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Nonexpansivity of the Newton's Cradle Impact Law

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The 3-ball Newton's cradle is used as a stepping stone to divulge the structure of impact laws. A continuous cone-wise linear impact law which maps the pre-impact contact velocities to the post-impact contact velocities is proposed for the 3-ball Newton's cradle. The proposed impact law is kinematically, kinetically, and energetically consistent. It reproduces all the classical experimental outcomes. Moreover, the impact law has the mathematical property of being non-expansive.

1 The 3-Ball Newton's Cradle

The 3-ball Newton's cradle consists of three balls of equal mass m with horizontal positions $\mathbf{q} = (q_1\ \ q_2\ \ q_3)^{\mathrm{T}}$ and velocities $\dot{\mathbf{q}} = \mathbf{u} = (u_1\ \ u_2\ \ u_3)^{\mathrm{T}}$, see Figure 1(a). The contact distances are given by $\mathbf{g} = (q_2 - q_1 - 2R\ \ q_3 - q_2 - 2R)^{\mathrm{T}}$, where R is the radius of the balls. The unilateral constraint $\mathbf{g} \geq \mathbf{0}$ expresses the impenetrability of the balls. The contact velocities are given by the relative velocities between the balls $\boldsymbol{\gamma} = (\gamma_1\ \ \gamma_2)^{\mathrm{T}} = (u_2 - u_1\ \ u_3 - u_2)^{\mathrm{T}}$. The pre- and post-impact velocities are designated by $\mathbf{u}^-$ and $\mathbf{u}^+$, respectively. Analogously, $\boldsymbol{\gamma}^-$ and $\boldsymbol{\gamma}^+$ designate the pre- and post-impact contact velocities.

The impact equations of the system can be written in the following matrix form

$$\mathbf{M}(\mathbf{u}^+ - \mathbf{u}^-) = \mathbf{W}\boldsymbol{\Lambda}, \tag{1}$$

$$\boldsymbol{\gamma}^\pm = \mathbf{W}^{\mathrm{T}}\mathbf{u}^\pm, \tag{2}$$

where $\boldsymbol{\Lambda} = (\Lambda_1\ \ \Lambda_2)^{\mathrm{T}}$ are the impulsive contact forces during the impact. The impulsive force Λ_1 acts between balls 1 and 2, while Λ_2 occurs between balls 2 and 3. The matrix $\mathbf{W} = (\partial\mathbf{g}/\partial\mathbf{q})^{\mathrm{T}}$ is the matrix of generalized force directions. For the 3-ball Newton's cradle, the mass matrix $\mathbf{M}$ and the matrix of generalized force directions $\mathbf{W}$ are

$$\mathbf{M} = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix} \quad \text{and} \quad \mathbf{W} = \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}. \tag{3}$$

We consider collisions for which both contacts are closed, i.e. $\mathbf{g} = \mathbf{0}$. The impact law for the 3-ball Newton's cradle can be expressed by a mapping S from pre- to post-impact contact velocities

$$\boldsymbol{\gamma}^+ = S(\boldsymbol{\gamma}^-). \tag{4}$$

The impact law (4) should be kinematically, kinetically, and energetically consistent [1]:

- Pre-impact contact velocities $\boldsymbol{\gamma}^-$ and post-impact contact velocities $\boldsymbol{\gamma}^+$ are called *kinematically admissible* or *kinematically consistent* if $\boldsymbol{\gamma}^- \leq \mathbf{0}$ and $\boldsymbol{\gamma}^+ \geq \mathbf{0}$, respectively.
- *Kinetic consistency* is required by the unilateral character of non-adhesive contacts which requires the contact forces to be non-negative, i.e. $\boldsymbol{\Lambda} \geq \mathbf{0}$.
- *Energetic consistency* means that there is no increase in energy during the impact. Let the kinetic energy before and after the impact be designated by $T^- = \frac{1}{2}\mathbf{u}^{-\mathrm{T}}\mathbf{M}\mathbf{u}^-$ and $T^+ = \frac{1}{2}\mathbf{u}^{+\mathrm{T}}\mathbf{M}\mathbf{u}^+$, respectively. Energetic consistency then requires that $T^+ \leq T^-$, which can be expressed in terms of pre- and post-impact contact velocities as $\|\boldsymbol{\gamma}^+\|^2_{\mathbf{G}^{-1}} \leq \|\boldsymbol{\gamma}^-\|^2_{\mathbf{G}^{-1}}$, where $\mathbf{G}^{-1}$ denotes the inverse of the Delassus operator $\mathbf{G} = \mathbf{W}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{W}$.

2 The Sequential Impact Law

We propose a continuous cone-wise linear impact mapping $S\colon \mathbb{R}^2 \to \mathbb{R}^2,\ \boldsymbol{\gamma}^- \mapsto \boldsymbol{\gamma}^+$ for the 3-ball Newton's cradle, i.e.

$$\boldsymbol{\gamma}^+ = S(\boldsymbol{\gamma}^-) = \mathbf{Q}_i\boldsymbol{\gamma}^-, \tag{5}$$

where $\mathbf{Q}_i \in \mathbb{R}^{2\times 2}$ are 2-by-2 matrices which apply in a corresponding cone in the (γ_1^-, γ_2^-)-plane.

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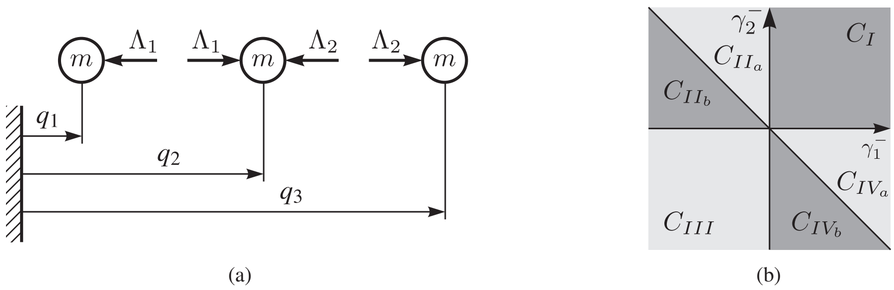

Fig. 1: *Left:* Newton's cradle with 3 balls of mass m. *Right:* The different cones in the (γ_1^-, γ_2^-)-plane.

We construct the matrices $\mathbf{Q}_i$ and their respective cones C_i by demanding the following properties of the impact law:

P1 The mapping is continuous, i.e. $\mathbf{Q}_i \mathbf{v}_i = \mathbf{Q}_{i+1} \mathbf{v}_i$ with $\mathbf{v}_i$ being the direction of the boundary half-line between the cones C_i and C_{i+1}.

P2 Conservation of energy holds, i.e. $\|\boldsymbol{\gamma}^+\|_{\mathbf{G}^{-1}} = \|\mathbf{Q}_i \boldsymbol{\gamma}^-\|_{\mathbf{G}^{-1}} = \|\boldsymbol{\gamma}^-\|_{\mathbf{G}^{-1}}$ for all matrices $\mathbf{Q}_i$. This implies energetic consistency.

P3 Each cone $\mathbf{Q}_i$ is mapped to the entire first quadrant, i.e. the cone C_i is spanned by the columns of $\mathbf{Q}_i^{-1}$. This implies kinematic consistency.

The pre-impact contact velocities in the first quadrant are positive meaning that no impact occurs. Therefore, $\mathbf{Q}_I$ is set to be the identity matrix. The properties **P1** to **P3** lead to the six cones C_i with $i \in \{I, II_a, II_b, III, IV_a, IV_b\}$ (see Figure 1(b)) together with their corresponding matrices

$$\begin{aligned} \mathbf{Q}_I &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{Q}_{II_a} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \mathbf{Q}_{II_b} = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \\ \mathbf{Q}_{III} &= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{Q}_{IV_a} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, \quad \mathbf{Q}_{IV_b} = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}. \end{aligned} \tag{6}$$

The symmetry of the problem appears in the symmetry between the matrices $\mathbf{Q}_{II_a}$ and $\mathbf{Q}_{IV_a}$ as well as between $\mathbf{Q}_{II_b}$ and $\mathbf{Q}_{IV_b}$. We call the impact law (5) the *Sequential Impact Law* because it is equivalent to a sequence of impacts between only two balls. Indeed it holds that $\mathbf{Q}_{II_b} = \mathbf{Q}_{IV_a}\mathbf{Q}_{II_a}$, $\mathbf{Q}_{IV_b} = \mathbf{Q}_{II_a}\mathbf{Q}_{IV_a}$, and $\mathbf{Q}_{III} = \mathbf{Q}_{IV_a}\mathbf{Q}_{II_a}\mathbf{Q}_{IV_a} = \mathbf{Q}_{II_a}\mathbf{Q}_{IV_a}\mathbf{Q}_{II_a}$, where $\mathbf{Q}_{II_a}$ and $\mathbf{Q}_{IV_a}$ describe the impact between only two of the three balls. It is shown in [2] that the impact mapping (5) is non-expansive in the metric $\mathbf{G}^{-1}$, i.e.

$$\|\boldsymbol{\gamma}_A^+ - \boldsymbol{\gamma}_B^+\|_{\mathbf{G}^{-1}} \le \|\boldsymbol{\gamma}_A^- - \boldsymbol{\gamma}_B^-\|_{\mathbf{G}^{-1}} \quad \forall \boldsymbol{\gamma}_A^-, \boldsymbol{\gamma}_B^- \in \mathbb{R}^2. \tag{7}$$

For the definition of non-expansivity we refer to [3]. The implications of this property on impact laws can be found in [4]. The 3-ball Newton's cradle can be fully described by a non-expansive cone-wise linear impact mapping that is composed by a series of single collisions between only two balls. Note that its phenomena cannot be described by the classical Newton's or Poisson's instantaneous impact law [5, 6].

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