 \quad (33)$$

with the fractional Riemann-Liouville integral

$$I^\alpha_{t_0+} q(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} q(\tau)\mathrm{d}\tau. \quad (34)$$

We consider the infinite state representation of (33)

$$\begin{cases} q(t) = \lim\limits_{\tau\to-\infty} q_{0,i}(\tau) + \displaystyle\int_0^\infty \mu_\alpha(\omega)\tilde y(\omega,t)\mathrm{d}\omega, \\ \dot{\tilde y}(\eta, t) = \tilde h_i(t, q(t)) - \eta \tilde y(\eta, t), \quad \eta \in (0,\infty), \\ \tilde y(\eta, 0) = 0, \quad \eta \in (0,\infty). \end{cases} \quad (35)$$

The integral with kernel $\mu_\alpha(\omega)$ has been approximated by a composite Gauss-Jacobi quadrature as above and the resulting system of one algebraic and N differential equations

$$\begin{cases} q(t) = \lim\limits_{\tau\to-\infty} q_{0,i}(\tau) + \displaystyle\sum_{j=1}^{N} \mu_\alpha(\eta_j)\tilde y_j(t) w(\eta_j), \\ \dot{\tilde y}_j(t) = \tilde h_i(t, q(t)) - \eta_j \tilde y_j(t), \quad j = 1,\dots,N, \\ \tilde y_j(0) = 0, \quad j = 1,\dots,N \end{cases} \quad (36)$$

has again been solved with `ode15s.m` in MATLAB. The absolute errors are shown in Figs. 3 and 4 as well.

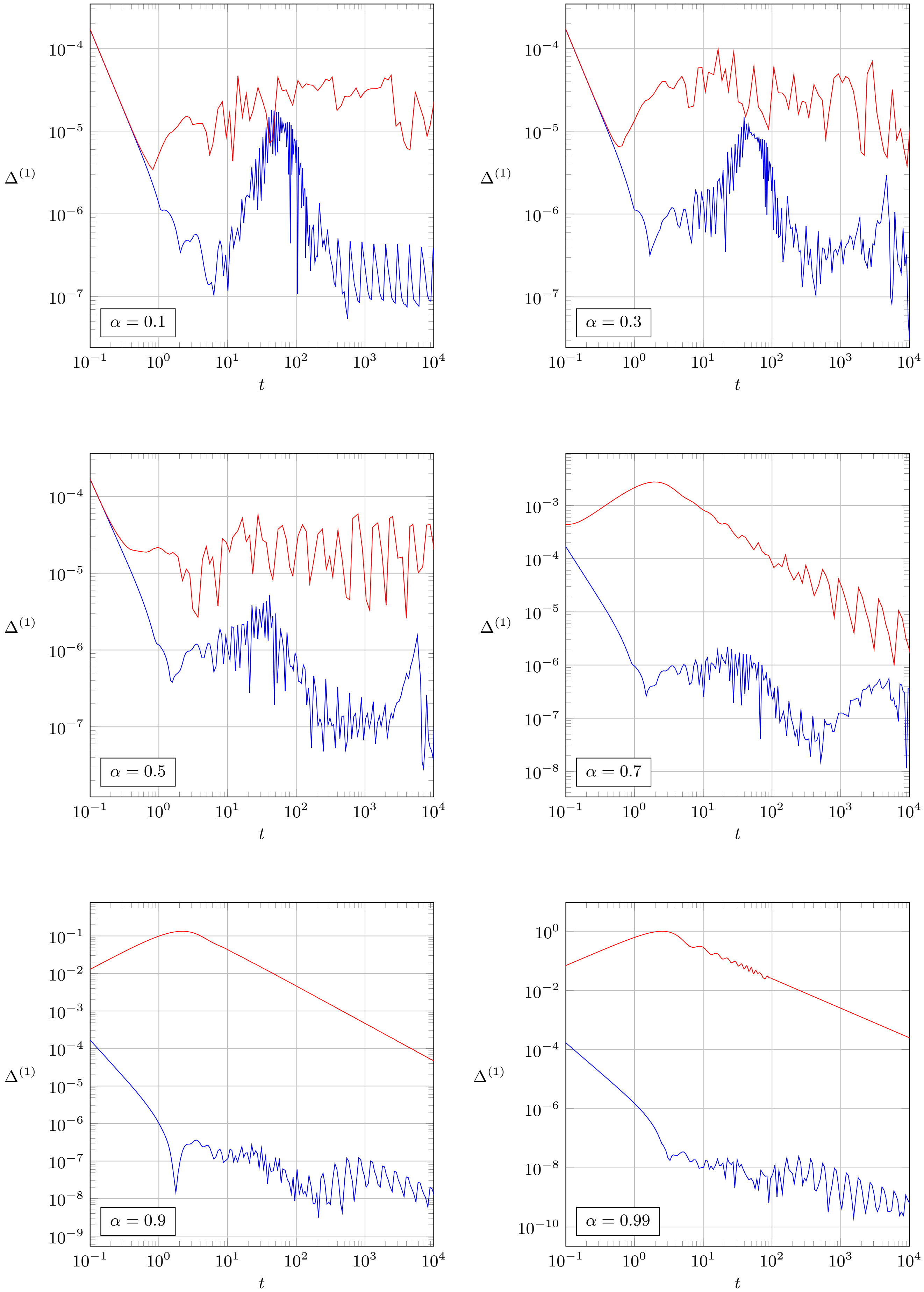

Fig. 2: Benchmark 1: Relative error of the numerical solution of (19) using (16) (blue) and (22) (red).

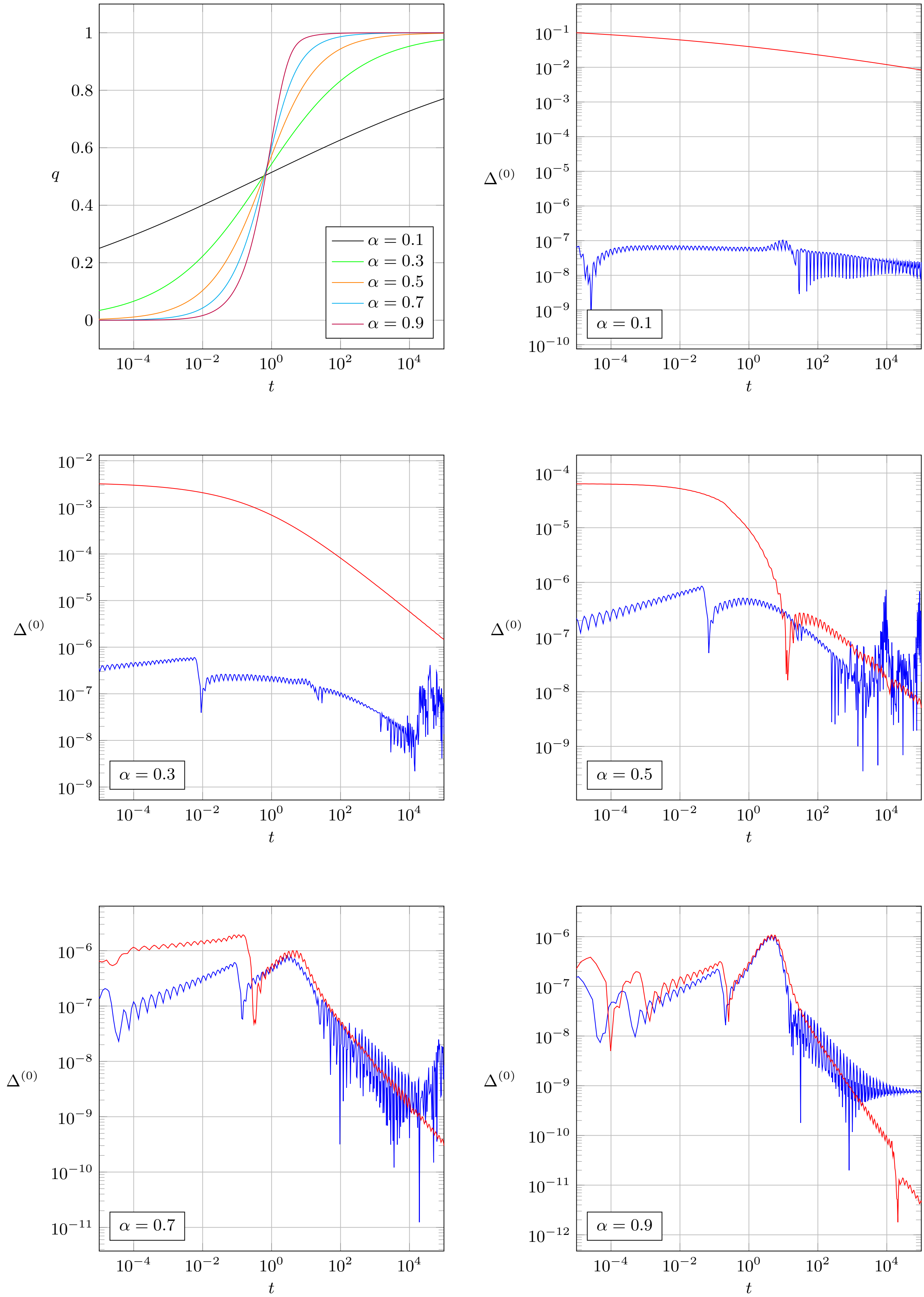

Fig. 3: Benchmark 2: Analytical solution (top left) and absolute error of the numerical solution of (28) with (29) using (24) (blue) and (36) (red).

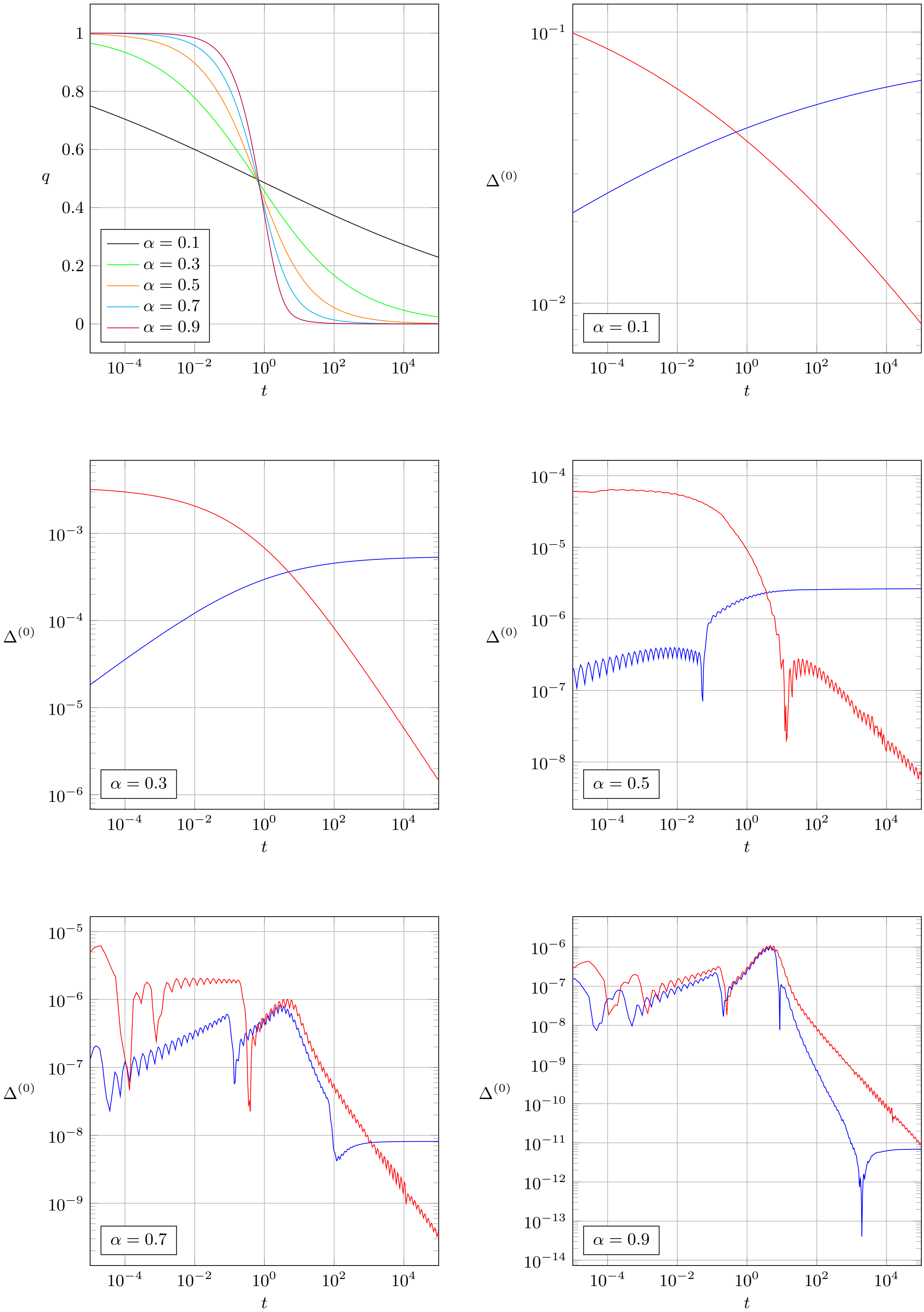

Fig. 4: Benchmark 3: Analytical solution (top left) and absolute error of the numerical solution of (28) with (30) using (24) (blue) and (36) (red).

5 Conclusion

The proposed numerical method for the simulation of fractionally damped mechanical systems and fractional differential equations is based on the infinite state representation of fractional derivatives and uses a reformulation which leads to an integrable kernel $K(\alpha, \omega)$, that is defined for $\alpha \in (0, 1)$ and $\omega \geq 0$ and does not have a weak singularity at $\omega = 0$. This can lead to more accurate results than using the infinite state representation in standard form as shown in the benchmark problems (Figs. 2 and 3). However, in Fig. 4 we see an example where both methods lead to large errors and the method proposed in this paper performs even worse for small values of α. The reason for this behaviour still has to be studied.

The method proposed in Sec. 3 can be generalized for mechanical systems with several degrees of freedom $f \geq 1$ of the form

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}{}^{C}D^{\alpha}_{t_0+}\dot{\mathbf{q}}(t) = \mathbf{h}(t, \mathbf{q}(t), \dot{\mathbf{q}}(t)) \in \mathbb{R}^f \qquad (37)$$

with regular $\mathbf{M}, \mathbf{C} \in \mathbb{R}^{f \times f}$, which leads to a $(2N + 2)f$-dimensional system of the form (16). Another generalization can be made by introducing a second fractional damping term $\tilde{c}{}^{C}D^{\tilde{\alpha}}_{t_0+}q(t),\ \tilde{c} > 0, \tilde{\alpha} \in (0, 1)$ in (13) which leads to an altered system (16) where the dimension remains unchanged. Finally, the method is even applicable in case of a variable-order fractional derivative. In this case, the matrices A in (16) and B in (24) are time-dependent.

Further research is needed to give a complete error analysis for the proposed scheme and a more general comparison to other methods. The influence of the choice of quadrature, time-stepping scheme and the value of r on the results still has to be investigated.

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