A novel Lyapunov-like method for the non-autonomous bouncing ball system

Thomas F. Heimsch*, Remco I. Leine*
**Department of Mechanical and Process Engineering, IMES, ETH Zürich, Switzerland*

Summary. The non-autonomous bouncing ball system consists of a point mass m in a constant gravitational field g, which bounces inelastically on a flat vibrating table. The presented Lyapunov-like method is set up for non-autonomous measure differential inclusions and constructs a decreasing step function W above the oscillating Lyapunov function V. Furthermore, it is proven that the attractivity of the equilibrium of the bouncing ball system is symptotic and a conservative estimate of the finite attraction time is given.

Aim

The main result of the paper is a novel Lyapunov-like method for the stability analysis of a class of non-autonomous measure differential inclusions. Systems which expose discontinuities in the state and/or vector field can be described by measure differential inclusions, a concept which describes the continuous dynamics as well as the impulsive dynamics with a single statement in terms of an inclusion and is able to describe accumulation phenomena with impact through an integration process. A Lyapunov-like technique is presented to prove global uniform attractive stability of the equilibria of non-autonomous measure differential inclusions in the sense of comparison functions. This theorem allows, in contrast to the classical direct method of Lyapunov, where the function V is required to be non-increasing, to choose among a more general class of Lyapunov candidate functions, which may also temporarily increase along solution curves, e.g. for the choice of Lyapunov functions with a clear physical meaning. In a brief communication of the authors [1], a sufficient condition for global symptotic attractive stability of the equilibrium of the bouncing ball system with a harmonically vibrating table is proved by using this method with a simple energy-like Lyapunov function and an upper-bound for the attraction time is given. In this paper, the presented Lyapunov-like method in [1] is generalized to the stability analysis of a class of measure differential inclusions and a Lyapunov technique to prove the conditional global uniform symptotic attractive stability of the equilibrium of the bouncing ball system with an arbitrary motion of the table (see also [2]).

The bouncing ball system

A standard problem of chaotic dynamics is a ball in a constant gravitational field bouncing inelastically on a flat vibrating table as depicted in Figure 1. We consider the vertical movement $e(t) \in \mathcal{C}^\infty$ of the table to be an analytic kinematic excitation. The vertical position and the velocity of the ball are addressed by the absolute coordinate $q(t)$ and $u(t)$, respectively. We describe the motion of the bouncing ball system with the state vector $\boldsymbol{x}(t)$ expressed in relative coordinates

$$\boldsymbol{x}(t) := \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} g_N(t) \\ \gamma_N(t) \end{bmatrix} = \begin{bmatrix} q(t) - e(t) \\ u(t) - \dot{e}(t) \end{bmatrix}, \tag{1}$$

such that the equilibrium position is located at the origin $\boldsymbol{x}^* = \boldsymbol{0}$. The non-impulsive dynamics of the ball is described by

$$m\dot{u}(t) = -mg + \lambda_N(t), \tag{2}$$

where mg is the weight of the ball and $\lambda_N(t)$ is the contact force between the ball and the table. The contact force is non-negative because the ball and the table can only push on each other in the absence of adhesion. The constitutive behaviour of the unilateral contact force λ_N is therefore described by Signorini's law as an inequality complementarity condition between the non-negative dual variables g_N and λ_N. The impulsive dynamics is described by the impact equation

$$m\left(u^+(t) - u^-(t)\right) = \Lambda_N(t), \tag{3}$$

where $\Lambda_N(t)$ is the contact impulse which causes an instantaneous velocity jump. Naturally, the contact impulse vanishes if the contact is open, i.e. for $g_N(t) > 0$, it holds that $\Lambda_N(t) = 0$. For time-instants for which the contact is closed ($g_N(t) = 0$) we will consider a Newton-type of restitution law expressed by the inequality complementarity condition

$$g_N(t) = 0: \quad \xi_N(t) \geq 0, \quad \Lambda_N(t) \geq 0, \quad \xi_N(t)\Lambda_N(t) = 0, \tag{4}$$

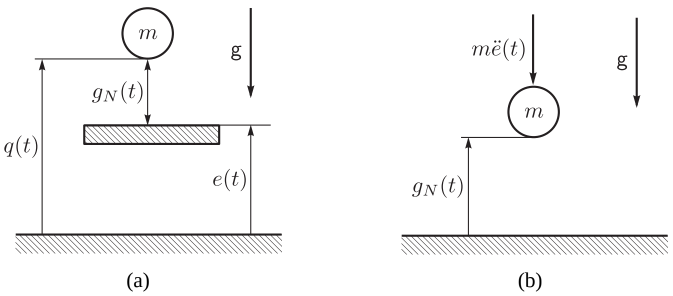

Figure 1: The bouncing ball system (a) and the equivalent forced system (b).

where $\xi_N(t) = \gamma_N^+(t) + \varepsilon(t)\gamma_N^-(t)$ and $\varepsilon(t) \in \mathcal{C}^0$ is Newton's coefficient of restitution with the restriction

$$0 \le \varepsilon(t) \le \bar{\varepsilon} < 1 \quad \forall t\,, \tag{5}$$

which we consider to be time-dependent. The inequality complementarity condition (4) implies that a positive contact impulse $\Lambda_N(t) > 0$ can only be transmitted by the contact if $\xi_N(t) = 0$, i.e. if Newton's restitution law $\gamma_N^+(t) = -\varepsilon(t)\gamma_N^-(t)$ holds. Similarly, if $\xi_N(t) > 0$, then the contact impulse $\Lambda_N(t)$ must vanish. Using the impact equation (3) together with $\Lambda_N(t) = 0$, we infer that there is no velocity jump ($u^+(t) = u^-(t)$). The relative velocity $\gamma_N(t)$ therefore also remains continuous and $\xi_N(t) > 0$ therefore implies that the momentarily closed contact will open ($\gamma_N(t) > 0$).
In the following, the kinematic excitation $e(t)$ of the table will be assumed to be analytic and to satisfy the bounds

$$a_{\min} \le \ddot{e}(t) \le a_{\max} \quad \forall t\,. \tag{6}$$

The velocity and the acceleration of the table are continuous and given by $\dot{e}(t)$ and $\ddot{e}(t)$, respectively. We say that the ball is in persistent contact with the table at time t_0 if $g_N(t) = 0$ on some time-interval $[t_0, t^*]$, and it therefore holds that $\gamma_N(t) = \dot{\gamma}_N(t) = 0$ for $t \in (t_0, t^*)$ from which we retrieve the contact force $\lambda_N(t)$ during persistent contact: $\lambda_N(t) = m\ddot{e}(t) + mg$. Detachment occurs at $t = t^*$ if $\dot{\gamma}_N(t) = 0$ can no longer be fulfilled, i.e. if $m\ddot{e}(t) + mg < 0$. We conclude that if the

$$\text{equilibrium condition:} \quad a_{\min} + g \ge 0 \tag{7}$$

holds, then a ball which is initially on the table will remain on the table for all future times. We will refer to this steady state behaviour as the equilibrium position of the ball.
Throughout the paper, illustrations are given based on a harmonic excitation $e(t) = -A\sin(\Omega t)$ with the amplitude A and the angular frequency Ω. For harmonic excitation it holds that $-a_{\min} = a_{\max} = A\Omega^2$, and we will use the ratio $\kappa := \frac{A\Omega^2}{g}$, which we call the relative acceleration of the table. Furthermore, the equilibrium condition (7) in the case of harmonic excitation reads as $g - A\Omega^2 \ge 0$ and can be expressed using the relative acceleration as $\kappa \le 1 =: \overline{\kappa}$.
The equation of motion (2) and the impact equation (3) together with the impact law (4) describe the motion of the ball at every time-instant. We use the concept of measure differential inclusions to describe the impulsive and non-impulsive dynamics in a unified way. The state $\boldsymbol{x}(t)$ of the dynamical system is interpreted as the result of an integration process over the differential measure $\mathrm{d}\boldsymbol{x}$, i.e. $\boldsymbol{x}^+(t) = \boldsymbol{x}^-(t_0) + \int_{[t_0,t]} \mathrm{d}\boldsymbol{x}$ for $t \ge t_0$ with $\mathrm{d}\boldsymbol{x} = \dot{\boldsymbol{x}}\mathrm{d}t + (\boldsymbol{x}^+ - \boldsymbol{x}^-)\,\mathrm{d}\eta$. The integration process takes the left limit $\boldsymbol{x}^-(t_0)$ of the initial value to the right limit $\boldsymbol{x}^+(t)$ of the final value over the compact interval $[t_0, t]$. The differential measure $\mathrm{d}\boldsymbol{x}$ contains a density $\dot{\boldsymbol{x}}(t)$ with respect to the differential Lebesgue measure $\mathrm{d}t$ and contains a density $\boldsymbol{x}^+ - \boldsymbol{x}^-$ with respect to the atomic measure $\mathrm{d}\eta$. The Lebesgue part $\dot{\boldsymbol{x}}\,\mathrm{d}t$ describes the continuous variation of $\boldsymbol{x}(t)$. The atomic part $(\boldsymbol{x}^+ - \boldsymbol{x}^-)\mathrm{d}\eta$ is used to describe discontinuities in $\boldsymbol{x}(t)$. The upper and lower limits of $\boldsymbol{x}(t)$ at impulsive time-instants t_n are denoted by $\boldsymbol{x}^+(t_n) := \lim_{t \downarrow t_n} \boldsymbol{x}(t)$ and $\boldsymbol{x}^-(t_n) := \lim_{t \uparrow t_n} \boldsymbol{x}(t)$, respectively. Note that $\int_I (\cdot)\mathrm{d}\eta = 0$ if the function $\boldsymbol{x}(t)$ is absolutely continuous on I. If $\mathrm{d}\boldsymbol{x}$ is integrated over a singleton $\{t_n\}$, then $\int_{\{t_n\}} (\cdot)\mathrm{d}t = 0$ and $\int_{\{t_n\}} \mathrm{d}\boldsymbol{x} = \boldsymbol{x}^+(t_n) - \boldsymbol{x}^-(t_n)$. The gap function $x_1(t) = g_N(t)$ is an absolutely continuous function in time and its differential measure only consists of a Lebesgue part: $\mathrm{d}x_1(t) = x_2(t)\mathrm{d}t$. The relative velocity $x_2(t) = \gamma_N(t)$ is considered to be a function of special locally bounded variation which is discontinuous at collision time-instants t_n. The equations (2) and (3) can be combined in a single equality of measures $\mathrm{d}\gamma_N(t) = -(g + \ddot{e}(t))\,\mathrm{d}t + \frac{1}{m}\mathrm{d}P_N(t)$, where $\mathrm{d}P_N(t) = \lambda_N(t)\mathrm{d}t + \Lambda_N(t)\mathrm{d}\eta$ contains the total contact percussion of the forces/impulses that act on the ball. The constitutive behaviour of the total contact percussion for a closed contact ($g_N(t) = 0$) can be expressed in the same way as in (4)

$$\xi_N(t) \ge 0\,, \quad \mathrm{d}P_N(t) \ge 0\,, \quad \xi_N(t)\mathrm{d}P_N(t) = 0\,, \quad \Leftrightarrow \quad -\mathrm{d}P_N \in N_{T_{\mathcal{K}}(g_N)}(\xi_N) = \begin{cases} \mathbb{R}_0^- & g_N = \xi_N = 0\,, \\ 0 & \text{else,} \end{cases} \tag{8}$$

where $T_{\mathcal{K}}(g_N)$ is the tangent cone on the set $\mathcal{K} = \{g_N \in \mathbb{R} \mid g_N \ge 0\}$ of admissible positions. The dynamics of the bouncing ball system can therefore be given in terms of a non-autonomous measure differential inclusion

$$\mathrm{d}\boldsymbol{x} \in \begin{bmatrix} x_2\mathrm{d}t \\ -(g + \ddot{e}(t))\,\mathrm{d}t + \frac{1}{m}N_{T_{\mathcal{K}}(g_N)}(\xi_N) \end{bmatrix} =: \mathrm{d}\boldsymbol{\Gamma}(t, \boldsymbol{x})\,, \tag{9}$$

with $\xi_N(t) = x_2^+(t) + \varepsilon(t)x_2^-(t)$. The system (9) has the admissible set $\mathcal{A} = \{\boldsymbol{x} \in \mathbb{R}^2 \mid x_1 \in \mathcal{K}\} = \{\boldsymbol{x} \in \mathbb{R}^2 \mid x_1 \ge 0\}$. Due to the choice of the state $\boldsymbol{x}(t)$ in (1), both the sets $\mathcal{A}$ and $\mathcal{K}$ are time-independent. The non-autonomicity of the system (9) is caused by explicit time-dependence of the table acceleration $\ddot{e}(t)$. In this respect, also note the equivalence with the forced system depicted in Figure 1(b).
We will use the notation $\boldsymbol{\varphi}(t, t_0, \boldsymbol{x}_0)$ for a solution curve $\boldsymbol{x}(t)$ with the initial condition $\boldsymbol{x}^-(t_0) = \boldsymbol{x}_0$. This solution is generally not unique in forward time. In [3] it has been proven that the solution of (9) is unique in forward time if the external excitation $e(t)$ of the table is an analytic function. For this reason we assume that $e(t)$ is analytic. Note that the solutions of the bouncing ball system are generally not unique in backward time. The bouncing ball system (9) is consistent in the sense that an admissible initial condition $\boldsymbol{x}_0 \in \mathcal{A}$ leads to an admissible solution curve $\boldsymbol{\varphi}(t, t_0, \boldsymbol{x}_0) \in \mathcal{A}$ for all $t \ge t_0$. We therefore have existence and uniqueness of solutions in forward time.

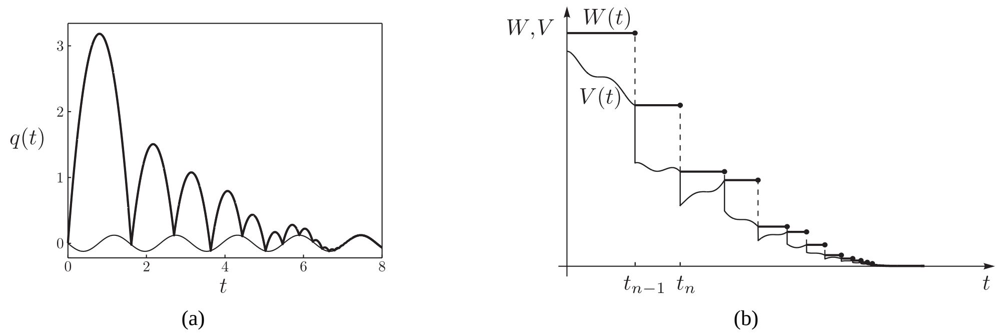

Figure 2: Sinusoidal excitation $e(t) = -A\sin(\Omega t)$: Trajectory of the bouncing ball system (a) and the corresponding Lyapunov function V and the step function W (b) for $\kappa = 0.2 < \kappa_{\mathrm{GUAS}}$, $\varepsilon = 0.8$, $t_0 = 0\,\mathrm{s}$, $g_N(t_0) = 0\,\mathrm{m}$, $\gamma_N^+(t_0) = 8\,\frac{\mathrm{m}}{\mathrm{s}}$.

Lyapunov stability of the equilibrium

In this section, a Lyapunov-like method for the stability analysis of non-autonomous measure differential inclusions of the form (9) is introduced. A point $\boldsymbol{x}^*$ is called an equilibrium point of $\mathrm{d}\boldsymbol{x}(t) \in \mathrm{d}\boldsymbol{\Gamma}(t,\boldsymbol{x})$ if there exists a solution curve such that $\boldsymbol{\varphi}(t, t_0, \boldsymbol{x}^*) = \boldsymbol{x}^*, \forall\, t \geq t_0$. In contrast to smooth dynamical systems (i.e. ODE's with a Lipschitz constant), the attractivity of an equilibrium point of a non-smooth system is not necessarily asymptotic as it might be reached in finite time. We will define global uniform attractive stability by making use of comparison functions as given in [4] and briefly recall the definitions. A continuous function $\alpha : [0, a) \to [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$. A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed r, the mapping $\beta(r, s)$ is decreasing with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$.

Definition 1 (Global Uniform Attractive Stability)
An equilibrium point $\boldsymbol{x}^*$ of (9) is called globally uniformly attractively stable if there exists a class $\mathcal{KL}$ function β such that each solution curve $\boldsymbol{\varphi}(\cdot, t_0, \boldsymbol{x}_0)$ for $\boldsymbol{x}_0 \in \mathcal{A}$ satisfies

$$\|\boldsymbol{\varphi}(t, t_0, \boldsymbol{x}_0) - \boldsymbol{x}^*\| \leq \beta(\|\boldsymbol{x}_0 - \boldsymbol{x}^*\|, t - t_0), \quad \textit{for almost all } t \geq t_0.$$

Stability properties are usually defined in terms of a Lyapunov ε–δ argument or, equivalently, by using comparison functions, see Appendix C.6 in [4]. The proof in [4], that the characterization with comparison functions implies the definition, is given for ordinary differential equations, but the proof does not use a solution concept and is therefore immediately valid for measure differential inclusions. The proof that the definition also implies the characterization is much more technical. Instead, we allow ourselves to take the characterization with comparison functions as *definition*. We now present a Lyapunov-like technique to prove global uniform attractive stability in the sense of Definition 1. Let t_0 denote the initial time-instant and $\boldsymbol{x}^-(t_0) = \boldsymbol{x}_0$ the initial condition. Doing so, we allow for a possible impulsive event at the initial time-instant. Let $\{t_n\}$ denote the sequence of time-instants $\{t_1, t_2, \cdots, t_\infty\}$ for which the solution curve $\boldsymbol{x}(t) := \boldsymbol{\varphi}(t, t_0, \boldsymbol{x}_0)$ is discontinuous for $t > t_0$. The solution $\boldsymbol{x}(t)$ has an accumulation point if t_∞ is finite.

Theorem 1
Let $\boldsymbol{x}^* = \mathbf{0}$ be an equilibrium point of (9). If there exists a positive definite function $V : \mathbb{R}^n \to \mathbb{R}_0^+ \cup \{+\infty\}$, being bounded on the admissible set $\mathcal{A}$ of (9), such that the step function $W(t)$ along solution curves of the system, defined by

$$W(t) = \begin{cases} \sup_{t\in[t_0,t_1]} V(\boldsymbol{x}^-(t)) & t \in [t_0, t_1] \\ V\left(\boldsymbol{x}^-(t_{n-1})\right) & t \in (t_{n-1}, t_n], n > 1 \\ 0 & t > t_\infty \end{cases}, \tag{10}$$

has the following properties

- $W(t_1) \leq \sigma(V(\boldsymbol{x}_0))$ *for some class $\mathcal{K}$ function σ,*
- $W(t)$ *is decreasing in time,*
- $W(t)$ *satisfies $V(\boldsymbol{x}(t)) \leq W(t)$ on each interval $t \in (t_{n-1}, t_n)$, $n > 1$,*
- $W(t)$ *converges to zero for $t \to t_\infty$, i.e. $\lim_{t\to t_\infty} W(t) = 0$,*

then the equilibrium $\boldsymbol{x}^*$ is globally uniformly attractively stable.

Proof: If the function V is positive definite and bounded on $\mathcal{A}$, then there exist functions α_1 and α_2 of class $\mathcal{K}_\infty$ such that

$$\alpha_1(\|\boldsymbol{x}\|) \leq V(\boldsymbol{x}) \leq \alpha_2(\|\boldsymbol{x}\|) \qquad \forall \boldsymbol{x} \in \mathcal{A}. \tag{11}$$

The function $W(t)$ therefore satisfies the inequality

$$W(t_0) = W(t_1) \leq \sigma(V(\boldsymbol{x}_0)) \leq \sigma(\alpha_2(\|\boldsymbol{x}_0\|)). \tag{12}$$

Define the comparison function

$$\beta_W(W(t_1), t-t_0) = \begin{cases} W(t_1) & t_0 \le t \le t_1, \\ \frac{W(t_n)-W(t_{n-1})}{t_n - t_{n-1}}(t-t_{n-1}) + W(t_{n-1}) & t_{n-1} \le t \le t_n, \\ 0 & t \ge t_\infty, \end{cases} \tag{13}$$

where the second case holds for all $n > 1$. Clearly, β_W is a continuous function in both arguments. For fixed $t - t_0$, the mapping $\beta_W(W(t_1), t-t_0)$ is upper-bounded by $\beta_W(W(t_1), t-t_0) \le W(t_1)$, i.e. a class $\mathcal{K}$ function with respect to $W(t_1)$. For fixed $W(t_1)$, the mapping $\beta_W(W(t_1), t-t_0)$ is non-increasing with respect to $t-t_0$ and $\beta_W(W(t_1), t-t_0) \to 0$ as $t \to t_\infty$. Hence, β_W is upper-bounded by a class $\mathcal{KL}$ function $\overline{\beta}_W(W(t_1), t-t_0) = \beta_W(W(t_1), t-t_0) + W(t_1)e^{t_0-t}$. It therefore holds that

$$\|\boldsymbol{x}(t)\| \le \alpha_1^{-1}\left(\overline{\beta}_W(\sigma(\alpha_2(\|\boldsymbol{x}_0\|)), t-t_0)\right) = \beta(\|\boldsymbol{x}_0\|, t-t_0), \tag{14}$$

for almost all $t \ge t_0$, where

$$\beta(x,t) = \alpha_1^{-1}\left(\overline{\beta}_W\left(\sigma\left(\alpha_2(x)\right)\right), t\right)$$

is a class $\mathcal{KL}$ function, which concludes the proof. □

We will use Theorem 1 to prove a sufficient condition for the global uniform attractive stability of the equilibrium of the bouncing ball system, see Proposition 1. Subsequently, we will prove that the attractivity is symptotic in Proposition 2.

Proposition 1 (Global uniform attractive stability)
Let the bouncing ball system (9) satisfy the bounds (6) on $e(t)$ and (5) on $\varepsilon(t)$ with $\mathrm{g} + a_{\min} > 0$. If it holds that

$$\alpha := \frac{\mathrm{g} + a_{\max}}{\mathrm{g} + a_{\min}} \bar{\varepsilon}^2 < 1\,, \tag{15}$$

then the equilibrium $\boldsymbol{x}^* = \mathbf{0}$ of the bouncing ball system is globally uniformly attractively stable.

Proof: Consider the Lyapunov candidate function

$$V(\boldsymbol{x}) = \frac{1}{2}x_2^2 + \tilde{g}x_1 + \Psi_{\mathcal{K}}(x_1) = \frac{1}{2}\gamma_N^2 + \tilde{g}g_N + \Psi_{\mathcal{K}}(g_N), \quad \text{with } \Psi_{\mathcal{K}}(g_N) = \begin{cases} 0 & g_N \in \mathcal{K}, \\ \infty & \text{else}, \end{cases} \tag{16}$$

where $\Psi_{\mathcal{K}}(g_N)$ is the indicator function on the admissible set $\mathcal{K} = \mathbb{R}_0^+$ and $\tilde{g} > 0$ is (for the moment) an arbitrary positive number. The function V is an energy-like Lyapunov function in terms of the relative coordinates g_N and γ_N. More specifically, if we take $\tilde{g} = g$, it is the total mechanical energy per unit mass of the equivalent forced system depicted in Figure 1(b). The indicator function $\Psi_{\mathcal{K}}(g_N)$ plays the role of a potential for the contact force and is necessary to make V a positive definite function. The bouncing ball system is consistent in the sense that solutions remain in the admissible set $\mathcal{A}$ for admissible initial conditions. It therefore holds that $\Psi_{\mathcal{K}}(x_1(t)) = 0$ for all $t \ge t_0$ along solution curves of the system. With some abuse of notation we define $V(t) = V(\boldsymbol{x}(t))$ to be the Lyapunov candidate function evaluated along a solution curve $\boldsymbol{x}(t)$. The differential measure of $V(t)$ is given by $\mathrm{d}V = \dot{V}\mathrm{d}t + (V^+ - V^-)\,\mathrm{d}\eta$, because $V(t)$ is a function of special locally bounded variation because of its dependence on $x_2(t)$. The function $V(t)$ is discontinuous at collision times t_n when the gap function $g_N(t_n)$ vanishes with $\gamma_N^-(t_n) < 0$. The jump height follows from the impact law (4), i.e.

$$V^+(t_n) - V^-(t_n) = \frac{1}{2}\gamma_N^+(t_n)^2 - \frac{1}{2}\gamma_N^-(t_n)^2 = -\frac{1}{2}\left(1 - \varepsilon(t_n)^2\right)\gamma_N^-(t_n)^2 \le -\frac{1}{2}\left(1 - \bar{\varepsilon}^2\right)\gamma_N^-(t_n)^2 < 0, \tag{17}$$

where the inequalities follow from the bound (5). This implies that V decreases over impacts. The time derivative of $V(t)$, i.e. $\dot{V} = \gamma_N\dot{\gamma}_N + \tilde{g}\gamma_N = -\gamma_N\ddot{e} + \gamma_N(\tilde{g} - g)$, depends explicitly on time and can be negative or positive such that the Lyapunov function may decrease or increase in between collisions. The maximal time derivative $\dot{V}$ is obtained if $\ddot{e}(t) = a_{\min}$ when $\gamma_N > 0$ and $\ddot{e}(t) = a_{\max}$ when $\gamma_N < 0$, see (6), which yields the conservative estimate

$$\dot{V} \le \begin{cases} (-a_{\min} + \tilde{g} - g)\,\gamma_N & \gamma_N \ge 0, \\ (-a_{\max} + \tilde{g} - g)\,\gamma_N & \gamma_N < 0, \end{cases} \quad \Leftrightarrow \quad \dot{V} \le \frac{a_{\max} - a_{\min}}{2}|\gamma_N| + \left(-\frac{a_{\max} + a_{\min}}{2} + \tilde{g} - g\right)\gamma_N. \tag{18}$$

We now choose $\tilde{g}$ such that the last term in (18) vanishes. This choice of $\tilde{g}$ still satisfies $\tilde{g} > 0$, i.e.

$$\tilde{g} = g + \frac{a_{\max} + a_{\min}}{2} > 0. \tag{19}$$

Define the step function $W(t)$ along solution curves $\boldsymbol{x}(t) = \boldsymbol{\varphi}(t, t_0, \boldsymbol{x}_0)$ of the system as in (10). The value of $W(t_1) = \sup_{t\in[t_0,t_1]} V(\boldsymbol{x}^-(t))$ is the maximum of $V(\boldsymbol{x}_0)$ and $\sup_{t\in(t_0,t_1)} V(t)$, where

$$V(t) = V^+(t_0) + \int_{t_0}^{t} \dot{V}(t)\,\mathrm{d}t \le V^+(t_0) + \int_{t_0}^{t_1} |\dot{V}(t)|\,\mathrm{d}t\,, \quad t \in (t_0, t_1). \tag{20}$$

Using $V^+(t_0) \leq V^-(t_0) = V(\boldsymbol{x}_0)$, together with (18), (19) and (20), we can give an upper-bound for $W(t_1)$:

$$W(t_1) \leq V(\boldsymbol{x}_0) + \frac{a_{\max} - a_{\min}}{2} \int_{t_0}^{t_1} |\gamma_N(t)| \,\mathrm{d}t. \tag{21}$$

The integral $\int_{t_0}^{t_1} |\gamma_N(t)|\mathrm{d}t$ is the total variation of the absolutely continuous function $g_N(t)$ on the time-interval $[t_0, t_1]$. On each non-impulsive interval, $g_N(t)$ is concave, because it holds that $\ddot{g}_N(t) = \dot{\gamma}_N(t) = -g - \ddot{e}(t) < 0$ due to the inequality

$$-g - a_{\max} \leq \dot{\gamma}_N(t) \leq -g - a_{\min} < 0. \tag{22}$$

The total variation of $g_N(t)$ on $[t_0, t_1]$ has therefore the upper-bound

$$\int_{t_0}^{t_1} |\gamma_N(t)|\mathrm{d}t \leq 2 \max_{[t_0,t_1]} g_N(t)\,. \tag{23}$$

The gap function $g_N(t)$ is smooth on (t_0, t_1) and can be written as a Taylor series at t_0 with Lagrange form of the remainder term as $g_N(t) = g_N(t_0) + \gamma_N^+(t_0)\,(t - t_0) + \frac{1}{2}\dot{\gamma}_N(\tilde{t})\,(t - t_0)^2$ for some $\tilde{t} \in (t_0, t)$. Using (22) we obtain the upper-bound

$$g_N(t) \leq g_N(t_0) + \gamma_N^+(t_0)\,(t - t_0) - \frac{1}{2}\,(g + a_{\min})\,(t - t_0)^2\,. \tag{24}$$

The function $g_N(t)$ on $[t_0, t_1]$ is therefore bounded from above by

$$\max_{[t_0,t_1]} g_N(t) \leq g_N(t_0) + \frac{1}{2}\frac{\gamma_N^+(t_0)^2}{g + a_{\min}} \leq \frac{1}{\tilde{g}}V(\boldsymbol{x}_0) + \frac{1}{g + a_{\min}}V(\boldsymbol{x}_0). \tag{25}$$

Using (21), (23) and (25), the value $W(t_1)$ is upper-bounded by the following class $\mathcal{K}_\infty$ function with respect to $V(\boldsymbol{x}_0)$:

$$W(t_1) \leq \left(1 + \frac{a_{\max} - a_{\min}}{\tilde{g}} + \frac{a_{\max} - a_{\min}}{g + a_{\min}}\right) V(\boldsymbol{x}_0). \tag{26}$$

The step function $W(t)$ is a left-continuous piecewise constant function with discontinuities at the collision time-instants $t = t_n$. The step height is $W(t_{n+1}) - W(t_n) = V^-(t_n) - V^-(t_{n-1})$, which can be interpreted as the cumulative change of V over one impact at the time-instant t_{n-1} and the subsequent non-impulsive interval (t_{n-1}, t_n), i.e.

$$W(t_{n+1}) - W(t_n) = \int_{[t_{n-1},t_n)} \mathrm{d}V = V^+(t_{n-1}) - V^-(t_{n-1}) + \int_{t_{n-1}}^{t_n} \dot{V}\mathrm{d}t. \tag{27}$$

Using (18) and (19), we can give an upper-bound for the last term in (27)

$$\int_{t_{n-1}}^{t_n} \dot{V}\mathrm{d}t \leq \frac{a_{\max} - a_{\min}}{2} \int_{t_{n-1}}^{t_n} |\gamma_N(t)| \,\mathrm{d}t, \tag{28}$$

where $\int_{t_{n-1}}^{t_n} |\gamma_N(t)|\mathrm{d}t$ is, again, the total variation of $g_N(t)$ on $[t_{n-1}, t_n]$. The steps in equations (22) to (25), which have been derived for the time-interval $[t_0, t_1]$, are now repeated for the time-interval $[t_{n-1}, t_n]$ using $g_N(t_{n-1}) = 0$:

$$\int_{t_{n-1}}^{t_n} |\gamma_N(t)|\mathrm{d}t = 2 \max_{[t_{n-1},t_n]} g_N(t) \leq \frac{\gamma_N^+(t_{n-1})^2}{g + a_{\min}} \leq \frac{\bar{\varepsilon}^2\gamma_N^-(t_{n-1})^2}{g + a_{\min}}\,. \tag{29}$$

With the conservative estimates (17) and (29) the step height (27) of $W(t)$ is bounded from above by

$$W(t_{n+1}) - W(t_n) \leq \left(-\frac{1}{2}\left(1 - \bar{\varepsilon}^2\right) + \frac{a_{\max} - a_{\min}}{2}\frac{\bar{\varepsilon}^2}{g + a_{\min}}\right) \gamma_N^-(t_{n-1})^2. \tag{30}$$

We now recall the definition of α from (15) and note that $0 \leq \alpha < 1$ under the conditions of (15). Substitution of α and $W(t_n) = V^-(t_{n-1}) = \frac{1}{2}\left(\gamma_N^-(t_{n-1})\right)^2$ in (30) gives an upper-bound for the discrete map $W(t_n) \mapsto W(t_{n+1})$

$$W(t_{n+1}) \leq \alpha W(t_n), \tag{31}$$

which is a contraction map because $|\alpha| < 1$ and $W(t) \geq 0\,\forall t$. This implies $\lim_{t\to t_\infty} W(t) = 0$ as required in Theorem 1. Lastly, we prove that the step function $W(t)$ forms an upper-bound for the Lyapunov function $V(t)$. Without loss of generality we consider $t \in (t_{n-1}, t_n)$. It holds that $W(t) = V^-(t_{n-1})$ and $V(t) = V^-(t_{n-1}) + \int_{[t_{n-1},t)} \mathrm{d}V$ and therefore

$$W(t) - V(t) = -\left(V^+(t_{n-1}) - V^-(t_{n-1})\right) - \int_{t_{n-1}}^{t} \dot{V}\mathrm{d}t \geq \frac{1}{2}(1 - \alpha)\gamma_N^-(t_{n-1})^2. \tag{32}$$

The inequality in (32) follows from using the same conservative estimate as in (28) and we note that $\int_{t_{n-1}}^{t} |\gamma_N(t)|\mathrm{d}t \leq \int_{t_{n-1}}^{t_n} |\gamma_N(t)|\mathrm{d}t$ for which we have the upper-bound (29). Hence, the difference $W - V$ can, using the definition of α, be bounded from below as stated in (32) which is non-negative under condition (15) meaning that $W(t) - V(t) \geq 0\,\forall t$. All the conditions of Theorem 1 are therefore satisfied which proves that the equilibrium $\boldsymbol{x}^* = \mathbf{0}$ of the bouncing ball system is globally uniformly attractively stable under condition (15). □

In the case of $e(t) = -A\sin(\Omega t)$, it holds that $a_{\max} = -a_{\min} = A\Omega^2$. The condition (15) can be expressed in terms of the relative acceleration $\kappa = \frac{A\Omega^2}{g}$ as $\kappa < \frac{1-\bar{\varepsilon}^2}{1+\bar{\varepsilon}^2} =: \kappa_{\mathrm{GUAS}}$. This is a sufficient condition for globally uniform attractive stability of the equilibrium $\boldsymbol{x}^* = \mathbf{0}$. In contrast to the classical direct method of Lyapunov, where the Lyapunov function V is required to be non-increasing, the function V may decrease and increase on the interval (t_{n-1}, t_n), but cumulatively, it decreases on the intervals between two consecutive impacts which is guaranteed by the condition $\kappa < \kappa_{\mathrm{GUAS}}$. Therefore, V can be regarded as a Lyapunov function in a generalized sense.
We might be tempted to think that the function W can be looked upon as a discrete-time Lyapunov function. A discrete-time Lyapunov function would be a (locally) positive definite function on the discrete state of the system at the impact time, which decreases under iterations of the impact map. Note, however, that the function W is constructed from the time-evolution of V along solution curves and is therefore not a function of the discrete state. For this reason, it cannot be regarded as a discrete-time Lyapunov function, although it surely is related to a discrete-time Lyapunov function of the impact map. But there is a more fundamental difference: a discrete-time Lyapunov function only gives information on the state at discrete time-instants and does not check whether the solution converges to zero in-between impacts. In contrast, the step function W is an upper-bound for V on the *whole* time-domain.

Proposition 2 (Symptotic attractive stability)
If the conditions of Proposition 1 are met, then the equilibrium $\boldsymbol{x}^* = \mathbf{0}$ of (9) is globally symptotically attractive.

Proof: The proposition uses the same conditions as Proposition 1 and we can therefore make use of the results of the proof of Proposition 1. Evaluation of (24) for $t = t_1$ with $g_N(t_1) = 0$ gives an inequality of the form $0 \leq f(t_1 - t_0)$ for the time lapse $t_1 - t_0 \geq 0$, where $f(t_1 - t_0)$ is a concave quadratic function with $g_N(t_0) \geq 0$. The time lapse $t_1 - t_0$ is therefore bounded from above by the positive root of $f(t_1 - t_0) = 0$, see first term in (35). Similarly, evaluation of the inequality (24) for the non-impulsive time-interval (t_n, t_{n+1}) with $g_N(t_n) = g_N(t_{n+1}) = 0$ and $\gamma_N^+(t_n) = -\varepsilon(t_n)\gamma_N^-(t_n)$ gives

$$t_{n+1} - t_n \leq \frac{-2\varepsilon(t_n)\gamma_N^-(t_n)}{g + a_{\min}} \leq \frac{2\bar{\varepsilon}\sqrt{2W(t_{n+1})}}{g + a_{\min}}\,, \tag{33}$$

which is an upper-bound for the time lapse between two consecutive collisions. The time-instant t_∞ is therefore bounded from above by the sum $t_\infty - t_1 = \sum_{n=1}^{\infty}(t_{n+1} - t_n)$ and (33). Recursive usage of (31) gives the upper-bound $W(t_{n+1}) \leq \alpha^n W(t_1)\ \forall n \geq 1$, where $0 \leq \alpha < 1$ due to (15). The sum is therefore bounded from above by the geometric series

$$t_\infty - t_1 \leq \frac{2\sqrt{2}\bar{\varepsilon}}{g + a_{\min}} \sum_{n=1}^{\infty} \sqrt{W(t_{n+1})} \leq \frac{2\sqrt{2}\bar{\varepsilon}}{g + a_{\min}} \sqrt{W(t_1)} \sum_{n=1}^{\infty} \sqrt{\alpha^n} = \frac{2\sqrt{2}\bar{\varepsilon}}{g + a_{\min}} \sqrt{W(t_1)} \frac{\alpha^{\frac{1}{2}}}{1 - \alpha^{\frac{1}{2}}}. \tag{34}$$

The upper-bound for the time lapse between the initial time t_0 and the accumulation point t_∞ is therefore given by

$$t_\infty - t_0 \leq \frac{\gamma_N^+(t_0) + \sqrt{\gamma_N^+(t_0)^2 + 2(g + a_{\min})g_N(t_0)}}{g + a_{\min}} + \frac{2\sqrt{2}\bar{\varepsilon}}{g + a_{\min}} \sqrt{W(t_1)} \frac{\alpha^{\frac{1}{2}}}{1 - \alpha^{\frac{1}{2}}} \tag{35}$$

with $W(t_1)$ bounded by (26) and $|\gamma_N^+(t_0)| \leq |\gamma_N^-(t_0)|$. Hence, for any bounded initial condition $\boldsymbol{x}_0$, the solution $\boldsymbol{\varphi}(t, t_0, \boldsymbol{x}_0)$ converges in a finite time $t_\infty - t_0$ to the equilibrium $\boldsymbol{x}^* = \mathbf{0}$. □

Propositions 1 and 2 consider the same system with identical assumptions. We can therefore summarize that if the conditions of Proposition 1 are met, then the equilibrium $\boldsymbol{x}^* = \mathbf{0}$ of (9) is globally uniformly symptotically attractively stable. The numerical simulations in Figure 2 illustrate the theoretical results for $\kappa < \kappa_{\mathrm{GUAS}}$. The trajectory of the ball shows that the solution is attracted to the equilibrium in finite time through an infinite number of impacts. The Lyapunov function V, evaluated along the solution curve, is oscillating but is bounded from above by the decreasing step function $W(t)$.

Conclusions

The proposed Lyapunov technique for non-smooth dynamical systems can be regarded as an extension of Lyapunov's direct method to Lyapunov functions which may also temporarily increase along solution curves. The merit of the proposed Lyapunov-like method is that it allows to choose more natural Lyapunov candidate functions, e.g. energy-like functions or other functions with a clear physical meaning. A sufficient condition for the global uniform symptotic attractive stability of the equilibrium of the bouncing ball system with an arbitrary motion of the table and a time-varying restitution coefficient is proved by using the presented Lyapunov-like method.

References

[1] T. Heimsch, R. I. Leine, Lyapunov stability theory for nonsmooth non-autonomous mechanical systems applied to the bouncing ball problem, Proceedings of the ASME 2009 International Design Engineering Technical Conferences & Computers and Information in Engineering Conference IDETC/CIE 2009, DETC/MSNDC-87185, San Diego, USA.

[2] R. I. Leine, T. F. Heimsch, Global uniform symptotic attractive stability of the non-autonomous bouncing ball system, Physica D, submitted 2010.

[3] M. Schatzman, Uniqueness and continuous dependence on data for one dimensional impact problems, Mathematical and Computational Modelling 28 (1998) 1–18.

[4] H. K. Khalil, Nonlinear Systems, Prentice-Hall, New Jersey, 1996.