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# Synchronization-based Estimation of the Maximal Lyapunov Exponent of Nonsmooth Systems

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## Abstract

The maximal Lyapunov exponent of a nonsmooth system is the lower bound for the proportional feedback gain necessary to achieve full state synchronization. In this paper, we prove this statement for the general class of nonsmooth systems in the framework of measure differential inclusions. The results are used to estimate the maximal Lyapunov exponent using chaos synchronization, which is illustrated on an impact oscillator.

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## 1. Introduction

The spectrum of Lyapunov exponents is an important characteristic of limit sets. It measures the exponential convergence or divergence of nearby trajectories, thereby capturing the sensitivity of solutions with respect to initial conditions<sup>1</sup>. An infinitesimal sphere of perturbed initial conditions will deform into an ellipsoid under the flow of a smooth dynamical system<sup>2</sup>. The Lyapunov exponents capture the average exponential growth or decay rate of the principal axes of the ellipsoid and the maximal Lyapunov exponent captures the long-term behavior of the dominating direction. A positive maximal Lyapunov exponent implies instability of the limit set (i.e., equilibrium, limit cycle, periodic or quasi-periodic solution) or it can be an indication for a chaotic attractor<sup>3,4</sup>.

The existence of the Lyapunov exponents is a subtle question for non-conservative systems<sup>5,6</sup>. The mathematic foundation for the existence is given by the multiplicative ergodic theorem of Oseledec<sup>7,8</sup>. It states that, if there exists an invariant measure of the flow, then the Lyapunov exponents exist for almost every point w.r.t. that measure.

Algorithms to find the spectrum of Lyapunov exponents of smooth systems are well established<sup>9–11</sup>. The spectrum can be computed numerically by linearizing the differential equations along the nominal solution. Time integration of the linearized equations yields the fundamental solution matrix from which the spectrum can be obtained. Lyapunov exponents can also be obtained from experimental time series of systems with unknown dynamics<sup>12–15</sup>.

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Dynamical systems with a discontinuous right-hand side exhibit discontinuities in the evolution of the fundamental solution matrix. The jumps can be described using a saltation matrix<sup>16</sup>, which has been used for example for the numerical computation of the Lyapunov exponents with jump conditions including the motion on sliding surfaces<sup>17</sup> or for Filippov-type systems with an emphasis on mechanical systems with Coulomb friction<sup>18</sup>. A model based algorithm for the calculation of the spectrum of Lyapunov exponents has been developed for dynamical systems with discontinuous motion<sup>19</sup>.

Two diffusively coupled identical smooth systems achieve synchronization despite the complicated dynamics of the individual systems if the coupling parameter is large enough<sup>20</sup>. The minimal value of the coupling parameter for which the synchronization set is (attractively) stable is determined by the maximal Lyapunov exponent of the individual systems. This relation arises from the competitive behavior of the separation due to the trajectory instability (dominated by the maximal Lyapunov exponent) and the convergence due to the coupling. Using this relation, the maximal Lyapunov exponent can be estimated by the critical coupling necessary for synchronization<sup>21</sup>. The method of estimating the maximal Lyapunov exponent using chaos synchronization has been considered for nonsmooth systems<sup>22</sup> and for discrete maps<sup>23</sup> assuming that the increase of the initial perturbation is uniform in time, which is only the case for linear time-invariant systems).

In this paper, we consider the class of nonsmooth systems with solutions of special locally bounded variation, which can be written in the framework of measure differential inclusions<sup>24</sup>. We prove for this general class of nonsmooth systems that the critical coupling is indeed given by the maximal Lyapunov exponent as long as it exists. The paper is organized as follows. We first restrict ourselves to smooth systems in Section 2 before we state the main result for nonsmooth systems in Section 3. The results are illustrated in Section 4 on an impact oscillator and conclusions are given in Section 5.

## 2. Smooth systems

The dynamics of a smooth system is given by

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t), \tag{1}$$

where the vector field  $\mathbf{f} : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n$  is continuously differentiable in its first argument and continuous in its second argument. We denote the solution of (1) for the initial conditions  $\mathbf{x}(t_0) = \mathbf{x}_0$  as  $\mathbf{x}(t) = \boldsymbol{\varphi}(t, \mathbf{x}_0, t_0)$ , where the dependence on initial conditions is written explicitly. We introduce the perturbed solution  $(\mathbf{x} + \Delta\mathbf{x})(t) = \boldsymbol{\varphi}(t, \mathbf{x}_0 + \kappa\mathbf{e}, t_0)$  obtained using the perturbed initial conditions  $\mathbf{x}_0 + \kappa\mathbf{e}$  with  $\|\mathbf{e}\| = 1$  and  $\kappa > 0$  small. The dynamics of the perturbation  $\Delta\mathbf{x}(t)$  is obtained as

$$\frac{d(\Delta\mathbf{x})}{dt} = \mathbf{f}(\mathbf{x} + \Delta\mathbf{x}, t) - \mathbf{f}(\mathbf{x}, t) = \mathbf{A}(t)\Delta\mathbf{x} + \mathbf{o}(\|\Delta\mathbf{x}\|), \tag{2}$$

where  $\mathbf{A}(t) := \left. \frac{\partial \mathbf{f}(\mathbf{x}, t)}{\partial \mathbf{x}} \right|_{\boldsymbol{\varphi}(t, \mathbf{x}_0, t_0)}$  is the linearization of the vector field  $\mathbf{f}$  along the unperturbed solution and  $\mathbf{o}$  denotes the (small) Landau-order symbol. The perturbation  $\Delta\mathbf{x}$  tends to zero for  $\kappa \rightarrow 0$ . Therefore, we introduce the normalized perturbation  $\boldsymbol{\xi}(t, \mathbf{e}, t_0) := \lim_{\kappa \rightarrow 0} \boldsymbol{\xi}_\kappa(t, \mathbf{e}, t_0)$ , where  $\boldsymbol{\xi}_\kappa(t, \mathbf{e}, t_0) := \frac{\Delta\mathbf{x}}{\|\Delta\mathbf{x}_0\|} = \frac{\boldsymbol{\varphi}(t, \mathbf{x}_0 + \kappa\mathbf{e}, t_0) - \boldsymbol{\varphi}(t, \mathbf{x}_0, t_0)}{\kappa}$ . The limit exists because  $\mathcal{O}(\|\Delta\mathbf{x}\|) = \mathcal{O}(\|\Delta\mathbf{x}(t_0)\|) = \mathcal{O}(\kappa)$ , where  $\mathcal{O}$  denotes the (big) Landau-order symbol. Taking the limit  $\kappa \rightarrow 0$  of (2), divided by  $\kappa$ , yields

$$\lim_{\kappa \rightarrow 0} \frac{d\boldsymbol{\xi}_\kappa}{dt} = \mathbf{A}(t)\boldsymbol{\xi}. \tag{3}$$

The vector field  $\mathbf{f}$  is continuously differentiable in its first argument, which implies local uniform convergence of  $\lim_{\kappa \rightarrow 0} \frac{d\boldsymbol{\xi}_\kappa}{dt}$ . Using Theorem 7.17 of Rudin<sup>25</sup> together with (3) yields

$$\frac{d\boldsymbol{\xi}}{dt} = \mathbf{A}(t)\boldsymbol{\xi}. \tag{4}$$

Let  $\boldsymbol{\Phi}(t, t_0)$  be the fundamental solution matrix, which is the solution to the matrix differential equation  $\frac{d\boldsymbol{\Phi}}{dt} = \mathbf{A}(t)\boldsymbol{\Phi}$  for the initial conditions  $\boldsymbol{\Phi}(t_0, t_0) = \mathbf{I}$ , where  $\mathbf{I}$  is the identity matrix. Then, the solution of the normalized perturbation  $\boldsymbol{\xi}$

can be written as  $\xi(t, e, t_0) = \Phi(t, t_0)e$ . Furthermore, let  $\sigma_i(\Phi) = \sqrt{\text{eig}_i(\Phi^T \Phi)}$  be the singular values of  $\Phi$ . The spectrum or Lyapunov exponents is given by  $\lambda_i = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \sigma_i$ , where the maximal Lyapunov exponent is denoted by  $\lambda_{\max} = \max_i \{\lambda_i\}$ . The largest singular value is the spectral norm, that is, the matrix norm induced by the Euclidean norm. The maximal Lyapunov exponent can therefore be defined by

$$\lambda_{\max} := \max_e \lim_{t \rightarrow \infty} \frac{1}{t} \ln \|\xi(t, e, t_0)\|. \tag{5}$$

Up to this point, we have discussed the perturbation dynamics of the smooth dynamical system (1) and we have defined the maximal Lyapunov exponent in (5) using the normalized perturbation  $\xi$ . In the following, we consider the error dynamics of two identical smooth systems with a unidirectional diffuse coupling. For this purpose, system (1) is accompanied by a replica together with a proportional error feedback as

$$\begin{aligned} \frac{dx}{dt} &= f(x, t), \\ \frac{dy}{dt} &= f(y, t) - k(y - x), \end{aligned} \tag{6}$$

where  $k \in \mathbb{R}$  is the coupling gain. The initial conditions are chosen as  $x(t_0) = x_0$  and  $y(t_0) = x_0 + \kappa e$ , which are the same initial conditions as for the perturbation dynamics. Let  $z := y - x$  denote the synchronization error of the coupled dynamics (6). Using the linearization  $A(t) = \left. \frac{\partial f(x, t)}{\partial x} \right|_{\varphi(t, x_0, t_0)}$ , the error dynamics is described by

$$\frac{dz}{dt} = A(t)z - kz + o(\|z\|). \tag{7}$$

The error  $z$  is of the order of the initial error  $z_0$ , which itself is proportional to  $\kappa$ . Analogously to the normalized perturbation  $\xi$ , we define the normalized synchronization error as  $\zeta(t, e, t_0) := \lim_{\kappa \rightarrow 0} \frac{z}{\kappa}$ . Taking the limit  $\kappa \rightarrow 0$  of (7), divided by  $\kappa$ , and using  $z \in \mathcal{O}(\kappa)$  yields

$$\frac{d\zeta}{dt} = A(t)\zeta - k\zeta. \tag{8}$$

By comparing (4) and (8), we find that

$$\zeta = \Phi(t, t_0)e^{-k(t-t_0)}e = \xi e^{-k(t-t_0)}. \tag{9}$$

Local synchronization of the coupled system (6) is achieved if there exists a constant  $c > 0$  such that  $\lim_{t \rightarrow \infty} \|z(t)\| = 0 \forall \|z_0\| < c$ . Hence, local synchronization is achieved if the limit of the normalized synchronization error  $\zeta(t, e, t_0)$  tends to zero as  $t \rightarrow \infty$  for all  $e$ . The ‘worst case’ of all possible limits can be written using (9) together with (5) as

$$\max_e \lim_{t \rightarrow \infty} \|\zeta\| = \max_e \lim_{t \rightarrow \infty} \|\xi\| e^{-k(t-t_0)} = \lim_{t \rightarrow \infty} e^{(\lambda_{\max} - k)(t-t_0)},$$

from which follows that local synchronization of the coupled system (6) is achieved if (and only if) the coupling gain  $k$  is larger than the maximal Lyapunov exponent  $\lambda_{\max}$ :

$$\lim_{t \rightarrow \infty} \|\zeta(t, e, t_0)\| = 0 \forall e \text{ if and only}^* \text{ if } k > \lambda_{\max}. \tag{10}$$

The star in (10) denotes that  $k = \lambda_{\max}$  is excluded in the converse. No statement can be made for the case where  $k$  is exactly equal to  $\lambda_{\max}$  because, for this case, the stability and attractivity is determined by higher-order terms.

**Remark 1.** Every strange attractor has a positive maximal Lyapunov exponent<sup>2</sup> and, thus, a positive critical coupling gain  $k_{\text{crit}}$ . The derived results are also applicable to other limit sets than chaotic attractors. For example, an attractive equilibrium has a negative maximal Lyapunov exponent and thus the critical coupling gain is negative as well.

If the initial conditions are exactly orthogonal to the direction corresponding to the maximal Lyapunov exponent, then the synchronization error tends to zero for a coupling gain  $k$  smaller than  $\lambda_{\max}$ . However, any slight error or uncertainty will turn any initial error in the direction of maximal expansion<sup>26,4</sup>, that is,

$$\exists e : \lim_{t \rightarrow \infty} \|\zeta(t, e, t_0)\| = 0 \Rightarrow \lim_{t \rightarrow \infty} \|\zeta(t, e, t_0)\| = 0 \text{ for almost all } e.$$

Therefore, in practice, local synchronization is achieved if and only if the coupling gain  $k$  is larger than the critical coupling gain  $k_{\text{crit}} = \lambda_{\max}$ .

### 3. Non-smooth systems

A broad class of non-smooth systems can be written in the form of a measure differential inclusion<sup>24</sup> as

$$\frac{dx}{d\mu} \in \mathcal{F}(x, t). \tag{11}$$

The solution for admissible initial conditions  $x^-(t_0) = x_0$  is denoted by  $x(t) = \varphi(t, x_0, t_0)$ . Solutions are generally discontinuous, but assumed to be of special locally bounded variation. The density  $\frac{dx}{d\mu}$  in (11) is the density of the measure  $dx$  with respect to a positive Radon measure  $d\mu$  and can be defined as<sup>27</sup>

$$\frac{dx}{d\mu}(t) := \lim_{\varepsilon \rightarrow 0} \frac{dx(I_\varepsilon)}{d\mu(I_\varepsilon)}, \tag{12}$$

where  $I_\varepsilon(t) := (t - \varepsilon, t + \varepsilon)$  with  $\varepsilon > 0$  is the open time interval centered at  $t$ . The solutions are absolutely continuous almost everywhere (a.e.) (i.e., for almost all  $t \in \mathbb{R}$  with respect to the Lebesgue measure) and have at most countably many discontinuities, at which the solutions are undefined. However, the one-sided limits  $x^-(t) = \lim_{\tau \rightarrow 0, \tau < 0} x(t + \tau)$  and  $x^+(t) = \lim_{\tau \rightarrow 0, \tau > 0} x(t + \tau)$  are well defined on the entire time axis.

As for the smooth case in Section 2, we consider a reference solution  $x(t) = \varphi(t, x_0, t_0)$  and a perturbed solution  $(x + \Delta x)(t) = \varphi(t, x_0 + \kappa e, t_0)$  for system (11), where the initial conditions  $x_0 + \kappa e$  with  $\|e\| = 1$  and  $\kappa > 0$  are assumed to be admissible for all  $\kappa \geq 0$ . Furthermore, we define the normalized perturbation as  $\xi(t, e, t_0) := \lim_{\kappa \rightarrow 0, \kappa \notin E_\kappa^\xi(t)} \xi_\kappa(t, e, t_0)$ , where  $\xi_\kappa(t, e, t_0) := \frac{\Delta x}{\kappa}$  and  $E_\kappa^\xi(t) := \{\kappa \in \mathbb{R}_0^+ \mid \xi_\kappa^-(t, e, t_0) \neq \xi_\kappa^+(t, e, t_0)\}$ . To ensure that the normalized perturbation exists a.e., we will make the following assumption.

**Assumption 1.** The difference between two solutions of system (11) is almost everywhere of the order of the initial difference, that is,  $\varphi(t, x_0 + \kappa e, t_0) - \varphi(t, x_0, t_0) \in \mathcal{O}(\kappa)$  a.e.

Assumption 1 together with  $\mathcal{O}(\kappa) \subset \mathcal{o}(1)$  implies continuous dependence on initial conditions, which can be written as  $\lim_{\kappa \rightarrow 0} \varphi(t, x_0 + \kappa e, t_0) - \varphi(t, x_0, t_0) = \mathbf{0} \forall e$ . Furthermore, continuous dependence on initial conditions implies uniqueness of solutions in forward time.

**Remark 2.** Assumption 1 is generally restrictive for non-smooth systems. For example, mechanical systems subjected to frictionless unilateral constraints do generally not have continuity on initial conditions<sup>28</sup>. Assumption 1 omits grazing trajectories but allows for accumulation points, which is a phenomenon describing (countably) infinitely many impact events in a finite time interval. The maximal Lyapunov exponent does not exist if Assumption 1 is not met such that no local synchronization can be achieved with a finite coupling gain  $k$ .

If the unperturbed solution has a discontinuity at time  $t$ , then the one-sided limits of  $\varphi(t, x_0 + \kappa e, t_0) - \varphi(t, x_0, t_0)$  are not necessarily of order  $\mathcal{O}(\kappa)$ , because the discontinuities of  $\varphi(t, x_0 + \kappa e, t_0)$  and  $\varphi(t, x_0, t_0)$  do generally not occur at the same points in time for  $\kappa > 0$ . However, Assumption 1 implies that the discontinuity points of  $\varphi(t, x_0 + \kappa e, t_0)$  tend to the ones of  $\varphi(t, x_0, t_0)$  such that  $\xi$  exists for almost all  $t \in \mathbb{R}$ , which is illustrated in Figure 1. It shows an unperturbed solution  $\varphi(t, x_0, t_0)$  with a discontinuity at  $t = 1$  and a kink at  $t = 5$ . The perturbed solutions  $\varphi(t, x_0 + \kappa e, t_0)$  tend a.e. to the unperturbed solutions for  $\kappa \rightarrow 0$ , that is,  $\lim_{\kappa \rightarrow 0} \Delta x = \mathbf{0}$  a.e. The jump in  $\varphi$  yields a peak in the normalized perturbation  $\xi$  (i.e.,  $\xi$  is undefined at this point) and the kink in  $\varphi$  yields a jump in  $\xi$ . Therefore, due to Assumption 1, the normalized perturbation  $\xi(t, e, t_0)$  exists for almost all  $t$ .

Since (11) has a unique solution in forward time by Assumption 1, the density  $\frac{dx}{d\mu}$  is unique as well and the dynamics can be written as a measure differential equation

$$\frac{dx}{d\mu} = f(x^-, t), \tag{13}$$

where  $f(x^-, t) \in \mathcal{F}(x, t)$ . The dynamics of the perturbation  $\Delta x$  is governed by  $\frac{d\Delta x}{d\mu} = f(x^- + \Delta x^-, t) - f(x^-, t)$ . Dividing by  $\kappa$  and substituting the definition of  $\xi_\kappa$  yields

$$\frac{d\xi_\kappa}{d\mu} = \frac{f(x^- + \kappa \xi_\kappa^-, t) - f(x^-, t)}{\kappa}. \tag{14}$$

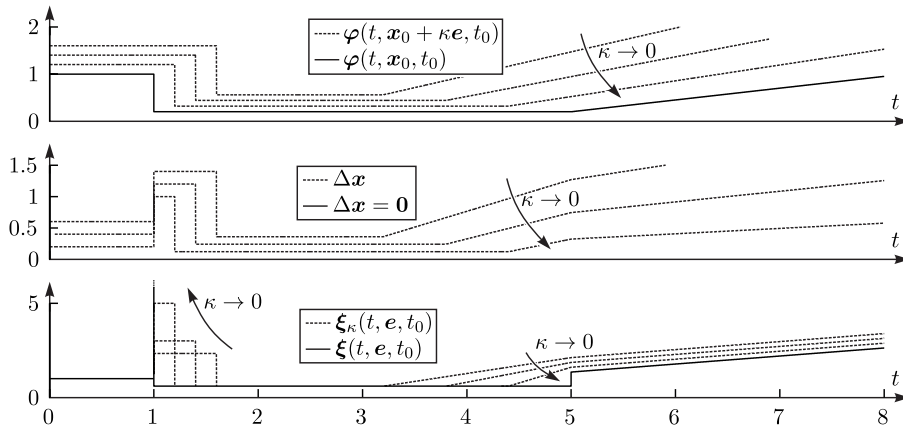


Fig. 1. Assumption 1 implies that the normalized perturbation  $\xi(t, e, t_0)$  exists for almost all  $t$ .

The convergence of  $\xi_\kappa$  to  $\xi$  is not uniform because the discontinuities of  $\varphi(t, \mathbf{x}_0 + \kappa \mathbf{e}, t_0)$  and  $\varphi(t, \mathbf{x}_0, t_0)$  do generally not coincide (see illustrative example depicted in Figure 1 at  $t = 1$ ). The density  $\frac{d\xi_\kappa}{d\mu}$  does therefore not converge to  $\frac{d\xi}{d\mu}$  at the discontinuity points such that we cannot simply take the limit of (14) as  $\kappa \rightarrow 0$ . In order to deal with this problem, we do not consider the density  $\frac{d\xi_\kappa}{d\mu}$  for a singleton  $\{t\}$ , but rather on an open time interval  $I_\varepsilon(t) = (t - \varepsilon, t + \varepsilon)$  with  $\varepsilon > 0$ . Using the notation  $(f \odot d\mu)(I_\varepsilon) = \int_{I_\varepsilon} f d\mu$ <sup>29</sup>, the ‘widened’ density  $\frac{d\xi_\kappa}{d\mu}$  for the interval  $I_\varepsilon$  is obtained as

$$\frac{d\xi_\kappa(I_\varepsilon)}{d\mu(I_\varepsilon)} = \frac{(f(\mathbf{x}^- + \kappa \xi_\kappa^-, t) \odot d\mu)(I_\varepsilon) - (f(\mathbf{x}^-, t) \odot d\mu)(I_\varepsilon)}{\kappa d\mu(I_\varepsilon)}. \tag{15}$$

The right-hand side of (15) can be considered as the time average over the interval  $I_\varepsilon$  of the one-sided directional derivative of  $f$  in the direction of  $\xi_\kappa^-$ . The vector field  $f$  is non-differentiable, but the one-sided directional derivative is nevertheless well-defined according to Assumption 1 if the solution  $\varphi$  is not discontinuous on the boundary of  $I_\varepsilon$ . Considering only values of  $\varepsilon \notin E_\varepsilon^\varphi := \{\varepsilon \in \mathbb{R}^+ \mid \varphi^-(t - \varepsilon, \mathbf{x}_0, t_0) \neq \varphi^+(t - \varepsilon, \mathbf{x}_0, t_0) \vee \varphi^-(t + \varepsilon, \mathbf{x}_0, t_0) \neq \varphi^+(t + \varepsilon, \mathbf{x}_0, t_0)\}$ , we obtain  $\lim_{\kappa \rightarrow 0, \varepsilon \notin E_\varepsilon^\varphi} \frac{d\xi_\kappa(I_\varepsilon)}{d\mu(I_\varepsilon)} = \frac{d\xi(I_\varepsilon)}{d\mu(I_\varepsilon)}$ , where  $E_\kappa^\xi(t \pm \varepsilon) = E_\kappa^\xi(t - \varepsilon) \cup E_\kappa^\xi(t + \varepsilon)$ .

In the following step, we let  $\varepsilon$  tend to zero while omitting the values  $\varepsilon \in E_\varepsilon^\varphi$  and use  $\lim_{\varepsilon \rightarrow 0} I_\varepsilon = \{t\}$  and (12) to obtain  $\lim_{\varepsilon \rightarrow 0, \varepsilon \notin E_\varepsilon^\varphi} \frac{d\xi(I_\varepsilon)}{d\mu(I_\varepsilon)} = \frac{d\xi}{d\mu}$ . Together with (15), the density  $\frac{d\xi_\kappa}{d\mu}$  is obtained as

$$\frac{d\xi_\kappa}{d\mu} = \lim_{\varepsilon \rightarrow 0, \varepsilon \notin E_\varepsilon^\varphi} \lim_{\kappa \rightarrow 0, \kappa \notin E_\kappa^\xi(t \pm \varepsilon)} \frac{(f(\mathbf{x}^- + \kappa \xi_\kappa^-, t) \odot d\mu)(I_\varepsilon) - (f(\mathbf{x}^-, t) \odot d\mu)(I_\varepsilon)}{\kappa d\mu(I_\varepsilon)}. \tag{16}$$

The maximal Lyapunov exponent for a non-smooth system is defined using the normalized perturbation  $\xi$  as

$$\lambda_{\max} := \max_e \limsup_{t \rightarrow \infty, t \notin E_t^\xi} \frac{1}{t} \ln \|\xi(t, e, t_0)\|, \tag{17}$$

where  $E_t^\xi$  is the discontinuity set of  $\xi$ .

In the following, we consider the synchronization problem in order to compare the maximal Lyapunov exponent to the critical coupling gain. The coupled dynamics consists of the non-smooth system (13) and a replica with an error feedback of the form

$$\begin{aligned} \frac{dx}{d\mu} &= f(\mathbf{x}^-, t), \\ \frac{dy}{d\mu} &= f(\mathbf{y}^-, t) - k(\mathbf{y} - \mathbf{x}) \frac{dt}{d\mu}. \end{aligned} \tag{18}$$

The same initial conditions as for the perturbation  $\Delta \mathbf{x}$  are chosen, that is,  $\mathbf{x}^-(t_0) = \mathbf{x}_0$  and  $\mathbf{y}^-(t_0) = \mathbf{x}_0 + \kappa \mathbf{e}$ . The dynamics of the synchronization error  $\mathbf{z} = \mathbf{y} - \mathbf{x}$  is obtained as

$$\frac{d\mathbf{z}}{d\mu} = \mathbf{f}(\mathbf{x}^- + \mathbf{z}^-, t) - \mathbf{f}(\mathbf{x}^-, t) - k\mathbf{z} \frac{dt}{d\mu}. \tag{19}$$

Analogously to the smooth case, we define  $\zeta(t, \mathbf{e}, t_0) := \lim_{\kappa \rightarrow 0, \kappa \notin E_\kappa^\zeta(t)} \zeta_\kappa(t, \mathbf{e}, t_0)$  as the normalized synchronization error, where  $\zeta_\kappa(t, \mathbf{e}, t_0) := \frac{\mathbf{z}}{\kappa}$  and  $E_\kappa^\zeta(t) := \{\kappa \in \mathbb{R}_0^+ \mid \zeta_\kappa^-(t, \mathbf{e}, t_0) \neq \zeta_\kappa^+(t, \mathbf{e}, t_0)\}$ . The limit  $\zeta$  exists a.e. according to Assumption 1. It will prove advantageous to introduce  $\hat{\zeta}_\kappa := \zeta_\kappa e^{k(t-t_0)}$  together with its limit  $\hat{\zeta} = \lim_{\kappa \rightarrow 0, \kappa \notin E_\kappa^\zeta(t)} \hat{\zeta}_\kappa = \zeta e^{k(t-t_0)}$ . The density of  $\hat{\zeta}_\kappa$  w.r.t.  $d\mu$  is obtained from (19) as

$$\frac{d\hat{\zeta}_\kappa}{d\mu} = \frac{\mathbf{f}(\mathbf{x}^- + \kappa e^{-k(t-t_0)} \hat{\zeta}_\kappa^-, t) - \mathbf{f}(\mathbf{x}^-, t)}{\kappa e^{-k(t-t_0)}}, \tag{20}$$

where  $\frac{d(\hat{\zeta}_\kappa e^{-k(t-t_0)})}{d\mu} = \frac{d(\hat{\zeta}_\kappa)}{d\mu} e^{-k(t-t_0)} - k \hat{\zeta}_\kappa e^{-k(t-t_0)} \frac{dt}{d\mu}$  has been used. As for the normalized perturbation, we cannot directly take the limit of (20) for  $\kappa \rightarrow 0$ . Therefore, the same steps as performed between (14) and (16) are applied to (20) in order to obtain the density  $\frac{d\hat{\zeta}}{d\mu}$  as

$$\frac{d\hat{\zeta}}{d\mu} = \lim_{\varepsilon \rightarrow 0, \varepsilon \notin E_\varepsilon^\zeta} \lim_{\kappa \rightarrow 0, \kappa \notin E_\kappa^\zeta(t \pm \varepsilon)} \frac{(\mathbf{f}(\mathbf{x}^- + \kappa e^{-k(t-t_0)} \hat{\zeta}_\kappa^-, t) \odot d\mu)(I_\varepsilon) - (\mathbf{f}(\mathbf{x}^-, t) \odot d\mu)(I_\varepsilon)}{\kappa e^{-k(t-t_0)} d\mu(I_\varepsilon)}. \tag{21}$$

Due to  $e^{-k(t-t_0)} > 0 \forall t \in \mathbb{R}$ , we can replace  $\kappa$  in (21) by  $\hat{\kappa} = \kappa e^{-k(t-t_0)} > 0$  and obtain

$$\frac{d\hat{\zeta}}{d\mu} = \lim_{\varepsilon \rightarrow 0, \varepsilon \notin E_\varepsilon^\zeta} \lim_{\hat{\kappa} \rightarrow 0, \hat{\kappa} \notin E_{\hat{\kappa}}^\zeta(t \pm \varepsilon)} \frac{(\mathbf{f}(\mathbf{x}^- + \hat{\kappa} \hat{\zeta}_\kappa^-, t) \odot d\mu)(I_\varepsilon) - (\mathbf{f}(\mathbf{x}^-, t) \odot d\mu)(I_\varepsilon)}{\hat{\kappa} d\mu(I_\varepsilon)}. \tag{22}$$

The initial perturbation  $\Delta \mathbf{x}^-(t_0) = \kappa \mathbf{e}$  and the initial synchronization error  $\mathbf{z}^-(t_0) = \kappa \mathbf{e}$  are identical and, thus, the initial normalized perturbation  $\xi^-(t_0, \mathbf{e}, t_0) = \mathbf{e}$  and the initial normalized synchronization error  $\zeta^-(t_0, \mathbf{e}, t_0) = \mathbf{e}$  are identical as well. Together with  $\hat{\zeta}^-(t_0, \mathbf{e}, t_0) = \zeta^-(t_0, \mathbf{e}, t_0) e^{k(t_0-t_0)} = \mathbf{e}$  we obtain that the initial conditions of  $\xi$  and  $\hat{\zeta}$  are identical. Furthermore, the dynamics of  $\xi$ , given by (16), and the dynamics of  $\hat{\zeta}$ , given by (22), are identical as well, which yields  $\xi(t, \mathbf{e}, t_0) = \hat{\zeta}(t, \mathbf{e}, t_0)$  a.e. Therefore, using  $\zeta(t, \mathbf{e}, t_0) = \hat{\zeta}(t, \mathbf{e}, t_0) e^{-k(t-t_0)}$ , the relation between the normalized disturbance  $\xi$  and the normalized synchronization error  $\zeta$  is obtained as

$$\zeta = \xi e^{-k(t-t_0)} \quad \text{a.e.} \tag{23}$$

The set of discontinuities  $E_t^\xi$  of  $\xi$  and  $E_t^\zeta$  of  $\zeta$  coincide. The only difference between the non-smooth result (23) and the smooth result (9) is that  $\zeta$  and  $\xi$  are discontinuous in the non-smooth case and, thus, not defined on the entire time axis. From (23) together with the definition of the maximal Lyapunov exponent in (17) we obtain the maximal increase of an infinitesimal disturbance along solutions as

$$\max_{\mathbf{e}} \lim_{t \rightarrow \infty, t \notin E_t^\zeta} \|\zeta\| = \max_{\mathbf{e}} \lim_{t \rightarrow \infty, t \notin E_t^\zeta} \|\xi\| e^{-k(t-t_0)} = \lim_{t \rightarrow \infty} e^{(\lambda_{\max} - k)(t-t_0)}.$$

Local synchronization of the coupled systems (18) is therefore achieved if (and only if) the coupling gain  $k$  is larger than the maximal Lyapunov exponent  $\lambda_{\max}$ , that is,

$$\lim_{t \rightarrow \infty, t \notin E_t^\zeta} \|\zeta(t, \mathbf{e}, t_0)\| = 0 \quad \forall \mathbf{e} \text{ if and only}^* \text{ if } k > \lambda_{\max}.$$

As for the smooth case, any slight error or perturbation will eventually turn any initial error in the direction of maximal expansion, which yields

$$\exists \mathbf{e} : \lim_{t \rightarrow \infty, t \notin E_t^\zeta} \|\zeta(t, \mathbf{e}, t_0)\| = 0 \quad \Rightarrow \quad \lim_{t \rightarrow \infty, t \notin E_t^\zeta} \|\zeta(t, \mathbf{e}, t_0)\| = 0 \text{ for almost all } \mathbf{e}.$$

The maximal Lyapunov exponent can be estimated using the minimal proportional feedback gain for which synchronization is achieved, that is,  $k_{\text{crit}} = \lambda_{\max}$ .

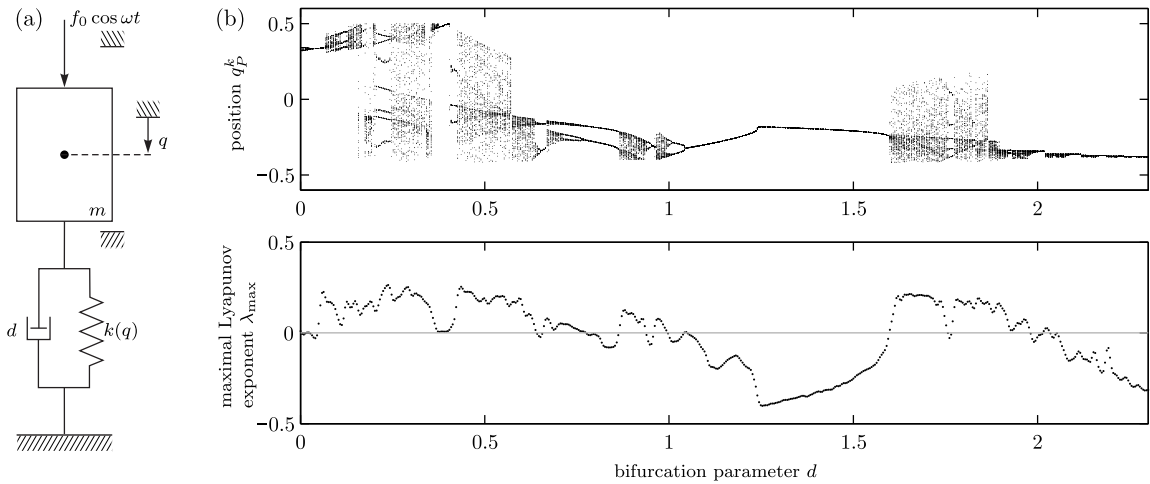


Fig. 2. (a) Duffing oscillator with two opposing geometric unilateral constraints; (b) Brute force diagram (top) and estimation of the maximal Lyapunov exponent  $\lambda_{\max}$  (bottom) with the viscous damping coefficient  $d$  as bifurcation parameter. The chaotic solutions correspond to  $\lambda_{\max} > 0$ , while the periodic windows correspond to  $\lambda_{\max} < 0$ .

#### 4. Numerical example

We apply the method of estimating the maximal Lyapunov exponent using chaos synchronization to the example of a mechanical Duffing-oscillator with two geometric unilateral constraints as depicted in Figure 2(a). The state vector  $\mathbf{x} = (q, u)^T$  consists of the coordinate  $q$  and velocity  $u$ . The impact oscillator is excited by an external, harmonic forcing  $f_0 \cos(\omega t)$ . The further system parameters are the mass  $m$ , the nonlinear stiffness coefficient  $k(q) = \bar{k}(q^2 - 1)$  and the viscous damping coefficient  $d$ .

The opposing unilateral constraints located at  $q = \pm q_c$  can be written as  $\mathbf{g} = (q + q_c, -q + q_c)^T \geq \mathbf{0}$ , which yields  $C = [-q_c, q_c]$  as the admissible set for  $q$ . The force law is given by Signorini's condition and the generalized Newton's impact law with a restitution coefficient  $e$  is chosen to describe the impact process<sup>30</sup>.

The viscous damping coefficient  $d$  is chosen as bifurcation parameter. A brute force diagram is generated for the system parameters  $m = \bar{k} = f_0 = \omega = 1, q_c = 0.5, e = 0.65$  for the range  $d \in [0, 2.3]$  and is depicted in Figure 2(b) (top). It is generated using a sweep down of the bifurcation parameter and shows the position  $q_p^k$  at the Poincaré sections (here, stroboscopic map with period time  $T = \frac{2\pi}{\omega}$ ). The critical coupling is determined numerically for the solutions depicted in the bifurcation diagram. The critical coupling is used as estimate of the maximal Lyapunov exponent of the corresponding solution and is depicted in Figure 2(b) (bottom). The estimated maximal Lyapunov exponent is positive for the chaotic attractors and negative ( $\lambda_{\max}$  at zero is not considered) during the periodic windows. The coupled system is considered to be synchronized if the synchronization error becomes smaller than a certain threshold and a long but finite time horizon is chosen. Therefore, the maximal Lyapunov exponent is generally overestimated, especially for small values of the damping coefficient  $d$ .

#### 5. Conclusions

The critical coupling gain for local synchronization of two identical nonsmooth systems with unidirectional diffuse coupling is given by the maximal Lyapunov exponent. This statement holds for the general class of nonsmooth systems with solutions of special locally bounded variation under the assumption that the maximal Lyapunov exponent exists. This relation can directly be used to estimate the maximal Lyapunov exponent of nonsmooth systems using chaos synchronization.



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