On the principle of Hamilton as variational inequality

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Summary. The classical form of Hamilton's principle holds for conservative systems with perfect bilateral constraints. In this paper we derive Hamilton's principle for perfect unilateral constraints (involving impulsive motion) using so-called weak and strong variations. The resulting principle has the form of a variational inequality.

Introduction

The classical principle of Hamilton is the variational problem

$$s(\boldsymbol{q}) = \int_{t_0}^{t_f} L(\boldsymbol{q}, \dot{\boldsymbol{q}}) \mathrm{d}t \rightarrow \text{stationary} \tag{1}$$

with boundary conditions $\boldsymbol{q}(t_0) = \boldsymbol{q}_0$ and $\boldsymbol{q}(t_f) = \boldsymbol{q}_f$ and Lagrange function $L(\boldsymbol{q}, \dot{\boldsymbol{q}}) = T(\boldsymbol{q}, \dot{\boldsymbol{q}}) - V(\boldsymbol{q})$, where T is a quadratic form in $\dot{\boldsymbol{q}}$. The stationarity condition $\delta s = 0$ of the principle of Hamilton leads after partial integration to the variational equality

$$\int_{t_0}^{t_f} \left(\frac{\partial L}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{\boldsymbol{q}}} \right) \delta \boldsymbol{q} \, \mathrm{d}t = 0 \quad \forall \delta \boldsymbol{q} \tag{2}$$

and therefore to the Euler-Lagrange equations $\frac{\partial L}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{\boldsymbol{q}}} = 0$ for a conservative system with perfect bilateral constraints. In this paper we derive Hamilton's principle for perfect unilateral constraints (involving impulsive motion) using so-called strong variations. The resulting principle has the form of a variational inequality and is valid for elastic impact laws.
Several attempts have been made in literature to generalize Hamilton's principle for mechanical systems with perfect unilateral constraints involving impulsive motion [3–7]. This has led to a number of different variants of Hamilton's principle, some expressed as variational inequalities. Up to now, the connection between these different principles has been missing. The current paper gives a unified framework of Hamilton's principle as variational inequality by using the concept of weak and strong virtual displacements. This conference proceeding is an abridged version of [2].

A mechanical system with unilateral constraints

Consider a mechanical system with n degrees of freedom and let $\boldsymbol{q}(t) \in \mathbb{R}^n$ describe the motion of the system on the compact time-interval $I = [t_0, t_f]$. The mechanical system is unilaterally constrained such that the generalized coordinates remain in an admissible set K, i.e. $\boldsymbol{q}(t) \in K$ for all t. The set K is closed, not necessarily convex, and we will assume that it is tangentially regular. The velocity $\boldsymbol{u}(t)$ is considered to be of bounded variation, which means that we can define the left limit $\boldsymbol{u}^-(t)$ and right limit $\boldsymbol{u}^+(t)$ for each time t and $\boldsymbol{u}(t) = \dot{\boldsymbol{q}}(t)$ for almost all t. Typically, we will allow for impulsive motion if $\boldsymbol{q}(t_c) \in \partial K$. On K we define the tangent cone $\mathcal{T}_K(\boldsymbol{q})$ and the normal cone $\mathcal{N}_K(\boldsymbol{q})$. Admissible motion implies that $-\boldsymbol{u}^-(t) \in \mathcal{T}_K(\boldsymbol{q}(t))$ and $\boldsymbol{u}^+(t) \in \mathcal{T}_K(\boldsymbol{q}(t))$ for all t.
The non-impulsive dynamics of the system is described by the equation of motion

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \boldsymbol{u}} \right)^{\mathrm{T}} - \left(\frac{\partial L}{\partial \boldsymbol{q}} \right)^{\mathrm{T}} = \boldsymbol{f}, \tag{3}$$

where $\boldsymbol{f}$ is the contact force enforcing the unilateral constraint $\boldsymbol{q} \in K$. The assumption of perfect unilateral constraints requires that the constraint forces are elements of the normal cone to constraint set K:

$$-\boldsymbol{f} \in \mathcal{N}_K(\boldsymbol{q}). \tag{4}$$

The latter can be expressed in a variational way as

$$\boldsymbol{f}^{\mathrm{T}} \delta \boldsymbol{q} \geq 0 \quad \forall \delta \boldsymbol{q} \in \mathcal{T}_K(\boldsymbol{q}), \tag{5}$$

which is the principle of d'Alembert-Lagrange in inequality form, i.e. the constraint forces produce a non-negative virtual work for admissible virtual displacements. The non-impulsive dynamics is therefore described by the variational inequality

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \boldsymbol{u}} - \frac{\partial L}{\partial \boldsymbol{q}} \right) \delta \boldsymbol{q} \geq 0 \quad \forall \delta \boldsymbol{q} \in \mathcal{T}_K(\boldsymbol{q}). \tag{6}$$

In the following we will use the generalized momentum $\boldsymbol{p}$ and the energy function H, given by

$$\boldsymbol{p}(\boldsymbol{q},\boldsymbol{u}) = \left(\frac{\partial L}{\partial \boldsymbol{u}}\right)^{\mathrm{T}}, \quad H(\boldsymbol{q},\boldsymbol{u}) = \frac{\partial L}{\partial \boldsymbol{u}}\boldsymbol{u} - L = T + V. \tag{7}$$

The impulsive dynamics is described the impact equation

$$\boldsymbol{p}_c^+ - \boldsymbol{p}_c^- = \boldsymbol{R}_c \tag{8}$$

where $\boldsymbol{p}_c^\pm = \boldsymbol{p}(\boldsymbol{q}(t_c),\boldsymbol{u}^\pm(t_c))$ and $\boldsymbol{R}_c$ is the contact impulsive force at the impact time t_c. The assumption of perfect unilateral constraints requires that also the impulsive forces obey

$$-\boldsymbol{R}_c \in \mathcal{N}_K(\boldsymbol{q}(t_c)). \tag{9}$$

The above description of a mechanical system with perfect unilateral constraints has to be completed with an impact law which relates the contact impulsive force $\boldsymbol{R}$ to kinematic quantities. Here, we will not adopt a specific impact law. If the energy is conserved by the impact law, then it holds that $H_c^+ = H_c^-$ where $H_c^\pm = H(\boldsymbol{q}(t_c),\boldsymbol{u}^\pm(t_c))$. Such kind of impact laws are generally referred to as elastic impact laws.

Strong virtual displacements

Consider the following two norms (see [1, 8])

$$\begin{aligned} &\text{weak norm} \quad \|\boldsymbol{y}\|_1 = \sum_{i=1}^{n} \max_{t\in I}|y_i(t)| + \operatorname*{ess\,sup}_{t\in I}|\dot{y}_i(t)|, \\ &\text{strong norm} \quad \|\boldsymbol{y}\|_0 = \sum_{i=1}^{n} \max_{t\in I}|y_i(t)|. \end{aligned} \tag{10}$$

In order to set up a virtual displacement we will consider the (hereafter called strong) family of comparison functions

$$\hat{\boldsymbol{q}}(\varepsilon,t) = \boldsymbol{q}(t-\hat{t}(\varepsilon,t)) + \varepsilon\boldsymbol{w}(t-\hat{t}(\varepsilon,t)), \tag{11}$$

which are parameterized by the variation parameter ε. The motion $\boldsymbol{q}(t)$ is contained in the family of comparison functions, such that $\hat{\boldsymbol{q}}(\varepsilon=0,t)=\boldsymbol{q}(t)$. The function $\hat{t}(\varepsilon,t)$, being continuous and differentiable with the property $\hat{t}(0,t)=0$, induces a virtual time-shift $\delta t(t) = \hat{t}_\varepsilon(0,t)\,\delta\varepsilon$. The function $\boldsymbol{w}$, being continuous, induces a value shift. The strong family of comparison functions $\hat{\boldsymbol{q}}(\varepsilon,t)$ converges to $\boldsymbol{q}(t)$ in the strong norm $\|\cdot\|_0$ in the sense that

$$\lim_{\varepsilon\downarrow 0} \|\hat{\boldsymbol{q}}(\varepsilon,\cdot) - \boldsymbol{q}\|_0 = 0 \tag{12}$$

because $\lim_{\varepsilon\downarrow 0}\hat{\boldsymbol{q}}(\varepsilon,t) = \hat{\boldsymbol{q}}(0,t) = \boldsymbol{q}(t)$, but it does not converge in the weak norm $\|\cdot\|_1$. The virtual displacement

$$\delta\boldsymbol{q}(t) = \left.\frac{\partial\hat{\boldsymbol{q}}(\varepsilon,t)}{\partial\varepsilon}\right|_{\varepsilon=0}\delta\varepsilon = -\boldsymbol{u}(t)\,\delta t(t) + \boldsymbol{w}(t)\,\delta\varepsilon \tag{13}$$

is discontinuous and not defined for those time-instants for which $\boldsymbol{u}(t)$ is discontinuous and not defined. Hence, the virtual displacement $\delta\boldsymbol{q}(t)$ is of bounded variation and admits for each $t\in I$ a left and a right limit

$$\delta\boldsymbol{q}^\pm(t) = -\boldsymbol{u}^\pm(t)\,\delta t(t) + \boldsymbol{w}(t)\,\delta\varepsilon. \tag{14}$$

The virtual displacement $\delta\boldsymbol{q}(t)$ should be understood as an infinitesimal difference between the comparison function $\hat{\boldsymbol{q}}(\varepsilon,t)$ and the function $\boldsymbol{q}(t)$ for the same value of t. Likewise, we can introduce the variation $\overline{\delta\boldsymbol{q}}(t) \approx \hat{\boldsymbol{q}}(\varepsilon,t+\delta t(t)) - \boldsymbol{q}(t)$, or more explicitly using (17)

$$\overline{\delta\boldsymbol{q}}(t) = \boldsymbol{q}_t(t)\,\delta t(t) + \hat{\boldsymbol{q}}_\varepsilon(0,t)\,\delta\varepsilon = \boldsymbol{u}(t)\,\delta t(t) + \delta\boldsymbol{q}(t) = \boldsymbol{w}(t)\,\delta\varepsilon, \tag{15}$$

for which also the time is varied. Clearly, the variation $\overline{\delta\boldsymbol{q}}(t)$ is continuous, because $\boldsymbol{w}(t)$ is continuous.

Weak virtual displacements

Consider the weak family of comparison functions

$$\hat{\boldsymbol{q}}(\varepsilon,t) = \boldsymbol{q}(t) + \varepsilon\boldsymbol{w}(t), \tag{16}$$

which only induce value shift but not a virtual time-shift. The weak family of comparison functions $\hat{\boldsymbol{q}}(\varepsilon,t) = \boldsymbol{q}(t)+\varepsilon\boldsymbol{w}(t)$ converges to $\boldsymbol{q}(t)$ in both the strong norm $\|\cdot\|_0$ and weak norm $\|\cdot\|_1$. The resulting virtual displacement

$$\delta\boldsymbol{q}(t) = \left.\frac{\partial\hat{\boldsymbol{q}}(\varepsilon,t)}{\partial\varepsilon}\right|_{\varepsilon=0}\delta\varepsilon = \boldsymbol{w}(t)\,\delta\varepsilon \tag{17}$$

is continuous in time. Clearly, the weak family of comparison functions is contained in the strong family of comparison functions.

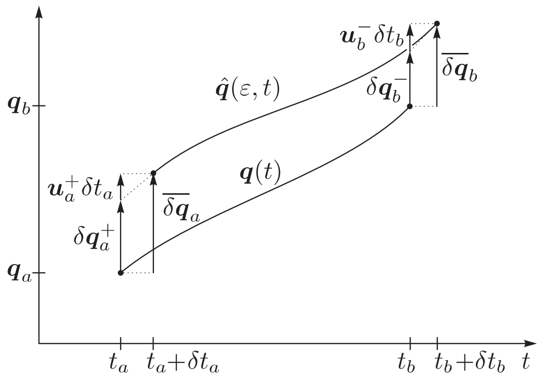

Figure 1: The general variation of a function $\boldsymbol{q}(t)$.

The general variation of the action integral with kink

The time-interval I will be split in two non-impulsive sub-intervals $I_1 = [t_0, t_c]$ and $I_2 = [t_c, t_f]$ and the variational conditions at the impact time t_c will be set up. This requires the variation of the action integral over time-intervals with variable begin or end time, i.e. a general variation of the action integral with kink.
We first consider a time-interval $[t_a, t_b] \subset I$ on which the motion $\boldsymbol{q}(t)$ is differentiable in its interior, i.e. $\boldsymbol{u}(t)$ is continuous on the open interval (t_a, t_b). The general variation of the action integral

$$s(\boldsymbol{q}) = \int_{t_a}^{t_b} L \, \mathrm{d}t, \tag{18}$$

where $L(\boldsymbol{q}, \boldsymbol{u})$ is the Lagrange function of a time-independent mechanical system, involves a variation of the begin point $\boldsymbol{q}(t_a) = \boldsymbol{q}_a$ and end point $\boldsymbol{q}(t_b) = \boldsymbol{q}_b$ as well as a variable begin time t_a and end time t_b, see Figure 1. We introduce the function

$$h(\varepsilon) = \int_{t_a(\varepsilon)}^{t_b(\varepsilon)} L(\hat{\boldsymbol{q}}(\varepsilon, t), \hat{\boldsymbol{u}}(\varepsilon, t)) \, \mathrm{d}t, \tag{19}$$

where $\hat{\boldsymbol{q}}(\varepsilon, t) = \boldsymbol{q}(t - \hat{t}(\varepsilon, t)) + \varepsilon \boldsymbol{w}(t - \hat{t}(\varepsilon, t))$ is the strong family of comparison functions. The general variation δs of (18) is defined as the ε-derivative $\delta s = h'(0) \, \delta\varepsilon$, i.e.

$$\begin{aligned} \delta s &= \int_{t_a}^{t_b} \delta L \, \mathrm{d}t + \Big[L \, \delta t \Big]_{t \downarrow t_a}^{t \uparrow t_b} \\ &= \int_{t_a}^{t_b} \left(\frac{\partial L}{\partial \boldsymbol{q}} \delta \boldsymbol{q} + \frac{\partial L}{\partial \boldsymbol{u}} \delta \boldsymbol{u} \right) \mathrm{d}t + \Big[L \, \delta t \Big]_{t \downarrow t_a}^{t \uparrow t_b} \\ &= \int_{t_a}^{t_b} \left(\frac{\partial L}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \boldsymbol{u}} \right) \delta \boldsymbol{q} \, \mathrm{d}t + \left[\frac{\partial L}{\partial \boldsymbol{u}} \delta \boldsymbol{q} + L \, \delta t \right]_{t \downarrow t_a}^{t \uparrow t_b} . \end{aligned} \tag{20}$$

The boundary terms in (20) are due to the variation of the begin time t_a and end time t_b (see Figure 1), and are expressed with left and right limits because the velocity $\boldsymbol{u}$ and therefore L may not exist for $t = t_a$ or $t = t_b$. Note that $\delta \boldsymbol{q}(t)$ is continuous in the open interval (t_a, t_b). Using the generalized momentum $\boldsymbol{p}$ and the Hamiltonian function H we can express the boundary terms in (20) as

$$\lim_{t \downarrow t_a} \left(\frac{\partial L}{\partial \boldsymbol{u}} \delta \boldsymbol{q} + L \, \delta t \right) = \boldsymbol{p}(\boldsymbol{q}_a, \boldsymbol{u}_a^+)^{\mathrm{T}} (\delta \boldsymbol{q}_a^+ + \boldsymbol{u}_a^+ \, \delta t_a) - H(\boldsymbol{q}_a, \boldsymbol{u}_a^+) \, \delta t_a, \tag{21}$$

$$\lim_{t \uparrow t_b} \left(\frac{\partial L}{\partial \boldsymbol{u}} \delta \boldsymbol{q} + L \, \delta t \right) = \boldsymbol{p}(\boldsymbol{q}_b, \boldsymbol{u}_b^-)^{\mathrm{T}} (\delta \boldsymbol{q}_b^- + \boldsymbol{u}_b^- \, \delta t_b) - H(\boldsymbol{q}_b, \boldsymbol{u}_b^-) \, \delta t_b, \tag{22}$$

in which the abbreviation $\delta \boldsymbol{q}^+(t_a) = \delta \boldsymbol{q}_a^+$, $\boldsymbol{u}^+(t_a) = \boldsymbol{u}_a^+$ etc. has been used. We recognize in (21) and (22) the variation $\overline{\delta \boldsymbol{q}}(t)$ of the begin and endpoint

$$\begin{aligned} \overline{\delta \boldsymbol{q}}_a &= \delta \boldsymbol{q}_a^+ + \boldsymbol{u}_a^+ \, \delta t_a, \\ \overline{\delta \boldsymbol{q}}_b &= \delta \boldsymbol{q}_b^- + \boldsymbol{u}_b^- \, \delta t_b, \end{aligned} \tag{23}$$

see Figure 1. The total variation (20) simplifies to

$$\delta s = \int_{t_a}^{t_b} \left(\frac{\partial L}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \boldsymbol{u}} \right) \delta \boldsymbol{q} \, \mathrm{d}t - (\boldsymbol{p}_a^+)^{\mathrm{T}} \, \overline{\delta \boldsymbol{q}}_a + (\boldsymbol{p}_b^-)^{\mathrm{T}} \, \overline{\delta \boldsymbol{q}}_b + H_a^+ \, \delta t_a - H_b^- \, \delta t_b, \tag{24}$$

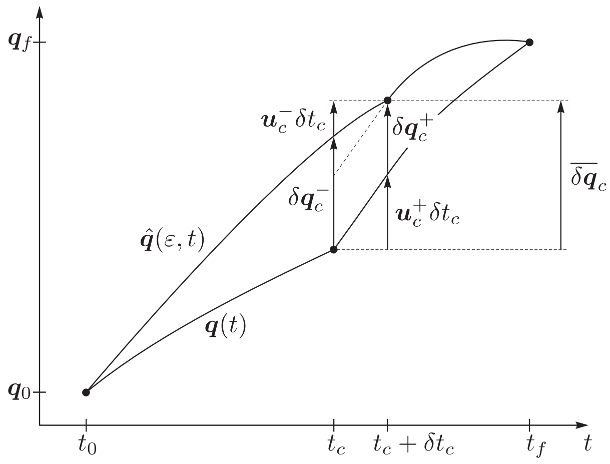

Figure 2: The general variation of a function $\boldsymbol{q}(t)$ with kink.

where the notation $\boldsymbol{p}_a^+ = \boldsymbol{p}(\boldsymbol{q}_a, \boldsymbol{u}_a^+)$, $H_a^+ = H(\boldsymbol{q}_a, \boldsymbol{u}_a^+)$ etc. has been used.
In the following we will consider solution curves $\boldsymbol{q}(t)$ which may have a kink at some point in time $t_c \in I$ (see Figure 2). The action integral, which is a Lebesgue integral over an interval I, can be decomposed into two differentiable parts

$$s(\boldsymbol{q}) = \int_I L \,\mathrm{d}t = \int_{I_1} L \,\mathrm{d}t + \int_{I_2} L \,\mathrm{d}t = s_1(\boldsymbol{q}) + s_2(\boldsymbol{q}), \tag{25}$$

where $I_1 = [t_0, t_c]$ and $I_2 = [t_c, t_f]$. The end points $t = t_0$ and $t = t_f$ are fixed and we require that the two differentiable parts of the function $\boldsymbol{q}(t)$ join continuously at $t = t_c$, but otherwise the point $t = t_c$ can move freely under virtual variations. The comparison function $\hat{\boldsymbol{q}}(\varepsilon, t)$ is fixed at the end points $t = t_0$ and $t = t_f$ and consists of two differentiable parts which join continuously at $t = t_c + \delta t_c$. The function $\overline{\delta \boldsymbol{q}}(t)$ is therefore continuous at $t = t_c$, whereas the variation $\delta \boldsymbol{q}(t)$ is discontinuous as $t = t_c$. The variation of the impact position $\overline{\delta \boldsymbol{q}}_c = \overline{\delta \boldsymbol{q}}(t_c)$ can be assessed from the left and from the right which gives the equality

$$\overline{\delta \boldsymbol{q}}_c = \delta \boldsymbol{q}_c^- + \boldsymbol{u}_c^- \,\delta t_c = \delta \boldsymbol{q}_c^+ + \boldsymbol{u}_c^+ \,\delta t_c, \tag{26}$$

with the notation $\delta \boldsymbol{q}^\pm(t_c) = \delta \boldsymbol{q}_c^\pm$, see Figure 2. It holds that $-\boldsymbol{u}_c^- \in \mathcal{T}_K(\boldsymbol{q}(t_c))$ and $\boldsymbol{u}_c^+ \in \mathcal{T}_K(\boldsymbol{q}(t_c))$. Therefore, if $\delta t_c < 0$ and $\delta \boldsymbol{q}_c^- \in \mathcal{T}_K(\boldsymbol{q}(t_c))$ we have $\overline{\delta \boldsymbol{q}}_c = \delta \boldsymbol{q}_c^- + \boldsymbol{u}_c^- \,\delta t_c \in \mathcal{T}_K(\boldsymbol{q}(t_c))$, whereas if $\delta t_c > 0$ and $\delta \boldsymbol{q}_c^+ \in \mathcal{T}_K(\boldsymbol{q}(t_c))$ we have $\overline{\delta \boldsymbol{q}}_c = \delta \boldsymbol{q}_c^+ + \boldsymbol{u}_c^+ \,\delta t_c \in \mathcal{T}_K(\boldsymbol{q}(t_c))$. Hence, if $\delta \boldsymbol{q}_c^\pm \in \mathcal{T}_K(\boldsymbol{q}(t_c))$ then it also holds that $\overline{\delta \boldsymbol{q}}_c \in \mathcal{T}_K(\boldsymbol{q}(t_c))$.
We introduce the functions

$$h_1(\varepsilon) = \int_{t_0}^{t_c(\varepsilon)} L(\hat{\boldsymbol{q}}(\varepsilon, t), \hat{\boldsymbol{u}}(\varepsilon, t)) \,\mathrm{d}t, \quad h_2(\varepsilon) = \int_{t_c(\varepsilon)}^{t_f} L(\hat{\boldsymbol{q}}(\varepsilon, t), \hat{\boldsymbol{u}}(\varepsilon, t)) \,\mathrm{d}t \tag{27}$$

such that $h(\varepsilon) = h_1(\varepsilon) + h_2(\varepsilon)$ and take the variation of the the action integral s by calculating the ε-derivatives $h_1'(\varepsilon)$ and $h_2'(\varepsilon)$ separately. In other words, the variation δs_1 has a variable end-point whereas δs_2 has a variable starting-point. Evaluation of $h_1'(\varepsilon)$ and $h_2'(\varepsilon)$ gives

$$\delta s_1 = h_1'(0)\delta\varepsilon = \int_{I_1} \left(\frac{\partial L}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \boldsymbol{u}} \right) \delta \boldsymbol{q} \,\mathrm{d}t + \left(\boldsymbol{p}_c^-\right)^{\mathrm{T}} \overline{\delta \boldsymbol{q}}_c - H_c^- \,\delta t_c, \tag{28}$$

$$\delta s_2 = h_2'(0)\delta\varepsilon = \int_{I_2} \left(\frac{\partial L}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \boldsymbol{u}} \right) \delta \boldsymbol{q} \,\mathrm{d}t - \left(\boldsymbol{p}_c^+\right)^{\mathrm{T}} \overline{\delta \boldsymbol{q}}_c + H_c^+ \,\delta t_c. \tag{29}$$

Addition of $h_1'(0)$ and $h_2'(0)$ yields $h'(0)$. The variation of s can be written as

$$\delta s = \int_I \left(\frac{\partial L}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \boldsymbol{u}} \right) \delta \boldsymbol{q} \,\mathrm{d}t - \left(\boldsymbol{p}_c^+ - \boldsymbol{p}_c^-\right)^{\mathrm{T}} \overline{\delta \boldsymbol{q}}_c + \left(H_c^+ - H_c^-\right) \delta t_c, \tag{30}$$

where $\boldsymbol{p}_c^\pm$ and $H_c^\pm$ are the pre- and post-impact values of the canonical variables.

The principle of Hamilton in inequality form

We now address the question of finding conditions for which the principle of Hamilton holds in inequality form. We will make use of the concepts of strong and weak local extrema of functionals (see [2] and references therein for more details). If $-\delta s \geq 0$ for all strong/weak $\delta \boldsymbol{q} \in \mathcal{T}_K(\boldsymbol{q})$, then we call $\boldsymbol{q}$ a strong/weak local extremal of the functional $-s(\boldsymbol{q})$ on K.

If $-\delta s \geq 0$ for all strong $\delta \boldsymbol{q} \in \mathcal{T}_K(\boldsymbol{q})$, then (30) yields

$$-\int_I \left(\frac{\partial L}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \boldsymbol{u}} \right) \delta \boldsymbol{q} \, \mathrm{d}t + \left(\boldsymbol{p}_c^+ - \boldsymbol{p}_c^- \right)^{\mathrm{T}} \overline{\delta \boldsymbol{q}}_c - \left(H_c^+ - H_c^- \right) \delta t_c \geq 0 \tag{31}$$

for all $\delta \boldsymbol{q}(t) \in \mathcal{T}_K(\boldsymbol{q}(t))$, where $t \in I \backslash \{t_c\}$, for all $\overline{\delta \boldsymbol{q}}_c \in \mathcal{T}_K(\boldsymbol{q}(t_c))$ and for all δt_c. This analysis suggests to make the following suppositions about the nature of the contact:

1. The supposition of perfect unilateral constraints. This implies the Euler-Lagrange equations as variational inequality as in (6) and therefore

$$-\int_I \left(\frac{\partial L}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \boldsymbol{u}} \right) \delta \boldsymbol{q} \, \mathrm{d}t \geq 0 \quad \forall \delta \boldsymbol{q}(t) \in \mathcal{T}_K(\boldsymbol{q}(t)). \tag{32}$$

2. The supposition of a perfect constraint remaining valid during the impact. This implies that the impulsive contact force $\boldsymbol{R}_c$ must be normal to the constraint in the sense that

$$-\boldsymbol{R}_c \in \mathcal{N}_K(\boldsymbol{q}(t_c)), \tag{33}$$

where $\boldsymbol{p}_c^+ - \boldsymbol{p}_c^- = \boldsymbol{R}_c$, which is equivalent to

$$\overline{\delta \boldsymbol{q}}_c^{\mathrm{T}} (\boldsymbol{p}_c^+ - \boldsymbol{p}_c^-) \geq 0 \quad \forall \overline{\delta \boldsymbol{q}}_c \in \mathcal{T}_K(\boldsymbol{q}(t_c)). \tag{34}$$

3. The supposition of energy preservation during impact, i.e.

$$H_c^+ - H_c^- = 0, \tag{35}$$

which implies an elastic impact law.

The last two suppositions are for the unconstrained case ($K = \mathbb{R}^n$, $\mathcal{N}_K(\boldsymbol{q}(t_c)) = \boldsymbol{0}$) exactly the so-called first and the second Weierstrass-Erdmann corner conditions. Given a functional $J(y) = \int_I f(x, y, y') \, \mathrm{d}x$, the first Weierstrass-Erdmann condition reads as

$$\lim_{x \downarrow x_c} f_{y'} = \lim_{x \uparrow x_c} f_{y'}, \qquad \text{or } \boldsymbol{p}_c^+ = \boldsymbol{p}_c^-, \tag{36}$$

and the second Weierstrass-Erdmann condition reads as

$$\lim_{x \downarrow x_c} f - y' f_{y'} = \lim_{x \uparrow x_c} f - y' f_{y'}, \qquad \text{or } H_c^+ = H_c^-. \tag{37}$$

The suppositions (32), (33) and (35) lead to the following theorem:

Theorem 1 (The Strong Principle of Hamilton in Inequality Form)
Consider a conservative Lagrangian mechanical system with perfect unilateral constraints and a non-dissipative impact law. A function $\boldsymbol{q}(t) \in K$ is a motion of the system if and only if it is a strong local extremal of the action integral, i.e.

$$-\delta \int_I L \, \mathrm{d}t \geq 0 \quad \forall \delta \boldsymbol{q} \in \mathcal{T}_K(\boldsymbol{q}) \text{ a.e. on } I, \quad \boldsymbol{q}(t_0) = \boldsymbol{q}_0, \ \boldsymbol{q}(t_f) = \boldsymbol{q}_f, \tag{38}$$

which is the strong principle of Hamilton in inequality form for impulsive motion.

Proof: If $\boldsymbol{q}(t)$ is a strong local extremal of the action integral then, by definition, $-\delta s \geq 0$ for all strong variations $\delta \boldsymbol{q} \in \mathcal{T}_K(\boldsymbol{q})$. Using (30) we obtain the variational problem (39):

$$-\int_I \left(\frac{\partial L}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \boldsymbol{u}} \right) \delta \boldsymbol{q} \, \mathrm{d}t + \left(\boldsymbol{p}_c^+ - \boldsymbol{p}_c^- \right)^{\mathrm{T}} \overline{\delta \boldsymbol{q}}_c - \left(H_c^+ - H_c^- \right) \delta t_c \geq 0 \tag{39}$$

for all $\delta \boldsymbol{q}(t) \in \mathcal{T}_K(\boldsymbol{q}(t))$, where $t \in I \backslash \{t_c\}$, for all $\overline{\delta \boldsymbol{q}}_c \in \mathcal{T}_K(\boldsymbol{q}(t_c))$ and for all δt_c. The variations $\delta \boldsymbol{q}$, $\overline{\delta \boldsymbol{q}}_c$ and δt_c are independent. Hence they yield two variational inequalities and one variational equality:

$$-\int_I \left(\frac{\partial L}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{\boldsymbol{q}}} \right) \delta \boldsymbol{q} \, \mathrm{d}t \geq 0 \quad \forall \delta \boldsymbol{q}(t) \in \mathcal{T}_K(\boldsymbol{q}(t)), \quad \text{almost everywhere on } I \tag{40}$$

$$\left(\boldsymbol{p}_c^+ - \boldsymbol{p}_c^- \right)^{\mathrm{T}} \overline{\delta \boldsymbol{q}}_c \geq 0 \quad \forall \overline{\delta \boldsymbol{q}}_c \in \mathcal{T}_K(\boldsymbol{q}(t_c)) \tag{41}$$

$$\left(H_c^+ - H_c^- \right) \delta t_c = 0 \quad \forall \delta t_c \tag{42}$$

From (40) we see that the Euler-Lagrange inequality (6) holds for almost all $t \in I$. Equation (41) requires that the unilateral constraints are perfect (in the sense of impulsive motion) and (42) that the impacts are non-dissipative. Consequently, if $\boldsymbol{q}$ is a strong extremal of the action integral, then $\boldsymbol{q}(t)$ is a motion of the system. The proof can easily be followed in the opposite direction. □

From the strong principle of Hamilton we can almost immediately come to a weak form of the principle of Hamilton by considering weak families of comparison functions. The weak virtual displacement does not induce a virtual time-shift and it therefore holds that $\delta t(t) = 0$. The weak principle of Hamilton as variational inequality does therefore not require the preservation of energy during the impact.

Theorem 2 (The Weak Principle of Hamilton in Inequality Form)
Consider a conservative Lagrangian mechanical system with perfect unilateral constraints. A function $\boldsymbol{q}(t) \in K$ is a motion of the system if and only if it is a weak local extremal of the action integral, i.e.

$$-\delta \int_I L \,\mathrm{d}t \geq 0 \quad \forall \delta \boldsymbol{q} = \boldsymbol{w} \delta \varepsilon \in \mathcal{T}_K(\boldsymbol{q}), \quad \boldsymbol{q}(t_0) = \boldsymbol{q}_0,\ \boldsymbol{q}(t_f) = \boldsymbol{q}_f, \tag{43}$$

which is the weak principle of Hamilton in inequality form for impulsive motion.

Proof: The proof is identical to the proof of the strong principle of Hamilton, but with $\delta t_c = 0$. □

Conclusions

In this paper we have derived two different forms of Hamilton's principle as variational inequality. The strong form of Hamilton's principle has been derived using the general variation of the functional which leads to 'generalized' Weierstrass-Erdmann corner conditions. The first 'generalized' Weierstrass-Erdmann condition demands that the contact impulses are from the normal cone, i.e. the supposition of perfect unilateral constraints. The second Weierstrass-Erdmann condition requires that the collisions are completely elastic, i.e. there is no energy loss during the impact. This form of the principle of Hamilton takes all neighbouring functions into consideration for the substationarity of the solution and is therefore a strong form of Hamilton's principle as variational inequality. A weak form of Hamilton's principle has been derived from the strong form by only considering weak varitions for which there is not virtual time-shift. The weak form of Hamilton's principle only requires that the contact impulses are from the normal cone. No restriction is posed in the weak form on the energy dissipation of the impact law.
The various forms of the principle of Hamilton as variational inequality which exist in literature can now be put within the context of weak and strong extrema. We conclude that Theorem 3 of [3] is the weak form of Hamilton's principle, while Theorem 4 of [3] (or Proposition 4 of [4]) is the strong form of Hamilton's principle. This insight clarifies why the various principles have different conditions on the impact law. The forms of the principle of Hamilton used by [5–7] assume strong variations and the authors of these works therefore state that the principles are valid for completely elastic impact.
What is the practical/theoretical relevance of the results and insight gained in this paper? An obvious application is the development of numerical schemes through a Ritz-type of method on the principle of Hamilton in inequality form. However, in the opinion of the authors, the relevance of the paper is more fundamental. One way to think about dynamics is in terms of variational principles. History proved that this way of thinking has been very rewarding. Variational principles form the foundation of Classical Analytical Mechanics and have been essential for the development of modern physics, e.g. quantum mechanics. Furthermore, variational principles give the link to optimization theory and put dynamics in an appropriate mathematical framework. Classical variational principles, however, are strictly valid for perfect bilateral constraints. For this reason, if we endeavour to develop a proper theoretical foundation for non-smooth dynamics, it is a promising step to go back to these principles and reformulate them in terms of variational inequalities.

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