Convergence based synchronization of unilaterally constrained multibody systems

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Summary. The convergence property of a dynamical system is a strong condition with various useful implications. A convergent system exhibits a bounded globally attractively stable solution and thus its asymptotic (or symptotic) behaviour is independent of initial conditions. This paper presents conditions for the convergence property of mechanical systems submitted to unilateral constraints. A key role plays the maximal monotonicity of the impact law, which is a stronger condition than dissipativity. The convergence property can directly be used for the design of a state observer for mechanical systems with unilateral constraints. These observers are based on master-slave synchronization and replicate the full state of the observed system for every choice of initial conditions using only the impact time instants. The results are illustrated with the example of a harmonically excited beam with unilateral support.

Introduction

In this paper we investigate the convergence property of non-smooth dynamical systems and its application to the design of a state observer in a master-slave synchronization setup. The concept of convergence from nonlinear stability theory has been developed in the 60s in the Russian research community [2] and is a topic of active research [10]. A system excited by an input (e.g. a forced mechanical system) is called convergent if it possesses a bounded globally attractively stable steady state solution thereby attracting all other solutions regardless of their initial conditions [6, 10]. Convergent dynamics is beneficial in many control problems such as stabilization, tracking control, output regulation problems, synchronization and observer design. The input to the system may originate from various sources, e.g. it may be an external command signal, a signal generated by the feedback part of a controller or a measured signal from the observed system.
Here, we consider the design of a state observer using a master-slave coupling of two identical Lagrangian systems with unilateral constraints. The convergent dynamics ensures full state synchronization for all initial conditions. The coupling is unidirectional and consists only of the impact time instants of the master system and therefore requiring only little effort in measuring. A similar approach has been proposed in [12], where a system theoretical framework has been adopted. The class of Lagrangian systems which we consider in this paper does however not require the strict passivity property as has been assumed in [12]. More precisely, the transfer matrix of the linear part of the system is in our case (in the absence of additional measurements) not strictly positive real [1].
Commonly used impact laws, such as the generalized Newton's impact law, enjoy the maximal monotonicity property as is shown in the accompanying paper [8]. In the first part of this paper it is shown that Lagrangian systems with switched frictionless unilateral constraints exhibit the convergence property if the impact law is maximal monotone. In the second part the use of the convergence property is shown for the design of a state observer based on master-slave synchronization. The theoretical results are illustrated using numerical simulations of a harmonically excited beam with a unilateral support. The slave system, acting as the state observer, reproduces the full state of the master system using only the impact time instants of the master system.

The convergence property

We briefly review the convergence property as defined in [6] for a time-invariant measure differential inclusion with inputs

$$\mathrm{d}\boldsymbol{x} \in \mathrm{d}\boldsymbol{\Gamma}(\boldsymbol{x}, \boldsymbol{w}), \tag{1}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$ is the state vector and $\boldsymbol{w} \in \mathbb{R}^d$ is the time-varying input. We will restrict the input $\boldsymbol{w}$ to the class $\mathbb{W}$ which at least guarantees that a solution $\boldsymbol{x}(t)$ of locally bounded variation exists.
A convergent system has a globally attractively[1] stable steady state solution $\bar{\boldsymbol{x}}_w(t)$, which is defined and bounded almost everywhere on the time axis [6]. The term *almost everywhere* captures that the solution is not defined at impact time instants and is referred to as a.e. in the following. This class of systems 'forgets' their initial conditions and all solutions converge to the same particular solution (see Figure 1). In addition, if the system is *uniformly* convergent, then the steady state solution $\bar{\boldsymbol{x}}_w$ is unique. Let us formally define the property of convergence for systems with inputs.

Definition 1
System (1) is said to be

- convergent *if there exists a solution* $\bar{\boldsymbol{x}}_w(t)$ *for every* $\boldsymbol{w}(t) \in \mathbb{W}$ *satisfying the following conditions:*
  1. $\bar{\boldsymbol{x}}_w(t)$ *is defined for almost all* $t \in \mathbb{R}$*,*
  2. $\bar{\boldsymbol{x}}_w(t)$ *is bounded for almost all* $t \in \mathbb{R}$*,*
  3. $\bar{\boldsymbol{x}}_w(t)$ *is globally attractively stable.*
- uniformly convergent *if it is convergent and* $\bar{\boldsymbol{x}}_w(t)$ *is globally uniformly attractively stable for every* $\boldsymbol{w}(t) \in \mathbb{W}$*.*

[1] The wording 'attractively stable' has been used instead of the usual term 'asymptotically stable', because attractivity of solutions in (measure) differential inclusions can be asymptotic or symptotic (finite-time attractivity), see [6].

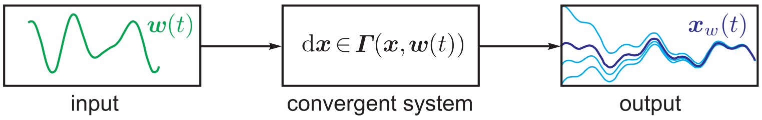

Figure 1: A convergent system has a globally attractive steady state solution $\boldsymbol{x}_w$ for every input $w(t)$ within a given class.

Convergence of a Lagrangian system with switched kinematic unilateral constraints

We show under which conditions mechanical systems exhibit the convergence property. More specifically, we consider forced linear time-invariant Lagrangian systems with positive definite system matrices, which are subjected to switched kinematic unilateral constraints. Let $\boldsymbol{q} \in \mathbb{R}^f$ be the generalized coordinates and $\boldsymbol{u} \in \mathbb{R}^f$ be the generalized velocities. The non-impulsive dynamics of the system is described the equation of motion together with the kinematic equation

$$\boldsymbol{M}\dot{\boldsymbol{u}} + \boldsymbol{C}\boldsymbol{u} + \boldsymbol{K}\boldsymbol{q} = \boldsymbol{W}\boldsymbol{\lambda} + \boldsymbol{f}(t), \quad \dot{\boldsymbol{q}} = \boldsymbol{u}, \quad \text{a.e.} \tag{2}$$

where the mass matrix $\boldsymbol{M}$ and the stiffness matrix $\boldsymbol{K}$ are symmetric and the time-dependent external forcing $\boldsymbol{f}(t)$ is bounded. The motion of the system is restricted by m unilateral constraints, which invoke unilateral constraint forces $\boldsymbol{\lambda} \in \mathbb{R}^m$ together with generalized force directions $\boldsymbol{W} \in \mathbb{R}^{f\times m}$, which we assume to be linearly independent.
Correspondingly, the impulsive dynamics is described by the impact equation

$$\boldsymbol{M}(\boldsymbol{u}^+ - \boldsymbol{u}^-) = \boldsymbol{W}\boldsymbol{\Lambda}, \tag{3}$$

where $\boldsymbol{u}^-$ and $\boldsymbol{u}^+$ denote pre- and post-impact velocities and $\boldsymbol{\Lambda} \in \mathbb{R}^m$ are the impulsive constraint forces.
The system, described by the equation of motion (2) together with the impact equation (3), needs to be complemented with a force law for the constraint forces $\boldsymbol{\lambda}$ and an impact law for the impulsive constraint forces $\boldsymbol{\Lambda}$. These constitutive laws depend on the type of unilateral constraint and typically operate on local kinematic quantities. We will assume that the constraint distances $\boldsymbol{g}$ are an affine function of the generalized positions and the constraint velocities $\boldsymbol{\gamma}$ are therefore a linear function of the generalized velocities, i.e. $\boldsymbol{\gamma} = \boldsymbol{W}^{\mathrm{T}}\boldsymbol{u}$.
Here, we will consider two different unilateral constraints. A *switched kinematic unilateral constraint* (also called one-way clutch or sprag clutch) imposes a kinematic unilateral constraint $\gamma_i \geq 0$ for contact i whenever the corresponding binary switching function $\chi_i = 1$ and imposes no constraint if $\chi_i = 0$. The kinematic unilateral constraint restricts the sign of the constraint velocity $\gamma_i \geq 0$, thereby allowing for relative motion in positive direction and blocking in the opposite direction. The force law is described by the inequality complementarity

$$-\lambda_i \in \begin{cases} \mathrm{Upr}(\gamma_i) & \text{if } \chi_i = 1, \\ 0 & \text{if } \chi_i = 0, \end{cases} \tag{4}$$

where Upr denotes the unilateral primitive [3]. It is defined as the subdifferential of the indicator function $\Psi_{\mathbb{R}_0^+}$ on the set $\mathbb{R}_0^+$, i.e. $\mathrm{Upr}(x) = \partial\Psi_{\mathbb{R}_0^+}$. This inequality complementarity is identical to $-\lambda_i \in \mathrm{Upr}(\gamma_i) \Leftrightarrow 0 \leq \lambda_i \perp \gamma_i \geq 0$. A *geometric unilateral constraint*, also known as impenetrability constraint, restricts the sign of a constraint distance $g_i \geq 0$. Its force law $0 \leq \lambda_i \perp g_i \geq 0$ (also referred to as Signorini's law) can be written on velocity level using the switched inequality complementarity (4) for which the switching function χ_i is determined by the constraint distance g_i as

$$\chi_i = \begin{cases} 1 & \text{if } g_i = 0, \\ 0 & \text{if } g_i > 0. \end{cases} \tag{5}$$

The impact law for the impulsive unilateral constraint forces $\boldsymbol{\Lambda}$ is given by the inclusion (see [8])

$$-\boldsymbol{\Lambda} \in \mathcal{H}(\bar{\boldsymbol{\gamma}}, \boldsymbol{\chi}(t)), \qquad \text{where } \bar{\boldsymbol{\gamma}} := \frac{1}{2}(\boldsymbol{\gamma}^+ + \boldsymbol{\gamma}^-). \tag{6}$$

The set-valued operator $\mathcal{H}$, defined as a function of the kinematic variable $\bar{\boldsymbol{\gamma}}$ and the binary switching functions $\boldsymbol{\chi}(t)$, is assumed to be maximal monotone with respect to the first argument and to fulfill the natural condition $\mathbf{0} \in \mathcal{H}(\mathbf{0}, \boldsymbol{\chi}(t))$. The maximal monotonicity property of $\mathcal{H}$ is given by the condition $(\boldsymbol{\Lambda}_1 - \boldsymbol{\Lambda}_2)^{\mathrm{T}}(\bar{\boldsymbol{\gamma}}_1 - \bar{\boldsymbol{\gamma}}_2) \leq 0$ for all pairs $(\boldsymbol{\Lambda}_1, \bar{\boldsymbol{\gamma}}_1)$ and $(\boldsymbol{\Lambda}_2, \bar{\boldsymbol{\gamma}}_2)$ which fulfill the impact law (6) for a fixed $\boldsymbol{\chi}(t)$. These assumptions are fulfilled by the generalized Newton's impact law [8] and generalized Poisson's impact law [7] with global coefficients of restitution. It is shown in [8] that the mapping $Z : \boldsymbol{u}^+ = Z(\boldsymbol{u}^-)$ from pre-impact to post-impact generalized velocities, defined by the impact equation (3) together with the impact law (6), is maximal non-expansive [11] in the metric $\boldsymbol{M}$, i.e.

$$\|\boldsymbol{u}_1^+ - \boldsymbol{u}_2^+\|_{\boldsymbol{M}}^2 \leq \|\boldsymbol{u}_1^- - \boldsymbol{u}_2^-\|_{\boldsymbol{M}}^2, \tag{7}$$

if and only if $\mathcal{H}$ is maximal monotone. This implies the dissipativity of the impact, assuming the natural condition $\mathbf{0} \in \mathcal{H}(\mathbf{0}, \boldsymbol{\chi}(t))$. The Lagrangian system (2)-(4),(6) is chosen as simple as possible in order to keep the further analysis concise. The switching functions $\boldsymbol{\chi}(t)$ must not allow for persistent contact for all times. In order to define the allowed class of switching functions, we introduce the notation $\tilde{a}_{\Delta t}(t) = \frac{1}{\Delta t}\int_t^{t+\Delta t} a(\tau)\,\mathrm{d}\tau$ for a given $\Delta t > 0$.

Definition 2
A function $\chi(t)$ *is of class* $\mathbb{K}$ *if for each* t *there exists a* $t^*(t) > t$ *and a number* $\Delta t > 0$ *independent of* t *such that* $\tilde{\chi}_{\Delta t}(t^*) = 0$. *Furthermore, a function* $\boldsymbol{\chi}(t)$ *is of class* $\mathbb{K}^n$ *if each component* $\chi_i(t)$ *is of class* $\mathbb{K}$.

We will make the following standing assumptions on the system:

A1 The function $\boldsymbol{\chi}(t)$ is of class $\mathbb{K}^n$.

A2 The function $\boldsymbol{\chi}(t)$ and the external forcing $\boldsymbol{f}(t)$ are such that there exists a bounded solution for almost all t.

A3 The impact map $\mathcal{H}(\bar{\gamma}, \boldsymbol{\chi}(t))$ is maximal monotone with respect to the first argument, $\mathbf{0} \in \mathcal{H}(\mathbf{0}, \boldsymbol{\chi}(t))$ and $\mathbf{0} = \mathcal{H}(\bar{\gamma}, \mathbf{0})$.

Theorem 1
System (2)-*(4)*,(6) *with the assumptions* A1-A3 *is uniformly convergent.*

Proof: According to Definition 1 we must show that there exists a bounded steady state solution a.e. which is globally uniformly attractively stable. The existence of a bounded steady state solution is directly given by assumption A2. Therefore, the proof is reduced to show that all solution curves of the system (2)-(4),(6) with the assumptions A1-A3 are globally uniformly attractively stable, which is also referred to as *incremental stability*.
Consider two arbitrary solutions $(\boldsymbol{q}_1(t), \boldsymbol{u}_1(t))$ and $(\boldsymbol{q}_2(t), \boldsymbol{u}_2(t))$ of system (2)-(4),(6). Let $\boldsymbol{e} = \boldsymbol{q}_1 - \boldsymbol{q}_2$ and $\boldsymbol{v} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ be the position and the velocity error between these two solutions, which we gather in the state vector $\boldsymbol{x}_e = \begin{pmatrix}\boldsymbol{e}^{\mathrm{T}} & \boldsymbol{v}^{\mathrm{T}}\end{pmatrix}^{\mathrm{T}}$ of the error dynamics. The Lyapunov function

$$V(\boldsymbol{x}_e) = \frac{1}{2}\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|^2_{\boldsymbol{M}} + \frac{1}{2}\|\boldsymbol{q}_1 - \boldsymbol{q}_2\|^2_{\boldsymbol{K}} = \frac{1}{2}\|\boldsymbol{v}\|^2_{\boldsymbol{M}} + \frac{1}{2}\|\boldsymbol{e}\|^2_{\boldsymbol{K}} \tag{8}$$

gives a notion of distance between these two solutions and it is a positive definite function, since $\boldsymbol{M}, \boldsymbol{K} > 0$. The two solutions agree if and only if $V = 0$. The differential measure of the Lyapunov function contains a density $\dot{V}$ with respect to the Lebesgue measure $\mathrm{d}t$ and a density $V^+ - V^-$ with respect to the atomic measure $\mathrm{d}\eta$. The density $\dot{V}$ is given by

$$\begin{aligned}\dot{V}(t, \boldsymbol{x}_e) &= (\boldsymbol{u}_1 - \boldsymbol{u}_2)^{\mathrm{T}}\boldsymbol{M}(\dot{\boldsymbol{u}}_1 - \dot{\boldsymbol{u}}_2) + (\boldsymbol{q}_1 - \boldsymbol{q}_2)^{\mathrm{T}}\boldsymbol{K}(\boldsymbol{u}_1 - \boldsymbol{u}_2)\\ &= (\boldsymbol{u}_1 - \boldsymbol{u}_2)^{\mathrm{T}}\left(-\boldsymbol{C}(\boldsymbol{u}_1 - \boldsymbol{u}_2) + \boldsymbol{W}(\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2)\right)\\ &= -\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|^2_{\boldsymbol{C}} + (\gamma_1 - \gamma_2)^{\mathrm{T}}(\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2).\end{aligned} \tag{9}$$

Hence, the positive definiteness of $\boldsymbol{C}$ and the maximal monotonicity of the force law (4) imply $\dot{V}(t, \boldsymbol{x}_e) \leq -W(\boldsymbol{x}_e)$ a.e., where $W(\boldsymbol{x}_e) = \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|^2_{\boldsymbol{C}}$ is a positive semi-definite function in $\boldsymbol{x}_e$.
The jump in the Lyapunov function at impulsive time-instants is given by

$$V^+ - V^- = \frac{1}{2}\|\boldsymbol{u}_1^+ - \boldsymbol{u}_2^+\|^2_{\boldsymbol{M}} - \frac{1}{2}\|\boldsymbol{u}_1^- - \boldsymbol{u}_2^-\|^2_{\boldsymbol{M}} \leq 0, \tag{10}$$

where the non-expansivity of the mapping Z (7) has been used, since the impact map $\mathcal{H}(\bar{\gamma}, \boldsymbol{\chi}(t))$ is monotone (assumption A3). Consequently, the Lyapunov function V can not increase neither during continuous nor discontinuous flow. Therefore, the equilibrium $\boldsymbol{x}_e = \mathbf{0}$ is uniformly stable [6]. From

$$V(\boldsymbol{x}_e(t)) - V(\boldsymbol{x}_e(t_0)) \leq -\int_{t_0}^{t} W(\boldsymbol{x}_e(\tau))\mathrm{d}\tau \tag{11}$$

and the fact that V is bounded from below, we deduce that

$$\lim_{t\to\infty}\int_{t_0}^{t} W(\boldsymbol{x}_e(\tau))\mathrm{d}\tau = \lim_{t\to\infty}\int_{t_0}^{t} \|\boldsymbol{v}(\tau)\|^2_{\boldsymbol{C}}\mathrm{d}\tau < \infty. \tag{12}$$

The solution $\boldsymbol{x}_e(t)$ is not continuous in time and we can therefore not invoke Barbalat's lemma [5]. However, the condition (12) and the equivalence of norms yield $\lim_{t\to\infty}\int_{t_0}^{t}\|\boldsymbol{v}(\tau)\|\mathrm{d}\tau < \infty$. The variation function $\mathrm{Var}_{\boldsymbol{e}}(t) = \mathrm{var}(\boldsymbol{e}, [t_0, t]) = \int_{t_0}^{t}\|\boldsymbol{v}\|\mathrm{d}t$ is absolutely continuous and monotonically increasing and, thus, $\boldsymbol{e}(t)$ attains a limit. Additionally, each level set of V is a compact positively invariant set, which implies boundedness of $\boldsymbol{e}(t)$ and $\boldsymbol{v}(t)$ in forward time. Therefore it holds that $\lim_{t\to\infty}\boldsymbol{e}(t) = \boldsymbol{c}$ for some $\boldsymbol{c}$ satisfying $\frac{1}{2}\|\boldsymbol{c}\|^2_{\boldsymbol{K}} \leq V(\boldsymbol{x}_e(t_0))$. Next, we prove that also $\lim_{t\to\infty}\boldsymbol{v} = \mathbf{0}$. The Lyapunov function V is bounded from below and non-increasing. Hence, the limit

$$V_\infty := \lim_{t\to\infty} V(\boldsymbol{x}_e(t)) = \lim_{t\to\infty}\left(\frac{1}{2}\|\boldsymbol{v}\|^2_{\boldsymbol{M}} + \frac{1}{2}\|\boldsymbol{e}\|^2_{\boldsymbol{K}}\right) \tag{13}$$

exists and lies in the interval $0 \leq V_\infty \leq V(\boldsymbol{x}_e(t_0))$. Using $\lim_{t\to\infty} \boldsymbol{e}(t) = \boldsymbol{c}$, we obtain the limit

$$\lim_{t\to\infty} \frac{1}{2}\|\boldsymbol{v}\|^2_{\boldsymbol{M}} = V_\infty - \frac{1}{2}\|\boldsymbol{c}\|^2_{\boldsymbol{K}}. \tag{14}$$

Hence, due to the positive definiteness of $\boldsymbol{M}$ and $\boldsymbol{C}$ and the equivalence of norms, also the limit $\lim_{t\to\infty} \|\boldsymbol{v}\|_{\boldsymbol{C}}$ exists. Together with the boundedness condition (12) we have $\lim_{t\to\infty} \boldsymbol{v} = \boldsymbol{0}$ and $V_\infty = \frac{1}{2}\|\boldsymbol{c}\|^2_{\boldsymbol{K}}$.
Up to now, we have proven that the error dynamics, governed by the equality of measures

$$\boldsymbol{M}\mathrm{d}\boldsymbol{v} + \boldsymbol{C}\boldsymbol{v}\mathrm{d}t + \boldsymbol{K}\boldsymbol{e}\mathrm{d}t = \boldsymbol{W}(\mathrm{d}\boldsymbol{P}_1 - \mathrm{d}\boldsymbol{P}_2), \tag{15}$$

is tending towards the limit point $(\boldsymbol{c}, \boldsymbol{0})$ for $t \to \infty$, where $\mathrm{d}\boldsymbol{P}_1 = \boldsymbol{\lambda}_1\mathrm{d}t + \boldsymbol{\Lambda}_1\mathrm{d}\eta$ and $\mathrm{d}\boldsymbol{P}_2 = \boldsymbol{\lambda}_2\mathrm{d}t + \boldsymbol{\Lambda}_2\mathrm{d}\eta$ are the constraint impulse measures. Integrating the equality of measures over the time interval $[t, t + \Delta t]$ yields

$$\boldsymbol{M}(\boldsymbol{v}^+(t+\Delta t) - \boldsymbol{v}^-(t)) + \boldsymbol{C}(\boldsymbol{e}(t+\Delta t) - \boldsymbol{e}(t)) + \boldsymbol{K}\int_t^{t+\Delta t} \boldsymbol{e}(\tau)\mathrm{d}\tau = \int_{[t,t+\Delta t]} \boldsymbol{W}(\mathrm{d}\boldsymbol{P}_1 - \mathrm{d}\boldsymbol{P}_2), \tag{16}$$

where $\Delta t > 0$ is arbitrary. It proves useful to introduce the quantity $\tilde{\boldsymbol{\lambda}}_{\Delta t}(t) = \frac{1}{\Delta t}\int_{[t,t+\Delta t]}(\mathrm{d}\boldsymbol{P}_1 - \mathrm{d}\boldsymbol{P}_2)$, which can be regarded as the average constraint force of the error dynamics over the time lapse $[t, t + \Delta t]$. Subsequently, we take the limit $t \to \infty$ and use $\boldsymbol{v}(t) \to 0$ and $\boldsymbol{e}(t) \to \boldsymbol{c}$ for $t \to \infty$. The integrated equality of measures, divided by Δt, yields

$$\boldsymbol{K}\boldsymbol{c} = \lim_{t\to\infty} \boldsymbol{W}\tilde{\boldsymbol{\lambda}}_{\Delta t}(t) = \lim_{t\to\infty} \sum_{i=1}^{m} \boldsymbol{w}_i \tilde{\lambda}_{\Delta t,i}(t), \tag{17}$$

which is, in an averaged sense, the equilibrium of forces at infinity. The columns $\boldsymbol{w}_i$ of $\boldsymbol{W}$ are linearly independent from which we deduce that each of the limits $\lim_{t\to\infty} \tilde{\lambda}_{\Delta t,i}(t)$ have to exist. Moreover, it holds that $\lambda_{1i}(t) = \lambda_{2i}(t) = 0$ and $\Lambda_{1i}(t) = \Lambda_{2i}(t) = 0$ for $\chi_i(t) = 0$ according to (4) and assumption A3. Taking a small enough Δt, we conclude that the limit of $\tilde{\lambda}_{\Delta t,i}(t)$ must vanish, since each switching function $\chi_i(t)$ is of class $\mathbb{K}$. According to (17) and $\boldsymbol{K} > 0$, we obtain $\boldsymbol{c} = \boldsymbol{0}$. Therefore, $\boldsymbol{x}_e(t) = \boldsymbol{0}$ is uniformly stable and globally attractive, which concludes the proof. □

Synchronization of master-slave systems

The convergence property is used to design a state observer using master-slave synchronization. The master system (index m) and the slave system (index s) are described by the set of equations (2)-(4),(6). The switching function is defined by (5), where the gap function of the master system is taken in both cases. Therefore, the master system is subjected to geometric unilateral constraints, whereas and the slave system is a perfect replica with one-way clutch constraints that are switched on when the corresponding contacts of the master system are closed. According to Theorem 1, the slave system is uniformly convergent. Therefore, the synchronization error tends to zero and the full state of the master system is reconstructed by the slave system using only the impact time instants.
We illustrate the synchronization based observer using the example of a harmonically excited beam with a unilateral support as depicted in Figure 2. The master system is subjected to a geometric unilateral constraint and the slave system has a switched kinematic unilateral constraint. The only coupling between the master system and the slave system is given through the collision time-instants t_i of the master system, captured by the binary switching function $\chi(t)$, and is unidirectional.
The vertical displacement $w(t, x)$ of the beam is measured with respect to a solid frame, which itself is excited by $e(t) = e_0 \cos\omega t$. The support is at the location x_c and the contact is closed for $w(t, x_c) = 0$. The beam is modeled as a plane linearized Euler-Bernoulli beam. In this case the virtual work is given by the Lagrange's central equation as

$$0 = \delta W = \int_0^l \Big\{\delta w'' EI w'' + \delta w A\rho\ddot{w} + \delta w d(\dot{w} - \dot{e})\Big\}\mathrm{d}x - \delta w(x_c)\lambda \quad \forall \delta w | \delta w(0) = 0. \tag{18}$$

The damping is modeled as linear viscous damping with the frame as reference. Even though there are physically more intuitive ways to model the damping effects, this choice is made since it leads to the same decay rate for all eigenmodes. The first three eigenmodes of the free beam are given by

$$v_i(x) = \cos\beta_i x - \cosh\beta_i x + \frac{\cos\beta_i l + \cosh\beta_i l}{\sin\beta_i l + \sinh\beta_i l}(-\sin\beta_i x + \sinh\beta_i x) \quad \text{for } i \in 1, 2, 3, \tag{19}$$

where β_i is the i-th root of $\cos\beta_i l \cosh\beta_i l + 1 = 0$. We proceed with the Galerkin approach by approximating the displacement $w(t, x)$ and the virtual displacements $\delta w(t, x)$ using the first three eigenmodes as

$$\begin{aligned} w(t,x) &\approx w_n(t,x) = \boldsymbol{v}(x)^{\mathrm{T}}\boldsymbol{q}(t) + e(t) + \text{const.}, \\ \delta w(t,x) &\approx \delta w_n(t,x) = \boldsymbol{v}(x)^{\mathrm{T}}\delta\boldsymbol{q}(t), \end{aligned} \tag{20}$$

where $\boldsymbol{q}(t)$ are the generalized coordinates and $\delta\boldsymbol{q}(t)$ the corresponding virtual displacements. Substituting the approximation (20) into (18) yields

$$0 = \delta\boldsymbol{q}^{\mathrm{T}}\Big(\underbrace{\int_0^l \boldsymbol{v}'' EI \boldsymbol{v}''^{\mathrm{T}} \mathrm{d}x}_{\boldsymbol{K}}\,\boldsymbol{q} + \underbrace{\int_0^l \boldsymbol{v}\, A\rho\, \boldsymbol{v}^{\mathrm{T}} \mathrm{d}x}_{\boldsymbol{M}}\,\ddot{\boldsymbol{q}} + \underbrace{\int_0^l \boldsymbol{v}\, A\rho\, \mathrm{d}x\, \ddot{e}}_{-\boldsymbol{f}(t)} + \underbrace{\int_0^l \boldsymbol{v}\, d\, \boldsymbol{v}^{\mathrm{T}} \mathrm{d}x}_{\boldsymbol{C}}\,\dot{\boldsymbol{q}} - \underbrace{\boldsymbol{v}(x_c)}_{\boldsymbol{W}}\,\lambda\Big) \quad \forall\, \delta\boldsymbol{q}. \tag{21}$$

The impacting beam model can therefore be written in the form (2)-(4),(6) with

$$\boldsymbol{M} = \rho A l \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; \boldsymbol{C} = dl \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; \boldsymbol{K} = \frac{EI}{l^3} \begin{pmatrix} (\beta_1 l)^4 & 0 & 0 \\ 0 & (\beta_2 l)^4 & 0 \\ 0 & 0 & (\beta_3 l)^4 \end{pmatrix},$$
$$\boldsymbol{W} = \boldsymbol{v}(x_c), \; f_i(t) = -2\frac{\rho A l}{\beta_i l}\frac{\cos\beta_i l + \cosh\beta_i l}{\sin\beta_i l + \sinh\beta_i l} e_0 \omega^2 \cos\omega t. \tag{22}$$

The kinematic unilateral constraint of the slave system is switched on ($\chi(t) = 1$), whenever the geometric unilateral constraint of the master system is closed ($g_m(t) = 0$). The impacts are modeled using the generalized Poisson's impact law [4], which is compatible with switched kinematic unilateral constraints in contrast to e.g. the generalized Newton's impact law. Furthermore, the generalized Poisson's impact law fulfills assumption A3 as it is shown in [7].

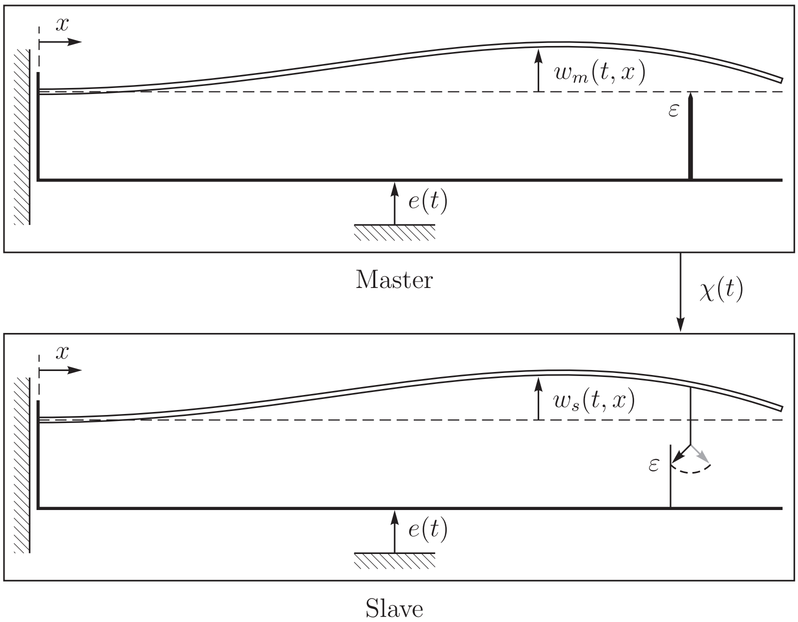

Figure 2: Master-slave system coupled by the binary switching function $\chi(t)$ for a harmonically excited beam with unilateral support.

We take the mechanical energy of the error dynamics as the positive definite Lyapunov function

$$V(t) = \frac{1}{2}\|\boldsymbol{u}_m - \boldsymbol{u}_s\|^2_{\boldsymbol{M}} + \frac{1}{2}\|\boldsymbol{q}_m - \boldsymbol{q}_s\|^2_{\boldsymbol{K}}. \tag{23}$$

According to Theorem 1 the Lyapunov function tends to zero as t tends to infinity and we have $\lim_{t\to\infty}\boldsymbol{q}_s(t) = \boldsymbol{q}_m(t)$ and $\lim_{t\to\infty}\boldsymbol{u}_s(t) = \boldsymbol{u}_m(t)$. Thus synchronization is achieved and the state of the slave system can be used as state observer of the master system.

The Lyapunov function can be split up into the energy contents of each eigenmode as

$$V(t) = \sum_{i=1}^{3} V_i(t), \quad \text{where } V_i(t) = \frac{1}{2}\rho A l (u_{mi} - u_{si})^2 + \frac{1}{2}\frac{EI}{l^3}(\beta_i l)^4 (q_{mi} - q_{si})^2. \tag{24}$$

The time evolution of the Lyapunov function $V(t)$ as well as the energy content $V_1(t)$, $V_2(t)$, $V_3(t)$ of the eigenmodes are depicted in Figure 3. The Lyapunov function is plotted on a logarithmic axis for a simulation with the set of parameters $EI = \rho A = l = 1$, $e_0 = 0.03$, $\omega = 9.5$, $d = 0.17$, $\varepsilon = 0.7$, $x_c = 0.868$ and the initial conditions $\boldsymbol{q}_{m0} = \begin{pmatrix} 0.05 & 0 & 0 \end{pmatrix}^{\mathrm{T}}$, $\boldsymbol{q}_{s0} = -0.5\begin{pmatrix} \beta_1^{-2} & \beta_2^{-2} & \beta_3^{-2} \end{pmatrix}^{\mathrm{T}}$, $\boldsymbol{u}_{m0} = \boldsymbol{u}_{s0} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}^{\mathrm{T}}$.

During non-impulsive motion the eigenmodes are decoupled and their energy is non-increasing due to the positive definiteness of the damping matrix $\boldsymbol{C}$. The only coupling of the eigenmodes is due to the support. At the time instants of an impact, energy may transfer between the first two eigenmodes. The energy content $V_1(t)$ or $V_2(t)$ of the first or second eigenmode generally increase during an impact, but the sum is strictly decreasing. The third eigenmode is not influenced by the impact since the support is chosen to be located at a node of this eigenmode and it is therefore completely decoupled from the other eigenmodes.

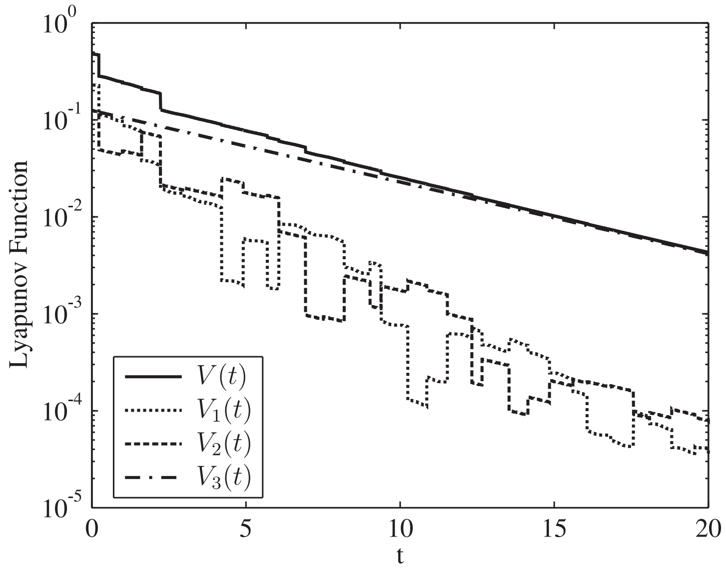

Figure 3: Lyapunov function for the impacting beam and the energy content in the modeled eigenmodes.

Conclusions

A convergence based design of a state observer for Lagrangian systems with unilateral constraints has been presented and has been applied to a unilaterally supported beam. The convergence property of the impacting beam model is due to the maximal monotonicity of the impact law and the internal damping. This property persists for an arbitrary number n of eigenmodes (or generally Ansatz functions) and other approaches for the internal damping as long as the system matrices remain positive definite. The synchronization does not rely on the impacts although the rate of synchronization is improved by the impacts as long as the master system is not blocked by a persistent contact.

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References

[1] Camibel, M. K. and van de Wouw, N. On the convergence of linear passive complementarity systems. *Proceedings of the 46th IEEE Conference on Decision and Control* (2007).

[2] Demidovich, B. P., *Lectures on Stability Theory (in Russian)*. Nauka, Moscow, 1967.

[3] Glocker, Ch. *Set-Valued Force Laws, Dynamics of Non-Smooth Systems*, vol. 1 of *Lecture Notes in Applied Mechanics*. Springer-Verlag, Berlin, 2001

[4] Glocker, Ch., Energetic consistency conditions for standard impacts. Part II: Poisson-type inequality impact laws. *Multibody System Dynamics* (2013), 1-65.

[5] Khalil, H. K. *Nonlinear Systems*. Prentice-Hall, New Jersey, 1996.

[6] Leine, R. I., and van de Wouw, N. *Stability and Convergence of Mechanical Systems with Unilateral Constraints*, vol. 36 of *Lecture Notes in Applied and Computational Mechanics*. Springer Verlag, Berlin, 2008.

[7] Leine, R. I., and Baumann, M. *A state observer for impact oscillators through master-slave synchronization.* to be submitted (2014).

[8] Leine, R. I., and Baumann, M. Variational analysis of inequality impact laws. *Proceedings of the ENOC 2014 Conference* (2014).

[9] Maschek, A.-K. Synchronization of non-smooth mechanical systems. Master's thesis, ETH Zurich, 2012.

[10] Pavlov, A., van de Wouw, N., and Nijmeijer, H. *Uniform Output Regulation of Nonlinear Systems. A Convergent Dynamics Approach.* Birkhäuser, Boston, 2006.

[11] Rockafellar, R. T. and Wets, R.J-B. *Variational Analysis*. Springer, Berlin, 1998.

[12] Tanwani, A., Brogliato, B. and Prieur, Ch. *Stability and observer design for Lur'e systems with multivalued, non-monotone, time-varying nonlinearities and state jumps.* submitted (2014).