

Synchronization-based state observer for impacting multibody systems using switched geometric unilateral constraints

Michael Baumann[†], Remco I. Leine^{*}

[†] Institute of Mechanical Systems
ETH Zurich
8092 Zurich, Switzerland
baumann@imes.mavt.ethz.ch

^{*} Institute for Nonlinear Mechanics
University of Stuttgart
70569 Stuttgart, Germany
remco.leine@inm.uni-stuttgart.de

ABSTRACT

We present a new design of a state observer for linear time-invariant multibody systems subjected to unilateral constraints using only the information of the impact time. It extends the approach presented in [4] with the new concept of switched geometric unilateral constraints. These constraints introduce constraint forces in the kinematic equation which render the generalized coordinates discontinuous. The introduction of position jumps improves the synchronization rate and expands the applicability of the observer.

A master–slave synchronization setup is used for which the unidirectional coupling between the master (observed system) and the slave system (observer) consists only of the impact time information. The dynamics of the slave system is shown to be attractively incrementally stable due to the switched constraints. The observer replicates the full state for all initial conditions, also in the presence of accumulations points (Zeno behavior) and in the vicinity of grazing impacts. The results are illustrated using two examples of impacting mechanical systems.

Keywords: observer design, master–slave synchronization, switched unilateral constraint, measure differential inclusion, non-smooth dynamics.

1 Introduction

In this paper we introduce a new type of constraints called switched geometric unilateral constraints, which allow for position jumps. Sufficient conditions are shown for which linear time-invariant systems subjected to switched geometric unilateral constraints are attractively incrementally stable. This result is used to design a state observer based on master–slave synchronization which uses only the Boolean information of the impact time instants. The proposed observer improves the observer presented in [4] by extending the applicability and increasing the synchronization rate.

The multibody systems considered in this paper are subjected to unilateral constraints. The states of these non-smooth systems are discontinuous and assumed to be special functions of locally bounded variation [2]. The dynamics can conveniently be described using the framework of measure differential inclusions [15, 14, 5, 1, 9] and phenomena such as accumulation points (Zeno behavior) do not have to be excluded.

The new concept of switched geometric unilateral constraints are accompanied by constraint forces which act on the kinematic equation. These constraints are used to improve the observer presented in [4] by extending the applicability and increasing the synchronization rate. Using these new type of constraints, the generalized coordinates become discontinuous, which raises the question of a suitable metric for the position jumps. The same problem arises when considering the Gear-Gupta-Leimkuhler method to enforce the geometric constraints when simulating mechanical system with impacts [8, 10, 19]. The GGL approach classically projects the generalized coordinates to the non-penetration constraint using the identity metric or the metric induced by the mass matrix, which is

non-energy consistent. Energy considerations imply the use of the stiffness matrix for the metric, which is only applicable if the stiffness matrix is known and positive definite.

The observer uses a master–slave synchronization setup and the decay of the synchronization error is based on the property of attractive incremental stability of the slave system. Incremental stability is a system property and several similar notions have been presented in the literature [21, 6, 7, 3]. Incremental stability is beneficial in many control problems such as stabilization, output regulation problems, synchronization and observer design [16, 20, 17, 11].

The proposed observer uses only the Boolean impact time information and no additional continuous measurement is necessary. Mechanical systems without any feedback are generally not strictly passive. More precisely, the transfer matrix of the linear part of the system is positive real, but not strictly positive real. Therefore, invariance like results are necessary for this class of mechanical systems to show the property of incremental stability.

This paper is organized as follows. The dynamics of the observed system is described in Section 2 using the framework of measure differential inclusions. The main result is presented in Section 3, where the design of the state observer is shown. Sufficient conditions are provided for which the dynamics of the observer is attractively incrementally stable. The results are illustrated with simulations of two examples of impact oscillators in Section 4 and final conclusions are given in Section 5.

2 Dynamics of the observed system

In this section we describe the dynamics of the system for which we will design a state observer. The dynamics is expressed in the form of a measure differential inclusion which allows us to describe the impulsive and non-impulsive dynamics in a combined form [15]. We consider an n -DOF linear time-invariant multibody system subjected to geometric unilateral constraints. The generalized coordinates $\mathbf{q}(t)$ are absolutely continuous in time and the generalized velocities $\mathbf{u}(t)$ are discontinuous due to the unilateral constraints and are assumed to be special functions of bounded variation [2]. The dynamics of the observed system is given by

$$\begin{aligned} d\mathbf{q} &= \mathbf{u} dt \\ \mathbf{M} d\mathbf{u} &= (-\mathbf{C}\mathbf{u} - \mathbf{K}\mathbf{q} + \mathbf{f}(t)) dt + \mathbf{W}\boldsymbol{\lambda} dt + \mathbf{W}\boldsymbol{\Lambda} d\eta. \end{aligned} \quad (1)$$

The differential measure $d\mathbf{q} = \dot{\mathbf{q}} dt$ has only a density with respect to the Lebesgue measure dt , whereas the differential measure $d\mathbf{u} = \dot{\mathbf{u}} dt + (\mathbf{u}^+ - \mathbf{u}^-) d\eta$ has additionally a density with respect to an atomic measure $d\eta$. The atoms correspond to the impact times and the atomic measure can be interpreted as the sum of Dirac point measures. See [5, 14, 1] for a detailed description of the framework of measure differential inclusions for mechanical systems.

The mass matrix $\mathbf{M} = \mathbf{M}^\top$ and the stiffness matrix $\mathbf{K} = \mathbf{K}^\top$ are assumed to be symmetric. The system matrices $\mathbf{M}, \mathbf{C}, \mathbf{K} \in \mathbb{R}^{n \times n}$ are positive definite and time invariant. The external forcing $\mathbf{f}(t)$ is independent of the state. The mechanical system (1) is subjected to m geometric unilateral constraints. The generalized force directions \mathbf{w}_i are collected in the matrix $\mathbf{W} = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m\} \in \mathbb{R}^{n \times m}$ and are assumed to be linearly independent and time invariant. The constraint forces $\boldsymbol{\lambda}$ and constraint impulses $\boldsymbol{\Lambda}$ are given by constitutive laws defined using the local kinematic quantities

$$\text{constraint distance: } \mathbf{g}(\mathbf{q}, t) = \mathbf{W}^\top \mathbf{q} + \boldsymbol{\nu}(t), \quad (2)$$

$$\text{constraint velocity: } \boldsymbol{\gamma}(\mathbf{u}, t) = \mathbf{W}^\top \mathbf{u} + \dot{\boldsymbol{\nu}}(t), \quad (3)$$

where $\boldsymbol{\nu}(t)$ is an absolutely continuous function in time.

The i -th geometric unilateral constraint restricts the sign of the constraint distance $g_i \geq 0$. Its force law $0 \leq \lambda_i \perp g_i \geq 0$, also referred to as Signorini's law, can be written on velocity level (see [9]) as

$$-\lambda_i \in \begin{cases} \partial\Psi_{\mathbb{R}_0^+}(\gamma_i) & \text{if } g_i(\mathbf{q}, t) = 0, \\ 0 & \text{if } g_i(\mathbf{q}, t) > 0, \end{cases} \quad (4)$$

where $\partial\Psi_{\mathbb{R}_0^+}$ is the subdifferential of the indicator function $\Psi_{\mathbb{R}_0^+}$ on the set \mathbb{R}_0^+ . Therefore, the admissible set of system (1) is given by $\mathcal{A} = \left\{ \begin{pmatrix} \mathbf{q} \\ \mathbf{u} \end{pmatrix} \in \mathbb{R}^{2n} \mid \mathbf{g}(\mathbf{q}) \geq \mathbf{0} \right\}$. The impact law for the impulsive unilateral constraint forces $\mathbf{\Lambda}$ is given by the inclusion

$$-\mathbf{\Lambda} \in \mathcal{H}_g(\bar{\gamma}). \quad (5)$$

The set-valued map \mathcal{H}_g puts a relationship between the dual variables $\bar{\gamma} := \frac{1}{2}(\gamma^+ + \gamma^-)$ and $\mathbf{\Lambda}$ (see [13]). The index indicates the dependence on the set of closed contacts (given by \mathbf{g}). It will become apparent in Subsection 3.2 that monotonicity of the impact map is a useful property of the impact map. The definition of monotone maps can be found in [18].

Definition 1. The set-valued map $\mathcal{H}_g(\bar{\gamma})$ is called *monotone* if it fulfills

$$(\mathbf{\Lambda}_1 - \mathbf{\Lambda}_2)^\top (\bar{\gamma}_1 - \bar{\gamma}_2) \leq 0 \quad \forall -\mathbf{\Lambda}_1 \in \mathcal{H}_g(\bar{\gamma}_1), \forall -\mathbf{\Lambda}_2 \in \mathcal{H}_g(\bar{\gamma}_2). \quad (6)$$

Commonly used impact laws such as the generalized Newton's impact law [13] or the generalized Poisson's impact law [4] are monotone under some mild assumptions. The monotonicity property will directly be used in Subsection 3.2 to show the decrease of the velocity error between the observer and the observed system.

3 Observer design

We present the design of a state observer for the class of mechanical systems described in Section 2. We assume that the model of the observed system is known, and the only available measurement is the time information when the impacts occur. We use a master-slave synchronization setup, where the observed system is the master system and state of the slave system (observer) is the estimate.

In the first step, the dynamics of the slave system is presented. Secondly, it is shown that the master and the slave system have the same solution if they are initialized with the same initial conditions. Finally, we show that all solutions of the slave system converge to each other and therefore also to the solution of the master system.

3.1 Observer dynamics

The model of the observer (slave system) is a replica of the observed system (master system) except for the constraints. The observer system is subjected to *switched geometric unilateral constraints* and the dynamics is described by

$$\begin{pmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{pmatrix} \begin{pmatrix} d\mathbf{q} \\ d\mathbf{u} \end{pmatrix} = \begin{pmatrix} \mathbf{K}\mathbf{u} \\ -\mathbf{C}\mathbf{u} - \mathbf{K}\mathbf{q} + \mathbf{f}(t) \end{pmatrix} dt + \begin{pmatrix} \mathbf{W}\boldsymbol{\sigma} \\ \mathbf{W}\boldsymbol{\lambda} \end{pmatrix} dt + \begin{pmatrix} \mathbf{W}\boldsymbol{\Sigma} \\ \mathbf{W}\boldsymbol{\Lambda} \end{pmatrix} d\eta. \quad (7)$$

The new concept of switched geometric unilateral constraints extends switched kinematic unilateral constraints (introduced in [4]) with position jumps by introducing the constraint forces $\boldsymbol{\sigma}$ and impulses $\boldsymbol{\Sigma}$ in the kinematic equation. Therefore, the generalized velocities $\mathbf{u}(t)$ and constraint distances $\gamma(\mathbf{u}(t), t)$ are generally no longer the time derivative of the generalized coordinates $\mathbf{q}(t)$ and constraint distances $\mathbf{g}(\mathbf{q}(t), t)$, respectively.

Furthermore, the generalized coordinates $\mathbf{q}(t)$ are no longer absolutely continuous, but they are assumed to be functions of special locally bounded variation.

The switched kinematic unilateral constraints with the constraint forces $\boldsymbol{\lambda}$ and impulses $\mathbf{\Lambda}$ are kinematic unilateral constraints, which are switched on and off by an external Boolean switching function $\boldsymbol{\chi}(t) : \mathbb{R} \rightarrow \{0, 1\}^m$. The i -th constraint imposes a kinematic unilateral constraint $\gamma_i \geq 0$ whenever the corresponding external Boolean switching function $\chi_i(t) = 1$. Its force law is described by the inequality complementarity

$$-\lambda_i \in \begin{cases} \partial\Psi_{\mathbb{R}_0^+}(\gamma_i) & \text{if } \chi_i(t) = 1, \\ 0 & \text{if } \chi_i(t) = 0. \end{cases} \quad (8)$$

The impact law for the constraint impulses $\mathbf{\Lambda}$ is given by the inclusion (see [13])

$$-\mathbf{\Lambda} \in \mathcal{H}_{\boldsymbol{\chi}(t)}(\bar{\boldsymbol{\gamma}}). \quad (9)$$

The operator $\mathcal{H}_{\boldsymbol{\chi}(t)}$ is chosen such that it is identical to the operator \mathcal{H}_g in (5) if the same contacts are closed or switched on, respectively.

The i -th switched geometric bilateral constraint ensures that the i -th constraint is closed when $\chi_i(t) = 1$ and imposes no constraint otherwise. Therefore, the constitutive laws for the constraint forces $\boldsymbol{\sigma}$ and the constraint impulses $\boldsymbol{\Sigma}$ of the switched geometric bilateral constraint are given by

$$-\sigma_i \in \begin{cases} \partial\Psi_{\{0\}}(g_i) & \text{if } \chi_i(t) = 1, \\ 0 & \text{if } \chi_i(t) = 0, \end{cases} \quad (10)$$

$$-\Sigma_i \in \begin{cases} \partial\Psi_{\{0\}}(g_i^+) & \text{if } \chi_i(t) = 1, \\ 0 & \text{if } \chi_i(t) = 0. \end{cases} \quad (11)$$

The switched geometric unilateral constraints are ‘time-triggered’ and $\boldsymbol{\chi}(t)$ is an external input independent of the constraint distances. The ‘state-triggered’ geometric unilateral constraints of the observed system, however, are closed if the corresponding constraint distances vanish. In order to relate these two types of constraints, we make the following definition.

Definition 2. The Boolean switching functions $\boldsymbol{\chi}(t)$ are called to be *generated* by $\mathbf{g}(t)$ if every component $\chi_i(t)$ of $\boldsymbol{\chi}(t)$ fulfills

$$\chi_i(t) = \begin{cases} 1 & \text{if } g_i(t) = 0, \\ 0 & \text{if } g_i(t) > 0. \end{cases} \quad (12)$$

If the observer is initialized with the same initial conditions as the observed system and the switching functions $\boldsymbol{\chi}(t)$ are generated by the constraint distances of the observed system, then both solutions are identical. This statement is the content of the following proposition.

Proposition 1. Let the master system be described by (1)–(5) and let the slave system be described by (7)–(11) together with (2)–(3). Let $\begin{pmatrix} \mathbf{q}_m(t) \\ \mathbf{u}_m(t) \end{pmatrix}$ be the solution of the master system for the initial conditions $\begin{pmatrix} \mathbf{q}_m(t_0) \\ \mathbf{u}_m(t_0) \end{pmatrix} = \begin{pmatrix} \mathbf{q}_0 \\ \mathbf{u}_0 \end{pmatrix} \in \mathcal{A}$ and let $\begin{pmatrix} \mathbf{q}_s(t) \\ \mathbf{u}_s(t) \end{pmatrix}$ be the solution of the slave system for the same initial conditions $\begin{pmatrix} \mathbf{q}_s(t_0) \\ \mathbf{u}_s(t_0) \end{pmatrix} = \begin{pmatrix} \mathbf{q}_0 \\ \mathbf{u}_0 \end{pmatrix}$. Then, both solutions are identical if the Boolean switching functions $\boldsymbol{\chi}(t)$ in (8)–(11) are generated by the constraint distances $\mathbf{g}_m(t)$ of the master system.

Proof. The only difference between the master and the slave system are the constraints. The condition that $\chi(t)$ is generated by \mathbf{g}_m directly implies that the force laws (4) and (8) as well as the impact laws (5) and (9) are identical. For the constraint forces $\boldsymbol{\sigma}$ and constraint impulses $\boldsymbol{\Sigma}$ we seek an explicit expression in order to show that both vanish for the considered solutions.

We define I as the index set of closed constraints at a certain point in time as $I(t) = \{i \mid \chi_i(t) = 0\}$ and introduce the following notation. A subscript I indicates that only the closed constraints are considered, e.g. $\mathbf{W}_I = \{\dots, \mathbf{w}_i, \dots\}$, $\mathbf{g}_{s,I} = (\dots, g_{s,i}, \dots)^\top$, where $i \in I$.

We multiply the kinematic equation in (7) from the left by $\mathbf{W}_I^\top \mathbf{K}^{-1}$ and obtain

$$\mathbf{W}_I^\top \left(\dot{\mathbf{q}}_s dt + \left(\mathbf{q}_s^+ - \mathbf{q}_s^- \right) d\eta \right) = \mathbf{W}_I^\top \mathbf{u}_s dt + \mathbf{W}_I^\top \mathbf{K}^{-1} \mathbf{W} \boldsymbol{\sigma} dt + \mathbf{W}_I^\top \mathbf{K}^{-1} \mathbf{W} \boldsymbol{\Sigma} d\eta.$$

Substituting (2)–(3) and using the absolute continuity of $\nu(t)$ yields

$$\dot{\mathbf{g}}_{s,I} dt + \left(\mathbf{g}_{s,I}^+ - \mathbf{g}_{s,I}^- \right) d\eta = \boldsymbol{\gamma}_{s,I} dt + \mathbf{W}_I^\top \mathbf{K}^{-1} \mathbf{W} \boldsymbol{\sigma} dt + \mathbf{W}_I^\top \mathbf{K}^{-1} \mathbf{W} \boldsymbol{\Sigma} d\eta.$$

The force law (10) (written on velocity level) and the impact law (11) imply $\dot{\mathbf{g}}_{s,I} = 0$ and $\mathbf{g}_{s,I}^+ = 0$, since the inverse of $\partial\Psi_{\{0\}}$ is the zero function, and we obtain $\boldsymbol{\sigma}_I = -\mathbf{L}_I^{-1} \boldsymbol{\gamma}_{s,I}$ and $\boldsymbol{\Sigma}_I = -\mathbf{L}_I^{-1} \mathbf{g}_{s,I}^-$, where $\mathbf{L}_I := \mathbf{W}_I^\top \mathbf{K}^{-1} \mathbf{W}_I$. We define the matrix \mathbf{P}_I as the matrix obtained by taking the identity matrix $\mathbf{P} \in \mathbb{R}^{m \times m}$ and removing all columns with an index $i \notin I$. The force and impact laws state $\sigma_i = \Sigma_i = 0 \forall i \notin I$, which implies $\boldsymbol{\sigma} = \mathbf{P}_I \boldsymbol{\sigma}_I$ and $\boldsymbol{\Sigma} = \mathbf{P}_I \boldsymbol{\Sigma}_I$. Together with $\boldsymbol{\gamma}_{s,I} = \mathbf{P}_I^\top \boldsymbol{\gamma}_s$ and $\mathbf{g}_{s,I}^- = \mathbf{P}_I^\top \mathbf{g}_s^-$, an explicit form for $\boldsymbol{\sigma}$ and $\boldsymbol{\Sigma}$ is obtained as

$$\boldsymbol{\sigma} = -\mathbf{P}_I \mathbf{L}_I^{-1} \mathbf{P}_I^\top \boldsymbol{\gamma}_s, \quad (13)$$

$$\boldsymbol{\Sigma} = -\mathbf{P}_I \mathbf{L}_I^{-1} \mathbf{P}_I^\top \mathbf{g}_s^-, \quad (14)$$

where the product of matrices $\mathbf{P}_I \mathbf{L}_I^{-1} \mathbf{P}_I^\top \in \mathbb{R}^{m \times m}$ is positive semi-definite.

The absolute continuity of the generalized coordinates \mathbf{q}_m of the master system, and therefore of the generalized constraint distances \mathbf{g}_m , implies $\mathbf{g}_{m,I}^- = \mathbf{0}$ for a given set I . Together with (14), we obtain that $\mathbf{q}_m^- = \mathbf{q}_s^-$ implies $\boldsymbol{\Sigma} = 0$, which yields $\mathbf{q}_m^+ = \mathbf{q}_s^+$ almost everywhere w.r.t. $d\eta$. Analogously, the absolute continuity of \mathbf{q}_m implies $\boldsymbol{\gamma}_{m,I} = 0$ for a given set I . Together with (13), we obtain that $\mathbf{u}_m = \mathbf{u}_s$ implies $\boldsymbol{\sigma} = 0$, which yields $\dot{\mathbf{q}}_m = \dot{\mathbf{q}}_s$ almost everywhere w.r.t. dt . Finally, we obtain $\begin{pmatrix} d\mathbf{q}_m \\ d\mathbf{u}_m \end{pmatrix} = \begin{pmatrix} d\mathbf{q}_s \\ d\mathbf{u}_s \end{pmatrix}$ together

with $\begin{pmatrix} \mathbf{q}_m(t_0) \\ \mathbf{u}_m(t_0) \end{pmatrix} = \begin{pmatrix} \mathbf{q}_s(t_0) \\ \mathbf{u}_s(t_0) \end{pmatrix}$, which concludes the proof. \square

In order that the observer (slave) provides an estimate of the observed system (master), the systems need to synchronize with each other. The master-slave synchronization is based on the attractive incremental stability of the observer, which is presented in the following subsection.

3.2 Attractive incremental stability

Attractive incremental stability (a.i.s.) is a stability property of dynamical systems which implies that all solution curves are globally uniformly attractively stable. Therefore, all solution curves approach each other and remain close in the sense of Lyapunov for all initial conditions and the information of the initial condition is lost. Here, we consider the definition of a.i.s. for measure differential inclusions as presented in [4]. Other notions of incremental stability have been presented in literature, see e.g. [3, 22].

In order to state the a.i.s. of the observer, we will make the following assumptions:

- A1 The switching functions $\chi(t)$ are generated by an absolutely continuous function.
- A2 The external forcing $\mathbf{f}(t)$ is bounded, i.e. $\sup_{t \in \mathbb{R}} \|\mathbf{f}(t)\| \leq f_{\max}$ for a given bound $f_{\max} < \infty$.
- A3 The impact map $\mathcal{H}_{\chi(t)}(\bar{\gamma})$ is monotone for any $\chi(t)$.

Assumption A1 guarantees that all intervals during which a constraint is switched off are open time intervals. This property is used for the existence of solutions, but this is not in the scope of this paper. The following theorem states the a.i.s. of the slave system under the previous assumptions and is an extension of the corresponding theorem in [4].

Theorem 1. System (7)–(11) together with (2)–(3) for given switching functions $\chi(t)$ is attractively incrementally stable if the Assumptions A1–A3 are fulfilled.

Proof. We will proof the property of a.i.s. by showing that all solution curves are globally uniformly attractively stable. Therefore, consider two arbitrary solutions $\begin{pmatrix} \mathbf{q}_1(t) \\ \mathbf{u}_1(t) \end{pmatrix}$ and $\begin{pmatrix} \mathbf{q}_2(t) \\ \mathbf{u}_2(t) \end{pmatrix}$ for given switching functions $\chi(t)$. The position and velocity errors are given by $\mathbf{e} = \mathbf{q}_1 - \mathbf{q}_2$ and $\mathbf{v} = \mathbf{u}_1 - \mathbf{u}_2$, where the index is used to distinguish between the two solutions. The error dynamics follows from (7) as

$$\begin{pmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{pmatrix} \begin{pmatrix} d\mathbf{e} \\ d\mathbf{v} \end{pmatrix} = \begin{pmatrix} \mathbf{K}\mathbf{v} \\ -\mathbf{C}\mathbf{v} - \mathbf{K}\mathbf{e} \end{pmatrix} dt + \begin{pmatrix} \mathbf{W}(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2) \\ \mathbf{W}(\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2) \end{pmatrix} dt + \begin{pmatrix} \mathbf{W}(\boldsymbol{\Sigma}_1 - \boldsymbol{\Sigma}_2) \\ \mathbf{W}(\boldsymbol{\Lambda}_1 - \boldsymbol{\Lambda}_2) \end{pmatrix} d\eta. \quad (15)$$

The constraints forces and impulses are given by (8)–(11), where the switching functions $\chi(t)$ are the same for both considered solutions. We introduce the Lyapunov function

$$V(\mathbf{e}, \mathbf{v}) = \frac{1}{2} \begin{pmatrix} \mathbf{e} \\ \mathbf{v} \end{pmatrix}^\top \begin{pmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{pmatrix} \begin{pmatrix} \mathbf{e} \\ \mathbf{v} \end{pmatrix} = \|\mathbf{v}\|_{\mathbf{M}}^2 + \frac{1}{2} \|\mathbf{e}\|_{\mathbf{K}}^2, \quad (16)$$

which is positive definite on the error states gives a notion of distance between these two solutions. We evaluate V along solutions $\mathbf{e}(t)$, $\mathbf{v}(t)$ and the differential measure dV contains the densities with respect to the measures dt and $d\eta$. Using the symmetry of \mathbf{K} and \mathbf{M} , the differential measure dV can be written as $dV = \begin{pmatrix} \bar{\mathbf{e}} \\ \bar{\mathbf{v}} \end{pmatrix}^\top \begin{pmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{pmatrix} \begin{pmatrix} d\mathbf{e} \\ d\mathbf{v} \end{pmatrix}$, where $\bar{\mathbf{e}} := \frac{1}{2}(\mathbf{e}^+ + \mathbf{e}^-)$ and $\bar{\mathbf{v}} := \frac{1}{2}(\mathbf{v}^+ + \mathbf{v}^-)$. Substituting the error dynamics (15) and using the local kinematic quantities (2)–(3), we obtain

$$\begin{aligned} dV &= \begin{pmatrix} \mathbf{e} \\ \mathbf{v} \end{pmatrix}^\top \begin{pmatrix} \mathbf{K}\mathbf{v} \\ -\mathbf{C}\mathbf{v} - \mathbf{K}\mathbf{e} \end{pmatrix} dt + \begin{pmatrix} \mathbf{W}^\top \mathbf{e} \\ \mathbf{W}^\top \mathbf{v} \end{pmatrix}^\top \begin{pmatrix} \boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2 \\ \boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2 \end{pmatrix} dt + \begin{pmatrix} \mathbf{W}^\top \bar{\mathbf{e}} \\ \mathbf{W}^\top \bar{\mathbf{v}} \end{pmatrix}^\top \begin{pmatrix} \boldsymbol{\Sigma}_1 - \boldsymbol{\Sigma}_2 \\ \boldsymbol{\Lambda}_1 - \boldsymbol{\Lambda}_2 \end{pmatrix} d\eta \\ &= -\|\mathbf{v}\|_{\bar{\mathbf{C}}}^2 dt + \begin{pmatrix} \mathbf{g}_1 - \mathbf{g}_2 \\ \boldsymbol{\gamma}_1 - \boldsymbol{\gamma}_2 \end{pmatrix}^\top \begin{pmatrix} \boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2 \\ \boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2 \end{pmatrix} dt + \begin{pmatrix} \bar{\mathbf{g}}_1 - \bar{\mathbf{g}}_2 \\ \bar{\boldsymbol{\gamma}}_1 - \bar{\boldsymbol{\gamma}}_2 \end{pmatrix}^\top \begin{pmatrix} \boldsymbol{\Sigma}_1 - \boldsymbol{\Sigma}_2 \\ \boldsymbol{\Lambda}_1 - \boldsymbol{\Lambda}_2 \end{pmatrix} d\eta, \end{aligned}$$

where $\bar{\mathbf{g}} := \frac{1}{2}(\mathbf{g}^+ + \mathbf{g}^-)$ and $\bar{\mathbf{C}} := \frac{1}{2}(\mathbf{C} + \mathbf{C}^\top)$ is the symmetric part of the damping matrix \mathbf{C} . The terms $(\mathbf{g}_1 - \mathbf{g}_2)^\top (\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)$ and $(\boldsymbol{\gamma}_1 - \boldsymbol{\gamma}_2)^\top (\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2)$ vanish, since the force laws (8) and (10) are given by complementarity conditions. Similarly, the impact law (11) yields $(\bar{\mathbf{g}}_1 - \bar{\mathbf{g}}_2)^\top (\boldsymbol{\Sigma}_1 - \boldsymbol{\Sigma}_2) = \left(\frac{1}{2}\mathbf{g}_1^- - \frac{1}{2}\mathbf{g}_2^-\right)^\top (\boldsymbol{\Sigma}_1 - \boldsymbol{\Sigma}_2)$. Therefore, the differential measure dV simplifies to

$$dV = -\|\mathbf{v}\|_{\bar{\mathbf{C}}}^2 dt + \left(\frac{1}{2}\mathbf{g}_1^- - \frac{1}{2}\mathbf{g}_2^-\right)^\top (\boldsymbol{\Sigma}_1 - \boldsymbol{\Sigma}_2) d\eta + (\bar{\boldsymbol{\gamma}}_1 - \bar{\boldsymbol{\gamma}}_2)^\top (\boldsymbol{\Lambda}_1 - \boldsymbol{\Lambda}_2) d\eta.$$

The constraint impulses Σ_1 and Σ_2 are explicitly derived in the proof of Proposition 1. Furthermore, the impact map $\mathcal{H}_{\chi(t)}(\bar{\gamma})$ in (9) is monotone according to Assumption A3, which implies $(\bar{\gamma}_1 - \bar{\gamma}_2)^\top (\Lambda_1 - \Lambda_2) \leq 0$ according to (6), and we obtain

$$dV \leq -\|\mathbf{v}\|_{\bar{\mathbf{C}}}^2 dt - \frac{1}{2} \|\mathbf{g}_1^- - \mathbf{g}_2^-\|_{\mathbf{P}_I \mathbf{L}_I^{-1} \mathbf{P}_I^\top}^2 d\eta. \quad (17)$$

The matrix $\bar{\mathbf{C}}$ is positive definite and $\mathbf{P}_I \mathbf{L}_I^{-1} \mathbf{P}_I^\top$ is positive semi-definite. Hence, we have $dV \leq 0$ and the equilibrium at the origin is uniformly stable [14]. Furthermore, since the Lyapunov function V is bounded from below and non-increasing, the limit

$$V_\infty := \lim_{t \rightarrow \infty} V(\mathbf{e}(t), \mathbf{v}(t)) = \lim_{t \rightarrow \infty} \left(\frac{1}{2} \|\mathbf{v}(t)\|_M^2 + \frac{1}{2} \|\mathbf{e}(t)\|_K^2 \right) \quad (18)$$

exists and lies in the interval $0 \leq V_\infty \leq V(\mathbf{e}^-(t_0), \mathbf{v}^-(t_0))$.

The Lyapunov function $V(\mathbf{e}(t), \mathbf{v}(t))$ tends to an absolutely continuous function (constant function) and both summands in (16) have a non-positive density with respect to $d\eta$. Therefore, we conclude that $\|\mathbf{v}(t)\|_M^2$, $\|\mathbf{e}(t)\|_K^2$ and, hence, $\mathbf{v}(t)$ and $\mathbf{e}(t)$ are asymptotically absolutely continuous as well. From (17) and (18) follows

$$V_\infty - V(\mathbf{e}^-(t_0), \mathbf{v}^-(t_0)) \leq - \lim_{t \rightarrow \infty} \int_{[t_0, t]} \|\mathbf{v}\|_{\bar{\mathbf{C}}}^2 dt - \lim_{t \rightarrow \infty} \int_{[t_0, t]} \frac{1}{2} \|\mathbf{g}_1^- - \mathbf{g}_2^-\|_{\mathbf{L}_{\chi(t)}^{-1}}^2 d\eta. \quad (19)$$

Since the left-hand side in (19) is finite, we deduce that

$$\lim_{t \rightarrow \infty} \int_{[t_0, t]} \|\mathbf{v}\|_{\bar{\mathbf{C}}}^2 dt < \infty. \quad (20)$$

We cannot invoke Barbalat's lemma [12] since the solution $\mathbf{v}(t)$ is not uniformly continuous in time. The extension of this lemma is presented in [4] for the class of asymptotically absolutely continuous functions to which $\mathbf{v}(t)$ belongs. Applying the extended Barbalat's lemma to (20) yields $\lim_{t \rightarrow \infty} \|\mathbf{v}(t)\|_{\bar{\mathbf{C}}}^2 = 0$. The positive definiteness of $\bar{\mathbf{C}}$ implies $\lim_{t \rightarrow \infty} \mathbf{v}(t) = \mathbf{0}$. Substituting the limit of \mathbf{v} into (18) yields $\lim_{t \rightarrow \infty} \mathbf{e}(t) = \mathbf{c}$ for some \mathbf{c} satisfying $V(\mathbf{e}^-(t_0), \mathbf{v}^-(t_0)) \geq \frac{1}{2} \|\mathbf{c}\|_K^2 \geq V_\infty \geq 0$. In the next step, we show that $\mathbf{c} = \mathbf{0}$. Therefore, we integrate the equality of measures (15) over a time interval $\Delta T = [t, t + \Delta t]$ with an arbitrary $\Delta t > 0$ and obtain

$$\begin{pmatrix} \mathbf{K} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{pmatrix} \begin{pmatrix} \mathbf{e}^+(t + \Delta t) - \mathbf{e}^-(t) \\ \mathbf{v}^+(t + \Delta t) - \mathbf{v}^-(t) \end{pmatrix} = \int_{\Delta T} \begin{pmatrix} \mathbf{K}\mathbf{v} \\ -\mathbf{C}\mathbf{v} - \mathbf{K}\mathbf{e} \end{pmatrix} dt + \int_{\Delta T} \begin{pmatrix} \mathbf{W}(d\Sigma_1 - d\Sigma_2) \\ \mathbf{W}(d\Lambda_1 - d\Lambda_2) \end{pmatrix},$$

where $d\Sigma_i := \sigma_i dt + \Sigma_i d\eta$, $d\Lambda_i := \lambda_i dt + \Lambda_i d\eta$ for $i \in \{1, 2\}$. It proves useful to introduce the quantities $\tilde{\Sigma}_{\Delta t}(t) = \frac{1}{\Delta t} \int_{\Delta T} (d\Sigma_1 - d\Sigma_2)$ and $\tilde{\Lambda}_{\Delta t}(t) = \frac{1}{\Delta t} \int_{\Delta T} (d\Lambda_1 - d\Lambda_2)$, which can be regarded as the average constraint forces of the error dynamics over the time lapse ΔT . Subsequently, we take the limit $t \rightarrow \infty$ and use $\mathbf{v}(t) \rightarrow \mathbf{0}$ and $\mathbf{e} \rightarrow \mathbf{c}$ for $t \rightarrow \infty$. The integrated equality of measures, divided by Δt , yields

$$\begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ -\mathbf{K}\mathbf{c} \end{pmatrix} + \lim_{t \rightarrow \infty} \begin{pmatrix} \mathbf{W}\tilde{\Sigma}_{\Delta t}(t) \\ \mathbf{W}\tilde{\Lambda}_{\Delta t}(t) \end{pmatrix},$$

which describes, in an averaged sense, the equilibrium of forces at infinity. Since the columns \mathbf{w}_i of \mathbf{W} are linearly independent, we obtain

$$\mathbf{K}\mathbf{c} = \lim_{t \rightarrow \infty} \sum_{i=1}^m \mathbf{w}_i \tilde{\Lambda}_{\Delta t, i}(t), \quad (21)$$

from which we deduce that each of the limits $\lim_{t \rightarrow \infty} \tilde{\Lambda}_{\Delta t, i}(t)$ has to exist. Let I be the set of constraints for which the limit $\lim_{t \rightarrow \infty} \tilde{\Lambda}_{\Delta t, i}(t)$ does not vanish for any Δt . The error in constraint distances for these constraints vanish due to the force and impact laws (10)–(11), i.e. $\mathbf{g}_{1,I} - \mathbf{g}_{2,I} = \mathbf{W}_I^T \mathbf{c} = \mathbf{0}$. We multiply equation (21) from the left by $\mathbf{W}_I^T \mathbf{K}^{-1}$ and obtain

$$\mathbf{0} = \mathbf{W}_I^T \mathbf{c} = \mathbf{W}_I^T \mathbf{K}^{-1} \lim_{t \rightarrow \infty} \sum_{i \in I} \mathbf{w}_i \tilde{\Lambda}_{\Delta t, i}(t) = \mathbf{L}_I \lim_{t \rightarrow \infty} \tilde{\Lambda}_{\Delta t, I}(t),$$

where the notation introduced in Proposition 1 has been used for \mathbf{L}_I and $\tilde{\Lambda}_{\Delta t, I}$. The positive definiteness of \mathbf{L}_I yields $\lim_{t \rightarrow \infty} \tilde{\Lambda}_{\Delta t, I}(t) = \mathbf{0}$ and we obtain $\mathbf{c} = \mathbf{0}$. Therefore, the equilibrium at the origin $\begin{pmatrix} \mathbf{e} \\ \mathbf{v} \end{pmatrix} = \mathbf{0}$ is globally uniformly attractively stable, which concludes the proof. \square

Theorem 1 together with Proposition 1 imply that any solution of the slave system approaches the solution of the master system and can thus be used as state estimate.

4 Examples

In Section 3 we have presented a new observer design for the class of mechanical systems shown in Section 2. The proposed observer uses only the information which constraints of the observed system are open or closed and no continuous measurement of the states is necessary. The synchronization of the unilaterally coupled systems is based on the attractive incremental stability of the observer dynamics. The results are illustrated using simulations of two examples of impact oscillators.

Example 1 (Double mass impact oscillator). The first example is a double mass impact oscillator, which consists of two masses coupled by spring-damper elements and the upper mass is harmonically forced. Figure 1 depicts the synchronization setup with the coupled master (left) and slave system (right). Both systems are described by (7)–(11) together with (2)–(3) subjected to switched geometric unilateral constraints. The switching functions $\chi(t)$ are generated by the constraint distances \mathbf{g}_m of the master system. Therefore, the (real) master system is subjected to geometric unilateral constraints, whereas the (artificial) slave system is a perfect replica subjected to switched geometric unilateral constraints that are switched on when the corresponding constraints of the master system are closed. The generalized Poisson’s impact law is chosen with a global coefficient of restitution of $\varepsilon = 0.8$.

The system matrices \mathbf{M} , \mathbf{C} and \mathbf{K} are all symmetric and positive definite. Assumption A1 is fulfilled, since the constraint distances of the master system are absolutely continuous. The external forcing is chosen as a harmonic function such that Assumption A2 is fulfilled. The impact law is described by a monotone impact map, which fulfills Assumption A3. Since the Assumptions A1–A3 from Section 3 are fulfilled, the slave system is attractively incrementally stable according to Theorem 1. Therefore, the synchronization error tends to zero and the slave system reproduces the full state of the master system using only the impact time instants.

The example is simulated for a certain choice of parameters, and the Lyapunov function (16) is shown in Figure 2 for the observer presented in this paper with position jumps (blue) as well as for the observer without position jumps presented in [4] (red). Both observers tend to the observed system, but the observer including position jumps has a higher synchronization rate.

Figure 3 shows the time evolution of the distance between the ground and the first mass. Depicted is the master system (black), the slave system including position jumps (blue)

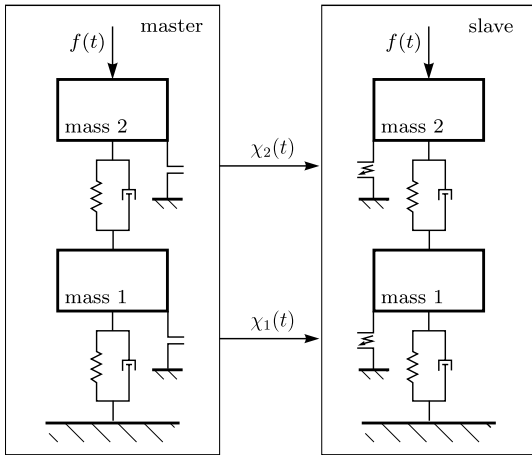


Figure 1. Master-slave system unidirectionally coupled by the Boolean switching functions $\chi(t)$.

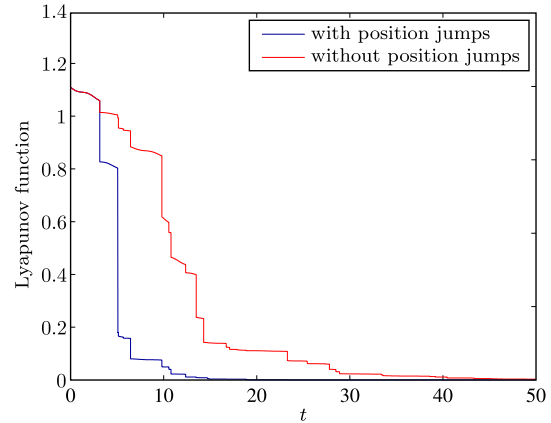


Figure 2. Lyapunov function with position jumps (blue) and without position jumps (red).

and the slave system without position jumps (red). The newly proposed observer tends faster to the master system since it additionally imposes a projection of the generalized coordinates. Both observers reproduce the full state of the observed system using only the impact time information also in the presence of accumulation points, which correspond to infinitely many impacts in a finite time interval (also called Zeno behavior).

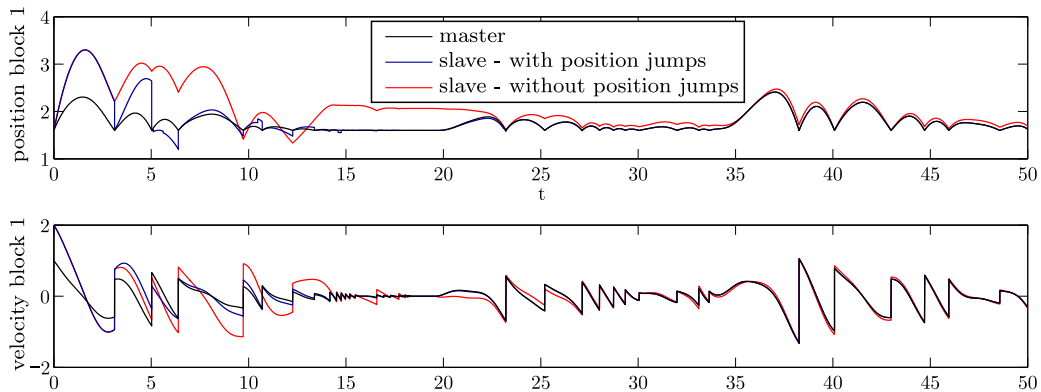


Figure 3. Time evolution of the generalized coordinate and velocity of the first mass of the double mass impact oscillator for the master system (black), slave system with position jumps (blue) and without position jumps (red).

Example 2 (Chain of rocking blocks). The second example consists of a chain of blocks which are connected by spring-damper elements as depicted in Figure 4. Each block can move vertically and rotate about its center of gravity. The lowest block is connected to the harmonically moving ground. Between each neighboring block and between the lowest block and the ground there are on both sides unilateral constraints. Only small deflections are considered such that the system can be considered linear time-invariant with symmetric and positive definite system matrices. The generalized Poisson's impact law is chosen with a global coefficient of restitution of $\varepsilon = 0.9$. The master system is subjected to geometric unilateral constraints and the slave system is subjected to switched geometric unilateral constraints. The slave system is a.i.s. according to Theorem 1.

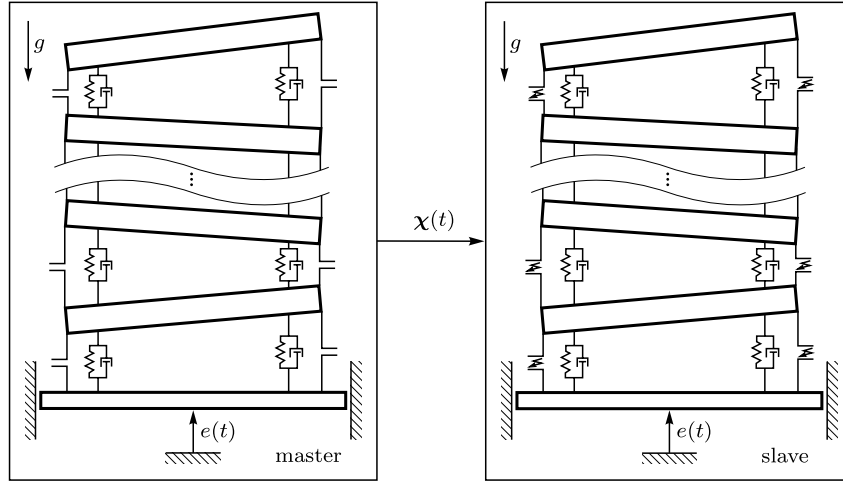


Figure 4. Master-slave system unilaterally coupled by the switching functions $\chi(t)$.

The coupled system is simulated with a chain consisting of 10 blocks, which corresponds to 20 degrees of freedom. As for Example 1, the observer presented in this paper with position jump as well as the observer without position jumps presented in [4] are implemented. The time evolution of the constraint distance and coordinate of a constraint (bottom right constraint in Figure 4) is shown in Figure 5 for the master system (black), slave system with position jumps (blue) and without position jumps (red). The Lyapunov function for both observers is depicted in Figure 6 using a logarithmic scale. During the time interval $4 \leq t \leq 8$, the Boolean switching functions $\chi(t)$ are generated by the master system, and the slave system (having switched constraints) acts as a state observer. For illustrative purposes, the switching functions are generated by the slave system itself for $t \in [0, 4)$. Therefore, the slave systems are subjected to geometric unilateral constraints during this time interval and there is only one solution visible since the solutions of the uncoupled slave systems are identical.

The slave systems are initialized using initial conditions which are very close to the initial conditions of the master system. The uncoupled case shows that the master and the slave system do generally not synchronize without any coupling. Furthermore, the system has extreme sensitivity on initial conditions such that the solutions will diverge for any small initial error. Therefore, there is no local synchronization and the zero-solution of the error dynamics is unstable in the uncoupled case.

5 Conclusions

An observer is proposed which uses the new concept of switched geometric unilateral constraints in order to extend the approach presented in [4]. These constraints introduce position jumps which are feasible since the observer is not a physical system. The new observer clearly improves the synchronization speed and it relaxes the assumptions on the switching functions, e.g. it allows for persistent constraints (constraints which close and do not open again).

The presented observer uses the property of attractive incremental stability and is based on master-slave synchronization. The observer uses only the information of the impact time instants and no continuous measurement is used. Furthermore, it reproduces the full state of the observed system for every initial condition and also in the presence of accumulation points.

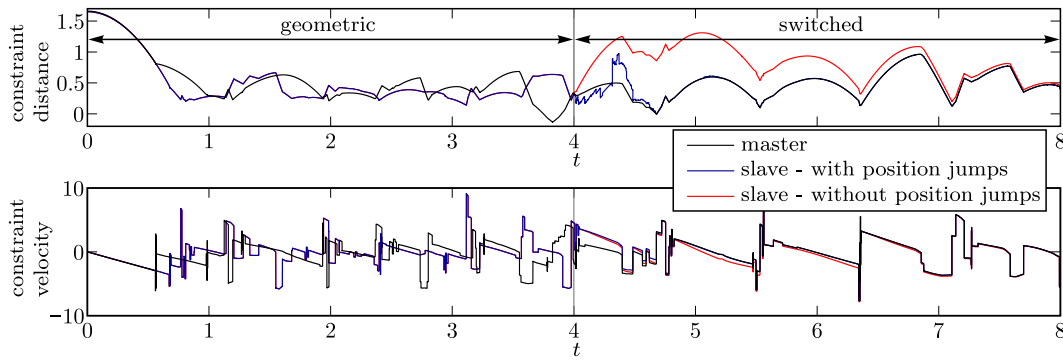


Figure 5. Time evolution of the constraint distance and coordinate of a constraint (bottom right in Figure 4) for the master system (black), slave system with position jumps (blue) and without position jumps (red). The systems are decoupled during the time interval $0 \leq t < 4$ for which there is generally no (local) synchronization.

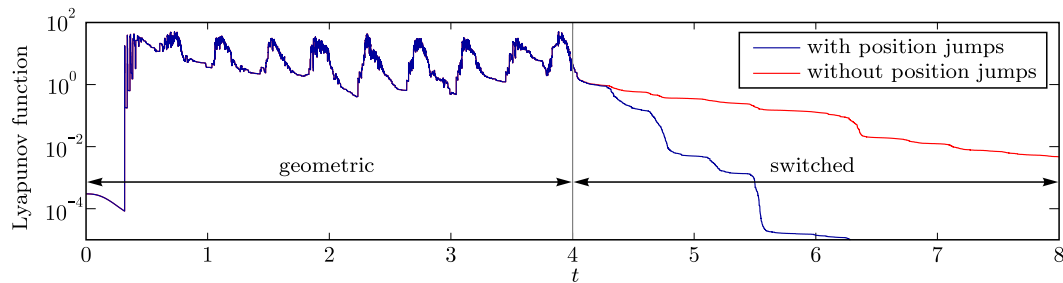


Figure 6. Lyapunov function for the coupled chain of rocking blocks system with position jumps (blue) and without position jumps (red) using a logarithmic scale.

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